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The C^*-Algebras of Completely Solvable Lie Groups are Solvable

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Abstract. We prove that if a connected and simply connected Lie group G admits connected closed normal subgroups $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_m = G$ with $\dim G_j = j$ for $j = 1, \dots, m$, then its group C^*-algebra has closed two-sided ideals $\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n = C^*(G)$ with $\mathcal{J}_j/\mathcal{J}_{j-1} \simeq \mathcal{C}_0(\Gamma_j, \mathcal{K}(\mathcal{H}_j))$ for a suitable locally compact Hausdorff space Γ_j and a separable complex Hilbert space $\mathcal{H}_j$, where $\mathcal{C}_0(\Gamma_j, \cdot)$ denotes the continuous mappings on Γ_j that vanish at infinity, and $\mathcal{K}(\mathcal{H}_j)$ is the C^*-algebra of compact operators on $\mathcal{H}_j$ for $j = 1, \dots, n$.

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Dedicated to Karl-Hermann Neeb on the occasion of his 60th birthday.

1. Introduction

In this paper we establish a direct relation between solvability properties of two very different mathematical objects:

1. A *completely solvable* Lie group G, i.e., a connected and simply connected (real) Lie group whose Lie algebra $\mathfrak{g}$ has a chain of ideals
   $\{0\} = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \cdots \subseteq \mathfrak{g}_m = \mathfrak{g}$ with $\dim(\mathfrak{g}_j/\mathfrak{g}_{j-1}) = 1$, $j = 1, \dots, m$.
   Clearly $m = \dim \mathfrak{g}$.
2. A *solvable* C^*-algebra $\mathcal{A}$, i.e., a C^*-algebra that has a finite chain of closed two-sided ideals $\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n = \mathcal{A}$ with $\mathcal{J}_j/\mathcal{J}_{j-1} \simeq \mathcal{C}_0(\Gamma_j, \mathcal{K}(\mathcal{H}_j))$ for suitable locally compact Hausdorff spaces Γ_j and separable complex Hilbert space $\mathcal{H}_j$, $j = 1, \dots, n$. The least number n for which there exists a chain of ideals as above is called the *length* of $\mathcal{A}$.

Here $\mathcal{C}_0(\Gamma_j, \cdot)$ denotes the continuous mappings on Γ_j that vanish at infinity, and $\mathcal{K}(\mathcal{H}_j)$ is the C^*-algebra of compact operators on $\mathcal{H}_j$. With the above terminology, one of the main results of this paper (Corollary 4.11) is that if a Lie group G is completely solvable, then its group C^*-algebra $C^*(G)$ is solvable. In the special case of nilpotent Lie groups, that result was obtained in [6, Th. 4.11] by a method that required global canonical coordinates on coadjoint orbits. (See also [17].)

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In contrast, the present approach relies on the continuous selections of polarizations constructed in [4] along with the technique of ultrafine layerings initiated in [10] and [9]. (See also [1] and [2].)

This result is remarkable since, as already noted in [12], the C^*-algebra of a solvable Lie group may not be solvable, a fact further discussed in [26]. A weaker condition, in which the successive quotients $\mathcal{J}_j/\mathcal{J}_{j-1}$ should be continuous-trace C^*-algebras, was verified in [9] for the wider class of exponential Lie groups.

Even in the special case when G is a nilpotent Lie group, the relation between the length of $C^*(G)$ and various algebraic data of G, such as the nilpotency step, remains to be clarified. For example, the length of $C^*(G)$ is equal to 1 if and only if G is commutative, irrespective of the dimension of G. Nontrivial examples already appear for 2-step nilpotent Lie groups: for every positive integer $n \geq 1$, if H_{2n+1} denotes the $(2n+1)$-dimensional Heisenberg group, then the length of $C^*(H_{2n+1})$ is equal to 2. (See for instance [3], [5], and the references therein.) Moreover, if G is the free 2-step nilpotent Lie group with 3 generators, then $\dim G = 6$ while the length of $C^*(G)$ is again equal to 2, cf. [13, Ex. 6.3.5] and also [4, Ex. 7.2]. Other examples can be found in [3].

The structure of the present paper is as follows: Section 2 contains preliminaries on the Pedersen ideal of a C^*-algebra and continuous selections of Haar measures on the closed subgroups of a locally compact group. Section 3 contains a construction of continuous fields of Hilbert spaces and our main technical result (Theorem 3.6) gives sufficient conditions for a continuous-trace subquotient of $C^*(G)$ to be Morita-equivalent to a commutative C^*-algebra if G is an exponential Lie group. This result is motivated by the continuous-trace subquotient conjecture [28, Conj. 4.18] to the effect that every continuous-trace subquotient of $C^*(G)$ is Morita-equivalent to a commutative C^*-algebra, which is still an open problem. (See also [9, Rem. 3.3].) Finally, Section 4 includes our main results (Theorem 4.10 and Corollary 4.11).

General notation. We denote the connected and simply connected Lie groups by upper case Roman letters and their Lie algebras by the corresponding lower case Gothic letters. By a solvable/nilpotent Lie group we always mean a *connected and simply connected* solvable/nilpotent Lie group, unless otherwise specified.

An exponential Lie group is a Lie group G whose corresponding exponential map $\exp_G\colon \mathfrak{g} \to G$ is bijective. All exponential Lie groups are solvable, and completely solvable Lie groups are exponential. (See for instance [23, §14.4].) We emphasize that in this paper, all completely solvable Lie groups are connected and simply connected.

For a Lie algebra $\mathfrak{g}$ with its linear dual space $\mathfrak{g}^*$ we denote by $\langle\cdot,\cdot\rangle\colon \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the corresponding duality pairing. We often denote the group actions simply by juxtaposition, and in particular this is the case for the coadjoint action $G \times \mathfrak{g}^* \to \mathfrak{g}^*$, $(g,\xi) \mapsto g\xi$.

For every C^*-algebra $\mathcal{A}$ we denote by $\mathrm{SQ}^{\mathrm{Tr}}(\mathcal{A})$ the set of its $*$-isomorphism classes of continuous-trace subquotients, i.e., $*$-isomorphism classes of C^*-algebras with continuous trace $\mathcal{J}_2/\mathcal{J}_1$, where $\mathcal{J}_1 \subseteq \mathcal{J}_2$ are closed two-sided ideals of $\mathcal{A}$. By abuse of notation, we write $\mathcal{S} \in \mathrm{SQ}^{\mathrm{Tr}}(\mathcal{A})$ for $\mathcal{S} := \mathcal{J}_2/\mathcal{J}_1$ as above. We refer to [6, Rem. 2.4] for the bijection between subquotients and locally closed subsets of the dual space $\widehat{\mathcal{A}}$ consisting of equivalence classes of irreducible $*$-representations of $\mathcal{A}$. We denote by $\mathrm{LC}^{\mathrm{Tr}}(\widehat{\mathcal{A}})$ the set of all locally closed subsets of $\widehat{\mathcal{A}}$ which are dual spaces of continuous-trace subquotients of $\mathcal{A}$. If $\mathcal{A}=C^*(G)$ for a solvable Lie group G, then we write $\mathrm{LC}^{\mathrm{Tr}}(\widehat{G})$ instead of $\mathrm{LC}^{\mathrm{Tr}}(\widehat{\mathcal{A}})$, because of the canonical identification of $\widehat{\mathcal{A}}$ with the unitary dual $\widehat{G}$ consisting of equivalence classes of irreducible unitary representations of G.

For every unitary representation π we denote by $[\pi]$ its unitary equivalence class. We use similar notation for $*$-representations of associative Banach $*$-algebras, in particular for C^*-algebras. Our general references for C^*-algebras and Morita equivalence are [11] and [29], and we refer to [27], [7], and [1] for representation theory of exponential Lie groups.

2. Preliminaries

2.1. On the smallest dense ideal of a continuous-trace C^*-algebra

Definition 2.1. Let $\mathcal{A}$ be any C^*-algebra with $\mathcal{A}^+:=\{a\in\mathcal{A}\mid 0\le a\}$. We define K_0^+ as the set of all $a\in\mathcal{A}^+$ for which there exist $b\in\mathcal{A}^+$ and a continuous function $\varphi\colon(0,\infty)\to[0,\infty)$ with compact support such that $a=\varphi(b)$. We also define

$$K_{\mathcal{A}}^+:=\{a\in\mathcal{A}^+\mid(\exists n\in\mathbb{N})(\exists a_1,\dots,a_n\in K_0^+)\quad a\le a_1+\dots+a_n\}.$$

Then $K_{\mathcal{A}}:=\operatorname{span}K_{\mathcal{A}}^+$ is called the *Pedersen ideal* of the C^*-algebra $\mathcal{A}$.

Definition 2.2. For any C^*-algebra $\mathcal{A}$ we denote by $J_{\mathcal{A}}$ the set of all $a\in\mathcal{A}$ for which $\sup\{\dim\pi(a)<\infty\mid[\pi]\in\widehat{\mathcal{A}}\}<\infty$ and the function $[\pi]\mapsto\operatorname{Tr}\pi(a)$ vanishes outside some quasi-compact set.

Definition 2.3. For any C^*-algebra $\mathcal{A}$ we denote by $\mathfrak{m}_{\mathcal{A}}^+$ the set of all $a\in\mathcal{A}^+$ for which the function $[\pi]\mapsto\operatorname{Tr}\pi(a)$ is continuous and finite on $\widehat{\mathcal{A}}$. We also denote by $\mathfrak{m}_{\mathcal{A}}$ the linear span of $\mathfrak{m}_{\mathcal{A}}^+$, which is a self-adjoint two-sided ideal of A with $\mathfrak{m}_A\cap\mathcal{A}^+=\mathfrak{m}_{\mathcal{A}}^+$.

Proposition 2.4. *The following assertions hold for any C^*-algebra $\mathcal{A}$:*

- (i) *The set $J_{\mathcal{A}}$ is a self-adjoint two-sided ideal of $\mathcal{A}$.*
- (ii) *The set $K_{\mathcal{A}}$ is the smallest element of the set of all dense self-adjoint two-sided ideals of $\mathcal{A}$, one has $K_{\mathcal{A}}\cap\mathcal{A}^+=K_{\mathcal{A}}^+$, and moreover $K_{\mathcal{A}}^+$ is dense in $\mathcal{A}^+$.*
- (iii) *If $\Phi\colon\mathcal{A}\to\mathcal{B}$ is any surjective $*$-morphism of C^*-algebras, then $\Phi(K_{\mathcal{A}})=K_{\mathcal{B}}$ and $\Phi(K_{\mathcal{A}}^+)=K_{\mathcal{B}}^+$.*
- (iv) *If $\mathcal{A}$ has continuous trace and every compact subset of $\widehat{\mathcal{A}}$ has finite covering dimension, then $K_{\mathcal{A}}=J_{\mathcal{A}}$.*
- (v) *The C^*-algebra $\mathcal{A}$ has continuous trace if and only if $K_{\mathcal{A}}\cap\mathcal{A}^+\subseteq\mathfrak{m}_{\mathcal{A}}^+$.*
- (vi) *If $\mathcal{A}$ has continuous trace and every compact subset of $\widehat{\mathcal{A}}$ has finite covering dimension, then $J_{\mathcal{A}}\cap\mathcal{A}^+\subseteq\mathfrak{m}_{\mathcal{A}}^+$.*

Proof. Assertion (i) is a direct consequence of the definition of $J_{\mathcal{A}}$.
Assertion (ii) follows by [25, Th. 5.6.1 and its proof].
Assertion (iii) follows by the previous assertion, since $\Phi(K_{\mathcal{A}}^+)=K_{\mathcal{B}}^+$.

Assertion (iv) follows by [15, Cor.2.8] along with the alternative description of the Pedersen ideal given in Assertion (ii) above.

Assertion (v) is a by-product of the proof of [19, Th. 17]. Specifically, if $\mathcal{A}$ has continuous trace, then $\mathfrak{m}_{\mathcal{A}}$ is dense in $\mathcal{A}$, hence $K_{\mathcal{A}} \subseteq \mathfrak{m}_{\mathcal{A}}$ by Assertion (ii), and then $K_{\mathcal{A}} \cap \mathcal{A}^+ \subseteq \mathfrak{m}_{\mathcal{A}} \cap \mathcal{A}^+ = \mathfrak{m}_{\mathcal{A}}^+$. Conversely, assume that $K_{\mathcal{A}} \cap \mathcal{A}^+ \subseteq \mathfrak{m}_{\mathcal{A}}^+$. Since $K_{\mathcal{A}}$ is spanned by its positive part $K_{\mathcal{A}} \cap \mathcal{A}^+$, we obtain $K_{\mathcal{A}} \subseteq \mathfrak{m}_{\mathcal{A}}$. We have that $K_{\mathcal{A}}$ is dense in $\mathcal{A}$ by Assertion (ii) above, hence also $\mathfrak{m}_{\mathcal{A}}$ is dense in $\mathcal{A}$, that is, $\mathcal{A}$ has continuous trace.

Assertion (vi) follows by Assertions (iv)–(v). ∎

Corollary 2.5. *Let $\mathcal{S}$ be a continuous-trace separable C^*-algebra, denote $\Gamma := \widehat{\mathcal{S}}$, and assume the following:*

- $\boldsymbol{\mathcal{H}}_\Gamma = (\mathcal{H}_\gamma)_{\gamma\in\Gamma}$ *is a continuous field of Hilbert spaces with its corresponding continuous field of elementary C^*-algebras $\mathcal{A}(\boldsymbol{\mathcal{H}}_\Gamma) = (\mathcal{K}(\mathcal{H}_\gamma))_{\gamma\in\Gamma}$ and continuous-trace C^*-algebra of global sections $\mathcal{A}_0(\boldsymbol{\mathcal{H}}_\Gamma)$.*
- $\pi_\gamma \colon \mathcal{S} \to \mathcal{B}(\mathcal{H}_\gamma)$ *is an irreducible $*$-representation with $[\pi_\gamma] = \gamma$ for every $\gamma \in \Gamma$.*
- *For every $a \in K_{\mathcal{S}}^+$ we have $(\pi_\gamma(a))_{\gamma\in\Gamma} \in \mathcal{A}_0(\boldsymbol{\mathcal{H}}_\Gamma)$.*

Then the mapping $\mathcal{S} \to \mathcal{A}_0(\boldsymbol{\mathcal{H}}_\Gamma)$, $a \mapsto (\pi_\gamma(a))_{\gamma\in\Gamma}$, is a well-defined $*$-isomorphism.

Proof. We consider the $*$-morphism

$$\Phi \colon \mathcal{S} \to \prod_\gamma \mathcal{K}(\mathcal{B}_\gamma), \quad \Phi(a) := (\pi_\gamma(a))_{\gamma\in\Gamma}$$

which takes values as indicated, since $\mathcal{S}$ is a liminal C^*-algebra by [11, Prop. 4.5.3(i)]. Moreover, we have $\operatorname{Ker}\Phi = \{0\}$ since $\widehat{\mathcal{S}} = \{[\pi_\gamma] \mid \gamma \in \Gamma\}$.

It remains to show that $\Phi(\mathcal{A}) = \mathcal{A}_0(\boldsymbol{\mathcal{H}}_\Gamma)$. To this end we recall from Proposition 2.4(ii) that $K_{\mathcal{S}}^+$ is dense in $\mathcal{S}^+$, while $\Phi(K_{\mathcal{S}}^+) \subseteq \mathcal{A}_0(\boldsymbol{\mathcal{H}}_\Gamma)$ by hypothesis, hence $\Phi(\mathcal{S}^+) \subseteq \mathcal{A}_0(\boldsymbol{\mathcal{H}}_\Gamma)$. Since every element of the C^*-algebra $\mathcal{S}$ is a linear combination of positive elements, we then obtain $\Phi(\mathcal{S}) \subseteq \mathcal{A}_0(\boldsymbol{\mathcal{H}}_\Gamma)$.

We now recall that $\mathcal{A}_0(\boldsymbol{\mathcal{H}}_\Gamma)$ is a liminal C^*-algebra whose spectrum is equal to Γ, cf. [11, 10.9.2], and for every $\gamma \in \Gamma$, $\mathcal{A}_0(\boldsymbol{\mathcal{H}}_\Gamma) \ni a \mapsto a(\gamma) \in \mathcal{K}(\mathcal{H}_\gamma)$ is the unique irreducible representation, modulo unitary equivalence, corresponding to γ. It follows that the C^*-subalgebra $\Phi(\mathcal{S}) \subseteq \mathcal{A}_0(\boldsymbol{\mathcal{H}}_\Gamma)$ is rich, hence $\Phi(S) = \mathcal{A}_0(\boldsymbol{\mathcal{H}}_\Gamma)$ by the noncommutative Stone-Weierstrass theorem [11, Lemma 11.1.4]. ∎

2.2. Chabauty-Fell topology on closed subgroups, Haar measures

Let G be a locally compact group with its unit element denoted by $\mathbf{1} \in G$ and its set of closed subgroups denoted by $\Sigma(G)$. The set $\Sigma(G)$ is regarded as a compact Hausdorff topological space, endowed with its Chabauty-Fell topology (cf. [22, Prop. 2(vi)] or [30, Cor. H.4]). We recall that a basis of open sets of the Chabauty-Fell topology consists of the sets

$$\mathcal{U}(C, \mathcal{T}) = \{K \in \Sigma(G) \mid K \cap C = \emptyset,\ K \cap A \neq \emptyset, \forall\, A \in \mathcal{T}\},$$

where C runs over the compact subspaces of G and $\mathcal{T}$ over finite families of nonempty open subsets of G.

We also define $\Sigma_0(G) := \{K \in \Sigma(G) \mid K \text{ is connected}\}$

and $\mathcal{T}(G) := \{(K, g) \in \Sigma(G) \times G \mid g \in K\}.$

One can prove that $\mathcal{T}(G)$ is a closed subset of $\Sigma(G) \times G$.

Remark 2.6. Assume that G is an exponential Lie group with its Lie algebra $\mathfrak{g}$. Let $\mathrm{Gr}(\mathfrak{g})$ be the Grassmann manifold of $\mathfrak{g}$, whose points are the linear subspaces of $\mathfrak{g}$. We also consider the subset $\mathrm{Gr}_{\mathrm{alg}}(\mathfrak{g})$ whose points are the subalgebras of the Lie algebra $\mathfrak{g}$. The smooth manifold $\mathrm{Gr}(\mathfrak{g})$ is compact and $\mathrm{Gr}_{\mathrm{alg}}(\mathfrak{g})$ is a closed subset, hence $\mathrm{Gr}_{\mathrm{alg}}(\mathfrak{g})$ is in turn a compact space. (See [4, Sect. 6].) Then the mapping $\mathrm{Gr}_{\mathrm{alg}}(\mathfrak{g}) \to \Sigma(G)$, $\mathfrak{k} \mapsto \exp_G(\mathfrak{k})$, is a homeomorphism from $\mathrm{Gr}_{\mathrm{alg}}(\mathfrak{g})$ onto $\Sigma_0(G)$. (See e.g., [18, Lemma 2.4].) ■

We denote by $\mathcal{C}_c(G)$ the set of $\mathbb{C}$-valued continuous functions on G having compact support and we fix a function $0 \le \varphi_0 \in \mathcal{C}_c(G)$ with $\varphi_0(\mathbf{1}) = 1$.
For every $K \in \Sigma(G)$ we introduce the following notation, cf. [29, Sect. C.1]:

- μ_K is the left-invariant Haar measure of K satisfying $\int\limits_K \varphi_0(k)\mathrm{d}\mu_K(k) = 1$.
- $\Delta_K \colon K \to \mathbb{R}_+^\times$ is the modular function of K, that is, the continuous group homomorphism satisfying

$$\int\limits_K \psi(k)\mathrm{d}\mu_K(k) = \Delta_K(k_0)\int\limits_K \psi(kk_0)\mathrm{d}\mu_K(k), \tag{1}$$

$$\int\limits_K \psi(k)\mathrm{d}\mu_K(k) = \int\limits_K \psi(k^{-1})\Delta_K(k^{-1})\mathrm{d}\mu_K(k) \tag{2}$$

  for every $k_0 \in K$ and $\psi \in \mathcal{C}_c(K)$.
- $\omega_K \colon G \to \mathbb{R}_+^\times$, $\omega_K(g) := \int\limits_K \varphi_0(gkg^{-1})\mathrm{d}\mu_K(k)$. Note that $\omega_K|_K = \Delta_K$, and $\omega_K(gh) = \omega_K(g)\omega_K(h)$ for every $g \in G$ and $h \in K$.
- $\rho_K \colon G \to \mathbb{R}_+^\times$, $\rho_K(g) := \Delta_G(g^{-1})\omega_K(g)$. Note that $\rho_K|_K = \Delta_K(\Delta_G|_K)^{-1}$, and $\rho_K(gh) = \rho_K(g)\rho_K(h)$ for every $g \in G$ and $h \in K$.
- ν_K is the positive Radon measure on G/K satisfying, for every $\varphi \in \mathcal{C}_c(G)$,

$$\int\limits_G \varphi(g)\rho_K(g)\mathrm{d}\mu_G(g) = \int\limits_{G/K}\Big(\int\limits_K \varphi(gk)\mathrm{d}\mu_K(k)\Big)\mathrm{d}\nu_K(gK). \tag{3}$$

We call the above family of measures $(\mu_K)_{K\in\Sigma(G)}$ a *Haar system* on the closed subgroups of G and for every $K \in \Sigma(G)$ we regard μ_K as a Radon measure on G supported by K.

We now give a version of [30, Lemma H.9] adapted to our purposes. For any locally compact Hausdorff space X we endow $\mathcal{C}_c(X)$ with the inductive limit topology as inductive limit of the Banach spaces $\mathcal{C}_C(X) := \{f \in \mathcal{C}_c(X) \mid \operatorname{supp} f \subseteq C\}$ for arbitrary compact $C \subseteq X$. (See e.g., [21, Ch. 4, Part 1, §2, Ex. (d)].)

Lemma 2.7. *If G is a locally compact group and $(\mu_K)_{K\in\Sigma(G)}$ is a Haar system on the closed subgroups of G, then the following assertions hold:*

(i) *The mapping $\mathcal{I}\colon \mathcal{C}_c(G)\to\mathcal{C}(\Sigma(G))$, $(\mathcal{I}(f))(K):=\int f\mathrm{d}\mu_K$ is well defined, linear, and continuous.*

(ii) *For every compact subset $C\subseteq G$ the mapping*

$$\widehat{\mathcal{I}}_C\colon \Sigma(G)\times\mathcal{C}_C(G)\to\mathbb{C},\quad \widehat{\mathcal{I}}_C(K,f):=\int f\mathrm{d}\mu_K$$

is continuous.

(iii) *The subset $Y:=\{(K,k)\in\Sigma(G)\times G\mid k\in K\}\subseteq\Sigma(G)\times G$ is closed and the function*

$$\widetilde{\Delta}\colon Y\to\mathbb{C},\quad \widetilde{\Delta}(K,k):=\Delta_K(k)$$

is continuous.

(iv) *If X is a locally compact space, then for every $F\in\mathcal{C}_c(X\times\Sigma(G)\times G)$ its corresponding function*

$$\widetilde{F}\colon X\times\Sigma(G)\to\mathbb{C},\quad \widetilde{F}(x,K):=\int F(x,K,k)\mathrm{d}\mu_K(k) \tag{4}$$

is continuous.

Proof. (i) For every $f\in\mathcal{C}_c(G)$ we have $\mathcal{I}(f)\in\mathcal{C}(\Sigma(G))$ by [16, Appendix] or [30, Lemma H.8], hence the mapping $\mathcal{I}$ in the statement is well defined. Moreover, the mapping $\mathcal{I}$ is clearly linear. Therefore, by [21, Ch. 4, Part 1, §1, Prop. 2], it suffices to prove that for an arbitrary compact subset $C\subseteq X$ the linear map $\mathcal{I}|_{\mathcal{C}_C(G)}\colon\mathcal{C}_C(G)\to\mathcal{C}(\Sigma(G))$ is continuous.

To this end we first select $0\le\varphi\in\mathcal{C}_c(G)$ with $\varphi|_C=1$. Then for every $f\in\mathcal{C}_C(G)$ we have $|f|\le\|f\|_\infty\varphi$. This further implies

$$|(\mathcal{I}(f))(K)|=\Bigl|\int f\mathrm{d}\mu_K\Bigr|\le\int|f|\mathrm{d}\mu_K\le\|f\|_\infty\int\varphi\mathrm{d}\mu_K=\|f\|_\infty(\mathcal{I}(\varphi))(K).$$

Now, since $0\le\mathcal{I}(\varphi)\in\mathcal{C}(\Sigma(G))$ and the space $\Sigma(G)$ is compact, it follows that

$$\|\mathcal{I}(\varphi)\|_\infty=\sup_{K\in\Sigma(G)}(\mathcal{I}(\varphi))(K)<\infty$$

hence the above inequalities imply

$$(\forall f\in\mathcal{C}_C(G))\quad \|\mathcal{I}(f)\|_\infty\le\|\mathcal{I}(\varphi)\|_\infty\|f\|_\infty. \tag{5}$$

This shows that the linear map $\mathcal{I}|_{\mathcal{C}_C(G)}\colon\mathcal{C}_C(G)\to\mathcal{C}(\Sigma(G))$ is continuous.

(ii) It is enough to prove that if $\lim\limits_{j\in J}K_j=K$ in $\Sigma(G)$ and $\lim\limits_{j\in J}f_j=f$ in $\mathcal{C}_C(G)$ then $\lim\limits_{j\in J}\widehat{\mathcal{I}}_C(K_j,f_j)=\widehat{\mathcal{I}}_C(K,f)$. For arbitrary $j\in J$ we have

$$\begin{aligned}|\widehat{\mathcal{I}}_C(K_j,f_j)-\widehat{\mathcal{I}}_C(K,f)|&\le|\widehat{\mathcal{I}}_C(K_j,f_j-f)|+|\widehat{\mathcal{I}}_C(K_j,f)-\widehat{\mathcal{I}}_C(K,f)|\\&=|(\mathcal{I}(f_j-f))(K_j)|+|(\mathcal{I}(f))(K_j)-(\mathcal{I}(f))(K)|\\&\le\|\mathcal{I}(\varphi)\|_\infty\|f_j-f\|_\infty+|(\mathcal{I}(f))(K_j)-(\mathcal{I}(f))(K)|,\end{aligned}$$

where we used (5).

Now, using the fact that $\lim_{j\in J} f_j = f$ in $\mathcal{C}_C(G)$ and $\mathcal{I}(f) \in \mathcal{C}(\Sigma(G))$, we obtain $\lim_{j\in J} \widehat{\mathcal{I}}_C(K_j, f_j) = \widehat{\mathcal{I}}_C(K, f)$.

(iii) See [30, Lemma H.9(c)].

(iv) Since $F \in \mathcal{C}_c(X \times \Sigma(G) \times G)$, there exists compact subsets $X_0 \subseteq X$ and $C \subseteq G$ with $\operatorname{supp} F \subseteq X_0 \times \Sigma(G) \times C$. This implies that the function

$$\widehat{F}\colon X \times \Sigma(G) \to \mathcal{C}_c(G), \quad \widehat{F}(x, K) := F(x, K, \cdot)$$

is well defined and continuous. Then, by (ii) above, the function

$$\widehat{\mathcal{I}}_C \circ (\mathrm{id}_{\Sigma(G)} \times \widehat{F})\colon X \times \Sigma(G) \to \mathbb{C},$$
$$(\widehat{\mathcal{I}}_C \circ (\mathrm{id}_{\Sigma(G)} \times \widehat{F}))(x, K) = \widehat{\mathcal{I}}_C(K, \widehat{F}(x, K))$$

is continuous as a composition of two continuous maps. Since

$$\widehat{\mathcal{I}}_C(K, \widehat{F}(x, K)) = \int F(x, K, k)\mathrm{d}\mu_K(k) = \widetilde{F}(x, K),$$

and this completes the proof. ∎

3. Continuous fields of Hilbert spaces defined by continuous selections of polarizations

In this section we obtain our main technical results that give sufficient conditions for triviality of certain continuous-trace subquotients of group C^*-algebras (Theorem 3.6 and Corollary 3.11).

3.1. Continuous fields of Hilbert spaces defined by continuous selections of subgroups

Let G be a locally compact group.

Hypothesis 3.1. Throughout this subsection we assume the following:

(i) $\Gamma \in \mathrm{LC}^{\mathrm{Tr}}(\widehat{G})$, Γ has finite covering dimension, and every compact subset of Γ has finite covering dimension.

(ii) We have a continuous mapping $\Gamma \to \Sigma(G)$, $\gamma \mapsto P(\gamma)$.

(iii) For every $\gamma \in \Gamma$, $\chi_\gamma\colon G \to \mathbb{T}$ is such that $\chi_\gamma|_{P(\gamma)}$ is a character and the map $\Gamma \times G \ni (\gamma, g) \mapsto \chi_\gamma(g) \in \mathbb{T}$ is continuous.

(iv) For every $\gamma \in \Gamma$, $\pi_\gamma = \mathrm{Ind}_{P(\gamma)}^G \chi_\gamma$ is an infinite-dimensional irreducible representation and $\gamma = [\pi_\gamma]$.

Remark 3.2. In Hypothesis 3.1, the set Γ is the spectrum of a continuous-trace separable C^*-algebra, therefore it is a second countable, locally compact Hausdorff space. (See [11, 3.3.4, 3.3.8, and 4.5.3].) By [30, Lemma 6.5], it follows that Γ is a (completely) metrizable space, therefore paracompact. ∎

For every $\gamma \in \Gamma$ we set $\mu_\gamma := \mu_{P(\gamma)}$, $\nu_\gamma := \nu_{G/P(\gamma)}$, $\rho_\gamma := \rho_{P(\gamma)}$, with $\mu_{P(\gamma)}$, $\nu_{G/P(\gamma)}$, and $\rho_{P(\gamma)}$ as in Section 2.

Define

$$\mathcal{D}(G,P(\gamma),\chi_\gamma)=\left\{\varphi\in\mathcal{C}(G)\;\middle|\;\begin{array}{l}(\forall g\in G,h\in P(\gamma))\ \varphi(gh)=\chi_\gamma(h)^{-1}\varphi(g),\\ q_\gamma(\text{supp } f)\text{ compact}\end{array}\right\},$$

where $q_\gamma\colon G\to G/P(\gamma)$, $g\mapsto\dot g$, is the quotient map, and consider the norm

$$\|f\|_\gamma:=\Big(\int\limits_{G/P(\gamma)}|\varphi(g)|^2\mathrm{d}\nu_\gamma(\dot g)\Big)^{1/2}.$$

The Hilbert space $\mathcal{H}_\gamma$, defined as the completion of $\mathcal{D}(G,P(\gamma),\chi_\gamma)$ with respect to the norm $\|\cdot\|_\gamma$, is the representation space of the induced representation π_γ.

We now consider the linear mapping $R_\gamma\colon\mathcal{C}_c(G)\to\mathcal{D}(G,P(\gamma),\chi_\gamma)$ given by

$$(R_\gamma\varphi)(g):=\int\limits_{P(\gamma)}\rho_\gamma(gh)^{-1/2}\varphi(gh)\chi_\gamma(h)\mathrm{d}\mu_\gamma(h). \tag{6}$$

We note that, if $k\in P(\gamma)$ and $g\in G$, then

$$\begin{aligned}(R_\gamma\varphi)(gk)&=\int\limits_{P(\gamma)}\rho_\gamma(gkh)^{-1/2}\varphi(gkh)\chi_\gamma(h)\mathrm{d}\mu_\gamma(h)\\ &=\int\limits_{P(\gamma)}\rho_\gamma(gh)^{-1/2}\varphi(gh)\chi_\gamma(k^{-1}h)\mathrm{d}\mu_\gamma(h)=\chi_\gamma(k)^{-1}(R_\gamma\varphi)(g),\end{aligned} \tag{7}$$

hence indeed $R_\gamma\varphi\in\mathcal{D}(G,P(\gamma),\chi_\gamma)$. The function

$$\Gamma\times G\ni(\gamma,g)\mapsto(R_\gamma\varphi)(g)\in\mathbb{C}$$

is continuous by Lemma 2.7 (iv). Moreover, the set $\{R_\gamma\varphi\mid\varphi\in\mathcal{C}_c(G)\}$ is dense in $\mathcal{H}_\gamma$. (See the proof of [29, Th. C.33].)

Remark 3.3. For a family of Hilbert spaces $(\mathcal{H}_\gamma)_{\gamma\in\Gamma}$, where Γ is a topological space, we will use the notion of continuity structure $F\subseteq\prod\limits_{\gamma\in\Gamma}\mathcal{H}_\gamma$ as in [14, I.1]. By definition, F is a complex linear subspace with the properties that the function $\gamma\mapsto\|v_\gamma\|_\gamma$ is continuous for every section $v=(v_\gamma)_{\gamma\in\Gamma}\in F$ and moreover the subset $\{v_\gamma\mid v\in F\}\subseteq\mathcal{H}_\gamma$ is dense for every $\gamma\in\Gamma$. That notion is equivalently described in [11, Prop. 10.2.3], which ensures that there is a space $\widetilde F$ of sections of $(\mathcal{H}_\gamma)_{\gamma\in\Gamma}$ such that $\boldsymbol{\mathcal{H}}_\Gamma=((\mathcal{H}_\gamma)_{\gamma\in\Gamma},\widetilde F)$ is a continuous field of Hilbert spaces and $F\subseteq\widetilde F$. Specifically, $\widetilde F$ is the space of sections that can be approximated by sections in F, uniformly on neighbourhoods of points in Γ. The space Γ is paracompact and has finite covering dimension. It follows by [11, Lemma 10.8.7] that $\boldsymbol{\mathcal{H}}_\Gamma$ is a trivial continuous field of Hilbert spaces, hence its corresponding continuous field of elementary C^*-algebras $\mathcal{A}(\boldsymbol{\mathcal{H}}_\Gamma)$ is trivial. ∎

Proposition 3.4. *The set of sections*

$$F:=\{(R_\gamma\varphi)_{\gamma\in\Gamma}\mid\varphi\in\mathcal{C}_c(G)\}\subseteq\prod_{\gamma\in\Gamma}\mathcal{H}_\gamma$$

defines a continuity structure on the field of Hilbert spaces $(\mathcal{H}_\gamma)_{\gamma\in\Gamma}$.

Proof. By the discussion above, it suffices to prove that for every $\varphi \in \mathcal{C}_c(G)$ the function $\gamma \mapsto \|R_\gamma\varphi\|_\gamma$ is continuous on Γ.

To this end let $K \subseteq G$ be a compact subset with $\operatorname{supp}\varphi \subseteq K$. Then there is a compact subset $K_\gamma \subseteq G/P(\gamma)$ with $q_\gamma(\operatorname{supp}(R_\gamma\varphi)) \subseteq K_\gamma$, and we have

$$\begin{aligned}\|R_\gamma\varphi\|_\gamma^2 &= \int\limits_{K_\gamma}\int\limits_{P(\gamma)} \rho_\gamma(gh)^{-1/2}\varphi(gh)\chi_\gamma(h)\overline{(R_\gamma\varphi)(g)}\mathrm{d}\mu_\gamma(h)\mathrm{d}\nu_\gamma(\dot g)\\ &\overset{(\dagger)}{=} \int\limits_{K_\gamma}\int\limits_{P(\gamma)} \rho_\gamma(gh))^{-1/2}\varphi(gh)\overline{(R_\gamma\varphi)(gh)}\mathrm{d}\mu_\gamma(h)\mathrm{d}\nu_\gamma(\dot g)\\ &\overset{(\ddagger)}{=} \int\limits_{G} \rho_\gamma(g)^{1/2}\varphi(g)\overline{(R_\gamma\varphi)(g)}\mathrm{d}\mu_G(g),\end{aligned}$$

where we have used (7) to get (†), and (‡) follows from (3). The required continuity property is then a consequence of the fact that the functions $(\gamma, g) \mapsto \rho_\gamma(g)^{1/2}$ and $(\gamma, g) \mapsto (R_\gamma\varphi)(g)$ are continuous, while $\varphi \in \mathcal{C}_c(G)$. ∎

Lemma 3.5. *For every $f \in C^*(G)$, $\phi, \psi \in \mathcal{C}_c(G)$ the function*

$$\gamma \mapsto (\pi_\gamma(f)R_\gamma\varphi \mid R_\gamma\psi)_\gamma$$

is continuous on Γ.

Proof. Recall that the induced representation $\pi_\gamma = \operatorname{Ind}_{P(\gamma)}^G\chi_\gamma$ is the unitary representation of G on $\mathcal{H}_\gamma$ given by

$$(\pi_\gamma(g)\phi)(s) = \big(\tfrac{\rho_\gamma(g^{-1}s)}{\rho_\gamma(s)}\big)^{1/2}\phi(g^{-1}s)$$

for every $\phi \in \mathcal{D}(G, P(\gamma), \chi_\gamma)$. (See [29, Thm. C.33].) Then for every function $\varphi \in \mathcal{C}_c(G)$ and $s, t \in G$ we have

$$\begin{aligned}(\pi_\gamma(t)R_\gamma\varphi)(s) &= \big(\tfrac{\rho_\gamma(t^{-1}s)}{\rho_\gamma(s)}\big)^{1/2}(R_\gamma\varphi)(t^{-1}s)\\ &= \big(\tfrac{\rho_\gamma(t^{-1}s)}{\rho_\gamma(s)}\big)^{1/2}\int\limits_{P(\gamma)} \rho_\gamma(t^{-1}sh)^{-1/2}\varphi(t^{-1}sh)\chi_\gamma(h)\mathrm{d}\mu_K(h)\\ &= \int\limits_{P(\gamma)} \rho_\gamma(sh)^{-1/2}\varphi(t^{-1}sh)\chi_\gamma(h)\mathrm{d}\mu_K(h),\end{aligned}$$

where we have used that $\rho_\gamma(gh) = \rho_\gamma(g)\rho_\gamma(h)$ when $h \in P(\gamma)$ and $g \in G$.

Consider first $f \in \mathcal{C}_c(G)$. For every $s \in G$ we have

$$\begin{aligned}(\pi_\gamma(f)R_\gamma\varphi)(s) &= \int\limits_G f(t)(\pi_\gamma(t)R_\gamma\varphi)(s)\mathrm{d}\mu_G(t)\\ &= \int\limits_G\int\limits_{P(\gamma)} f(t)\rho_\gamma(sh)^{-1/2}\varphi(t^{-1}sh)\chi_\gamma(h)\mathrm{d}\mu_\gamma(h)\mathrm{d}\mu_G(t).\end{aligned}$$

It follows that

$$\begin{aligned}&(\pi_\gamma(f)R_\gamma\varphi \mid R_\gamma\psi)_\gamma\\ &= \int\limits_{G/P(\gamma)}\int\limits_G\int\limits_{P(\gamma)} f(t)\rho_\gamma(sh)^{-1/2}\varphi(t^{-1}sh)\chi_\gamma(h)\overline{(R_\gamma\psi)(s)}\mathrm{d}\mu_\gamma(h)\mathrm{d}\mu_G(t)\mathrm{d}\nu_\gamma(\dot{s}).\end{aligned}$$

By (7), $\chi_\gamma(h)\overline{(R_\gamma\psi)(s)} = \overline{(R_\gamma\psi)(sh)}$. Since f,φ,ψ have compact support in G, and $R_\gamma\psi$ has compact support in G modulo $P(\gamma)$ we can write

$$\begin{aligned}&(\pi_\gamma(f)R_\gamma\varphi \mid R_\gamma\psi)_\gamma\\ &= \int\limits_G\int\limits_{G/P(\gamma)}\int\limits_{P(\gamma)} f(t)\rho_\gamma(sh)^{-1/2}\varphi(t^{-1}sh)\overline{(R_\gamma\psi)(sh)}\mathrm{d}\mu_\gamma(h)\mathrm{d}\nu_\gamma(\dot{s})\mathrm{d}\mu_G(t)\\ &= \int\limits_G\int\limits_G f(t)\rho_\gamma(s)^{1/2}\varphi(t^{-1}s)\overline{(R_\gamma\psi)(s)}\mathrm{d}\mu_G(s)\mathrm{d}\mu_G(t)\end{aligned}$$

Then, since f, φ are continuous and have compact support and the functions $(\gamma,s)\mapsto\overline{(R_\gamma\psi)(s)}$ and $(\gamma,s)\mapsto\rho_\gamma(s)$ are continuous, we get the continuity in the statement for $f\in\mathcal{C}_c(G)$.

Assume now that $f\in C^*(G)$. Then for every $\epsilon>0$ there is $f_\epsilon\in\mathcal{C}_c(G)$ such that $\|f-f_\epsilon\|<\epsilon$. Fix $\gamma_0\in\Gamma$. Then, since the function $\gamma\to\|R_\gamma\phi\|_\gamma$ is continuous on Γ for every $\phi\in\mathcal{C}_0(G)$, we have

$$|(\pi_\gamma(f)R_\gamma\varphi \mid R_\gamma\psi)_\gamma - (\pi_\gamma(f_\epsilon)R_\gamma\varphi \mid R_\gamma\psi)_\gamma| \le \epsilon\|R_\gamma\varphi\|_\gamma\|R_\gamma\psi\|_\gamma \le C_\Omega\epsilon,$$

for γ in a compact neighbourhood Ω of γ_0, where C_Ω is a constant that depends only on Ω, φ and ψ. Therefore, the function $\gamma\mapsto(\pi_\gamma(f)R_\gamma\varphi \mid R_\gamma\psi)_\gamma$ can be approximated by continuous function on compact neighbourhoods of γ_0, hence it is continuous at γ_0. Since γ_0 is arbitrary we get the continuity in the statement. ■

Theorem 3.6. *In the Hypothesis* 3.1 *above, if* $\mathcal{S}\in\mathrm{SQ}^{\mathrm{Tr}}(C^*(G))$ *satisfies* $\Gamma=\widehat{\mathcal{S}}$*, then there is a* $*$*-isomorphism*

$$\mathcal{S}\xrightarrow{\sim}\mathcal{A}(\boldsymbol{\mathcal{H}}_\Gamma),\quad f\mapsto(\pi_\gamma(f))_{\gamma\in\Gamma}$$

and the C^**-algebra* $\mathcal{S}$ *is Morita equivalent to a commutative* C^**-algebra.*

Proof. By hypothesis, every compact subset of $\widehat{\mathcal{S}}$ has finite covering dimension, hence $J_{\mathcal{S}}=K_{\mathcal{S}}$ by Proposition 2.4(iv). Therefore, by Corollary 2.5, it suffices to prove that for arbitrary $f\in K_{\mathcal{S}}^+$ we have $(\pi_\gamma(f))_{\gamma\in\Gamma}\in\mathcal{A}(\boldsymbol{\mathcal{H}}_\Gamma)$.

Let $\mathcal{J}_1\subseteq\mathcal{J}_2$ be closed two-sided ideals of $C^*(G)$ with $\mathcal{S}=\mathcal{J}_2/\mathcal{J}_1$. For every $f\in J_{\mathcal{S}}=K_{\mathcal{S}}$, there is $f_0\in K_{\mathcal{J}_2}$ such that $f=f_0+\mathcal{J}_2$, by Proposition 2.4, since Γ has finite covering dimension. Since $\pi_\gamma(J_2)=0$ for every $\gamma\in\Gamma$, it follows by Lemma 3.5 that, for every $\varphi,\psi\in\mathcal{C}_c(G)$, the function

$$\gamma\mapsto\langle\pi_\gamma(f)R_\gamma\varphi,R_\gamma\psi\rangle=\langle\pi_\gamma(f_0)R_\gamma\varphi,R_\gamma\psi\rangle$$

is continuous on Γ.

On the other hand, as $f\in J_{\mathcal{S}}$, the function $\gamma\mapsto\mathrm{Tr}\,(\pi_\gamma(f))$ is continuous on Γ, and has compact support. By [14, Cor., page 260] we obtain $(\pi_\gamma(f))_{\gamma\in\Gamma}\in\mathcal{A}(\boldsymbol{\mathcal{H}}_\Gamma)$, which completes the proof. ■

3.2. The case of exponential Lie groups

In this subsection we denote by G an exponential Lie group with its Lie algebra $\mathfrak{g}$. We first recall the definition of the Bernat-Kirillov correspondence.

Definition 3.7. For any $\xi \in \mathfrak{g}^*$ we define

$$\begin{aligned} \mathfrak{S}(\xi) &:= \{\mathfrak{p} \in \mathrm{Gr}_{\mathrm{alg}}(\mathfrak{g}) \mid [\mathfrak{p},\mathfrak{p}] \subseteq \mathrm{Ker}\,\xi\}, \\ \mathfrak{P}(\xi) &:= \{\mathfrak{p} \in \mathfrak{S}(\xi) \mid 2\dim\mathfrak{p} = \dim\mathfrak{g} + \dim\mathfrak{g}(\xi)\}. \\ \mathfrak{I}(\xi) &:= \{\mathfrak{p} \in \mathfrak{S}(\xi) \mid [\mathrm{Ind}_P^G(\chi_\xi|_P)] \in \widehat{G}.\} \end{aligned}$$

The elements of $\mathfrak{P}(\xi)$ are called *polarizations* at $\xi \in \mathfrak{g}^*$. Here we used the notation

$$\chi_\xi \colon G \to \mathbb{T}, \quad \chi_\xi(x) := \mathrm{e}^{\mathrm{i}\langle \xi, x\rangle}$$

hence $\chi_\xi|_P \colon P \to \mathbb{T}$ is a continuous morphism of Lie groups for every $\mathfrak{p} \in \mathfrak{S}(\xi)$ with its corresponding connected subgroup $P \subseteq G$.

Remark 3.8. In the setting of Definition 3.7, the following assertions hold:

(i) For every $\xi \in \mathfrak{g}^*$ one has $\mathfrak{I}(\xi) = \{\mathfrak{p} \in \mathfrak{P}(\xi) \mid P.\xi = \xi + \mathfrak{p}^\perp\} \neq \emptyset$ and $\mathfrak{I}(\xi) \subseteq \mathfrak{P}(\xi) \subseteq \mathfrak{S}(\xi)$.

(ii) If $\xi \in \mathfrak{g}^*$, then $[\mathrm{Ind}_{P_1}^G(\chi_\xi|_{P_1})] = [\mathrm{Ind}_{P_2}^G(\chi_\xi|_{P_2})] \in \widehat{G}$ for all $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{I}(\xi)$.

(iii) If $\xi_j \in \mathfrak{g}^*$ and $\mathfrak{p}_j \in \mathfrak{I}(\xi_j)$ for $j = 1, 2$, then $[\mathrm{Ind}_{P_1}^G(\chi_{\xi_1}|_{P_1})] = [\mathrm{Ind}_{P_2}^G(\chi_{\xi_2}|_{P_2})]$ if and only if there exists $x \in G$ with $\xi_2 = x.\xi_1$.

See [7, Ch. VI, §§2–3] for proofs of these assertions.

As proved in [7, Ch. VI, §2], the Bernat-Kirillov correspondence

$$\kappa_G \colon \mathfrak{g}^*/G \to \widehat{G}, \ \ \kappa_G(G.\xi) = [\mathrm{Ind}_P^G(\chi_\xi|_P)] \text{ for all } \xi \in \mathfrak{g}^* \text{ and } \mathfrak{p} \in \mathfrak{I}(\xi)$$

is well defined and bijective. ∎

For the exponential group G above, we let $q \colon \mathfrak{g}^* \to \mathfrak{g}^*/G$ be the quotient map defined by the coadjoint action of G and denote simply by $\kappa := \kappa_G \colon \mathfrak{g}^*/G \to \widehat{G}$ the Kirillov-Bernat bijection.

We assume that G satisfies the following hypothesis.

Hypothesis 3.9. (i) $\Gamma \in \mathrm{LC}^{\mathrm{Tr}}(\widehat{G})$ and $(q \circ \kappa)^{-1}(\Gamma) \subseteq \mathfrak{g}^* \setminus [\mathfrak{g},\mathfrak{g}]^\perp$.

(ii) $\sigma \colon \Gamma \to q^{-1}(\Gamma)$ is a continuous mapping with $q(\sigma(\ell)) = \ell$ for all $\ell \in q^{-1}(\Gamma)$.

(iii) $\mathfrak{p} \colon \Gamma \to \mathrm{Gr}(\mathfrak{g})$ is a continuous mapping such that $\mathfrak{p}(\gamma) \in \mathfrak{P}(\sigma(\gamma))$ and the real polarization $\mathfrak{p}(\gamma)$ satisfies Pukánszky's condition $P(\gamma)\sigma(\gamma) = \sigma(\gamma) + \mathfrak{p}(\gamma)^\perp$ for every $\gamma \in \Gamma$, where $P(\gamma)$ is the connected closed subgroup of G that corresponds to $\mathfrak{p}(\gamma)$

Lemma 3.10. *If the exponential Lie group G and the subset set $\Gamma \subseteq \widehat{G}$ satisfy Hypothesis* 3.9 *above, then they satisfy Hypothesis* 3.1 *as well.*

Proof. In Hypothesis 3.9, the topological space Γ is paracompact and its covering dimension is finite. Moreover, the covering dimension of every compact subset of Γ is finite.

In fact, Γ is the spectrum of a continuous-trace separable C^*-algebra, and paracompact, as above. Moreover, the mapping $\sigma\colon\Gamma\to q^{-1}(\Gamma)$ is a homeomorphism (with its inverse $q|_{q^{-1}(\Gamma)}$), hence it suffices to show that $q^{-1}(\Gamma)$, regarded as a topological subspace of $\mathfrak{g}^*$, has finite covering dimension. We have seen above that Γ is a locally compact Hausdorff space, hence $q^{-1}(\Gamma)$ is in turn a locally compact Hausdorff space, and then it is a locally closed subset of $\mathfrak{g}^*$ by [30, Lemma 1.26]. That is,

$$q^{-1}(\Gamma)=E\cap D$$

for a suitable closed subset $E\subseteq\mathfrak{g}^*$ and a suitable open subset $D\subseteq\mathfrak{g}^*$. The covering dimension of the closed subset E is finite since it is less than the covering dimension of $\mathfrak{g}^*$, cf. [24, Ch. 3, Prop. 1.5 and Th. 2.7]. Moreover, since we have seen above that Γ is a metrizable space, it follows that both $q^{-1}(\Gamma)$ and its subspace E are metrizable. Then $D\cap E$ is an open subset of the metrizable space E, hence $D\cap E$ is an F_σ-subset of E. The space E is metrizable, hence is a normal space, and then the covering dimension of its F_σ-subset $D\cap E$ is less than the covering dimension of E by [24, Ch. 3, Cor. 6.3]. Consequently, the covering dimension of $E\cap D$ is finite. That is, $q^{-1}(\Gamma)$ has finite covering dimension.

Moreover, since Γ is a Hausdorff space, every compact subset of Γ is closed, hence its covering dimension is less than the covering dimension of Γ by [24, Ch. 3, Prop. 1.5] again.

By Remark 2.6 and Hypothesis 3.9 we have that the mapping

$$\Gamma\to\Sigma(G),\quad \gamma\mapsto P(\gamma)$$

is continuous where $\Sigma(G)$ is endowed with the Chabauty-Fell topology.

Let
$$\chi_\gamma\colon G\to\mathbb{T},\quad \chi_\gamma(g)=\mathrm{e}^{\mathrm{i}\langle\sigma(\gamma),\log_G g\rangle},$$

and denote by χ_γ also its restriction to $P(\gamma)$, that is, the character of $P(\gamma)$, so that $\chi_\gamma|_{P(\gamma)}\in\mathrm{Hom}(P(\gamma),\mathbb{T})$. Then χ_γ satisfies (iii) in Hypothesis 3.1.

Finally, the representation $\pi_\gamma:=\mathrm{Ind}_{P(\gamma)}^G(\chi_\gamma|_{P(\gamma)})\colon G\to\mathcal{B}(\mathcal{H}_\gamma)$ is irreducible since the real polarization $\mathfrak{p}(\gamma)$ satisfies Pukánszky's condition, cf. [7, Ch. VI, Prop. 3.2]. ■

Corollary 3.11. *Assume the exponential Lie group G and the subset $\Gamma\subseteq\widehat{G}$ satisfy the Hypothesis 3.9 above. If $\mathcal{S}\in\mathrm{SQ}^{\mathrm{Tr}}(C^*(G))$ satisfies $\Gamma=\widehat{\mathcal{S}}$, then we have the $*$-isomorphism*

$$\mathcal{S}\overset{\sim}{\longrightarrow}\mathcal{A}(\boldsymbol{\mathcal{H}}_\Gamma),\quad f\mapsto(\pi_\gamma(f))_{\gamma\in\Gamma}$$

and the C^*-algebra $\mathcal{S}$ is Morita equivalent to a commutative C^*-algebra.

Proof. This is a direct consequence of Lemma 3.10 and Theorem 3.6. ■

4. Main results

In order to state our main results (Theorem 4.10 and Corollary 4.11) we need first to recall the setting of [4, Sect. 6].

We start with a simple lemma.

Lemma 4.1. *Let $\mathcal{V}$ be an $\mathbb{R}$-linear space with its complexification $\mathcal{V}_{\mathbb{C}}$ and define the antilinear map $C\colon \mathcal{V}_{\mathbb{C}} \to \mathcal{V}_{\mathbb{C}}$, $C(x+\mathrm{i}y) := x - \mathrm{i}y$. Then the following assertions hold.*

(i) *For every $\mathbb{C}$-linear subspace $\mathcal{W} \subseteq \mathcal{V}$ satisfying $C(\mathcal{W}) = \mathcal{W}$, if we denote $\mathcal{X} := \mathcal{W} \cap \mathcal{V}$ then $\mathcal{W} = \mathcal{X}_{\mathbb{C}} = \mathcal{X} + \mathrm{i}\mathcal{X}$.*

(ii) *If $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{V}$ are $\mathbb{R}$-linear subspaces, then*

$$\mathcal{X} \subsetneqq \mathcal{Y} \iff \mathcal{X}_{\mathbb{C}} \subsetneqq \mathcal{Y}_{\mathbb{C}}; \tag{8}$$

$$(\mathcal{X} \cap \mathcal{Y})_{\mathbb{C}} = \mathcal{X}_{\mathbb{C}} \cap \mathcal{Y}_{\mathbb{C}}; \tag{9}$$

$$(\mathcal{X} + \mathcal{Y})_{\mathbb{C}} = \mathcal{X}_{\mathbb{C}} + \mathcal{Y}_{\mathbb{C}}.$$

Proof. Straightforward. ■

Definition 4.2. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If $\mathcal{V}$ is a $\mathbb{K}$-linear space and $B\colon \mathcal{V} \times \mathcal{V} \to \mathbb{K}$ is a skew-symmetric $\mathbb{K}$-bilinear functional, we denote

$$N(B) := \mathcal{V}^{\perp_B} := \{v_0 \in \mathcal{V} \mid (\forall v \in \mathcal{V}) \quad B(v_0, v) = \{0\}\}.$$

We now assume $m := \dim_{\mathbb{K}} \mathcal{V} < \infty$ and we fix a sequence of linear subspaces

$$\{0\} = \mathcal{V}_0 \subsetneqq \mathcal{V}_1 \subsetneqq \cdots \subsetneqq \mathcal{V}_m = \mathcal{V}$$

with $\dim_{\mathbb{K}} \mathcal{V}_j = j$ for $j = 0, \dots, m$, set $B_j := B|_{\mathcal{V}_j \times \mathcal{V}_j}$ for $j = 0, \dots, m$, and define

$$\mathfrak{p}(B) := N(B_1) + \cdots + N(B_m) \in \mathrm{Gr}(\mathcal{V}). \tag{10}$$

For every $\mathbb{K}$-linear subspace $\mathcal{W} \subseteq \mathcal{V}$ we define

$$\mathrm{jump}\, \mathcal{W} := \{j \in \{1, \dots, m\} \mid \mathcal{V}_j \not\subset \mathcal{V}_{j-1} + \mathcal{W}\}.$$

We now set $\mathfrak{p}^0(B) := \mathcal{V}$. Inductively, assume $k \geq 0$ is an integer and we have already defined the linear subspaces $\mathfrak{p}^0(B) \supsetneqq \cdots \supsetneqq \mathfrak{p}^k(B)$ of $\mathcal{V}$. If the condition $\mathfrak{p}^k(B) \not\perp_B \mathfrak{p}^k(B)$ is satisfied, then we define

$$i_{k+1}(B) := \min\{i \in \{0, \dots, m\} \mid \mathcal{V}_i \cap \mathfrak{p}^k(B) \not\perp_B \mathfrak{p}^k(B)\}, \tag{11}$$

$$\mathfrak{p}^{k+1}(B) := (\mathcal{V}_{i_{k+1}(B)} \cap \mathfrak{p}^k(B))^{\perp_B} \cap \mathfrak{p}^k(B) \tag{12}$$

Moreover, we define

$$j_{k+1}(B) := \min\{j \in \{0, \dots, m\} \mid \mathcal{V}_j \cap \mathfrak{p}^k(B) \not\subset \mathfrak{p}^{k+1}(B)\}. \tag{13}$$

Denoting $d := \dim_{\mathbb{K}}(\mathcal{V}/N(B))$ we have $d = 2\dim_{\mathbb{K}}(\mathcal{V}/\mathfrak{p}(B))$ and

$$\mathcal{V} = \mathfrak{p}^0(B) \supsetneqq \cdots \supsetneqq \mathfrak{p}^{d/2}(B) = \mathfrak{p}(B),$$

cf., [4, Lemmas 5.3 and 5.5].

Remark 4.3. In Definition 4.2 we have

$$i_k(B), j_k(B) \in \mathrm{jump}\, N(B), \quad i_k(B) < j_k(B), \quad i_k(B) < i_{k+1}(B)$$

cf. [4, Lemmas 5.6].

Moreover, the mapping

$$\{1,\dots,d\} \to \operatorname{jump} N(B) \setminus \operatorname{jump} \mathfrak{p}(B), \quad k \mapsto i_k(B)$$

is a well-defined increasing bijection by [4, Lemmas 5.7], and the mapping

$$\mathbf{j}(B)\colon \{1,\dots,d\} \to \operatorname{jump} \mathfrak{p}(B), \quad k \mapsto j_k(B)$$

is a well-defined bijection, cf., [4, Lemmas 5.8].

Lemma 4.4. *Let $\mathcal{V}$ be an $\mathbb{R}$-linear space with a family of $\mathbb{R}$-linear subspaces*

$$\{0\} = \mathcal{V}_0 \subsetneqq \mathcal{V}_1 \subsetneqq \cdots \subsetneqq \mathcal{V}_m = \mathcal{V}$$

with $\dim \mathcal{V}_j = j$ for $j = 1,\dots,m$, with their corresponding complexified spaces

$$\{0\} = \mathcal{V}_{0,\mathbb{C}} \subsetneqq \mathcal{V}_{1,\mathbb{C}} \subsetneqq \cdots \subsetneqq \mathcal{V}_{m,\mathbb{C}} = \mathcal{V}_{\mathbb{C}}$$

If $B\colon \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ is a skew-symmetric $\mathbb{R}$-bilinear functional, we define the skew-symmetric $\mathbb{R}$-bilinear functionals $B_j := B|_{\mathcal{V}_j\times\mathcal{V}_j}\colon \mathcal{V}_j\times\mathcal{V}_j\to\mathbb{R}$ for $j=0,\dots,m$, with their corresponding skew-symmetric $\mathbb{C}$-bilinear extensions $B_{\mathbb{C}}\colon \mathcal{V}_{\mathbb{C}}\times\mathcal{V}_{\mathbb{C}}\to\mathbb{C}$ and $B_{j,\mathbb{C}} := B_{\mathbb{C}}|_{\mathcal{V}_{j,\mathbb{C}}\times\mathcal{V}_{j,\mathbb{C}}}\colon \mathcal{V}_{j,\mathbb{C}}\times\mathcal{V}_{j,\mathbb{C}}\to\mathbb{C}$ for $j=0,\dots,m$. Then the following assertions hold:

(i) *We have $N(B_{j,\mathbb{C}}) = N(B_j)_{\mathbb{C}}$ for $j=1,\dots,m$.*

(ii) *We have $\mathfrak{p}^k(B_{\mathbb{C}}) = \mathfrak{p}^k(B)_{\mathbb{C}}$, $i_k(B_{\mathbb{C}}) = i_k(B)$, and $j_k(B_{\mathbb{C}}) = j_k(B)$ for $k=1,\dots,d/2$.*

(iii) *For every $\mathbb{R}$-linear subspace $\mathcal{W}\subseteq\mathcal{V}$ we have $\operatorname{jump}\mathcal{W} = \operatorname{jump}\mathcal{W}_{\mathbb{C}}$.*

Proof. Straightforward, using Lemma 4.1. ∎

Let G be a completely solvable Lie group with its Lie algebra $\mathfrak{g}$ and fix a Jordan-Hölder sequence, that is, a chain of ideals

$$\{0\} = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \cdots \subseteq \mathfrak{g}_m = \mathfrak{g} \tag{14}$$

with $\dim \mathfrak{g}_j = j$ for $j = 0,1,\dots,m$. We denote by $\langle\cdot,\cdot\rangle\colon \mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$ and we define

$$(\forall \xi\in\mathfrak{g}^*) \quad B_\xi\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}, \quad B_\xi(x,y) := \langle \xi, [x,y]\rangle.$$

Definition 4.5. We introduce the sets

$$J_m := \{\mathbf{k} = (k_1,\dots,k_m)\in\mathbb{N}^m \mid 0\le k_j\le j \text{ for } j=0,\dots,m\} \tag{15}$$

and

$$\Xi_{\mathbf{k}} := \{\xi\in\mathfrak{g}^* \mid \dim\mathfrak{g}_j(\xi|_{\mathfrak{g}_j}) = k_j \text{ for } j=1,\dots,m\} \tag{16}$$

for $\mathbf{k} = (k_1,\dots,k_m)\in J_m$, where $\mathfrak{g}_j(\xi|_{\mathfrak{g}_j}) = \{x\in\mathfrak{g}_j \mid (\forall y\in\mathfrak{g}_j)\ \langle\xi,[x,y] = 0\}$ for $j=1,\dots,m$. It is clear that we have the partition

$$\mathfrak{g}^* = \bigsqcup_{\mathbf{k}\in J_m} \Xi_{\mathbf{k}}. \tag{17}$$

where some of the sets in the right-hand side may be empty. This partition is called the *fine layering* of $\mathfrak{g}^*$ and the G-invariant sets $\Xi_{\mathbf{k}}$ are called the *fine layers of* $\mathfrak{g}^*$ with respect to the Jordan-Hölder sequence (14).

For arbitrary $\xi \in \mathfrak{g}^*$, if we set $\mathfrak{p}_{\rm alg}(\xi) := \mathfrak{p}(B_\xi)$ we have $\mathfrak{g}(\xi) \subseteq \mathfrak{p}_{\rm alg}(\xi)$, therefore $\operatorname{jump} \mathfrak{p}_{\rm alg}(\xi) \subseteq \operatorname{jump} \mathfrak{g}(\xi)$. If we denote $d := \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{g}(\xi)) = \frac{1}{2}\dim(G\xi) \in \mathbb{N}$, then, by Remark 4.3, the elements of $\operatorname{jump} \mathfrak{p}_{\rm alg}(\xi)$ can be labeled via the bijective map

$$\mathbf{j}(B)\colon \{1,\dots,d\} \to \operatorname{jump} \mathfrak{p}_{\rm alg}(\xi), \quad k \mapsto j_k(B).$$

The multi-index $\mathbf{k} = (k_1,\dots,k_m) \in J_m$ with $\dim \mathfrak{g}_j(\xi|_{\mathfrak{g}_r}) = k_r$ for $r = 1,\dots,m$ is uniquely determined by the pair $(e,\mathbf{j})$, where $e = \operatorname{jump} \mathfrak{g}(\xi) \subseteq \{1,\dots,m\}$ with $\operatorname{card} e = 2d$ and $\mathbf{j} = \mathbf{j}(B)\colon \{1,\dots,d\} \to e$ is the injective map above, cf. [4, Lemma 5.9].

Definition 4.6. Let $\lambda_1,\dots\lambda_m\colon \mathfrak{g} \to \mathbb{R}$ be the roots with respect to the Jordan-Hölder series (14). Using the notation in Definition 4.2 for $\mathcal{V} = \mathfrak{g}$ and $\mathcal{V}_j = \mathfrak{g}_j$ for $j = 1,\dots,m$, we define

$$(\forall \xi \in \mathfrak{g}^*) \quad \mathbf{t}(\xi) := \{k \in \{1,\dots,d/2\} \mid \mathfrak{p}^{k-1}(B_\xi) \cap \operatorname{Ker} \lambda_{i_k} = \mathfrak{p}^k(B_\xi)\}.$$

Thus, if we denote by J the set of all triples $(e,\mathbf{j},\mathbf{t})$, where $e \subseteq \{1,\dots,m\}$, $\operatorname{card} e$ is an even integer, $\mathbf{j}\colon \{1,\dots,(\operatorname{card} e)/2\} \to e$ is an injective map, and $\mathbf{t} \subseteq \{1,\dots,(\operatorname{card} e)/2\}$, then we obtain from (17) the *ultrafine layering*

$$\mathfrak{g}^* = \bigsqcup_{(e,\mathbf{j},\mathbf{t})\in J} \Xi_{(e,\mathbf{j},\mathbf{t})} \tag{18}$$

where, again, some sets in the right-hand side may be empty. Here

$$\Xi_{(e,\mathbf{j},\mathbf{t})} := \{\xi \in \Xi_{\mathbf{k}} \mid \mathbf{t}(\xi) = \mathbf{t}\},$$

where $\mathbf{k} \in J_m$ corresponds to the pair $(e,\mathbf{j})$ as explained above.

Remark 4.7. The layers $\Xi_{(e,\mathbf{j},\mathbf{t})}$ in (18) are G-invariant subsets of $\mathfrak{g}^*$. Therefore, we can use the quotient map $q\colon \mathfrak{g}^* \to \mathfrak{g}^*/G$ corresponding to the coadjoint action and the Kirillov-Bernat bijection $\kappa\colon \mathfrak{g}^*/G \to \widehat{G}$, to define the partition

$$\widehat{G} = \bigsqcup_{(e,\mathbf{j},\mathbf{t})\in J} \Gamma_{(e,\mathbf{j},\mathbf{t})}.$$

where some of the sets $\Gamma_{(e,\mathbf{j},\mathbf{t})} := \kappa(q(\Xi_{(e,\mathbf{j},\mathbf{t})}))$ in the right-hand side may be empty. This partition is called the *ultrafine layering* of the unitary dual $\widehat{G}$ and the sets $\Gamma_{\mathbf{k}}$ are called the *ultrafine layers of* $\widehat{G}$ with respect to the Jordan-Hölder sequence (14). ∎

The ultrafine layering is built directly from the Jordan-Hölder sequence (14), which in turn exists since the Lie algebra $\mathfrak{g}$ is completely solvable. For more general solvable Lie algebras, such a construction is not so direct as it involves passing to complexifications in order to accommodate the complex-valued roots of the Lie algebra under consideration. In order to be able to use results from the earlier literature, we show below that these two approaches lead to the same result in the present framework of completely solvable Lie algebras.

Lemma 4.8. *Let $\lambda_1,\dots\lambda_m\colon \mathfrak{g} \to \mathbb{R}$ be the roots with respect to the Jordan-Hölder series* (14). *For every $\xi \in \mathfrak{g}^*$ we denote by $\xi_{\mathbb{C}}\colon \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$ its $\mathbb{C}$-linear extension with the corresponding $\mathbb{C}$-bilinear functional*

$$B_{\xi_{\mathbb{C}}}\colon \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C},\ B_{\xi_{\mathbb{C}}}(v,w) := \xi_{\mathbb{C}}([v,w]).$$

Using the notation in Definition 4.2 for $\mathcal{V} = \mathfrak{g}$ and $\mathcal{V}_j = \mathfrak{g}_j$ for $j = 1, \dots, m$, we get

$$\mathbf{t}(\xi) = \{k \in \{1, \dots, d/2\} \mid \mathfrak{p}^{k-1}(B_{\xi_{\mathbb{C}}}) \cap \operatorname{Ker} \lambda_{i_k,\mathbb{C}} = \mathfrak{p}^k(B_{\xi_{\mathbb{C}}})\}$$

for every $\xi \in \mathfrak{g}^*$.

Proof. For arbitrary $k \in \{1, \dots, d/2\}$ we have

$$\mathfrak{p}^{k-1}(B_\xi)_{\mathbb{C}} = \mathfrak{p}^{k-1}(B_{\xi_{\mathbb{C}}}) \text{ and } \mathfrak{p}^k(B_\xi)_{\mathbb{C}} = \mathfrak{p}^k(B_{\xi_{\mathbb{C}}})$$

by Lemma 4.4(ii). On the other hand $(\operatorname{Ker} \lambda_{i_k})_{\mathbb{C}} = \operatorname{Ker} \lambda_{i_k,\mathbb{C}}$. Therefore, using equations (8)–(9) in Lemma 4.1, we obtain

$$\mathfrak{p}^{k-1}(B_\xi) \cap \operatorname{Ker} \lambda_{i_k} = \mathfrak{p}^k(B_\xi) \iff \mathfrak{p}^{k-1}(B_{\xi_{\mathbb{C}}}) \cap \operatorname{Ker} \lambda_{i_k,\mathbb{C}} = \mathfrak{p}^k(B_{\xi_{\mathbb{C}}})$$

which implies the assertion. ■

Remark 4.9. Lemma 4.8 above shows that the ultrafine layering from [8, Th. 2.8] and and [9, Thm.1.5 and 3.1] coincides with the one introduced in Definition 4.6 above. ■

Theorem 4.10. *If G is a completely solvable Lie group and $\mathcal{S}$ is a subquotient of $C^*(G)$ such that $\widehat{\mathcal{S}}$ is included in one of the ultrafine layers with respect to a Jordan-Hölder sequence, then $\mathcal{S} \in \mathrm{SQ}^{\mathrm{Tr}}(C^*(G))$ and there exists a $*$-isomorphism $\mathcal{S} \simeq \mathcal{C}_0(\widehat{\mathcal{S}}, \mathcal{K}(\mathcal{H}))$ for a suitable separable Hilbert space $\mathcal{H}$.*

Corollary 4.11. *The C^*-algebra of every completely solvable Lie group is solvable.*

Remark 4.12. In the special case of nilpotent Lie groups, the ultrafine layering coincides with the fine layering. Therefore, in this case, the result of Corollary 4.11 coincides with the result obtained earlier earlier in [6, Th. 4.11] with a completely different proof.

Proof of Theorem 4.10. We denote $\Gamma := \widehat{\mathcal{S}}$. Since Γ is contained in a ultrafine layer, we have either $(q \circ \kappa)^{-1}(\Gamma) \subseteq [\mathfrak{g}, \mathfrak{g}]^\perp$ or $(q \circ \kappa)^{-1}(\Gamma) \subseteq \mathfrak{g}^* \setminus [\mathfrak{g}, \mathfrak{g}]^\perp$. In the first of these cases the subquotient Γ is commutative, so the assertion follows at once.

Now let us assume $(q \circ \kappa)^{-1}(\Gamma) \subseteq \mathfrak{g}^* \setminus [\mathfrak{g}, \mathfrak{g}]^\perp$. We check that all the conditions in Hypothesis 3.9 are satisfied.

We first recall from Remark 4.9 that, by Lemma 4.8, the ultrafine layering from [8, Th. 2.8] and [9, Thm.1.5 and 3.1] coincides with the one introduced in Definition 4.6 above.

Condition (i) follows from [9, Thm. 3.1] and Remark 4.9. Condition (ii) follows again by Remark 4.9 and [8, Th. 2.8] or [10], which shows that every ultrafine layer in $\mathfrak{g}^*$ has a cross-section.

For condition (ii) we note that $\mathfrak{p}_{\mathrm{alg}} \colon \mathfrak{g}^* \to \mathrm{Gr}(\mathfrak{g})$ is the Vergne polarization mapping. Thus for every ξ, $\mathfrak{p}_{\mathrm{alg}}(\xi)$ is a real polarization at ξ that satisfies Pukánszky's condition. (See [7, Th. 4.3.6 and Cor. 4.3.7] or [20, Cor. 3.2].) On the other hand, the mapping $\xi \to \mathfrak{p}_{\mathrm{alg}}(\xi)$ is continuous on every fine layer $\Xi_{\mathbf{k}} \subseteq \mathfrak{g}^*$ by [4, Th. 6.5], hence also on Γ.

Thus all the conditions in Hypothesis 3.9 are satisfied, and then Corollary 3.11 is applicable. Therefore $\mathcal{S}$ is $*$-isomorphic to the C^*-algebra of sections of a trivial continuous field of infinite-dimensional separable Hilbert spaces. This directly implies the assertion. ■

Proof of Corollary 4.11. We fix a Jordan-Hölder sequence in $\mathfrak{g}$. By Remark 4.9 and [9, Th. 3.1], there exists a finite increasing family of G-invariant dense open subsets

$$\emptyset = \Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega_N = \mathfrak{g}^*$$

such that for every $r \in \{1, \dots, N\}$ there exists $(e, \mathbf{j}, \mathbf{t}) \in J$ with $\Omega_r \setminus \Omega_{r-1} \subseteq \Xi_{(e,\mathbf{j},\mathbf{t})}$. For $r = 0, \dots, N$, let $\mathcal{J}_r \subseteq C^*(G)$ be the closed two-sided ideal corresponding to the open subset $\kappa(q(\Omega_r)) \subseteq \widehat{G}$, that is, $\widehat{\mathcal{J}_r} = \kappa(q(\Omega_r))$. Then we have

$$\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_N = C^*(G).$$

Moreover, denoting $\mathcal{S}_r := \mathcal{J}_r/\mathcal{J}_{r-1}$, we have $\widehat{\mathcal{S}_r} = \kappa(q(\Omega_r \setminus \Omega_{r-1}))$ is contained in an ultrafine layer. Therefore, by using Theorem 4.10, we obtain a $*$-isomorphism $\mathcal{S}_r \simeq \mathcal{C}_0(\widehat{\mathcal{S}}, \mathcal{K}(\mathcal{H}_r))$ for a suitable separable Hilbert space $\mathcal{H}_r$. This shows that the C^*-algebra $C^*(G)$ is solvable. ■

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