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Local Properties of the Schrödinger Algebra in $(n+1)$-Dimensional Space-Time

Xiaomin Tang*, **Peng Wang**

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Abstract. We investigate local properties of the Schrödinger algebra in $(n+1)$-dimensional space-time of Schrödinger Lie groups. Specifically, for any positive integer n, it initiates the study of 2-local derivations of this Lie algebra, denoted by $\mathcal{S}_n$. The main result establishes that every 2-local derivation on $\mathcal{S}_n$ is actually a derivation.

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Key Words: Schrödinger algebra, derivation, 2-local derivation.

1. Introduction

It is well known that the derivation algebra and automorphism group of an algebra A are crucial in understanding the structure of A. As generalizations of derivations and automorphisms, the study of local derivations and automorphisms was initiated by Kadison [8] and Larson, Sourour [9]. They assumed that these mappings are also linear. Dropping the assumption of linearity, Šemrl [12] introduced the concept of 2-local derivations. The notion of 2-local derivations is indeed important and interesting for an algebra. The central problem in this area is to determine all 2-local derivations and examine whether they are necessarily (global) derivations. In this paper, we mainly investigate related problems for Lie algebras.

All 2-local derivations on several important classes of Lie algebras have been determined. We first review some results on 2-local derivations of Lie algebras. In [2], the authors prove that every 2-local derivation on a semi-simple Lie algebra $\mathcal{L}$ is a derivation, and that each finite-dimensional nilpotent Lie algebra of dimension greater than two admits a 2-local derivation that is not a derivation. In [4], the authors study 2-local derivations on some infinite-dimensional Lie algebras; specifically, they prove that all 2-local derivations on both the Witt algebra and the positive Witt algebra are (global) derivations, and provide an example of an infinite-dimensional Lie algebra that admits a 2-local derivation which is not a derivation. In [3, 21], the authors prove that every 2-local derivation on certain generalized Witt algebras (or their Borel subalgebras) is a derivation. In [13], the author proves that every 2-local derivation on the W-algebra $W(2,2)$ is a derivation. In [19], the authors prove

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that every 2-local derivation on the Jacobson-Witt algebras is a derivation. In [6], the authors prove that every 2 -Local derivation on the planar Galilean conformal algebra is a derivation. In particular, recently in [15] and [14] we prove that every 2-local derivation on the Schrödinger algebra $\mathcal{S}_n$ in $(n+1)$-dimensional space-time of Schrödinger Lie groups is a derivation for $n=1$ and $n=2$, respectively. However, for general $n \geq 3$, the problem becomes more complicated and remains open. The present paper focuses on the general case for any positive integer n.

The Schrödinger Lie group describes the symmetries of the free particle Schrödinger equation, see [11]. For any positive integer n, the Lie algebra $\mathcal{S}_n$ in $(n+1)$-dimensional space-time of the Schrödinger Lie group is called the Schrödinger algebra; see [7]. The Schrödinger algebra $\mathcal{S}_n$ is a non-semisimple Lie algebra that also plays an important role in theoretical physics. Recently, there has been a series of papers studying the structure and representation theory of the Schrödinger algebra $\mathcal{S}_1$ in $(1+1)$-dimensional space-time, see [1, 17, 7, 20, 18, 5]. Representations and (bi)derivations over $\mathcal{S}_n$ were studied in [10, 16].

First, let us recall the definition of the Schrödinger algebra $\mathcal{S}_n$ in $(n+1)$-dimensional space-time by [7, 10] as follows. Throughout this paper, we denote by $\mathbb{C}$ the set of all complex numbers. More precisely, we have

Definition 1.1. The *Schrödinger algebra* $\mathcal{S}_n$ is a Lie algebra with a $\mathbb{C}$-basis

$$\{e, f, h, z, x_i, y_i, s_{jk}(=-s_{kj}) \mid 1 \leqslant i \leqslant n, 1 \leqslant j < k \leqslant n\}$$

equipped with the following non-trivial commutation relations

$$\begin{aligned}
&[e,f]=h, \quad [h,e]=2e, \quad [f,h]=2f,\\
&[x_i,y_i]=z, \quad [h,x_i]=x_i, \quad [h,y_i]=-y_i,\\
&[e,y_i]=x_i, \quad [f,x_i]=y_i,\\
&[s_{jk},x_i]=\delta_{ki}x_j-\delta_{ji}x_k, \quad [s_{jk},y_i]=\delta_{ki}y_j-\delta_{ji}y_k,\\
&[s_{jk},s_{lm}]=\delta_{lk}s_{jm}+\delta_{jm}s_{kl}+\delta_{mk}s_{lj}+\delta_{lj}s_{mk},
\end{aligned}$$

where δ is the Kronecker Delta defined as 1 for $i=j$ and as 0 otherwise. ■

It is readily apparent that the Schrödinger algebra $\mathcal{S}_n$ constitutes a finite-dimensional Lie algebra of dimension $\frac{n^2+3n+8}{2}$ that is neither semisimple nor solvable. Rather, it constitutes the semidirect product Lie algebra $\mathcal{S}_n = (\mathfrak{sl}_2 \oplus \mathfrak{so}_n) \ltimes \mathfrak{h}_n$ where $\mathfrak{sl}_2 = \mathrm{Span}_{\mathbb{C}}\{e,f,h\}$ is the 3-dimensional simple Lie algebra,

$$\mathfrak{so}_n = \mathrm{Span}_{\mathbb{C}}\{s_{jk} \mid 1 \leqslant j < k \leqslant n\}$$

is the orthogonal Lie algebra and $\mathfrak{h}_n = \mathrm{Span}_{\mathbb{C}}\{z, x_i, y_i \mid 1 \leqslant i \leqslant n\}$ is the Heisenberg Lie algebra. For further details, the reader is referred to [10].

In this paper, we shall investigate 2-local derivations of the Schrödinger algebra in $(n+1)$-dimensional space-time. We prove that every 2-local derivation on $\mathcal{S}_n$ constitutes a derivation for any integer n.

2. Preliminaries

In this section, we recall some definitions and known results for studying 2-local derivations on Schrödinger algebra $\mathcal{S}_n$.

A linear transformation D on a Lie algebra L is called a *derivation* on L if D satisfies

$$D([\mu,\nu]) = [D(\mu),\nu] + [\mu, D(\nu)], \quad \forall \mu,\nu \in L.$$

The set of all derivations on L is denoted by $\mathrm{Der}(L)$. Clearly, $\mathrm{Der}(L)$ is a vector space. For any $\mu \in L$, it is easy to see that the linear transformation $\mathrm{ad}\mu$ on L is a derivation on L where

$$\mathrm{ad}\mu(\nu) = [\mu,\nu], \quad \forall \nu \in L.$$

Such derivations of this form are called *inner derivations*, and the set of all inner derivations on L is denoted by $\mathrm{Ider}(L)$. Clearly, $\mathrm{Ider}(L)$ is a subspace of $\mathrm{Der}(L)$. Furthermore, we have the following.

Lemma 2.1. (see [16]) *For any integer $n > 2$, one has $\mathrm{Der}(\mathcal{S}_n) = \mathrm{Ider}(\mathcal{S}_n) \oplus \mathbb{C}\sigma$ where σ is an outer derivation of $\mathcal{S}_n$ determined by*

$$\sigma(z) = z,\ \sigma(x_i) = \frac{1}{2}x_i,\ \sigma(y_i) = \frac{1}{2}y_i,\ \sigma(e) = \sigma(f) = \sigma(h) = \sigma(s_{jk}) = 0,$$

$$(1 \leqslant i \leqslant n,\ 1 \leqslant j < k \leqslant n).$$

Note that $\mathrm{ad}z = 0$. By Lemma 2.1 one has the following lemma.

Lemma 2.2. *For any integer $n > 2$, we have*

$$\mathrm{Der}(\mathcal{S}_n) = \mathrm{Span}_{\mathbb{C}}\{\mathrm{ad}e, \mathrm{ad}f, \mathrm{ad}h, \mathrm{ad}x_i, \mathrm{ad}y_i, \mathrm{ad}s_{jk}, \sigma \mid 1 \leqslant i \leqslant n, 1 \leqslant j < k \leqslant n\}.$$

Definition 2.3. Let L be a Lie algebra. A map (not linear in general) $\Delta : L \longrightarrow L$ is called a 2-local derivation on L if for every $\mu,\nu \in L$, there exists a derivation $D_{\mu,\nu}$ (depending on μ,ν) on L such that $\Delta(\mu) = D_{\mu,\nu}(\mu)$ and $\Delta(\nu) = D_{\mu,\nu}(\nu)$. ■

Let Δ be a 2-local derivation on L, then for any $t \in \mathbb{C}$ and $\mu \in L$ we have

$$\Delta(t\mu) = D_{\mu,t\mu}(t\mu) = tD_{\mu,t\mu}(\mu) = t\Delta(\mu).$$

For any $\mu,\nu \in \mathcal{S}_n$, by Lemma 2.2 and above Definition 2.3, we always assume that

$$\begin{aligned} D_{\mu,\nu} = a_e(\mu,\nu)\mathrm{ad}e + a_f(\mu,\nu)\mathrm{ad}f + a_h(\mu,\nu)\mathrm{ad}h + \sum_{i=1}^{n}(a_i(\mu,\nu)\mathrm{ad}x_i \\ +a_{-i}(\mu,\nu)\mathrm{ad}y_i) + \sum_{1\leqslant j<k\leqslant n} a_{j,k}(\mu,\nu)\mathrm{ad}s_{jk} + a(\mu,\nu)\sigma \end{aligned} \tag{1}$$

where $a_e, a_f, a_h, a_i, a_{-i}, a_{j,k}$ and a are complex-valued functions on $\mathcal{S}_n \times \mathcal{S}_n$. For the sake of convenience, we shall omit the subscripts (μ,ν) after them for the most part in what follows, except where doing so risks causing confusion.

3. 2-Local derivations on the Schrödinger algebra

In this section, we will prove that every 2-local derivation on the Schrödinger algebra $\mathcal{S}_n$ is a derivation. To obtain this result, we first have to prove a few lemmas.
For $n > 2$ and $\mu \in \mathcal{S}_n$, by Definition 1.1 we always denote

$$\mu = \mu_e e + \mu_f f + \mu_h h + \mu_z z + \sum_{i=1}^{n}(\mu_i x_i + \mu_{-i} y_i) + \sum_{1\leqslant j<k\leqslant n} \mu_{j,k} s_{jk}, \tag{2}$$

for some $\mu_e, \mu_f, \mu_h, \mu_z, \mu_i, \mu_{-i}, \mu_{j,k} \in \mathbb{C}$.

Lemma 3.1. *Let Δ be a 2-local derivation on $\mathcal{S}_n$ $(n>2)$. For any $\mu \in \mathcal{S}_n$, we have the following results.*

- *If $\Delta(e)=0$, then*
$$D_{e,\mu} = a_e \mathrm{ad} e + \sum_{i=1}^{n} a_i \mathrm{ad} x_i + \sum_{1\leqslant j<k\leqslant n} a_{j,k} \mathrm{ad} s_{jk} + a\sigma; \tag{3}$$
- *If $\Delta(f)=0$, then*
$$D_{f,\mu} = a_f \mathrm{ad} f + \sum_{i=1}^{n} a_{-i} \mathrm{ad} y_i + \sum_{1\leqslant j<k\leqslant n} a_{j,k} \mathrm{ad} s_{jk} + a\sigma; \tag{4}$$
- *If $\Delta(h)=0$, then*
$$D_{h,\mu} = a_h \mathrm{ad} h + \sum_{1\leqslant j<k\leqslant n} a_{j,k} \mathrm{ad} s_{jk} + a\sigma; \tag{5}$$
- *If $\Delta(z)=0$, then*
$$D_{z,\mu} = a_e \mathrm{ad} e + a_f \mathrm{ad} f + a_h \mathrm{ad} h + \sum_{i=1}^{n} (a_i \mathrm{ad} x_i + a_{-i} \mathrm{ad} y_i) + \sum_{1\leqslant j<k\leqslant n} a_{j,k} \mathrm{ad} s_{jk}; \tag{6}$$
- *If $\Delta(x_\alpha)=0$ with $\alpha \in \{1,2,...,n\}$, then*
$$D_{x_\alpha,\mu} = a_e \mathrm{ad} e + a_h \mathrm{ad} h + \sum_{i=1}^{n} a_i \mathrm{ad} x_i + \sum_{\substack{1\leqslant i\leqslant n\\ i\neq\alpha}} a_{-i} \mathrm{ad} y_i + \sum_{\substack{1\leqslant j<k\leqslant n\\ j,k\neq\alpha}} a_{j,k} \mathrm{ad} s_{jk} - 2a_h\sigma; \tag{7}$$
- *If $\Delta(f+x_\alpha)=0$ with $\alpha \in \{1,2,...,n\}$, then*
$$D_{f+x_\alpha,\mu} = a_f \mathrm{ad}(f+x_\alpha) + \sum_{\substack{1\leqslant i\leqslant n\\ i\neq\alpha}} a_{-i} \mathrm{ad} y_i + \sum_{\substack{1\leqslant j<k\leqslant n\\ j,k\neq\alpha}} a_{j,k} \mathrm{ad} s_{jk}; \tag{8}$$
- *If $\Delta(h+z+s_{\iota\beta})=0$ when $\iota \in \{1,2,...,n-1\}$ and $\beta \in \{\iota+1,\iota+2,...,n\}$, then*
$$D_{h+z+s_{\iota\beta},\mu} = a_h \mathrm{ad} h + \sum_{\substack{1\leqslant j<k\leqslant n\\ j,k\neq\iota,\beta}} a_{j,k} \mathrm{ad} s_{jk} + a_{\iota,\beta} \mathrm{ad} s_{\iota\beta}, \tag{9}$$

where $a_e, a_f, a_h, a_i, a_{-i}, a_{j,k}$ and a are given by (1).

Proof. By (1) we have

$$\begin{aligned}\Delta(e) &= D_{e,\mu}(e)\\ &= [a_e e + a_f f + a_h h + \textstyle\sum_{i=1}^{n}(a_i x_i + a_{-i} y_i) + \displaystyle\sum_{1\leqslant j<k\leqslant n} a_{j,k} s_{jk}, e] + a\sigma(e)\\ &= -a_f h + 2a_h e - \textstyle\sum_{i=1}^{n} a_{-i} x_i.\end{aligned}$$

If $\Delta(e)=0$, then from the above equation we know that

$$a_f = a_h = a_{-i} = 0\,(1\leqslant i\leqslant n).$$

This, together with (1), implies that (3) holds. Similarly, we can obtain conclusions (4)–(7). We next prove conclusions (8) and (9) as follows.

If $\Delta(f+x_\alpha)=0$ for any integer $\alpha\,(1\leqslant\alpha\leqslant n)$, by taking $\mu=f+x_\alpha,\nu=\mu$ in (1) we have

$$\begin{aligned}\Delta(f+x_\alpha)&=D_{f+x_\alpha,\mu}(f+x_\alpha)\\&=[a_e e+a_f f+a_h h+\textstyle\sum_{i=1}^n(a_i x_i+a_{-i}y_i)\\&\qquad+\displaystyle\sum_{1\leqslant j<k\leqslant n}a_{j,k}s_{jk},f+x_\alpha]+a\sigma(f+x_\alpha)\\&=a_e h-2a_h f-a_{-\alpha}z+(\tfrac{1}{2}a+a_h)x_\alpha+(a_f-a_\alpha)y_\alpha\\&\qquad+\textstyle\sum_{i=1}^{\alpha-1}(a_{i,\alpha}x_i-a_i y_i)-\sum_{i=\alpha+1}^n(a_{\alpha,i}x_i+a_i y_i).\end{aligned}$$

Then by $\Delta(f+x_\alpha)=0$ we know that

$$\begin{gathered}a_e=a_h=a_{-\alpha}=a=0,\quad a_\alpha=a_f,\\a_i=0\,(1\leqslant i\leqslant n,i\neq\alpha),\\a_{j,\alpha}=0\,(1\leqslant j\leqslant\alpha-1),\\a_{\alpha,k}=0\,(\alpha+1\leqslant k\leqslant n).\end{gathered}$$

Thus, we obtain (8).

Finally, if $\Delta(h+z+s_{\iota\beta})=0$ for two positive integers $\iota\,(1\leqslant\iota\leqslant n-1)$ and $\beta\,(\iota+1\leqslant\beta\leqslant n)$, by taking $\mu=h+z+s_{\iota\beta},\nu=\mu$ in (1) we have

$$\begin{aligned}\Delta(h+z+s_{\iota\beta})&=D_{h+z+s_{\iota\beta},\mu}(h+z+s_{\iota\beta})\\&=[a_e e+a_f f+a_h h+\textstyle\sum_{i=1}^n(a_i x_i+a_{-i}y_i)\\&\qquad+\displaystyle\sum_{1\leqslant j<k\leqslant n}a_{j,k}s_{jk},h+z+s_{\iota\beta}]+a\sigma(h+z+s_{\iota\beta})\\&=2a_f f-2a_e e-\textstyle\sum_{i=1}^n a_i x_i-a_\beta x_\iota+a_\iota x_\beta+\sum_{i=1}^n a_{-i}y_i\\&\qquad-a_{-\beta}y_\iota+a_{-\iota}y_\beta+az+\textstyle\sum_{j=1}^{\iota-1}a_{j,\iota}s_{j\beta}-\sum_{k=\beta+1}^n a_{\beta,k}s_{\iota k}\\&\qquad+\displaystyle\sum_{\substack{1\leqslant j\leqslant\beta-1\\ j\neq\iota}}a_{j,\beta}s_{\iota j}+\sum_{\substack{\iota+1\leqslant k\leqslant n\\ k\neq\beta}}a_{\iota,k}s_{\beta k}.\end{aligned}$$

Then by $\Delta(h+z+s_{\iota\beta})=0$ we know that

$$\begin{gathered}a_f=a_e=a=a_i=a_{-i}=0\,(1\leqslant i\leqslant n),\\a_{j,\iota}=0\,(1\leqslant j\leqslant\iota-1),\\a_{\beta,k}=0\,(\beta+1\leqslant k\leqslant n),\\a_{j,\beta}=0\,(1\leqslant j\leqslant\beta-1,j\neq\iota),\\a_{\iota,k}=0\,(\iota+1\leqslant k\leqslant n,k\neq\beta)\end{gathered}$$

which yields (9). The proof is complete. ∎

Lemma 3.2. *Let Δ be a 2-local derivation on $\mathcal{S}_n(n>2)$ with the property $\Delta(e)=\Delta(f)=0$. Then $\Delta(h)=0$.*

Proof. In view of $\Delta(e)=\Delta(f)=0$, take $\mu=h$ in (3) and (4) respectively, then we have

$$\Delta(h)=D_{e,h}(h)=-2a_e e-\sum_{i=1}^n a_i x_i$$

and

$$\Delta(h)=D_{f,h}(h)=2a_f f+\sum_{i=1}^n a_{-i}y_i.$$

Therefore, the above two equations imply that $\Delta(h)=0$. ∎

Lemma 3.3. *Let Δ be a 2-local derivation on $\mathcal{S}_n (n > 2)$ with the property that $\Delta(e) = \Delta(f) = 0$. For any $\mu \in \mathcal{S}_n$, we have*

$$\Delta(\mu) = \vartheta_z z + \sum_{i=1}^{n}(\vartheta_i x_i + \vartheta_{-i} y_i) + \sum_{1\leqslant j<k\leqslant n} \vartheta_{jk} s_{jk}, \tag{10}$$

where $\vartheta_z, \vartheta_i, \vartheta_{-i}, \vartheta_{jk}$ are complex numbers depending only on μ.

Proof. Since $\Delta(e) = \Delta(f) = 0$, it follows by Lemma 3.2 that $\Delta(h) = 0$. For any but fixed $\mu \in \mathcal{S}_n$ given by (2), then by (3)-(5) we have

$$\begin{aligned}\Delta(\mu) &= D_{e,\mu}(\mu)\\ &= [a_e e + \textstyle\sum_{i=1}^n a_i x_i + \sum\limits_{1\leqslant j<k\leqslant n} a_{j,k} s_{jk}, \mu] + a\sigma(\mu)\\ &= *e + *h + *z + \textstyle\sum_{i=1}^n *x_i + \sum_{i=1}^n *y_i + \sum\limits_{1\leqslant j<k\leqslant n} *s_{jk},\end{aligned} \tag{11}$$

$$\begin{aligned}\Delta(\mu) &= D_{f,\mu}(\mu)\\ &= [a_f f + \textstyle\sum_{i=1}^n a_{-i} y_i + \sum\limits_{1\leqslant j<k\leqslant n} a_{j,k} s_{jk}, \mu] + a\sigma(\mu)\\ &= *f + *h + *z + \textstyle\sum_{i=1}^n *x_i + \sum_{i=1}^n *y_i + \sum\limits_{1\leqslant j<k\leqslant n} *s_{jk}\end{aligned} \tag{12}$$

and

$$\begin{aligned}\Delta(\mu) &= D_{h,\mu}(\mu)\\ &= [a_h h + \sum\limits_{1\leqslant j<k\leqslant n} a_{j,k} s_{jk}, \mu] + a\sigma(\mu)\\ &= *e + *f + \vartheta_z z + \textstyle\sum_{i=1}^n (\vartheta_i x_i + \vartheta_{-i} y_i) + \sum\limits_{1\leqslant j<k\leqslant n} \vartheta_{jk} s_{jk},\end{aligned} \tag{13}$$

where the symbol $*$ represents some complex number and $\vartheta_z, \vartheta_i, \vartheta_{-i}, \vartheta_{jk}$ $(1 \leqslant i \leqslant n$, $1 \leqslant j < k \leqslant n)$ are complex numbers depending only on μ.

Thus, by (11)–(13) we see that (10) holds. ■

Lemma 3.4. *Let Δ be a 2-local derivation on $\mathcal{S}_n (n > 2)$ with the property that $\Delta(e) = \Delta(f) = \Delta(z) = 0$. Then, for any $\mu \in \mathcal{S}_n$, we have*

$$D_{h+z,\mu} = a_h \mathrm{ad} h + \sum_{1\leqslant j<k\leqslant n} a_{j,k} \mathrm{ad} s_{jk} \tag{14}$$

where $a_h, a_{j,k}$ are given by (1).

Proof. In view of $\Delta(e) = \Delta(f) = 0$, by Lemma 3.2 one has $\Delta(h) = 0$. Taking $\mu = h + z$ in (5) we have

$$\begin{aligned}\Delta(h+z) &= D_{h,h+z}(h+z)\\ &= [a_h h + \sum\limits_{1\leqslant j<k\leqslant n} a_{j,k} s_{jk}, h+z] + a\sigma(h+z)\\ &= az.\end{aligned} \tag{15}$$

Since $\Delta(z) = 0$, by taking $\mu = h + z$ in (6) we have

$$\begin{aligned}\Delta(h+z) &= D_{z,h+z}(h+z)\\ &= [a_e e + a_f f + a_h h + \textstyle\sum_{i=1}^n (a_i x_i + a_{-i} y_i) + \sum\limits_{1\leqslant j<k\leqslant n} a_{j,k} s_{jk}, h+z]\\ &= 2a_f f - 2a_e e + \textstyle\sum_{i=1}^n (a_{-i} y_i - a_i x_i).\end{aligned}$$

This, together with (15), yields $$\Delta(h+z)=0. \tag{16}$$

Now by taking $\mu=h+z, \nu=\mu$ in (1) one has

$$\begin{aligned}\Delta(h+z)&=D_{h+z,\mu}(h+z)\\&=[a_e e+a_f f+a_h h+\textstyle\sum_{i=1}^n(a_i x_i+a_{-i}y_i)\\&\quad+\sum_{1\leqslant j<k\leqslant n}a_{j,k}s_{jk}, h+z]+a\sigma(h+z)\\&=2a_f f-2a_e e+az+\textstyle\sum_{i=1}^n(a_{-i}y_i-a_i x_i),\end{aligned}$$

which together with (16) implies $a_e=a_f=a=a_i=a_{-i}=0\,(1\leqslant i\leqslant n)$. Therefore, by (1) we obtain (14). ■

Lemma 3.5. *Let Δ be a 2-local derivation on $\mathcal{S}_n (n>2)$ such that*

$$\Delta(e)=\Delta(f)=\Delta(z)=\Delta(f+x_\alpha)=0 \;\; \alpha=1,2,...,n.$$

Then for any $\mu\in\mathcal{S}_n$, we have

$$D_{e+y_\alpha,\mu}=a_e\mathrm{ad}(e+y_\alpha)+\sum_{\substack{1\leqslant i\leqslant n\\ i\neq\alpha}}a_i\mathrm{ad}x_i+\sum_{\substack{1\leqslant j<k\leqslant n\\ j,k\neq\alpha}}a_{j,k}\mathrm{ad}s_{jk}, \quad \alpha=1,2,...,n \tag{17}$$

where $a_e, a_i, a_{j,k}$ are given in (1).

Proof. By taking $\mu=e+y_\alpha$ in (8) for $\alpha\in\{1,\cdots,n\}$ we have

$$\begin{aligned}\Delta(e+y_\alpha)&=D_{f+x_\alpha,e+y_\alpha}(e+y_\alpha)\\&=[a_f(f+x_\alpha)+\sum_{i=1}^{\alpha-1}a_{-i}y_i+\sum_{i=\alpha+1}^{n}a_{-i}y_i+\sum_{\substack{1\leqslant j<k\leqslant n\\ j,k\neq\alpha}}a_{j,k}s_{jk}, e+y_\alpha]\\&=a_f z-a_f h-\sum_{i=1}^{\alpha-1}a_{-i}x_i-\sum_{i=\alpha+1}^{n}a_{-i}x_i.\end{aligned} \tag{18}$$

Since $\Delta(e)=\Delta(f)=\Delta(z)=0$, by Lemma 3.4 we know that (14) holds. By taking $\mu=e+y_\alpha$ in (14) for $\alpha\in\{1,\cdots,n\}$ one has

$$\begin{aligned}\Delta(e+y_\alpha)&=D_{h+z,e+y_\alpha}(e+y_\alpha)\\&=[a_h h+\sum_{1\leqslant j<k\leqslant n}a_{j,k}s_{jk}, e+y_\alpha]\\&=2a_h e+\textstyle\sum_{i=1}^{\alpha-1}a_{i,\alpha}y_i-a_h y_\alpha-\sum_{i=\alpha+1}^{n}a_{\alpha,i}y_i.\end{aligned}$$

This, together with (18), yields

$$\Delta(e+y_\alpha)=0. \tag{19}$$

By taking $\mu=e+y_\alpha, \nu=\mu$ in (1) we have

$$\begin{aligned}\Delta(e+y_\alpha)&=D_{e+y_\alpha,\mu}(e+y_\alpha)\\&=[a_e e+a_f f+a_h h+\textstyle\sum_{i=1}^n(a_i x_i+a_{-i}y_i)\\&\quad+\sum_{1\leqslant j<k\leqslant n}a_{j,k}s_{jk}, e+y_\alpha]+a\sigma(e+y_\alpha)\\&=2a_h e-a_f h+a_\alpha z+(a_e-a_{-\alpha})x_\alpha+(\tfrac{1}{2}a-a_h)y_\alpha\\&\quad+\textstyle\sum_{i=1}^{\alpha-1}(a_{i,\alpha}y_i-a_{-i}x_i)-\sum_{i=\alpha+1}^{n}(a_{-i}x_i+a_{\alpha,i}y_i).\end{aligned}$$

Then by (19) we deduce

$$\begin{gathered} a_f = a_h = a_\alpha = a = 0, \quad a_{-\alpha} = a_e \\ a_{-i} = 0\,(1 \leqslant i \leqslant n, i \neq \alpha), \\ a_{j,\alpha} = 0\,(1 \leqslant j \leqslant \alpha - 1), \\ a_{\alpha,k} = 0\,(\alpha + 1 \leqslant k \leqslant n). \end{gathered}$$

Therefore, by (1) we get (17). The proof is completed. ∎

Lemma 3.6. *Let Δ be a 2-local derivation on $\mathcal{S}_n(n > 2)$ such that*

$$\Delta(e) = \Delta(f) = \Delta(z) = \Delta(f + x_\alpha) = \Delta(h + z + s_{\iota\beta}) = 0$$

for all $\alpha = 1, 2, ..., n$; $\iota = 1, 2, ..., n - 1$; $\beta = \iota + 1, \iota + 2, ..., n$. Then $\Delta(\mu) = 0$ for any $\mu \in \mathcal{S}_n$.

Proof. Let μ be an arbitrary but fixed element of $\mathcal{S}_n$ as given in (2). In view of $\Delta(e) = \Delta(f) = \Delta(z) = \Delta(f + x_\alpha) = \Delta(h + z + s_{\iota\beta}) = 0$ for $\alpha = 1, 2, ..., n$; $\iota = 1, 2, ..., n - 1$; $\beta = \iota + 1, \iota + 2, ..., n$, then by (14), (8), (17) and (9) we have

$$\begin{aligned} \Delta(\mu) &= D_{h+z,\mu}(\mu) \\ &= [a_h h + \sum_{1 \leqslant j < k \leqslant n} a_{j,k} s_{jk}, \mu] \\ &= *e + *f + \textstyle\sum_{i=1}^n *x_i + \sum_{i=1}^n *y_i + \displaystyle\sum_{1 \leqslant j < k \leqslant n} *s_{jk}, \end{aligned} \tag{20}$$

$$\begin{aligned} \Delta(\mu) &= D_{f+x_\alpha,\mu}(\mu) \\ &= [a_f(f + x_\alpha) + \textstyle\sum_{i=1}^{\alpha-1} a_{-i} y_i + \sum_{i=\alpha+1}^n a_{-i} y_i + \displaystyle\sum_{\substack{1 \leqslant j < k \leqslant n \\ j,k \neq \alpha}} a_{j,k} s_{jk}, \mu] \\ &= 2a_f \mu_h f - a_f \mu_h x_\alpha + \sum_{\substack{1 \leqslant i \leqslant n \\ i \neq \alpha}} *x_i + *h + *z + \textstyle\sum_{i=1}^n *y_i + \displaystyle\sum_{1 \leqslant j < k \leqslant n} *s_{jk}, \end{aligned} \tag{21}$$

$$\begin{aligned} \Delta(\mu) &= D_{e+y_\alpha,\mu}(\mu) \\ &= [a_e(e + y_\alpha) + \textstyle\sum_{i=1}^{\alpha-1} a_i x_i + \sum_{i=\alpha+1}^n a_i x_i + \displaystyle\sum_{\substack{1 \leqslant j < k \leqslant n \\ j,k \neq \alpha}} a_{j,k} s_{jk}, \mu] \\ &= -2a_e \mu_h e + a_e \mu_h y_\alpha + \sum_{\substack{1 \leqslant i \leqslant n \\ i \neq \alpha}} *y_i + *h + *z + \textstyle\sum_{i=1}^n *x_i + \displaystyle\sum_{1 \leqslant j < k \leqslant n} *s_{jk}, \end{aligned} \tag{22}$$

and

$$\begin{aligned} \Delta(\mu) &= D_{h+z+s_{\iota\beta},\mu}(\mu) \\ &= [a_h h + \sum_{\substack{1 \leqslant j < k \leqslant n \\ j,k \neq \iota,\beta}} a_{j,k} s_{jk} + a_{\iota,\beta} s_{\iota\beta}, \mu] \\ &= *e + *f + \textstyle\sum_{i=1}^n *x_i + \sum_{i=1}^n *y_i + \displaystyle\sum_{\substack{1 \leqslant j < k \leqslant n \\ j \neq \iota,\, k \neq \beta}} *s_{jk} + \\ &\quad + \sum_{\substack{\iota+1 \leqslant k \leqslant n \\ k \neq \beta}} *s_{\iota k} + \sum_{\substack{1 \leqslant j \leqslant \beta - 1 \\ j \neq \iota}} *s_{j\beta} \end{aligned} \tag{23}$$

where the symbol $*$'s represent some complex numbers.

From (20) and Lemma 3.3, we get

$$\Delta(\mu) = \sum_{i=1}^n (\vartheta_i x_i + \vartheta_{-i} y_i) + \sum_{1 \leqslant j < k \leqslant n} \vartheta_{jk} s_{jk}.$$

This, together with (21), deduces $a_f\mu_h = 0$. Thereupon we obtain

$$\Delta(\mu) = \sum_{i=1}^{n} \vartheta_{-i} y_i + \sum_{1\leqslant j<k\leqslant n} \vartheta_{jk} s_{jk}.$$

By (22) and the above equation we have $a_e\mu_h = 0$. Then we obtain

$$\Delta(\mu) = \sum_{1\leqslant j<k\leqslant n} \vartheta_{jk} s_{jk}.$$

This, together with (23), yields that $\Delta(\mu) = \sum_{1\leq j<k\leq n} \vartheta_{jk} s_{jk}$ does not have an $s_{\iota\beta}$ term. Let ι, β go through the possible values, then we see that $\Delta(\mu) = 0$. The proof is complete. ■

Lemma 3.7. *Let Δ be a 2-local derivation on $\mathcal{S}_n (n > 2)$ with the property that $\Delta(e) = \Delta(f) = \Delta(x_1) = 0$, then $\Delta(z) = 0$.*

Proof. Since $\Delta(x_1) = 0$, by taking $\mu = f + x_1$ in (7) we have

$$\begin{aligned}\Delta(f + x_1) &= D_{x_1, f+x_1}(f + x_1)\\ &= [a_e e + a_h h + \textstyle\sum_{i=1}^{n} a_i x_i + \sum_{i=2}^{n} a_{-i} y_i\\ &\quad + \sum_{1<j<k\leqslant n} a_{j,k} s_{jk}, f + x_1] - 2a_h \sigma(f + x_1)\\ &= a_e h - 2a_h f - \textstyle\sum_{i=1}^{n} a_i y_i.\end{aligned}$$

In view of $\Delta(e) = \Delta(f) = 0$, by Lemma 3.3 and the above equation we know that

$$\Delta(f + x_1) = -\sum_{i=1}^{n} a_i y_i. \tag{24}$$

By Lemma 3.2 we have $\Delta(h) = 0$. By taking $\mu = f + x_1$ in (5) we obtain

$$\begin{aligned}\Delta(f + x_1) &= D_{h, f+x_1}(f + x_1)\\ &= [a_h h + \sum_{1\leqslant j<k\leqslant n} a_{j,k} s_{jk}, f + x_1] + a\sigma(f + x_1)\\ &= -2a_h f + (a_h + \tfrac{1}{2}a)x_1 - \textstyle\sum_{k=2}^{n} a_{1,k} x_k.\end{aligned}$$

This with (24) implies $\Delta(f + x_1) = 0$. Then by (8) one has

$$\begin{aligned}\Delta(z) &= D_{f+x_1, z}(z)\\ &= [a_f(f + x_1) + \textstyle\sum_{i=2}^{n} a_{-i} y_i + \sum_{1<j<k\leqslant n} a_{j,k} s_{jk}, z]\\ &= 0.\end{aligned}$$

The proof is completed. ■

Lemma 3.8. *Let Δ be a 2-local derivation on $\mathcal{S}_n (n > 2)$ with the property that $\Delta(e) = \Delta(f) = \Delta(x_\alpha) = 0$ for any $\alpha = 1, 2, ..., n$. Then $\Delta(\mu) = 0$ for all $\mu \in \mathcal{S}_n$.*

Proof. The proof of this lemma is divided into three steps.

Step 1. $\Delta(f+x_\alpha)=0$ for all $\alpha=1,2,...,n$.
Since $\Delta(x_\alpha)=0$, by taking $\mu=f+x_\alpha$ in (7) we have

$$\begin{aligned}\Delta(f+x_\alpha)&=D_{x_\alpha,f+x_\alpha}(f+x_\alpha)\\&=[a_e e+a_h h+\textstyle\sum_{i=1}^n a_i x_i+\displaystyle\sum_{\substack{1\leqslant i\leqslant n\\ i\neq\alpha}} a_{-i}y_i\\&\qquad+\sum_{\substack{1\leqslant j<k\leqslant n\\ j,k\neq\alpha}} a_{j,k}s_{jk},f+x_\alpha]-2a_h\sigma(f+x_\alpha)\\&=a_e h-2a_h f-\textstyle\sum_{i=1}^n a_i y_i.\end{aligned}$$

In view of $\Delta(e)=\Delta(f)=0$, then by Lemma 3.3 and the above equation we know that

$$\Delta(f+x_\alpha)=-\sum_{i=1}^n a_i y_i. \tag{25}$$

It follows from Lemma 3.2 that $\Delta(h)=0$. By letting $\mu=f+x_\alpha$ in (5) we obtain

$$\begin{aligned}\Delta(f+x_\alpha)&=D_{h,f+x_\alpha}(f+x_\alpha)\\&=[a_h h+\sum_{1\leqslant j<k\leqslant n} a_{j,k}s_{jk},f+x_\alpha]+a\sigma(f+x_\alpha)\\&=-2a_h f+(a_h+\tfrac{1}{2}a)x_\alpha+\textstyle\sum_{j=1}^{\alpha-1} a_{j,\alpha}x_j-\sum_{k=\alpha+1}^n a_{\alpha,k}x_k.\end{aligned}$$

This, together with (25), gives $\Delta(f+x_\alpha)=0$.
Step 2. For all $\iota=1,2,...,n-1$ and $\beta=\iota+1,\iota+2,...,n$ we have

$$\Delta(h+z+s_{\iota\beta})=\sum_{j=1}^{\iota-1} a_{j,\iota}s_{j\beta}-\sum_{k=\beta+1}^n a_{\beta,k}s_{\iota k}+\sum_{\substack{1\leqslant j\leqslant\beta-1\\ j\neq\iota}} a_{j,\beta}s_{\iota j}+\sum_{\substack{\iota+1\leqslant k\leqslant n\\ k\neq\beta}} a_{\iota,k}s_{\beta k}. \tag{26}$$

By Lemma 3.7 we know that $\Delta(z)=0$. This, together with $\Delta(e)=\Delta(f)=0$, by using Lemma 3.4 with taking $\mu=h+z+s_{\iota\beta}$ in (14) for $\iota=1,2,...,n-1$ and $\beta=\iota+1,\iota+2,...,n$, gives

$$\begin{aligned}\Delta(h+z+s_{\iota\beta})&=D_{h+z,h+z+s_{\iota\beta}}(h+z+s_{\iota\beta})\\&=[a_h h+\sum_{1\leqslant j<k\leqslant n} a_{j,k}s_{jk},h+z+s_{\iota\beta}],\end{aligned}$$

and therefore we obtain (26).

Step 3. $\Delta(h+z+s_{\iota\beta})=0$ for all $\iota=1,2,...,n-1$; $\beta=\iota+1,\iota+2,...,n$.
By Step 1, $\Delta(f+x_\alpha)=0$ for all $\alpha=1,2,...,n$. By letting $\mu=h+z+s_{\iota\beta}$ and $\alpha=\iota,\beta$ in (8) respectively, we have

$$\begin{aligned}\Delta(h+z+s_{\iota\beta})&=D_{f+x_\iota,h+z+s_{\iota\beta}}(h+z+s_{\iota\beta})\\&=[a_f(f+x_\iota)+\sum_{\substack{1\leqslant i\leqslant n\\ i\neq\iota}} a_{-i}y_i+\sum_{\substack{1\leqslant j<k\leqslant n\\ j,k\neq\iota}} a_{j,k}s_{jk},h+z+s_{\iota\beta}]\\&=2a_f f-a_f x_\iota+a_f x_\beta-a_{-\beta}y_\iota+\sum_{\substack{1\leqslant i\leqslant n\\ i\neq\iota}} a_{-i}y_i\\&\quad-\textstyle\sum_{k=\beta+1}^n a_{\beta,k}s_{\iota k}+\displaystyle\sum_{\substack{1\leqslant j\leqslant\beta-1\\ j\neq\iota}} a_{j,\beta}s_{\iota j};\end{aligned} \tag{27}$$

$$\begin{aligned}\Delta(h+z+s_{\iota\beta}) &= D_{f+x_\beta,h+z+s_{\iota\beta}}(h+z+s_{\iota\beta})\\ &= [a_f(f+x_\beta)+\sum_{\substack{1\leqslant i\leqslant n\\ i\neq\beta}} a_{-i}y_i+\sum_{\substack{1\leqslant j<k\leqslant n\\ j,k\neq\beta}} a_{j,k}s_{jk}, h+z+s_{\iota\beta}]\\ &= 2a_f f-a_f x_\beta-a_f x_\iota+a_{-\iota}y_\beta+\sum_{\substack{1\leqslant i\leqslant n\\ i\neq\beta}} a_{-i}y_i\\ &\quad+\textstyle\sum_{j=1}^{\iota-1} a_{j,\iota}s_{j\beta}+\sum_{\substack{\iota+1\leqslant k\leqslant n\\ k\neq\beta}} a_{\iota,k}s_{\beta k}.\end{aligned} \tag{28}$$

Combining (26)–(28) we know that

$$\Delta(h+z+s_{\iota\beta})=-\sum_{k=\beta+1}^{n} a_{\beta,k}s_{\iota k}+\sum_{\substack{1\leqslant j\leqslant\beta-1\\ j\neq\iota}} a_{j,\beta}s_{\iota j}=\sum_{j=1}^{\iota-1} a_{j,\iota}s_{j\beta}+\sum_{\substack{\iota+1\leqslant k\leqslant n\\ k\neq\beta}} a_{\iota,k}s_{\beta k},$$

which implies that $\Delta(h+z+s_{\iota\beta})$ does not have an $s_{\iota\beta}$ term, and furthermore $\Delta(h+z+s_{\iota\beta})=0$ since $\iota\neq\beta$.

Finally, Lemma 3.6 with conclusions of Steps 1-3 gives $\Delta(\mu)=0$ for all $\mu\in\mathcal{S}_n$. ■

Now we state our main theorem as follows.

Theorem 3.9. *Every 2-local derivation on the Schrödinger algebra $\mathcal{S}_n$ is a derivation.*

Proof. Since the conclusion for $n=1,2$ are already given by [14, 15], so we assume that $n>2$ below. Let Δ be a 2-local derivation on $\mathcal{S}_n$. Then there exists a derivation $D_{e,f}$ on $\mathcal{S}_n$ such that $\Delta(e)=D_{e,f}(e)$ and $\Delta(f)=D_{e,f}(f)$. Set $\Delta_1=\Delta-D_{e,f}$. Then Δ_1 is a 2-local derivation on $\mathcal{S}_n$ such that $\Delta_1(e)=\Delta_1(f)=0$. Thus, by Lemma 3.2 we know that $\Delta_1(h)=0$.

By (5) we have

$$\begin{aligned}\Delta_1(x_1) &= D_{h,x_1}(x_1)\\ &= a_h(h,x_1)\mathrm{ad}h(x_1)+\sum_{1\leqslant j<k\leqslant n} a_{j,k}(h,x_1)\mathrm{ad}s_{jk}(x_1)+a(h,x_1)\sigma(x_1)\\ &= a_h(h,x_1)x_1+a(h,x_1)\sigma(x_1)-\textstyle\sum_{i=2}^{n} a_{1,i}(h,x_1)x_i\\ &= (2a_h(h,x_1)+a(h,x_1))\sigma(x_1)+\textstyle\sum_{i=2}^{n} a_{1,i}(h,x_1)\mathrm{ad}s_{1i}(x_1).\end{aligned} \tag{29}$$

Set $\Delta_2=\Delta_1-(2a_h(h,x_1)+a(h,x_1))\sigma-\sum_{i=2}^{n} a_{1,i}(h,x_1)\mathrm{ad}s_{1i}$. Then Δ_2 is also a 2-local derivation. By (29) we easily get $\Delta_2(x_1)=0$ and

$$\Delta_2(e)=\Delta_1(e)-(2a_h(h,x_1)+a(h,x_1))\sigma(e)-\textstyle\sum_{i=2}^{n} a_{1,i}(h,x_1)\mathrm{ad}s_{1i}(e)=0,$$
$$\Delta_2(f)=\Delta_1(f)-(2a_h(h,x_1)+a(h,x_1))\sigma(f)-\textstyle\sum_{i=2}^{n} a_{1,i}(h,x_1)\mathrm{ad}s_{1i}(f)=0.$$

Now we first prove the following claim.

Claim. *Suppose that ς is an integer with $2\leqslant\varsigma\leqslant n-1$. Let Δ_ς be a 2-local derivation on $\mathcal{S}_n$ such that $\Delta_\varsigma(e)=\Delta_\varsigma(f)=\Delta_\varsigma(x_1)=\Delta_\varsigma(x_2)=\cdots=\Delta_\varsigma(x_{\varsigma-1})=0$. Set $\Delta_{\varsigma+1}=\Delta_\varsigma-\sum_{i=\varsigma+1}^{n} a_{\varsigma,i}(x_1,x_\varsigma)\mathrm{ad}s_{\varsigma i}$. Then $\Delta_{\varsigma+1}$ is also a 2-local derivation and it satisfies $\Delta_{\varsigma+1}(e)=\Delta_{\varsigma+1}(f)=\Delta_{\varsigma+1}(x_1)=\Delta_{\varsigma+1}(x_2)=\cdots=\Delta_{\varsigma+1}(x_\varsigma)=0$.*

Clearly, $\Delta_{\varsigma+1}$ is a 2-local derivation. By direct checking we have

$$\Delta_{\varsigma+1}(e)=\Delta_\varsigma(e)-\textstyle\sum_{i=\varsigma+1}^{n}a_{\varsigma,i}(x_1,x_\varsigma)\mathrm{ad}s_{\varsigma i}(e)=0,$$
$$\Delta_{\varsigma+1}(f)=\Delta_\varsigma(f)-\textstyle\sum_{i=\varsigma+1}^{n}a_{\varsigma,i}(x_1,x_\varsigma)\mathrm{ad}s_{\varsigma i}(f)=0,$$

$$\begin{aligned}
&\Delta_{\varsigma+1}(x_1)=\Delta_\varsigma(x_1)-\textstyle\sum_{i=\varsigma+1}^n a_{\varsigma,i}(x_1,x_\varsigma)\mathrm{ad}s_{\varsigma i}(x_1)=0,\\
&\Delta_{\varsigma+1}(x_2)=\Delta_\varsigma(x_2)-\textstyle\sum_{i=\varsigma+1}^n a_{\varsigma,i}(x_1,x_\varsigma)\mathrm{ad}s_{\varsigma i}(x_2)=0,\\
&\cdots,\\
&\Delta_{\varsigma+1}(x_{\varsigma-1})=\Delta_\varsigma(x_{\varsigma-1})-\textstyle\sum_{i=\varsigma+1}^n a_{\varsigma,i}(x_1,x_\varsigma)\mathrm{ad}s_{\varsigma i}(x_{\varsigma-1})=0.
\end{aligned}$$

We only need to prove that $\Delta_{\varsigma+1}(x_\varsigma)=0$. By Lemma 3.7 with $\Delta_\varsigma(e)=\Delta_\varsigma(f)=\Delta_\varsigma(x_1)=0$, we get $\Delta_\varsigma(z)=0$. By letting $\mu=x_\varsigma$ in (14), it follows from Lemma 3.4 that

$$\begin{aligned}
\Delta_\varsigma(x_\varsigma)&=D_{h+z,x_\varsigma}(x_\varsigma)\\
&=[a_h(h+z,x_\varsigma)h+\sum_{1\leqslant j<k\leqslant n}a_{j,k}(h+z,x_\varsigma)s_{jk},x_\varsigma]\\
&=a_h(h+z,x_\varsigma)x_\varsigma+\textstyle\sum_{i=1}^{\varsigma-1}a_{i,\varsigma}(h+z,x_\varsigma)x_i-\sum_{i=\varsigma+1}^n a_{\varsigma,i}(h+z,x_\varsigma)x_i.
\end{aligned}\tag{30}$$

Since $\Delta_\varsigma(x_1)=\Delta_\varsigma(x_2)=\cdots=\Delta_\varsigma(x_{\varsigma-1})=0$, then by (7) we get

$$\begin{aligned}
\Delta_\varsigma(x_\varsigma)&=D_{x_\alpha,x_\varsigma}(x_\varsigma)\\
&=[a_e(x_\alpha,x_\varsigma)e+a_h(x_\alpha,x_\varsigma)h+\textstyle\sum_{i=1}^n a_i(x_\alpha,x_\varsigma)x_i+\displaystyle\sum_{\substack{1\leqslant i\leqslant n\\ i\neq\alpha}}a_{-i}(x_\alpha,x_\varsigma)y_i\\
&\qquad+\sum_{\substack{1\leqslant j<k\leqslant n\\ j,k\neq\alpha}}a_{j,k}(x_\alpha,x_\varsigma)s_{jk},x_\varsigma]-2a_h(x_\alpha,x_\varsigma)\sigma(x_\varsigma)\\
&=-a_{-\varsigma}(x_\alpha,x_\varsigma)z+\sum_{\substack{1\leqslant i\leqslant\varsigma-1\\ i\neq\alpha}}a_{i,\varsigma}(x_\alpha,x_\varsigma)x_i-\textstyle\sum_{i=\varsigma+1}^n a_{\varsigma,i}(x_\alpha,x_\varsigma)x_i
\end{aligned}\tag{31}$$

for $\alpha=1,2,...,\varsigma-1$. Combining (30) and (31) we obtain that

$$\Delta_\varsigma(x_\varsigma)=-\sum_{i=\varsigma+1}^n a_{\varsigma,i}(x_1,x_\varsigma)x_i=\sum_{i=\varsigma+1}^n a_{\varsigma,i}(x_1,x_\varsigma)\mathrm{ad}s_{\varsigma i}(x_\varsigma).$$

This deduces that $\Delta_{\varsigma+1}(x_\varsigma)=0$. Then, Claim 1 is proven. By virtue of Claim 1, we obtain a succession of 2-local derivations $\Delta_3,\Delta_4,...,\Delta_n$ iteratively. In particular, Δ_n satisfies $\Delta_n(e)=\Delta_n(f)=\Delta_n(x_1)=\Delta_n(x_2)=\cdots=\Delta_n(x_{n-1})=0$. We shall now proceed to examine $\Delta_n(x_n)$. By (14) and (7) we have

$$\begin{aligned}
\Delta_n(x_n)&=D_{h+z,x_n}(x_n)\\
&=[a_h(h+z,x_n)h+\sum_{1\leqslant j<k\leqslant n}a_{j,k}(h+z,x_n)s_{jk},x_n]\\
&=a_h(h+z,x_n)x_n+\textstyle\sum_{i=1}^{n-1}a_{i,n}(h+z,x_n)x_i,
\end{aligned}\tag{32}$$

$$\begin{aligned}
\Delta_n(x_n)&=D_{x_\alpha,x_n}(x_n)\\
&=[a_e(x_\alpha,x_n)e+a_h(x_\alpha,x_n)h+\textstyle\sum_{i=1}^n a_i(x_\alpha,x_n)x_i+\displaystyle\sum_{\substack{1\leqslant i\leqslant n\\ i\neq\alpha}}a_{-i}(x_\alpha,x_n)y_i\\
&\qquad+\sum_{\substack{1\leqslant j<k\leqslant n\\ j,k\neq\alpha}}a_{j,k}(x_\alpha,x_n)s_{jk},x_n]-2a_h(x_\alpha,x_n)\sigma(x_n)\\
&=-a_{-n}(x_\alpha,x_n)z+\sum_{\substack{1\leqslant i\leqslant n-1\\ i\neq\alpha}}a_{i,n}(x_\alpha,x_n)x_i
\end{aligned}$$

for $\alpha=1,2,...,n-1$. (33)

Combining (32) and (33) we know that $\Delta_n(x_n)=0$. Then by Lemma 3.8 we see that $\Delta_n(\mu)=0$ for all $\mu\in\mathcal{S}_n$. But by backtracking we know

$$\begin{aligned}\Delta_n &= \Delta_{n-1} - a_{n-1,n}(x_1,x_{n-1})\mathrm{ad}s_{n-1,n}\\ &= \Delta_2 - \textstyle\sum_{\varsigma=2}^{n-1}\sum_{i=\varsigma+1}^{n} a_{\varsigma,i}(x_1,x_\varsigma)\mathrm{ad}s_{\varsigma i}\\ &= \Delta_1 - (2a_h(h,x_1)+a(h,x_1))\sigma - \textstyle\sum_{i=2}^{n} a_{1,i}(h,x_1)\mathrm{ad}s_{1i}\\ &\qquad - \textstyle\sum_{\varsigma=2}^{n-1}\sum_{i=\varsigma+1}^{n} a_{\varsigma,i}(x_1,x_\varsigma)\mathrm{ad}s_{\varsigma i}\\ &= \Delta - D_{e,f} - (2a_h(h,x_1)+a(h,x_1))\sigma - \textstyle\sum_{i=2}^{n} a_{1,i}(h,x_1)\mathrm{ad}s_{1i}\\ &\quad - \textstyle\sum_{\varsigma=2}^{n-1}\sum_{i=\varsigma+1}^{n} a_{\varsigma,i}(x_1,x_\varsigma)\mathrm{ad}s_{\varsigma i}.\end{aligned}$$

Therefore, from $\Delta_n=0$ we see that

$$\begin{aligned}\Delta &= D_{e,f} + (2a_h(h,x_1)+a(h,x_1))\sigma + \sum_{i=2}^{n} a_{1,i}(h,x_1)\mathrm{ad}s_{1i}\\ &\quad + \sum_{\varsigma=2}^{n-1}\sum_{i=\varsigma+1}^{n} a_{\varsigma,i}(x_1,x_\varsigma)\mathrm{ad}s_{\varsigma i}\end{aligned}$$

is a derivation. The proof is complete. ∎

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Xiaomin Tang, Peng Wang, School of Mathematical Science, Heilongjiang University, Harbin, P. R. China; x.m.tang@163.com, 2996630677@qq.com.

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