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Gelfand Pairs and Corwin-Greenleaf Multiplicity Function

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Abstract. Let (K, N) be a nilpotent Gelfand pair and let $G := K \ltimes N$ be the semidirect product associated with (K, N). Let $\pi \in \widehat{G}$ be a generic representation of G and let $\tau \in \widehat{K}$. The Kirillov-Lipsman's orbit method suggests that the multiplicity $m_\pi(\tau)$ of an irreducible K-module τ occurring in the restriction of $\pi|_K$ can be linked to (the number of K-orbits) the Corwin-Greenleaf multiplicity function (C.G.M.F for short). Under some assumptions on the pair (K, N), in this work we focus on the connection between the geometric number C.G.M.F and the multiplicity $(m_\pi(.))$. In the geometric counterpart we give a necessary and sufficient conditions associated with the C.G.M.F. Moreover, we prove that this function is bounded for a special class of subgroups of G.

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1. Introduction and background

Due to the broad scope of the topic touched on by this paper, we begin with a relatively extensive history and background section. Let G be a Lie group with Lie algebra $\mathfrak{g}$. It is well-known in representation theory that the set of all equivalence classes of strongly continuous irreducible unitary representations of G is called the *unitary dual* of G and is denoted by $\widehat{G}$. That is why we may simplify the notation a little writing $\pi \in \widehat{G}$ instead of $[\pi] \in \widehat{G}$ for its equivalence class. In the representation theory, specifying all irreducible unitary representations of G, namely, an explicit description of the unitary dual $\widehat{G}$ of G, is a fundamental problem. It would be difficult to solve this problem, whereas, this problem has been investigated for some cases. Here, briefly we recall the so-called *inner hull-kernel topology* (Fell's topology), which was introduced by Fell on $\mathcal{A}(G)$, the set of all equivalence classes of unitary representations of G ([14]). For each subsets S and T of $\mathcal{A}(G)$, then we say that S is weakly contained in T if

$$\bigcap_{\pi \in S} \ker \pi \supseteq \bigcap_{\tau \in T} \ker \tau.$$

Note that when S is weakly contained in T we write $S \prec T$. Now, for each $\pi, \rho \in \mathcal{A}(G)$, we write $\pi \prec \rho$, when π is weakly contained in ρ and $\pi \preceq \rho$, when π is contained (subrepresentation of) in ρ.

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For each finite set $\mathcal{V}$ of nonvoid open subset of $\widehat{G}$, let:

$$\mathcal{U}(\mathcal{V}) = \Big\{\pi \in \mathcal{A}(G) \mid \operatorname{supp}(\pi) \cap U \neq \emptyset \ \ \forall U \in \mathcal{V}\Big\},$$

where
$$\operatorname{supp}(\pi) := \{\tau \in \widehat{G} | \ \tau \prec \pi\}$$

denote the support of the representation π in $\widehat{G}$. Then $\mathcal{U}(\mathcal{V})$ form a basis for the inner hull-kernel topology and when restricted to $\widehat{G}$, it coincides with the *dual topology* (hull-kernel topology).

One can declare that one of the important problem related to the unitary dual $\widehat{G}$ of G in harmonic analysis (representation theory) is the so-called *orbit method.* This method (or orbit theory) is the way to describe (determine) unitary representations $\pi \in \widehat{G}$ using a geometric objet which the so-called *coadjoint orbit.* This algorithm (method), first developed by Alexandre Kirillov in the early 1960's. In the setting of nilpotent groups and in the special case of nilpotent, simply connected Lie groups G the orbit philosophy produces a nice picture between the unitary dual $\widehat{G}$ of G and its space of coadjoint orbits $\mathfrak{g}^*/G$. The orbit method was later extended to solvable groups by Bertram Kostant, Lajos Pukánszky, Jean-Marie Souriau and others. More recently, Michel Duflo, David Vogan and others have studied the case of reductive groups.

A direct problem related to the orbit theory is to show that the restriction of representations to subgroups can be reformulated in terms of its counterpart in the geometry of coadjoint orbits. Let us start with the representations multiplicity. To clarify the situation, let H be a subgroup of G (Lie group) and $\pi \in \widehat{G}$. Then the decomposition of the restriction representation $\pi|_H$ into the direct integral of irreducible unitary representations of H is given by:

$$\pi|_H \simeq \int_{\widehat{H}}^{\oplus} m_\pi(\tau)\tau d\mu(\tau),$$

where μ is a Borel measure on the spectrum $\widehat{H}$ of H and $m_\pi : \widehat{H} \longrightarrow \mathbb{N} \cup \{\infty\}$ is the multiplicity function. This decomposition is sometimes refer to *branching rule* of $\pi|_H$.

To the above problem (branching rule), we attach its geometric part. Corwin and Greenleaf have proved in the setting of connected and simply connected nilpotent Lie groups, that the above multiplicity $m_\pi(\tau)$ coincides almost everywhere with the "mod H" intersection number $\chi(\mathcal{O}^G, \mathcal{O}^H)$ defined by

$$\chi(\mathcal{O}^G, \mathcal{O}^H) := \sharp\Big(\big(\mathcal{O}^G \cap q^{-1}(\mathcal{O}^H)\big)/H\Big),$$

where $\mathcal{O}^G \subset \mathfrak{g}^*$ and $\mathcal{O}^H \subset \mathfrak{h}^*$ are the coadjoint orbits corresponding to $\pi \in \widehat{G}$ and $\tau \in \widehat{H}$, respectively (under the Kirillov-orbit method $\widehat{G} \simeq \mathfrak{g}^*/G$) and $q : \mathfrak{g}^* \longrightarrow \mathfrak{h}^*$ is the natural projection. The number of H-orbits in $\mathcal{O}^G \cap q^{-1}(\mathcal{O}^H)$ is sometimes referred as the Corwin-Greenleaf multiplicity function χ (see [11]). Moreover, recently Toshiyuki Kobayashi proved a geometric criterion in ([21]) for the bounded multiplicity property of *small* infinite-dimensional representations of real reductive Lie groups in both induction and restriction, as a refinement of the general criterion in Kobayashi-Oshima ([24]).

Our interest to this problem is motivated by recent multiplicity-free results in the orbit method obtained by Kobayashi and Nasrin (see [22],[23]), (also one can see [11], [29]). With the above discussions we are closely interested to this genre of problems in the setting of a class of semidirect products Lie groups. Results in this direction are given for some cases of semidirect products (see, [1], [2]).

Now we clarify to which class of semidirect products we are interested. Let N be a simply connected nilpotent Lie group, and let K be a compact subgroup of $\mathrm{Aut}(N)$, the group of authomorphisms of N. We say that the pair (K,N) is a nilpotent Gelfand pair (n.G.p) if $L^1_K(N)$, the algebra of integrable K-bi-invariant functions on N commutes under convolution. In this case N is necessarily two-step or abelian (see, [5]).

Throughout this paper, we assume that (K,N) is a n.G.p. with N two-step and $[\mathfrak{n},\mathfrak{n}]=\mathfrak{z}$, with $\mathfrak{z}$ which is the center of $\mathfrak{n}$. We fix a K-invariant inner product $\langle .,.\rangle_{\mathfrak{n}}$ on $\mathfrak{n}$ and write

$$\mathfrak{n}=\mathcal{V}\oplus\mathfrak{z},\ \ \mathcal{V}=\mathfrak{z}^{\perp}.$$

The subspaces $\mathcal{V}$ and $\mathfrak{z}$ are K-invariant and the Lie bracket amounts to an anti-symmetric bilinear mapping $\mathcal{V}\times\mathcal{V}\longrightarrow\mathfrak{z}$. From [15], we recall the following Definition.

Definition 1.1. A nilpotent Gelfand pair (K,N) is said to be *irreducible* if K acts irreducibly on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$. ■

Note that the condition $[\mathfrak{n},\mathfrak{n}]=\mathfrak{z}$ holds for irreducible n.G.p's. Now, letting

$$G:=K\ltimes N,$$

with Lie algebra $\mathfrak{g}=\mathfrak{k}\ltimes\mathfrak{n}\simeq\mathfrak{k}\oplus\mathfrak{n}$. In the spirit of the Kirillov-orbit method, R. Lipsman established a nice description (parametrization) of the unitary dual $\widehat{G}$ of G by a subspace $\mathfrak{g}^{\ddagger}/G$ (will be defined in section 4) of $\mathfrak{g}^*/G$, the space of coadjoint orbits of G (see, [26]). This parametrization

$$\widehat{G}\simeq\mathfrak{g}^{\ddagger}/G$$

we call it Kirillov-Lipsman orbit method (or Kirillov-Lipsman orbit mapping). The above orbit mapping is a homeomorphism in some cases of Lie groups ([3], [4], [12], [31], [30]). A more details about this orbit mapping will be given below in section 4. Using Mackey theory, then it is well-known in representation theory that each $\pi\in\widehat{G}$ is determined by a pair of parameters, which are so-called Mackey's parameters ([27], [28]) (see, section 3 for a full discussion). Let $\pi\in\widehat{G}$ be a *generic* representation of G (see, Definition 3.1) and let $\tau\in\widehat{K}$. To these representations we associate, respectively, the admissible coadjoint orbits $\mathcal{O}^G\subset\mathfrak{g}^*$ and $\mathcal{O}^K\subset\mathfrak{k}^*$ (via the Kirillov-Lipsman orbit method). As a first result in our work, we give a necessary and sufficient conditions to obtain a non-zero representation multiplicity ($m_\pi(\tau_\lambda)\neq 0$) (see, Theorem 3.2). In the second result, we show that in general, one has

$$\chi(\mathcal{O}^G,\mathcal{O}^K)\neq m_\pi(\tau)$$

(see, Theorem 4.5). Finally, for a spacial class of subgroups of G we show that the Corwin-Greenleaf multiplicity function is bounded under some assumptions (see, Theorem 4.6).

The rest of this paper is organized as follows. Section 2 summarizes background concerning the coadjoint orbits of the group $G = K \ltimes N$. The unitary dual $\widehat{G}$ of G is described using Mackey theory and the proof of Theorem 3.2 are given in section 3. In the last section, we recall a detailed analysis about the Kirillov-Lipsman orbit method and we give an explicit expression for the *generic* coadjoint orbit of G using the moment map associated to the so-called nilpotent orbit $\mathcal{O}^N$ of N. The main results on the Corwin-Greenleaf multiplicity function for the group G are derived.

2. Coadjoint orbits of $G = K \ltimes N$

We fix some of our notations that will be used throughout this paper.

- The script letter indicates the Lie algebra for the corresponding Lie group.
- Let (K, N) be a n.G.p. with N two-step such that $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{z}$.
- Let $k \cdot x$ denote the result of applying $k \in K$ to $x \in N$, and the derived action of K on $\mathfrak{n}$ is written $A \cdot X$ for $A \in K$ and $X \in \mathfrak{n}$.
- Elements of $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{n}$ will be written as $(U, (x, t))$ where $U \in \mathfrak{k}$ and $(x, t) \in \mathfrak{n} = \mathcal{V} \oplus \mathfrak{z}$.
- We write $\mathrm{Ad}_H, \mathrm{Ad}^*_H$, respectively, the adjoint and the coadjoint representations for a such Lie group H.

Here, we give some computations to obtain the coadjoint orbits of G. A direct computation, one obtains that the adjoint action of G is

$$\begin{aligned} \mathrm{Ad}_G(k, (x, t))(U, (a, z)) \;=\; & \Big(\mathrm{Ad}_K(k)U, k \cdot (a, z) - (\mathrm{Ad}_K(k)U) \cdot (x, t) \\ & + \; [(x, t), k \cdot (a, z)] - \frac{1}{2}[(x, t), (\mathrm{Ad}_K(k)U) \cdot (x, t)]\Big). \end{aligned}$$

Identify the Lie algebra $\mathfrak{n}$ with its vector dual space $\mathfrak{n}^*$ through the K-invariant inner product $\langle ., . \rangle_{\mathfrak{n}}$. The coadjoint actions of N and $\mathfrak{n}$ on $\mathfrak{n}^*$ are defined by

$$(\mathrm{Ad}^*_N(x, t)(a, z))(x', t') = (a, z)(\mathrm{Ad}_N\left((x, t)^{-1}\right)(x', t'))$$
$$(\mathrm{ad}^*_N(x, t)(a, z))(x', t') = -(a, z)(\mathrm{ad}_N(x, t)(x', t')).$$

Since N is 2-step, the identification of N with $\mathfrak{n}$ (under the exponential map $\exp : \mathfrak{n} \longrightarrow N$) allows us to write

$$\mathrm{Ad}^*_N(x, t)(a, z) \;=\; (a, z) + \mathrm{ad}^*_N(x, t)(a, z). \tag{1}$$

Then, each linear functional $\psi \in \mathfrak{g}^*$ can be identified with an element $(U, (x, t)) \in \mathfrak{g}$ such that

$$\psi(U', x', t') \;=\; \langle (U, (x, t)), (U', (x', t')) \rangle$$

for $(U', (x', t')) \in \mathfrak{g}$. Write points $\psi \in \mathfrak{g}^*$ as $\psi = (\nu, (a, z))$ where $\nu \in \mathfrak{k}^*$ and $(a, z) \in \mathfrak{n}^*$. That is

$$\psi(U', (x', t')) = \nu(U') + a(x') + z(t').$$

Following [7], we define a map $\times : \mathfrak{n} \times \mathfrak{n}^* \longrightarrow \mathfrak{k}^*$ by

$$\Big((x,t) \times (a,z)\Big)(U) := (a,z)(U \cdot (x,t)) = -\Big(U \cdot (a,z)\Big)(x,t)$$

for $U \in \mathfrak{k}, (x,t) \in \mathfrak{n}$ and $(a,z) \in \mathfrak{n}^*$. The map $\times : \mathfrak{n} \times \mathfrak{n}^* \longrightarrow \mathfrak{k}^*$ satisfies the equivariance property

$$\mathrm{Ad}_K^*(k)\Big((x,t) \times (a,z)\Big) = \Big(k \cdot (x,t)\Big) \times \Big(k \cdot (a,z)\Big).$$

Then the coadjoint action of G on $\mathfrak{g}^*$ is given by

$$\mathrm{Ad}_G^*(k,(x,t))(\nu,(a,z)) =$$
$$\Big(\mathrm{Ad}_K^*(k)\nu + (x,t) \times (k \cdot (a,z)) + \frac{1}{2}(x,t) \times \mathrm{ad}_N^*(x,t)(k \cdot (a,z)), \mathrm{Ad}_N^*(x,t)(k \cdot (a,z))\Big).$$

According to ([7]), one obtains the following description of the coadjoint orbits $\mathcal{O}^G_{(\nu,(a,z))}$ of G:

$$\mathcal{O}^G_{(\nu,(a,z))} =$$
$$\Big\{k \cdot \Big(\nu + (x,t) \times (a,z) + \frac{1}{2}(x,t) \times \mathrm{ad}_N^*(x,t)(a,z), \mathrm{Ad}_N^*(x,t)(a,z)\Big)\Big| k \in K, (x,t) \in \mathfrak{n}\Big\}$$

where
$$\begin{aligned} k \cdot \Big(\nu,(a,z)\Big) &:= \Big(\mathrm{Ad}_K^*(k)\nu, k \cdot (a,z)\Big) \\ &= \mathrm{Ad}_G^*(k,0)\Big(\nu,(a,z)\Big) \\ &=: \mathrm{Ad}_K^*(k)\Big(\nu,(a,z)\Big). \end{aligned}$$

In the next section, we give a full description of the unitary dual $\widehat{G}$ of G and we show a necessary and sufficient condition for the non-zero representations multiplicity.

3. Non-zero multiplicity for generic representations of $G = K \ltimes N$

We use the notations of the previous sections. We start this section by a detailed description for the unitary dual $\widehat{G}$ of G using Mackey's theory (see [27], [28]). Let $(\pi, \mathcal{H}_\pi) \in \widehat{N}$ be an unitary irreducible representation of N. Note that the group K acts on the unitary dual $\widehat{N}$ of N via

$$k \cdot \pi := \pi \circ k^{-1}$$

and there exists a unitary representation $W_\pi : K_\pi \longrightarrow U(\mathcal{H}_\pi)$ of the stabilizer K_π of π in K. The representation W_π intertwining $k \cdot \pi$ with π, i.e.

$$(k \cdot \pi)(x,t) = W_\pi(k^{-1})\pi(x,t)W_\pi(k)$$

for all $k \in K_\pi, (x,t) \in N$. Using the fact that (K,N) is a n.G.p, then the intertwining representation W_π is multiplicity free for each $\pi \in \widehat{N}$ (see [10]).

Let ρ be any unitary irreducible representation of K_π, Mackey theory ensures that

$$\pi_{(\rho,\pi)} := \mathrm{ind}_{K_\pi \ltimes N}^{K \ltimes N}\Big((k,(x,t)) \longmapsto \rho(k) \otimes \pi(x,t)W_\pi(k)\Big)$$

is an irreducible unitary representation of $G = K \ltimes N$ and that all irreducible unitary representations of G have this form (up to unitary equivalence).

Moreover, $\pi_{(\rho,\pi)} = \pi_{(\rho',\pi')}$ if and only if the pairs (ρ,π) and (ρ',π') *differ by the action of* K, in the sense that

$$\pi' = k_0 \cdot \pi,\ \rho' = k_0 \cdot \rho$$

for some $k_0 \in K$ where $(k_0 \cdot \rho)(k) := \rho(k_0^{-1} k k_0)$.

Let $\mathcal{O}_\pi^N \subset \mathfrak{n}^* = \mathcal{V}^* \oplus \mathfrak{z}^*$ denote the $\mathrm{Ad}_N^*(N)$-orbit of $\pi \in \widehat{N}$ via the Kirillov-orbit method (see [20]). Using the fact that N is a two-step nilpotent Lie group, then one has type I representations which are non-trivial on the center $Z := \exp(\mathfrak{z})$, for which $\mathcal{O}_\pi^N$ is an affine subspace of $\mathfrak{n}^*$. On the other hand the type II representations have Z in their kernel, and act as a characters on N/Z. The coadjoint orbit associated with type II representation is a single point. Then we write

$$\widehat{N} = \widehat{N}^I \cup \widehat{N}^{II}.$$

Let $\pi \in \widehat{N}^I$ be a type I representation and $\mathcal{O}_\pi^N$ be the corresponding coadjoint orbit via the Kirillov-orbit method (see [20]). Take any linear form ℓ in $\mathcal{O}_\pi^N$ and let,

$$\mathfrak{a}_\pi := \Big\{ x \in \mathcal{V} |\ \ell([x, \mathfrak{n}]) = 0 \Big\},\ \mathfrak{w}_\pi := \mathfrak{a}_\pi^\perp \cap \mathcal{V}.$$

These spaces do not depend on the choice of $\ell \in \mathcal{O}_\pi^N$ and according to [9], the coadjoint orbit $\mathcal{O}_\pi^N$ contains a unique point ℓ_π, which is *aligned* in the sense that $\ell_\pi|_{\mathfrak{w}_\pi} = 0$ (see Definition 3.1 in [9]). An important worth mentioning here is that this unique aligned point ℓ_π has the property that the stabilizer K_π of π is the stabilizer of ℓ_π in K.

According to (Definition 4.1 in [9]), we recall that the moment map $\Phi_\pi : \mathcal{O}_\pi^N \longrightarrow \mathfrak{k}_\pi^*$ associated with the coadjoint orbit $\mathcal{O}_\pi^N$ is given by:

$$\Phi_\pi(\mathrm{Ad}_N^*(x,t)\ell_\pi)(U) = -\frac{1}{2}\ell_\pi\big[(x,t), U \cdot (x,t)\big] \tag{2}$$

for $U \in \mathfrak{k}_\pi, (x,t) \in \mathfrak{n}$ (where ℓ_π is the unique aligned point in $\mathcal{O}_\pi^N$). Using the K-invariant inner product $\langle .,. \rangle_{\mathfrak{n}}$ (to identify $\mathfrak{n}$ with $\mathfrak{n}^*$), one can regard the aligned point ℓ_π $(\pi \in \widehat{N}^I)$ as an element in $\mathfrak{n}$. Letting $\ell_{(a_0,z_0)} := a_0 + z_0 \simeq (a_0, z_0), a_0 \in \mathcal{V}, z_0 \in \mathfrak{z}$. Since ℓ_π is aligned then $a_0 \in \mathfrak{a}_\pi$, and by observing that $\mathfrak{a}_\pi$ is determined by z_0, then the decomposition $\mathcal{V} = \mathfrak{a}_\pi \oplus \mathfrak{w}_\pi$ becomes

$$\mathcal{V} = \mathfrak{a}_{z_0} \oplus \mathfrak{w}_{z_0}.$$

Here, we turn our attention to the description of a so-called generic dual $(\widehat{G})_{gen} \subset \widehat{G}$ of G (will be defined below).

Definition 3.1. A representation $\pi_{(\rho,\pi)} \in \widehat{G}$ is called *generic* if the Mackey's parameter $(\rho,\pi) \in \widehat{K_\pi} \times \widehat{N}^I$. ■

So, the generic dual $(\widehat{G})_{gen}$ of G is defined as the set of all generic representations of G and in this work, we are interested to this class of (generic) representations of G (interesting case). More precisely, we write

$$(\widehat{G})_{gen} := \Big\{ \pi_{(\rho,\pi)} := \mathrm{ind}_{K_\pi \ltimes N}^{K \ltimes N}(\rho \otimes \pi \circ W_\pi) |\ \pi \in \widehat{N}^I, \rho \in \widehat{K_\pi} \Big\}.$$

Let $\ell_{(a_0,z_0)} = (a_0, z_0) \in \mathfrak{a}_{z_0} \oplus \mathfrak{z}$ be as above (an aligned point in the coadjoint orbit $\mathcal{O}^N_{(a_0,z_0)}$ of N passing through the linear form $\ell_{(a_0,z_0)}$. Let $H := K_{(a_0,z_0)}$ denote the stabilizer of $\ell_{(a_0,z_0)}$ in K and let ρ_μ be an unitary irreducible representation of H with highest weight μ. By the above analysis we show that the generic representations of G have the form:

$$\pi^\mu_{(a_0,z_0)} := \mathrm{ind}_{H\ltimes N}^{K\ltimes N}(\rho_\mu \otimes \pi_{(a_0,z_0)} \circ W_{(a_0,z_0)}),$$

where $a_0 \in \mathfrak{a}_{z_0}, 0 \neq z_0 \in \mathfrak{z}$ and $\rho_\mu \in \widehat{H}$ (here, $\pi_{(a_0,z_0)}$ is denoted in the sense $\pi_{(a_0,z_0)} \longleftrightarrow \ell_{(a_0,z_0)}$ via the Kirillov correspondence).

Theorem 3.2. *Let $\tau_\lambda \in \widehat{K}$ and $\pi^\mu_{(a_0,z_0)} \in (\widehat{G})_{gen}$. Then the following statements are equivalent:*

(1) $m_{\pi^\mu_{(a_0,z_0)}}(\tau_\lambda) \neq 0$.

(2) $\mathrm{supp}(\tau_\lambda|_H) \cap S^\mu_{(a_0,z_0)} \neq \emptyset$,

where $S^\mu_{(a_0,z_0)} := \mathrm{supp}(\rho_\mu \otimes W_{(a_0,z_0)})$.

Proof. Let us assume that $m_{\pi^\mu_{(a_0,z_0)}}(\tau_\lambda) \neq 0$. Then, one gets the following:

$$\tau_\lambda \preceq \pi^\mu_{(a_0,z_0)}|_K \simeq \mathrm{ind}_H^K(\rho_\mu \otimes W_{(a_0,z_0)}).$$

It follows that,
$$\tau_\lambda \preceq \sum_{\rho_\nu \in S^\mu_{(a_0,z_0)}} \mathrm{ind}_H^K(\rho_\nu).$$

Hence, we can conclude that there exists $\rho_\nu \in S^\mu_{(a_0,z_0)}$, such that,

$$\tau_\lambda \preceq \mathrm{ind}_H^K(\rho_\nu).$$

Equivalently,
$$\rho_\nu \preceq \tau_\lambda|_H.$$

(Frobenius reciprocity for compact groups). This shows that,

$$\mathrm{supp}(\tau_\lambda|_H) \cap S^\mu_{(a_0,z_0)} \neq \emptyset.$$

Conversely, let us suppose that,

$$\mathrm{supp}(\tau_\lambda|_H) \cap S^\mu_{(a_0,z_0)} \neq \emptyset.$$

This means that there exists $\rho_\nu \in \widehat{H}$, such that,

$$\tau_\lambda \preceq \mathrm{ind}_H^K(\rho_\nu) \tag{3}$$

and
$$\rho_\nu \preceq \rho_\mu \otimes W_{(a_0,z_0)}. \tag{4}$$

By the continuity of induction ($\rho \longmapsto \mathrm{ind}_H^K(\rho)$) for the inner-hull-kernel topology (see, [14]), then we show from (4), that the induced representation $\mathrm{ind}_H^K(\rho_\nu)$ is weakly contained in $\mathrm{ind}_H^K(\rho_\mu \otimes W_{(a_0,z_0)})$ and we write for this

$$\mathrm{ind}_H^K(\rho_\nu) \prec \mathrm{ind}_H^K(\rho_\mu \otimes W_{(a_0,z_0)}). \tag{5}$$

Comparing (3) and (5), we observe that τ_λ is weakly contained in the induced representation $\mathrm{ind}_H^K(\rho_\mu \otimes W_{(a_0,z_0)})$. Since τ_λ is an unitary irreducible representation of the compact Lie group K, then representation theory for compact Lie groups tells us that τ_λ is contained (appears or subrepresentation of) in $\mathrm{ind}_H^K(\rho_\mu \otimes W_{(a_0,z_0)})$ and we write

$$\tau_\lambda \quad \preceq \quad \mathrm{ind}_H^K(\rho_\mu \otimes W_{(a_0,z_0)}). \tag{6}$$

Now, recall that the restriction representation $\pi^\mu_{(a_0,z_0)}|_K$ is given by

$$\pi^\mu_{(a_0,z_0)}|_K \quad = \quad \mathrm{ind}_H^K(\rho_\mu \otimes W_{(a_0,z_0)}). \tag{7}$$

By the facts (6) and (7), we conclude that τ_λ is contained (is a subrepresentation of) in $\pi^\mu_{(a_0,z_0)}|_K$. Thus,

$$m_{\pi^\mu_{(a_0,z_0)}}(\tau_\lambda) \neq 0. \qquad \blacksquare$$

In the sequel of this paper, we will study the Corwin-Greenleaf multiplicity function $\chi(.,.)$ in the setting of a so-called *generic orbits* of G. Here, we give a necessary and sufficient condition to obtain $\chi(.,.) \neq 0$. Furthermore, we compare this geometric number (the Corwin-Greenleaf multiplicity function) with the representation multiplicity. This is the subject of the following last section.

4. Admissible coadjoint orbits and Corwin-Greenleaf multiplicity function

We continue to use the notations of the previous sections. In the spirit of the orbit method in representation theory, R. Lipsman established a bijection between a class of coadjoint orbits of $G = K \ltimes N$ and the unitary dual $\widehat{G}$ (see, [26]). For $\psi \in \mathfrak{g}^*$, let G_ψ be the stabilizer in G of ψ. The linear form ψ is called admissible if there exists a unitary character χ of the identity component of G_ψ such that $d\chi = i\psi|_{\mathfrak{g}_\psi}$. Let $\mathfrak{g}^\ddagger \subset \mathfrak{g}^*$ denote the set of all the admissible linear forms on $\mathfrak{g}$ and let $\mathfrak{g}^\ddagger/G$ denote the space of admissible coadjoint orbits of G. R. Lipsman established a nice parametrization of the unitary dual $\widehat{G}$ of G via $\mathfrak{g}^\ddagger/G$. More precisely, we have a bijection

$$\widehat{G} \simeq \mathfrak{g}^\ddagger/G.$$

Indeed: According to Lipsman (see, [10, p. 23]) (compare [26]), for every admissible linear form $\psi \in \mathfrak{g}^\ddagger$, we can construct an irreducible unitary representation π_ψ by holomorphic induction and every irreducible representation of G arises in this manner. Then we get a map from the set $\mathfrak{g}^\ddagger$ of the admissible linear forms onto the dual space $\widehat{G}$ of G. Note that π_ψ is equivalent to $\pi_{\psi'}$ if and only if ψ and ψ' are in the same G-orbit. That is, the orbit mapping (Lipsman's mapping)

$$\mathcal{K}^L : \mathfrak{g}^\ddagger/G \ni \mathcal{O}^G_\psi \longleftrightarrow \pi_\psi \in \widehat{G},$$

yields a bijection between admissible coadjoint orbits in $\mathfrak{g}^\ddagger$ and irreducible unitary representations of G.

Let $\rho_\mu \in \widehat{H}$ with highest weight μ and $0 \neq z_0 \in \mathfrak{z}, a_0 \in \mathfrak{a}_{z_0}$ be as above. Then we take the linear form $\psi^\mu_{(a_0,z_0)} := (\mu, a_0, z_0) \in \mathfrak{g}^*$ and let $\mathcal{O}^G_{(\mu,a_0,z_0)}$ denote the coadjoint orbit of G passing through the linear form $\psi^\mu_{(a_0,z_0)}$. From section 1, we recall that

$$\mathcal{O}^G_{(\mu,a_0,z_0)} = \left\{ \begin{array}{l} k\cdot\Big(\mu+(x,t)\times(a_0,z_0)+\frac{1}{2}\big((x,t)\times \mathrm{ad}^*_N(x,t)(a_0,z_0)\big), \mathrm{Ad}^*_N(x,t)(a_0,z_0)\Big) \\ \Big|\ k \in K,\ (x,t) \in N \end{array} \right\}.$$

We show that $\psi^\mu_{(a_0,z_0)}$ is admissible in the sense of Lipsman. Indeed: Let $G(\psi^\mu_{(a_0,z_0)})$, $K(\psi^\mu_{(a_0,z_0)})$ and $N(\psi^\mu_{(a_0,z_0)})$ denotes the stabilizers of $\psi^\mu_{(a_0,z_0)}$ in G, K and N, respectively. By a direct calculation we show that

$$G(\psi^\mu_{(a_0,z_0)}) = K(\psi^\mu_{(a_0,z_0)}) \ltimes N(\psi^\mu_{(a_0,z_0)}).$$

Then the linear form $\psi^\mu_{(a_0,z_0)}$ is admissible in the sense of Lipsman ([26], Lemma 4.2).

Let $\pi^\mu_{(a_0,z_0)} \in (\widehat{G})_{gen}$ be as before and let $\tau_\lambda \in \widehat{K}$ with highest weight λ. To these representations, we attach respectively the generic coadjoint orbits $\mathcal{O}^G_{(\mu,a_0,z_0)}$ and the coadjoint orbit $\mathcal{O}^K_\lambda$ via the Kirillov-Lipsman orbit method

$$\widehat{G} \simeq \mathfrak{g}^\ddagger/G.$$

The following Lemma has some of use in the sequel of this paper.

Lemma 4.1. *Let*
$$\mathfrak{h}^\circ := \Big\{\varphi \in \mathfrak{k}^* :\ \varphi|_{\mathfrak{h}} = 0\Big\}$$
be the annihilator of $\mathfrak{h}$ *in* $\mathfrak{k}^*$. *Then for each* φ *in* $\mathfrak{h}^\circ$, *there exists* $(x,t) \in \mathfrak{n} = \mathcal{V} \oplus \mathfrak{z}$, *such that*

$$\varphi = (x,t) \times (a_0, z_0).$$

Proof. Let $\Theta_{(a_0,z_0)} : \mathfrak{k} \longrightarrow \mathfrak{n}^*$ be the linear map defined by

$$\Theta_{(a_0,z_0)}(U) := -U \cdot (a_0, z_0)\ \ \forall U \in \mathfrak{k}.$$

We easily see that
$$\mathfrak{h} = \ker(\Theta_{(a_0,z_0)}).$$

Now, for $(x,t) \in \mathfrak{n}$, we express the element $(x,t) \times (a_0,z_0)$ in terms of the map $\Theta_{(a_0,z_0)}$. The dual $\Theta^*_{(a_0,z_0)} : \mathfrak{n} \longrightarrow \mathfrak{k}^*$ of $\Theta_{(a_0,z_0)}$ is given by the relation

$$\Theta^*_{(a_0,z_0)}(x,t)(U) = \Theta_{(a_0,z_0)}(U)(x,t) = -(U \cdot (a_0,z_0))(x,t) = (a_0,z_0)(U \cdot (x,t)).$$

It follows that $\Theta^*_{(a_0,z_0)}(x,t) = (x,t) \times (a_0,z_0)$ for all $(x,t) \in \mathfrak{n}$.

Let $\mathrm{q}_{(a_0,z_0)} : \mathfrak{k}^* \longrightarrow \mathfrak{h}^* = \mathfrak{k}^*_{(a_0,z_0)}$ be the projection map. Then we have

$$\mathfrak{h}^\circ = \ker(\mathrm{q}_{(a_0,z_0)}),$$

(where $\mathfrak{h}^\circ$ is the annihilator of $\mathfrak{h}$ in $\mathfrak{k}^*$). By the Lagrange multipliers we conclude our result. Indeed, for $\varphi \in C^\infty(\mathfrak{k})$, the element $U \in \mathfrak{h} = \Theta^{-1}_{(a_0,z_0)}(\{0\}) \subset \mathfrak{k}$ is a critical point of the map

$$\varphi|_{\mathfrak{h}} : \mathfrak{h} \longrightarrow \mathbb{R}$$

if and only if there exists $(x_0,t_0) \in \mathfrak{n}^{**} \simeq \mathfrak{n}$, such that U is a critical point of the map $\varphi - (x_0,t_0) \circ \Theta_{(a_0,z_0)}$.

In our setting when $\varphi : \mathfrak{k} \longrightarrow \mathbb{R}$ is a real linear form on $\mathfrak{k}$, we find that $\varphi \in \mathfrak{h}^\circ$ if and only if there exists $(x_0, t_0) \in \mathfrak{n}$, such that

$$\varphi = (x_0, t_0) \circ \Theta_{(a_0,z_0)} = \Theta^*_{(a_0,z_0)}(x_0, t_0) = (x_0, t_0) \times (a_0, z_0).$$

Hence the lemma is proven. ■

Let K_{z_0} be the stabilizer of the aligned point $\ell_{z_0} := (0, z_0) \in \mathfrak{n}^*$ in K and let

$$\Phi_{z_0} : \mathcal{O}^N_{z_0} := \mathcal{O}^N_{(0,z_0)} \longrightarrow \mathfrak{k}^*_{z_0}$$

be the moment map associated with the coadjoint orbit $\mathcal{O}^N_{z_0}$ of N passing through the aligned linear form $(0, z_0)$ under the action of K_{z_0} (here $(0, z_0) \longleftrightarrow \ell_{(0,z_0)}$ under the identification $\mathfrak{n} \simeq \mathfrak{n}^*$).

Now, we are able to address the following result.

Theorem 4.2. *Let $\lambda \in \mathfrak{k}^*$ and $\mu \in \mathfrak{k}^*_{z_0}$ (any two linear forms). Then the following statements are equivalent:*

- $\chi(\mathcal{O}^G_{(\mu,z_0)}, \mathcal{O}^K_\lambda) \neq 0$.
- $\mathcal{C}_{(\lambda,\mu)} := \Big\{\nu \in \mathrm{Image}(\Phi_{z_0}) \,|\, \mu + \nu \in \mathrm{q}_{z_0}(\mathcal{O}^K_\lambda)\Big\}$ *is a non-empty subset in* $\mathfrak{k}^*_{z_0}$, *where* $\mathrm{q}_{z_0} : \mathfrak{k}^* \longrightarrow \mathfrak{k}^*_{z_0}$ *is the projection map.*

Proof. Let us assume that $\mathcal{C}_{(\lambda,\mu)} \neq \emptyset$. Then there exist $\nu_0 \in \mathrm{Image}(\Phi_{z_0})$ and $k_0 \in K$, such that

$$\mu + \nu_0 \;\; = \;\; \mathrm{q}_{z_0}(\mathrm{Ad}^*_K(k_0)\lambda).$$

Lemma 4.1, tells us that there exists $t_0 \in \mathfrak{z}$ such that

$$\mu + (t_0 \times z_0) + \nu_0 = \mathrm{Ad}^*_K(k_0)\lambda. \tag{8}$$

Now, let us clarify the following important fact below. From (2), one can write

$$\begin{aligned}\Phi_{z_0}\big(\mathrm{Ad}^*_N(x)(0, z_0)\big)(U) &:= -\frac{1}{2}(0, z_0)\big([x, U \cdot x]\big)\\ &= -\frac{1}{2}z_0\big([x, U \cdot x]\big) = \frac{1}{2}\Big((x, t) \times \mathrm{ad}^*_N(x, t)(0, z_0)\Big)(U) =: \frac{1}{2}(x \bullet x)(U)\end{aligned}$$

for all $U \in \mathfrak{k}_{z_0}$. Then we observe that

$$\begin{aligned}\Phi_{z_0}\big(\mathrm{Ad}^*_N(x)(0, z_0)\big) &= \frac{1}{2}\Big((x, t) \times \mathrm{ad}^*_N(x, t)(0, z_0)\Big)\Big|_{\mathfrak{h}} && (9)\\ &= \frac{1}{2}\,\mathrm{q}_{z_0}\Big((x, t) \times \mathrm{ad}^*_N(x, t)(0, z_0)\Big) && (10)\\ &= \frac{1}{2}\,\mathrm{q}_{z_0}\Big((x \bullet x)\Big), && (11)\end{aligned}$$

where $\mathrm{q}_{z_0} : \mathfrak{k}^* \longrightarrow \mathfrak{k}^*_{z_0}$ denote the naturel projection. Here, we recall that

$$\mathcal{O}^G_{(\mu,z_0)} = \Big\{k\cdot\big(\mu + \underbrace{(x, t) \times (0, z_0)}_{=:t\times z_0} + \frac{1}{2}(x \bullet x), \mathrm{Ad}^*_N(x, t)(0, z_0)\big) \,\big|\, k\in K,\ (x, t)\in N\Big\}.$$

Now, using the fact that $\nu_0 \in \mathrm{Image}(\Phi_{z_0})$, then from (9), we may assume that there exists $x_0 \in \mathcal{V}$ such that

$$\nu_0 = \Phi_{z_0}(\mathrm{Ad}_N^*(x_0)(0,z_0)) = \mathrm{q}_{z_0}\big(\frac{1}{2}(x_0 \bullet x_0)\big) = (t_1 \times z_0) + \frac{1}{2}(x_0 \bullet x_0)$$

for some $t_1 \in \mathfrak{z}$. Finally, the fact (8), gives us:

$$\mu + \Big((t_0 + t_1) \times z_0\Big) + \frac{1}{2}(x_0 \bullet x_0) = \mu + (t_0 \times z_0) + \nu_0 = \mathrm{Ad}_K^*(k_0)\lambda.$$

It follows that $$\mathcal{O}^G_{(\mu,z_0)} \cap \mathrm{pr}^{-1}(\mathcal{O}^K_\lambda) \neq \emptyset,$$

This implies that, $$\chi(\mathcal{O}^G_{(\mu,z_0)}, \mathcal{O}^K_\lambda) \neq 0$$

and this is what we wanted.

Conversely, if $\chi(\mathcal{O}^G_{(\mu,z_0)}, \mathcal{O}^K_\lambda) \neq 0$. Then there exist $k_0 \in K$ and $(x_0, t_0) \in \mathfrak{n} \simeq N$, such that

$$\mu + t_0 \times z_0 + \frac{1}{2}(x_0 \bullet x_0) = \mathrm{Ad}_K^*(k_0)\lambda.$$

It follows that $$\mu + \mathrm{q}_{z_0}\left(\frac{1}{2}(x_0 \bullet x_0)\right) = \mathrm{q}_{z_0}(\mathrm{Ad}_K^*(k_0)\lambda),$$

since $\mathrm{q}_{z_0}(t_0 \times z_0) \equiv 0$. By observing that

$$\mathrm{q}_{z_0}\left(\frac{1}{2}(x_0 \bullet x_0)\right) = \Phi_{z_0}\Big(\mathrm{Ad}_N^*(x_0)(0,z_0)\Big),$$

we deduce that there exists $\nu \in \mathrm{Image}(\Phi_{z_0})$, such that

$$\mu + \nu \in \mathrm{q}_{z_0}(\mathcal{O}^K_\lambda).\ i.e;\ \mathcal{C}_{(\lambda,\mu)} \neq \emptyset.$$

This completes the proof. ∎

A fundamental result due to I. M. Gelfand ([17]) implies that (K, N) is a Gelfand pair if and only if each irreducible unitary representation π of $G = K \ltimes N$ has at most a one dimensional space of K-fixed vectors, so that the multiplicity of the trivial representation 1_K of K in $\pi|_K$ is 0 or 1. Hence it is reasonable to give the following result.

Theorem 4.3. *There exists* $(a,z) \in \mathfrak{n}$ $((a,z) \longleftrightarrow \ell_{(a,z)} \in \mathfrak{n}^*$ *aligned form), such that*

$$m_{\pi^0_{(a,z)}}(1_K) = 0,$$

where $$\pi^0_{(a,z)} := \mathrm{ind}^G_{K_{(a,z)} \ltimes N}(1_{K_{(a,z)}} \otimes \pi_{(a,z)} W_{(a,z)}) \in \widehat{G}.$$

Proof. According to the above notations, let $\pi_{(a,z)} \in \widehat{N}^I$ and ρ be any irreducible representation of the stabilizer $K_{(a,z)}$ of (a,z) in K. Then Mackey's theory tells us that $\tilde{\pi}_{(\rho,(a,z))} := \rho \otimes \pi_{(a,z)} W_{(a,z)}$ is an irreducible representation of $K_{(a,z)} \ltimes N$ and the induced representation

$$\pi_{(\rho,(a,z))} := \mathrm{ind}^{K \ltimes N}_{K_{(a,z)} \ltimes N}(\tilde{\pi}_{(\rho,(a,z))})$$

is irreducible for $G = K \ltimes N$ (see, [28]).

Furthermore, the restriction representation $\pi_{(\rho,(a,z))}|_K$ of $\pi_{(\rho,(a,z))}$ to the subgroup K is given by

$$\pi_{(\rho,(a,z))}|_K \simeq \mathrm{ind}_{K_{(a,z)}}^{K}(\tilde{\pi}_{(\rho,z)}|_{K_{(a,z)}}) \simeq \mathrm{ind}_{K_{(a,z)}}^{K}(\rho \otimes W_{(a,z)}).$$

By applying Frobenius reciprocity for compact groups, one gets

$$m_{\mathrm{ind}_{K_{(a,z)}}^{K}(\rho\otimes W_{(a,z)})}(1_K) = m_{\rho\otimes W_{(a,z)}}(1_{K_{(a,z)}}).$$

Since $1_{K_{(a,z)}}$ has multiciplicity 1 in $\rho \otimes \overline{\rho}$ and multiplicity 0 in $\rho \otimes \rho^{'}$ for $\rho^{'}$ not equivalent to $\overline{\rho}$, the conjugate representation for ρ. Then we deduce that

$$\begin{aligned} m_{\pi_{(\rho,(a,z))}}(1_K) &= m_{\mathrm{ind}_{K_{(a,z)}}^{K}(\rho\otimes W_{(a,z)})}(1_K) && (12)\\ &= m_{W_{(a,z)}}(\overline{\rho}). && (13)\end{aligned}$$

Now, from the fact that (K,N) is a Gelfand pair, then G. Carcanno ([10]) tells us that $W_{(a,z)}$ is a multiplicity free representation of $K_{(a,z)}$ (see also [5]). To conclude it suffices to take $(a_0,z_0) \in \mathfrak{n} \simeq \mathfrak{n}^*$, such that the trivial representation $1_{K_{(a_0,z_0)}}$ of $K_{(a_0,z_0)}$ is not a subrepresentation of $W_{(a_0,z_0)}$ and we put:

$$\pi^0_{(a_0,z_0)} := \mathrm{ind}_{K_{(a_0,z_0)}\ltimes N}^{K\ltimes N}(1_{K_{(a_0,z_0)}} \otimes \pi_{(a_0,z_0)} W_{(a_0,z_0)}).$$

From the fact (12), one can conclude that $m_{\pi^0_{(a_0,z_0)}}(1_K) = 0$.

This completes our proof. ■

Remark 4.4. Let $(a_0,z_0) \in \mathfrak{n}$ be as in the above proof and let $\mathcal{O}^G_{(0,a_0,z_0)}$ be the coadjoint orbit of G associated to the unitary irreducible representation

$$\pi^0_{(a_0,z_0)} = \mathrm{ind}_{K_{(a_0,z_0)}\ltimes N}^{K\ltimes N}(1_{K_{(a_0,z_0)}} \otimes \pi_{(a_0,z_0)} W_{(a_0,z_0)})$$

of G (given as above) via the Kirillov-Lipsman's orbit method. By a direct calculation one can see that

$$\chi(\mathcal{O}^G_{(0,a_0,z_0)}, \{0\}) \neq 0,$$

here, $\{0\}$ is the coadjoint orbit associated to the trivial representation 1_K. ■

Combining Theorem 4.3 and Remark 4.4, the following result is immediate.

Theorem 4.5. *In general, one has* $m_\pi(\tau) \neq \chi(\mathcal{O}^G_\pi, \mathcal{O}^K_\tau)$, *where* $\mathcal{O}^G_\pi$ *and* $\mathcal{O}^K_\tau$ *are the coadjoint orbits of* G *associated to* $\pi \in \widehat{G}$ *and* $\tau \in \widehat{K}$, *respectively.*

Let $(a_0,z_0) \in \mathfrak{n}^*$ be an aligned form in $\mathfrak{n}^*$ (as before) and we put $H := K_{(a_0,z_0)}$ and $G^{'} := H \ltimes N$. Let $\mathcal{O}^{G^{'}}_{(\varphi,a_0,z_0)}$ passing through the linear form

$$(\varphi,(a_0,z_0)) \in (\mathfrak{g}^{'})^* \simeq \mathfrak{h}^* \oplus \mathfrak{n}^*.$$

In the setting of this class of groups we will show that the Corwin-Greenleaf multiplicity function is bounded under some assumption on the linear form $\varphi \in \mathfrak{h}^*$.

Theorem 4.6. *Let φ be a central element of $\mathfrak{h}^*$. Then*

$$\chi(\mathcal{O}^{G'}_{(\varphi,a_0,z_0)}, \mathcal{O}^{H}_{\nu}) \leq 1,$$

for each coadjoint orbit $\mathcal{O}^H_\nu \subset \mathfrak{h}^*$.

Proof. First of all, we observe that the coadjoint orbit $\mathcal{O}^{G'}_{(\varphi,a_0,z_0)}$ through the linear form $(\varphi,(a_0,z_0)) \in (\mathfrak{g}')^* \simeq \mathfrak{h}^* \oplus \mathfrak{n}^*$ is given by

$$\mathcal{O}^{G'}_{(\varphi,a_0,z_0)} := \Big\{k \cdot \Big(\varphi + \frac{1}{2}\,\mathrm{q}_{(a_0,z_0)}\big((v \bullet v)\big), \mathrm{Ad}^*_N(v)(a_0,z_0)\Big) \,|\, k \in H, v \in \mathcal{V}\Big\},$$

since we have $\mathrm{q}_{(a_0,z_0)}\big((v,z) \times (a_0,z_0)\big) = (v,z) \times (a_0,z_0)|_{\mathfrak{h}} = 0$ for all $(v,z) \in \mathfrak{n}$, where $H = K_{(a_0,z_0)}$ and $G' = H \ltimes N$ (as given before). Now, let us assume that $\chi(\mathcal{O}^{G'}_{(\varphi,a_0,z_0)}, \mathcal{O}^H_\nu) \neq 0$ and let any tow elements

$$\Psi = k \cdot \Big(\varphi + \frac{1}{2}\,\mathrm{q}_{(a_0,z_0)}\big((v \bullet v)\big), \mathrm{Ad}^*_N(v)(a_0,z_0)\Big) \in \mathcal{O}^{G'}_{(\varphi,a_0,z_0)} \cap \mathrm{pr}^{-1}(\mathcal{O}^H_\nu)$$

and

$$\Psi' = h \cdot \Big(\varphi + \frac{1}{2}\,\mathrm{q}_{(a_0,z_0)}\big((w \bullet w)\big), \mathrm{Ad}^*_N(w)(a_0,z_0)\Big) \in \mathcal{O}^{G'}_{(\varphi,a_0,z_0)} \cap \mathrm{pr}^{-1}(\mathcal{O}^H_\nu),$$

for some $k,h \in H$ and $v,w \in \mathcal{V}$, (here,

$$\mathrm{pr} : (\mathfrak{g}')^* \longrightarrow \mathfrak{h}^* \quad \text{and} \quad \mathrm{q}_{(a_0,z_0)} : \mathfrak{k}^* \longrightarrow \mathfrak{h}^* = \mathfrak{k}^*_{(a_0,z_0)}$$

are the projections maps). It follows that both $\mathrm{pr}(\Psi)$ and $\mathrm{pr}(\Psi')$ are contained in the same H-orbit $\mathcal{O}^H_\nu$. Therefore, we obtain:

$$\frac{1}{2}\,\mathrm{q}_{(a_0,z_0)}\big((v \bullet v)\big), \frac{1}{2}\,\mathrm{q}_{(a_0,z_0)}\big((w \bullet w)\big) \in \mathcal{O}^H_{\nu-\varphi},$$

since φ is a central element of $\mathfrak{h}^*$. From the fact (9), one gets

$$\Phi_{(a_0,z_0)}(\mathrm{Ad}^*_N(v)(a_0,z_0)), \Phi_{(a_0,z_0)}(\mathrm{Ad}^*_N(w)(a_0,z_0)) \in \mathcal{O}^H_{\nu-\varphi}.$$

Using the fact that the moment map $\Phi_{(a_0,z_0)}$ is one-to-one on H-orbits (see [9], Lemma 4.3), then we deduce that the vectors $\mathrm{Ad}^*_N(v)(a_0,z_0)$ and $\mathrm{Ad}^*_N(w)(a_0,z_0)$ are contained in the same H-orbit in $\mathcal{V}$, i.e., there exists $h_0 \in H$, such that $\mathrm{Ad}^*_N(w)(a_0,z_0) = h_0 \cdot \mathrm{Ad}^*_N(v)(a_0,z_0)$. One can conclude that the H-action on $\mathcal{O}^{G'}_{(\varphi,a_0,z_0)} \cap \mathrm{pr}^{-1}(\mathcal{O}^H_\nu)$ is transitive and hence

$$\chi(\mathcal{O}^{G'}_{(\varphi,a_0,z_0)}, \mathcal{O}^H_\nu) = 1.$$

This finishes our proof. ∎

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