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Local Noncommutative De Leeuw Theorems Beyond Reductive Lie Groups

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Abstract. Let Γ be a discrete subgroup of a unimodular locally compact group G. M. Caspers et al. [*Local and multilinear noncommutative de Leeuw theorems*, Math. Ann. 388 (2024) 4251–4305] showed that the L_p-norm of a Fourier multiplier $m\colon G \to \mathbb{C}$ on Γ can be bounded locally by its L_p-norm on G, modulo a constant $c(A)$ which depends on the support A of $m|_\Gamma$. In the context where G is a connected Lie group with Lie algebra $\mathfrak{g}$, we develop tools to find explicit bounds on $c(A)$. We show that the problem reduces to:

(1) The adjoint representation of the semisimple quotient $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$ of $\mathfrak{g}$ by the radical $\mathfrak{r} \subseteq \mathfrak{g}$ (which was handled in the paper of M. Caspers et al. cited above).

(2) The action of $\mathfrak{s}$ on a set of real irreducible representations that arise from quotients of the commutator series of $\mathfrak{r}$.

In particular, we show that $c(G) = 1$ for unimodular connected solvable Lie groups.

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Dedicated to Karl-Hermann Neeb on the occasion of his 60th birthday.

1. Introduction

For a unimodular locally compact group G, the the left regular representation L of G on $L_2(G)$, defined by $(L(s)\psi)(t) = \psi(s^{-1}t)$, is unitary. For $f \in L_1(G)$ we define $L(f)\psi(s) = \int_G f(t)\psi(t^{-1}s)\mu(dt)$ by integration against the Haar measure μ. The group von Neumann algebra $\mathcal{L}(G) := L(G)''$ then admits a unique normal semi-finite faithful trace τ, which satisfies

$$\tau(L(f)^*L(f)) = \langle f, f\rangle_{L_2(G)} \tag{1}$$

for $f \in L_1(G) \cap L_2(G)$. This gives rise to the L_p-norm $\|x\|_p := \tau(|x|^p)^{1/p}$, and the noncommutative L_p-space $L_p(\widehat{G})$ is defined as the completion of the space $\{x \in \mathcal{L}(G)\,;\, \|x\|_p < \infty\}$ with respect to the norm $\|\,.\,\|_p$.

A continuous bounded function $m\colon G \to \mathbb{C}$ is a p*-multiplier* if there exists a bounded linear map $T_m\colon L_p(\widehat{G}) \to L_p(\widehat{G})$ such that $T_m L(f) = L(mf)$ for all f in the twofold convolution product $C_c(G) * C_c(G)$ of the compactly supported continuous functions [1, 4, 14]. If it exists, it is unique because the set $\{L(f)\,;\, f \in C_c(G) * C_c(G)\}$ is dense in $L_p(G)$.

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To justify the above notations and terminology, note that if G is abelian, then $L_p(\widehat{G})$ coincides with the space of L_p-functions on the Pontryagin dual $\widehat{G}$ of G. The map T_m is then obtained by applying the Fourier transform from G to $\widehat{G}$, multiplying with m, and then applying the inverse Fourier transform.

In general, the problem of determining whether a given function m is a Fourier multiplier is quite nontrivial even in the commutative case [7, 8, 16]. In the noncommutative case, recent progress was obtained in [6, 9, 10, 11, 12, 13], see also [3] for a more extensive introduction to this topic.

In this connection, it is natural to study the relation between the L_p-norm

$$\|T_m \colon L_p(\widehat{G}) \to L_p(\widehat{G})\|$$

of a Fourier multiplier $m\colon G \to \mathbb{C}$ and the L_p-norm

$$\|T_{m|_\Gamma} \colon L_p(\widehat{\Gamma}) \to L_p(\widehat{\Gamma})\|$$

of its restriction $m_\Gamma \colon \Gamma \to \mathbb{C}$ to a discrete subgroup $\Gamma \subseteq G$. It is a celebrated theorem of DeLeeuw [15] that these two norms are the same for the inclusion $\mathbb{Z} \subset \mathbb{R}$. In the noncommutative case, the following result in this direction was obtained in [3], relying heavily on on techniques from [2].

Definition 1.1. Let G be a locally compact group with left Haar measure μ, and let $F \subseteq G$ be a subset. For an open, relatively compact subset $U \subseteq G$, we define

$$\delta_F(U) := \frac{\mu(\cap_{g\in F}\, gUg^{-1})}{\mu(U)}.$$

For an open neighbourhood basis $\mathcal{V}$ of the identity of G that consists of symmetric sets $U = U^{-1}$, we define $\delta_F(\mathcal{V}) := \liminf_{U\in\mathcal{V}} \delta_F(U)$. Then δ_F is defined as the supremum of $\delta_F(\mathcal{V})$ over all open, symmetric neighbourhood bases $\mathcal{V}$ of the identity. ■

For a subset $A \subseteq G$, we further define

$$c(A) := \inf\{\delta_F^{\frac{1}{2}}\,;\; F \subseteq A, F \text{ finite}\}. \tag{2}$$

We then have the following bound on $\|T_{m|_\Gamma} \colon L_p(\widehat{\Gamma}) \to L_p(\widehat{\Gamma})\|$ in terms of $c(\operatorname{supp}(m|_\Gamma))$ [3, Theorem A].

Theorem 1.2 (Local noncommutative De Leeuw Theorem)**.** *Let G be a unimodular second countable locally compact group, and let $m\colon G \to \mathbb{C}$ be a bounded, continuous function. Then for every $1 \leq p < \infty$ we have*

$$c(\operatorname{supp}(m|_\Gamma)) \cdot \|T_{m|_\Gamma} \colon L_p(\widehat{\Gamma}) \to L_p(\widehat{\Gamma})\| \leq \|T_m \colon L_p(\widehat{G}) \to L_p(\widehat{G})\|. \tag{3}$$

The strength of this result is that the constants δ_F, and hence the constants $c(A)$, can be determined explicitly in many interesting situations. For real reductive Lie groups G (which are automatically unimodular), the following explicit bound on δ_F was obtained in [3, Theorem B]. It results in nontrivial bounds for (3) in the *local* case, where $m\colon G \to \mathbb{C}$ is compactly supported.

Let $\mathfrak{g}$ be the Lie algebra of G, let $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation, and let $B_\theta(x, y) = -B(x, \theta y)$ be the inner product derived from the invariant bilinear form B and the Cartan involution θ.

Theorem 1.3 (Reductive groups). *Let G be a real reductive Lie group, and let $F \subseteq G$ be a relatively compact, symmetric subset that contains the identity. Then*

$$\delta_F \geq \rho^{-d/2}, \tag{4}$$

where $\rho := \sup_{g \in F} \|\mathrm{Ad}_g\|$ is the maximal norm of F in the adjoint representation with respect to the inner product B_θ, and d is the maximal dimension of a nilpotent adjoint orbit.

The aim of the present paper is to develop methods for bounding δ_F for connected Lie groups G beyond the class of reductive Lie groups. In view of Theorem 1.2, this immediately translates to bounds on the L_p norm of a Fourier multiplier $m|_\Gamma$ for discrete subgroups $\Gamma \subseteq G$. We show that $c(G) = 1$ for unimodular connected solvable Lie groups (Corollary 5.5). More generally, for an arbitrary connected Lie group G, we decompose δ_F into a bound that depends only on the adjoint representation of the Levi subalgebra $\mathfrak{s} \subseteq \mathfrak{r}$ (where Theorem 1.3 applies), and bounds that arise from the decomposition of the radical $\mathfrak{r} \subseteq \mathfrak{g}$ into irreducible representations for the Levi factor $\mathfrak{s}$. We describe the remaining open problems, and indicate a direction of further research.

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2. Linearization

Let G be a Lie group, and let (π, V) be a continuous representation of G on a finite-dimensional real vector space V. We denote the Lebesgue measure on V by Λ. A subset $F \subseteq G$ is called *symmetric* if $F = F^{-1}$, and $U \subseteq V$ is called symmetric if $U = -U$. The following is a linearized version of Definition 1.1.

Definition 2.1. For a relatively compact subset $F \subseteq G$ and a relatively compact open neighbourhood $U \subseteq V$ of $0 \in V$, we define

$$\delta_F^V(U) := \frac{\Lambda\big(\bigcap_{g \in F} \pi(g)(U)\big)}{\Lambda(U)}. \tag{5}$$

For a neighbourhood basis $\mathcal{V}$ of $0 \in V$ that consists of symmetric open sets, we set $\delta_F^V(\mathcal{V}) := \liminf_{U \in \mathcal{V}} \delta_F^V(U)$, and we define δ_F^V to be the supremum of $\delta_F^V(\mathcal{V})$ over all such neighbourhood bases. ■

According to the following minor generalization of [3, Prop. 8.5], it suffices to consider the above linearized version of δ_F for the adjoint representation.

Proposition 2.2 (Linearization). *For a Lie group G, we have $\delta_F = \delta_F^{\mathfrak{g}}$ for the adjoint representation $\mathfrak{g}$ of G.*

Proof. Since the exponential map $\exp\colon \mathfrak{g} \to G$ is a local diffeomorphism around zero, we may assume without loss of generality that the sets $U_G \subseteq G$ in the neighbourhood basis $\mathcal{V}_G$ for $1 \in G$ correspond to sets $U_{\mathfrak{g}} \subseteq \mathfrak{g}$ in a neighbourhood basis $\mathcal{V}_{\mathfrak{g}}$ for $0 \in \mathfrak{g}$ by $U_G = \exp(U_{\mathfrak{g}})$, and that the restriction $\exp\colon U_{\mathfrak{g}} \to U_G$ is a diffeomorphism. Since the exponential map is G-equivariant (so $\exp \circ \mathrm{Ad}_g(x) = g \exp(x) g^{-1}$ for all $g \in G$ and $x \in \mathfrak{g}$), we have

$$\frac{\mu(\bigcap_{g \in F} g U_G g^{-1})}{\mu(U_G)} = \frac{\mu \circ \exp(\bigcap_{g \in F} \mathrm{Ad}_g(U_{\mathfrak{g}}))}{\mu \circ \exp(U_{\mathfrak{g}})}.$$

It remains to compare $\mu \circ \exp$ to the Lebesgue measure on small zero-neighbourhoods.

Let vol_G be a left invariant volume form on G such that $\mu(U) = \int_U \mathrm{vol}_G$ for open relatively compact subsets $U \subseteq G$, and let $\mathrm{vol}_{\mathfrak{g}}$ be a translation invariant volume form on $\mathfrak{g}$ such that $\exp^* \mathrm{vol}_G|_0 = \mathrm{vol}_{\mathfrak{g}}|_0$ at the origin. Then $\exp^* \mathrm{vol}_G = \nu \mathrm{vol}_{\mathfrak{g}}$ for a smooth function $\nu\colon \mathfrak{g} \to \mathbb{R}$ with $\nu(0) = 1$. (In fact, $\nu(x) = \det(D^L_x \exp)$ is the determinant of the left logarithmic derivative

$$D^L \exp_x = \frac{\mathrm{Id}_{\mathfrak{g}} - \exp(-\mathrm{ad}_x)}{\mathrm{ad}_x}$$

of the exponential map [3, Prop. 8.5], but we will not need this fact here.) Since ν is smooth on $\mathfrak{g}$, there exists a radius $R > 0$ and a constant $C > 0$ such that $|\nu(x) - 1| \leq C\|x\|$ for all x in the ball $B_R(0)$ of radius R around zero. If $U_{\mathfrak{g}}$ is contained in $B_\varepsilon(0)$ with $0 < \varepsilon < R$, we then have

$$(1-\varepsilon)\Lambda(W) < \mu \circ \exp(W) \leq (1+\varepsilon)\Lambda(W)$$

for all open $W \subseteq U_{\mathfrak{g}}$. It follows that

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right) \frac{\Lambda(\cap_{g\in F} \mathrm{Ad}_g(U_{\mathfrak{g}}))}{\Lambda(U_{\mathfrak{g}})} \leq \frac{\mu \circ \exp(\cap_{g\in F} \mathrm{Ad}_g(U_{\mathfrak{g}}))}{\mu \circ \exp(U_{\mathfrak{g}})} \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right) \frac{\Lambda(\cap_{g\in F} \mathrm{Ad}_g(U_{\mathfrak{g}}))}{\Lambda(U_{\mathfrak{g}})}.$$

Since any ball $B_\varepsilon(0)$ contains an open set $U_{\mathfrak{g}}$ from $\mathcal{V}$, the result follows. ■

3. Trivial bounds and the Reduction Lemma

After proving a trivial bound on δ^V_F, we prove a Reduction Lemma that bounds δ^V_F in terms of δ^W_F and $\delta^{V/W}_F$ for every subrepresentation $W \subseteq V$. If G is a connected Lie group with Lie algebra $\mathfrak{g}$, then every ideal $\mathfrak{h} \subseteq \mathfrak{g}$ is a subrepresentation of the adjoint representation. In particular, the Reduction Lemma can be used to bound $\delta^{\mathfrak{g}}_F$ in terms of $\delta^{\mathfrak{h}}_F$ and $\delta^{\mathfrak{g}/\mathfrak{h}}_F$.

3.1. The trivial bound

We equip V with an inner product, and define

$$\rho := \sup_{g\in F} \|\pi(g)\|. \tag{6}$$

We then obtain the following trivial bound on δ^V_F.

Proposition 3.1 (Trivial bound)**.** *For every relatively compact, symmetric subset $F \subseteq G$, we have*

$$\delta^V_F \geq \rho^{-\dim(V)}. \tag{7}$$

Proof. Let $\mathcal{V}$ be the neighbourhood basis consisting of open balls $B_r(0)$ of radius r. Since $\pi(g)(B_r(0)) \subseteq B_{\rho r}(0)$, we have $B_r(0) \subseteq \pi(g^{-1})B_{\rho r}(0)$ for all $g \in F$. Since F is symmetric ($F = F^{-1}$), we have $B_r(0) \subseteq \bigcap_{g\in F} \pi(g)B_{\rho r}(0)$, and

$$\frac{\Lambda(\bigcap_{g\in F} \pi(g)B_r(0))}{\Lambda(B_r(0))} \geq \frac{\Lambda(B_{r/\rho}(0))}{\Lambda(B_r(0))} = \rho^{-\dim(V)}$$

as required. ■

Remark 3.2. Note that unlike the quantity δ^V_F, the bound (7) depends on the choice of inner product.

If A is a symmetric, relatively compact subset of a Lie group G, then the constant $c(A)$ from (2) satisfies $c(A) \geq \rho^{-\frac{1}{2}\dim(\mathfrak{g})}$ for $\rho = \sup_{g\in A} \|\mathrm{Ad}_g\|$. In particular, this shows that if m is compactly supported, the bound (3) is not vacuous.

3.2. The Reduction Lemma

If a subrepresentation $W \subseteq V$ admits a complementary G-representation (so that V decomposes as $V = W \oplus W'$), then it is not hard to see that

$$\delta_F^V \geq \delta_F^W \delta_F^{V/W}.$$

By the following lemma, this remains true even if W is *not* complemented as a G-representation.

Lemma 3.3 (Reduction Lemma). *Let $F \subseteq G$ be a relatively compact, symmetric subset. Let $W \subseteq V$ be a subrepresentation, and let $\mathcal{V}_{V/W}$ and $\mathcal{V}_W$ be open neighbourhood bases of the origin in V/W and W, respectively. Then there exists an open neighbourhood basis $\mathcal{V}$ of the origin in V such that*

$$\delta_F^V(\mathcal{V}) \geq \delta_F^{V/W}(\mathcal{V}_{V/W})\delta_F^W(\mathcal{V}_W), \tag{8}$$

and such that $\mathcal{V}_V$ is symmetric if $\mathcal{V}_{V/W}$ and $\mathcal{V}_W$ are symmetric.

We introduce some notation that will be useful in the proof. For a real representation (π, V_π) and an open subset $A \subseteq V_\pi$, we set $A^F := \bigcap_{g\in F} \big(\pi(g)(A)\big)$. Further, we choose an inner product on V_π and set

$$A^r := \bigcap_{x\in B_r(0)} (A + x),$$

where $B_r(0)$ is the ball of radius r around 0. Note that $A = \bigcup_{r>0} A^r$ since A is open, so we find

$$\lim_{r\downarrow 0} \Lambda(A^r) = \Lambda(A). \tag{9}$$

More generally, for relatively compact subsets $F \subseteq G$ we have:

Proposition 3.4. $\lim_{r\downarrow 0} \Lambda((A^r)^F) = \Lambda(A^F)$.

Proof. Since $\pi(g)B_r(0) \subseteq B_{\rho r}(0)$ for all $g \in F$, we have

$$\begin{aligned}(A^r)^F &= \bigcap_{g\in F}\bigcap_{x\in B_r(0)} \pi(g)(A + x) = \bigcap_{x\in B_r(0) \text{ and } g\in F} \pi(g)(A) + \pi(g)x \\ &\supseteq \bigcap_{y\in B_{\rho r}(0)}\bigcap_{g\in F} \pi(g)(A) + y = (A^F)^{\rho r}.\end{aligned}$$

Since $(A^F)^{\rho r} \subseteq (A^r)^F \subseteq A^F$, equation (9) yields $\lim_{r\downarrow 0} \Lambda((A^r)^F) = \Lambda(A^F)$. ∎

Proof of Lemma 3.3. Fix an inner product on V, and identify $W^\perp$ with V/W by means of the quotient map. For $U_W \in \mathcal{V}_W$, $U_{V/W} \in \mathcal{V}_{V/W}$ and $\varepsilon > 0$, we set

$$U_\varepsilon := U_W \times \varepsilon U_{V/W} \subseteq W \times V/W \simeq V.$$

Let P_W and $P_{W^\perp}$ be the orthogonal projections onto W and $W^\perp$, respectively. For $w \in W$ and $w' \in W^\perp$, we then have

$$\pi(g)(w \oplus \varepsilon w') = (\pi_W(g)w + \varepsilon P_W \pi(g) w') \oplus \varepsilon P_{W^\perp}\pi(g)w' \tag{10}$$

in $W \oplus W^\perp$.

Without loss of generality, we may assume that $U_{V/W}$ is contained in a ball of radius 1. Let $r := \varepsilon\rho$, where $\rho := \sup_{g\in F} \|\pi(g)\|$ as before. For $w \in ((U_W)^r)^F$ and $w' \in U^F_{V/W}$, we then have $\pi(g)w \in (U_W)^r$ and $\|\varepsilon P_W \pi(g)w'\| \leq \varepsilon\rho$, so the first component of (10) is contained in U_W. Since $[P_{W^\perp}\pi(g)w'] = \pi_{V/W}(g)([w'])$ in V/W, the second component of (10) is contained in $\varepsilon U_{V/W}$. It follows that

$$\pi(g)\big(((U_W)^r)^F \times \varepsilon U^F_{V/W}\big) \subseteq U_W \times \varepsilon U_{V/W}$$

for all $g \in F$. Since F is symmetric, we conclude that

$$((U_W)^r)^F \times \varepsilon U^F_{V/W} \subseteq \bigcap_{g\in F} \pi(g)(U_W \times \varepsilon U_{V/W}) = U^F_\varepsilon .$$

So
$$\frac{\Lambda(U^F_\varepsilon)}{\Lambda(U_\varepsilon)} \geq \frac{\varepsilon\Lambda\big(((U_W)^r)^F\big)\Lambda(U^F_{V/W})}{\varepsilon\Lambda(U_W)\Lambda(U_{V/W})},$$

and since $\lim_{r\downarrow 0} \Lambda((U_W)^r)^F = \Lambda(U^F_W)$ by Proposition 3.4, we conclude that

$$\liminf_{\varepsilon\downarrow 0} \delta^V_F(U_\varepsilon) \geq \delta^W_F(U_W)\delta^{V/W}_F(U_{V/W}). \tag{11}$$

The neighbourhood basis $\mathcal{V}_V$ is constructed as follows. Select for every $n \in \mathbb{N}$ an open set $U^n_W \in \mathcal{V}_W$ that is contained in a ball of radius $1/n$, and such that $U^n_W \subseteq U^{n-1}_W$ for $n > 1$. Select $U^n_{V/W}$ in a similar fashion, and select $0 < \varepsilon_n \leq 1$ such that $\varepsilon_n \leq \varepsilon_{n-1}$ for $n \in \mathbb{N}$, and

$$\delta^V_F(U_{\varepsilon_n}) \geq (1 - \tfrac{1}{n})\delta^W_F(U^n_W)\delta^{V/W}_F(U^n_{V/W}).$$

Then $\mathcal{V}_V = \{U^n_W \times \varepsilon_n U^n_{V/W}\,;\, n \in \mathbb{N}\}$ is a symmetric neighbourhood basis of $0 \in V$ that satisfies (8). ∎

3.3. Connected Lie groups

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{r} \subseteq \mathfrak{g}$ be the radical of $\mathfrak{g}$, i.e. the maximal solvable ideal. Then $\mathfrak{r}$ integrates to a closed, connected, normal Lie subgroup $R \subseteq G$, and the semisimple quotient $S := G/R$ has Lie algebra $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$ [17, Thm. 3.18.13].

Let $F \subseteq G$ be a relatively compact, symmetric subset. Using the Reduction Lemma 3.3, we obtain a bound

$$\delta^{\mathfrak{g}}_F \geq \delta^{\mathfrak{s}}_{[F]}\delta^{\mathfrak{r}}_F \tag{12}$$

with two types of contributions. Since $R \subseteq G$ acts trivially on $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$, the adjoint action of G on $\mathfrak{s}$ factors through the adjoint representation of $S = G/R$. The first factor $\delta^{\mathfrak{s}}_{[F]}$ therefore depends only on the adjoint representation of S, and on the image $[F] \subseteq S$ of $F \subseteq G$ under the quotient map $G \to S$. Since S is semisimple, the factor $\delta^{\mathfrak{s}}_{[F]}$ can be bounded using Theorem 1.3.

In the remainder of this paper we therefore focus on the second factor $\delta^{\mathfrak{r}}_F$, which depends only on the adjoint action of G on the subrepresentation $\mathfrak{r} \subseteq \mathfrak{g}$.

4. Solvable Lie groups and Lie algebras

Let G be a connected Lie group that acts by automorphisms on a solvable Lie algebra $\mathfrak{r}$, and let $F \subseteq G$ be a relatively compact, symmetric subset. We use the structure theory of solvable Lie algebras to further decompose $\delta_F^{\mathfrak{r}}$ into factors that depend only on the semisimple quotient $S = G/R$. The main tool for this is Lemma 4.6, an extension of the Reduction Lemma 3.3 that allows us to shift characters from a subrepresentation $W \subseteq V$ to the quotient V/W. We apply this to the G-representation $\mathfrak{r}$, whose subrepresentations are essentially given by Lie's Theorem.

4.1. Lie's Theorem for real solvable Lie algebras

In order to set notation, we briefly review Lie's Theorem for the adjoint representation of a solvable Lie algebra.

4.1.1. Root spaces. For a Lie algebra $\mathfrak{r}$, the commutator series is defined inductively by $\mathfrak{r}^0 := \mathfrak{r}$, and $\mathfrak{r}^{k+1} := [\mathfrak{r}^k, \mathfrak{r}^k]$ for $k > 0$. We say that $\mathfrak{r}$ is r-step solvable if $\mathfrak{r}^r = \{0\}$, in which case we have a finite filtration

$$\mathfrak{r} = \mathfrak{r}^0 \supset \mathfrak{r}^1 \supset \ldots \supset \mathfrak{r}^{r-1} \supset \mathfrak{r}^r = \{0\}. \tag{13}$$

Since $\mathfrak{r}^k$ is an ideal in $\mathfrak{r}$ [17, Thm. 3.7.1], we obtain a representation (π^k, V^k) of $\mathfrak{r}$ by setting $V^k := \mathfrak{r}^k/\mathfrak{r}^{k+1}$, and defining $\pi^k \colon \mathfrak{r} \to \mathrm{End}(V^k)$ by

$$\pi^k(x)([v]) := [\mathrm{ad}_x(v)] \tag{14}$$

for $x \in \mathfrak{r}$ and $[v] \in V^k$. Note that $\mathfrak{r}^k$ acts trivially on $\mathfrak{r}^k/\mathfrak{r}^{k+1}$, so the representation $\pi^k \colon \mathfrak{r} \to \mathrm{End}(V^k)$ factors through the quotient $\mathfrak{r}/\mathfrak{r}^k$. In particular, (π^0, V^0) is the trivial representation.

Let (π, V) be any representation of $\mathfrak{r}$, and let $\lambda \in \mathfrak{r}^*$. If

$$V_\lambda := \{v \in V\,;\, \pi(x)v = \lambda(x)v \text{ for all } x \in \mathfrak{r}\} \tag{15}$$

is nonzero, then λ is called a *weight* of V, and V_λ is called a *weight space.* Since every weight $\lambda \colon \mathfrak{r} \to \mathbb{C}$ is a Lie algebra homomorphism into an abelian Lie algebra, it vanishes on $[\mathfrak{r}, \mathfrak{r}]$, and we can identify the set of weights with a subset $W(V)$ of $(\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}])^*$. The weights of (V^k, π^k) are called the *roots of order* k. We denote the set of k^{th} order roots by $\Phi^k(\mathfrak{r}) := W(V^k)$, and we define

$$\Phi(\mathfrak{r}) := \{(k, \lambda)\,;\, k \in \{0, \ldots, r-1\}, \lambda \in \Phi^k(\mathfrak{r})\} \tag{16}$$

to be the set of all roots, labelled by k. (This is because the same root λ can occur in different representations $V^k \neq V^{k'}$.)

4.1.2. Automorphisms. Let G be a (not necessarily connected) Lie group with Lie algebra $\mathfrak{g}$, let $\mathfrak{r}$ be a solvable Lie algebra, and let $\alpha \colon G \times \mathfrak{r} \to \mathfrak{r}$ be a continuous action of G on $\mathfrak{r}$ by automorphisms. We write $\alpha_g(x)$ for the action of $g \in G$ on $x \in \mathfrak{r}$. The automorphisms α_g of $\mathfrak{r}$ restrict to automorphisms on $\mathfrak{r}^k$ and $\mathfrak{r}^{k+1}$, so that $V^k = \mathfrak{r}^k/\mathfrak{r}^{k+1}$ is a G-representation, which we denote by $\pi_G^k \colon G \to \mathrm{GL}(V^k)$. If $\lambda \in \Phi^k(\mathfrak{r})$ is a root of order k, and V_λ^k is its weight space, then $\lambda \circ \alpha_{g^{-1}}$ is a root of order k as well, and $\pi_G^k(g)$ maps V_λ^k to $V^k_{\lambda \circ \alpha_{g^{-1}}}$.

Indeed, for $[v] \in V^k_\lambda$ and $x \in \mathfrak{r}$, we have

$$\begin{aligned} \mathrm{ad}_x\big(\pi^k_G(g)([v])\big) &= \mathrm{ad}_x([\alpha_g(v)]) = [\alpha_g([\alpha_{g^{-1}}(x), v])] \\ &= \lambda\big(\alpha_{g^{-1}}(x)\big)[\alpha_g v] = \lambda \circ \alpha_{g^{-1}}(x)\big(\pi^k_G(g)([v])\big). \end{aligned}$$

Since the action of G on the discrete set $\Phi^k(\mathfrak{r}) \subseteq (\mathfrak{r}/[\mathfrak{r},\mathfrak{r}])^*$ of roots is continuous, the connected identity component G_0 fixes the weights, and every V^k_λ is a G_0-subrepresentation, which we denote by $\pi^k_\lambda\colon G_0 \to \mathrm{GL}(V^k_\lambda)$.

4.1.3. Lie's Theorem for the adjoint representation. Let $\mathfrak{r}$ be a solvable real Lie algebra, and let $\mathfrak{r}_\mathbb{C} := \mathfrak{r} \otimes_\mathbb{R} \mathbb{C}$ be its complexification.

Theorem 4.1 (Lie's Theorem (Thm. 3.7.3 in [17]))**.** *Every finite dimensional complex representation (V,π) of $\mathfrak{r}_\mathbb{C}$ has a weight space.*

We will need the following slight reformulation of Lie's Theorem in the particular context of the adjoint representation. Although it follows directly from Lie's Theorem, it is instructive to give a direct proof.

Theorem 4.2 (Lie's Theorem, special case)**.** *The $\mathfrak{r}_\mathbb{C}$-representation $V^k = \mathfrak{r}^k_\mathbb{C}/\mathfrak{r}^{k+1}_\mathbb{C}$ decomposes as a direct sum of weight spaces,*

$$V^k = \bigoplus_{\lambda \in \Phi^k(\mathfrak{r}_\mathbb{C})} V^k_\lambda. \tag{17}$$

Proof. Suppose that $\mathfrak{r}_\mathbb{C}$ is r-step solvable. If $r=1$, then the statement holds because the action of $\mathfrak{r}_\mathbb{C}$ on V^0 is trivial. Suppose by induction that the statement holds for all Lie algebras that are $(r-1)$-step solvable. Then V^0 is trivial, and for $k \geq 1$, the space V^k decomposes into weight spaces

$$V^k = \bigoplus_{\mu \in \Phi^{k-1}(\mathfrak{r}^1_\mathbb{C})} V^{k-1}_\mu \tag{18}$$

as an $\mathfrak{r}^1_\mathbb{C}$-representation, because $V^k(\mathfrak{r}_\mathbb{C}) = \mathfrak{r}^k_\mathbb{C}/\mathfrak{r}^{k+1}_\mathbb{C}$ is identical to $V^{k-1}(\mathfrak{r}^1_\mathbb{C}) = (\mathfrak{r}^1_\mathbb{C})^{k-1}/(\mathfrak{r}^1_\mathbb{C})^k$, and $\mathfrak{r}^1_\mathbb{C}$ is $(r-1)$-step solvable. Let $R \subseteq \mathrm{Aut}(\mathfrak{r})$ be the closed subgroup of $\mathrm{GL}(\mathfrak{r})$ generated by $\exp(\mathfrak{r})$. Since R is connected and acts on $\mathfrak{r}^1_\mathbb{C}$ by automorphisms, it respects the weight space decomposition (18). In particular, the weight spaces V^{k-1}_μ are subrepresentations of the $\mathfrak{r}_\mathbb{C}$-representation (π^k, V^k). Since the commutator

$$[\pi^k(x)|_{V^{k-1}_\mu}, \pi^k(y)|_{V^{k-1}_\mu}] = \mu([x,y])\mathrm{Id}_{V^{k-1}_\mu}$$

has zero trace for all $x, y \in \mathfrak{r}$, we conclude that $\mu \in (\mathfrak{r}^1_\mathbb{C})^*$ is identically zero. So $V^k(\mathfrak{r}_\mathbb{C}) = V^{k-1}(\mathfrak{r}^1_\mathbb{C})$ is trivial as an $\mathfrak{r}^1_\mathbb{C}$-representation, and the $\mathfrak{r}_\mathbb{C}$-representation (π^k, V^k) factors through $\mathfrak{r}_\mathbb{C}/\mathfrak{r}^1_\mathbb{C}$. Because this is an abelian Lie algebra, V^k decomposes into root spaces V^k_λ for a set of k^{th} order roots $\lambda \in (\mathfrak{r}_\mathbb{C}/[\mathfrak{r}_\mathbb{C},\mathfrak{r}_\mathbb{C}])^*$. ■

Since $\mathfrak{r}_\mathbb{C}$ is the complexification of the real Lie algebra $\mathfrak{r}$, the complex conjugation $\overline{\lambda}(x) := \overline{\lambda(\overline{x})}$ permutes the k^{th} order roots, and we have $\overline{V^k_\lambda} = V^k_{\overline{\lambda}}$ for the weight spaces of the $\mathfrak{r}_\mathbb{C}$-representation $V^k(\mathfrak{r}_\mathbb{C}) = \mathfrak{r}^k_\mathbb{C}/\mathfrak{r}^{k+1}_\mathbb{C}$. It follows that the real $\mathfrak{r}$-representation $V^k(\mathfrak{r}) = \mathfrak{r}^k/\mathfrak{r}^{k+1}$ does not necessarily decompose into weight spaces.

Rather, the isotypical components are

$$U_\lambda^k := \left(V_\lambda^k \oplus V_{\overline{\lambda}}^k\right) \cap (\mathfrak{r}^k/\mathfrak{r}^{k+1}) \quad \text{if } \lambda \neq \overline{\lambda}, \tag{19}$$

$$U_\lambda^k := V_\lambda^k \cap (\mathfrak{r}^k/\mathfrak{r}^{k+1}) \quad \text{if } \lambda = \overline{\lambda}. \tag{20}$$

The real $\mathfrak{r}$-representation $V^k(\mathfrak{r}) = \mathfrak{r}^k/\mathfrak{r}^{k+1}$ then decomposes as

$$V^k(\mathfrak{r}) = \bigoplus_{\{\lambda,\overline{\lambda}\}\subseteq\Phi^k(\mathfrak{r}_{\mathbb{C}})} U_\lambda^k. \tag{21}$$

4.2. Shifting characters

In order to take full advantage of the decomposition (21), we will need a generalization of the Reduction Lemma 3.3 that allows us to shift characters from a subrepresentation $W \subseteq V$ to the quotient V/W.

4.2.1. Motivating example. The motivating example is the representation $(\pi, \mathbb{R}^2)$ of $G = (\mathbb{R}^{>0}, \cdot\,)$ defined by

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

Here, the bound $\delta_F^{\mathbb{R}^2} \geq \rho^{-2}$ obtained from the Reduction Lemma is clearly suboptimal. Indeed, intersecting the neighbourhood $\pi(G)B_\varepsilon(0) = \{(x,y) \in \mathbb{R}^2;\ |xy| < \frac{1}{2}\varepsilon^2\}$ with the relatively compact square $C_R = \{(x,y) \in \mathbb{R}^2\,;\ |x| < R,\ |y| < R\}$ with sides $R \gg \varepsilon$, one obtains open neighbourhoods $U_{R,\varepsilon} = C_R \cap (\pi(G)B_\varepsilon(0))$ with volume

$$\Lambda(U_{R,\varepsilon}) = 2\varepsilon^2\Big(1 + \log\Big(\frac{2R^2}{\varepsilon^2}\Big)\Big)$$

that satisfy $\bigcap_{g\in F} \pi(g)U_{R,\varepsilon} \supseteq U_{R/\rho,\varepsilon}$. For $R = \sqrt{\varepsilon}$, the neighbourhood basis consisting of $U_{\sqrt{\varepsilon},\varepsilon}$ then yields the sharp bound

$$\delta_F^{\mathbb{R}^2} \geq \lim_{\varepsilon\downarrow 0} \frac{\Lambda(U_{\rho^{-1}\sqrt{\varepsilon},\varepsilon})}{\Lambda(U_{\sqrt{\varepsilon},\varepsilon})} = \lim_{\varepsilon\downarrow 0} \frac{2\varepsilon^2(1+\log(\frac{2\varepsilon}{\rho^2\varepsilon^2}))}{2\varepsilon^2(1+\log(\frac{2\varepsilon}{\varepsilon^2}))} = 1.$$

4.2.2. Neighbourhoods of logarithmic volume. Now let V_1 and V_2 be real vector spaces of dimension d_1 and d_2, respectively, and let $U \subseteq V_1$ and $V \subseteq V_2$ be relatively compact open subsets. Mimicking the above construction, we define $U \times_{R_1,R_2,\varepsilon} V \subseteq V_1 \times V_2$ for $\varepsilon \leq R_1^{d_1}R_2^{d_2}$ by

$$U \times_{R_1,R_2,\varepsilon} V := \{(\alpha u, \beta v)\,;\ u \in U, v \in V, |\alpha| \leq R_1, |\beta| \leq R_2, |\alpha^{d_1}\beta^{d_2}| \leq \varepsilon\}. \tag{22}$$

These sets are almost invariant under the $\mathbb{R}^{>0}$-action on $U \times V$ which is defined by $\mu \cdot (u,v) := (\mu^{\frac{1}{d_1}}u, \mu^{-\frac{1}{d_2}}v)$ in the following sense. For a symmetric set $F \subseteq \mathbb{R}^{>0}$, we set $D := \sup_{\mu\in F}|\mu|$. Then $\mu \cdot (U \times_{R_1,R_2,\varepsilon} V) \subseteq U \times_{D^{1/d_1}R_1, D^{1/d_2}R_2,\varepsilon} V$ for all $\mu \in F \subseteq \mathbb{R}^+$, with the same ε on both sides.

Definition 4.3. A subset U of a real vector space is *balanced* if $\alpha U \subseteq U$ for all real numbers α with $|\alpha| \leq 1$.

Lemma 4.4. *If U and V are balanced, then the inequality $|\alpha^{d_1}\beta^{d_2}| \leq \varepsilon$ in (22) can be replaced by $|\alpha^{d_1}\beta^{d_2}| = \varepsilon$.*

Proof. If $\alpha^{d_1} \leq \varepsilon/R_2^{d_2}$, then $(\alpha u, \beta v) = (\widetilde{\alpha}\widetilde{u}, \widetilde{\beta}\widetilde{v})$ with $\widetilde{\alpha} = (\varepsilon/R_2^{d_2})^{1/d_1}$, $\widetilde{\beta} = R_2$, $\widetilde{u} = (\alpha/\widetilde{\alpha})u$ and $\widetilde{v} = (\beta/\widetilde{\beta})v$. Then $\widetilde{u} \in U$ and $\widetilde{v} \in V$ because U and V are balanced, $|\widetilde{\alpha}| \leq R_1$ because $\varepsilon \leq R_1^{d_1}R_2^{d_2}$, and $|\widetilde{\alpha}^{d_1}\widetilde{\beta}^{d_2}| = \varepsilon$ by definition.

If $\alpha^{d_1} \geq \varepsilon/R_2^{d_2}$, then $(\alpha u, \beta v) = (\alpha u, \widetilde{\beta}\widetilde{v})$ for $\widetilde{\beta} = (\varepsilon/\alpha^{d_1})^{1/d_2}$ and $\widetilde{v} = (\beta/\widetilde{\beta})v$. Then $\widetilde{v} \in V$ because V is balanced and $(\beta/\widetilde{\beta})^{d_2} = \alpha^{d_1}\beta^{d_2}/\varepsilon \leq 1$, and $|\widetilde{\beta}| \leq R_2$ because $\alpha^{d_1} \geq \varepsilon/R_2^{d_2}$. ■

Lemma 4.5. *If U and V are balanced and $0 < \varepsilon < R_1^{d_1}R_2^{d_2}$, then*

$$\Lambda(U \times_{R_1,R_2,\varepsilon} V) = \varepsilon\Big(1 + \log\Big(\frac{R_1^{d_1}R_2^{d_2}}{\varepsilon}\Big)\Big)\Lambda(U)\Lambda(V).$$

Proof. Scaling V_1 by R_1 and V_2 by R_2, we find

$$\Lambda(U \times_{R_1,R_2,\varepsilon} V) = R_1^{d_1}R_2^{d_2}v(\varepsilon/R_1^{d_1}R_2^{d_2}), \tag{23}$$

where $v(\varepsilon) := \Lambda(U \times_{1,1,\varepsilon} V)$. For an ascending sequence $\varepsilon = \alpha_0^{d_1} < \ldots < \alpha_N^{d_1} = 1$ and the corresponding descending sequence $\beta_i^{d_2} = \varepsilon/\alpha_i^{d_1}$, we find

$$A_N \subseteq U \times_{1,1,\varepsilon} V \subseteq B_N,$$

where $A_N = \bigcup_{n=0}^{N-1}(\alpha_n U) \times (\beta_{n+1}V)$ and $B_N = \bigcup_{n=0}^{N-1}(\alpha_{n+1}U) \times (\beta_n V)$. (The second inclusion uses that both U and V are balanced). To find an underestimate for $v(\varepsilon)$, write A_N as the disjoint union

$$A_N = (\alpha_0 U) \times (\beta_1 V) \sqcup \bigsqcup_{n=0}^{N-1}(\alpha_{n+1}U \setminus \alpha_n U) \times (\beta_{n+1}V).$$

Since U is balanced, $\alpha_n U \subseteq \alpha_{n+1}U$, so that

$$\Lambda(A_N) = \Lambda(U)\Lambda(V)\Big(\alpha_0^{d_1}\beta_1^{d_2} + \sum_{n=0}^{N-1}\beta_{n+1}^{d_2}(\alpha_{n+1}^{d_1} - \alpha_n^{d_1})\Big).$$

Setting $x_n = \alpha_n^{d_1}$ and $y(x) = \frac{\varepsilon}{x}$, one sees that as the subdivision becomes finer, this converges to $\Lambda(U)\Lambda(V)(\varepsilon + \int_\varepsilon^1 \frac{\varepsilon}{x})dx$. Since the upper bound from B_N gives the same result, we have $v(\varepsilon) = \varepsilon(1 + \log(\frac{1}{\varepsilon}))\Lambda(U)\Lambda(V)$, which combined with (23) yields the desired result. ■

4.2.3. Rescaling representations by a character. Let G be a Lie group, and (π, V) a real representation of dimension d. Let $\chi\colon G \to \mathbb{R}^+$ be a character. Then $\pi^\chi(g) := \chi^{1/d}(g)\pi(g)$ is again a representation, and

$$\det(\pi^\chi(g)) = \chi(g)\det(\pi(g)).$$

The following lemma allows one to split a representation (π, V) into a subrepresentation $W \subseteq V$ and a quotient V/W, and simultaneously twist π_W and $\pi_{V/W}$ by a character χ in opposite directions.

Lemma 4.6 (Shifting characters)**.** *Let $W \subseteq V$ be a (real) subrepresentation, and let $\mathcal{V}_W$ and $\mathcal{V}_{V/W}$ be open neighbourhood bases in W and V/W consisting of balanced sets. Let $F \subseteq G$ be a relatively compact, symmetric subset. Then*

$$\delta_F^\pi \geq \delta_F^{\pi_W^{\chi^{-1}}}(\mathcal{V}_W) \cdot \delta_F^{\pi_{V/W}^{\chi}}(\mathcal{V}_{V/W}) \tag{24}$$

for any continuous character $\chi\colon G \to \mathbb{R}^+$.

Proof. Choose an inner product on V, and identify V/W with $W^\perp \subseteq V$. Then let $d_1 = \dim(W)$ and $d_2 = \dim(V/W)$. Define $\rho := \sup_{g\in F} \|\pi(g)\|$ and $D := \sup_{g\in F} |\chi(g)|$.

Let $U \in \mathcal{V}_U$, let $V \in \mathcal{V}_{V/W}$, and suppose that both are contained in a ball with radius 1. Let $(\alpha u, \beta v)$ be an element of $(U_r)^F \times_{R_1,R_2,\varepsilon} V^F$ for

$$r = \rho R_2^{1+\frac{d_2}{d_1}} (D/\varepsilon)^{1/d_1}. \tag{25}$$

Then
$$\pi(g)\begin{pmatrix}\alpha u\\ \beta v\end{pmatrix} = \begin{pmatrix}\alpha'\pi_W^{\chi^{-1}}(g)u + \beta P_W\pi(g)v\\ \beta'\pi^{\chi}_{V/W}(g)v\end{pmatrix} \tag{26}$$

with $\alpha' = \alpha\chi^{1/d_1}(g)$ and $\beta' = \beta\chi(g)^{-1/d_2}$, and P_W the orthogonal projection onto W. If $\alpha \neq 0$, this can be written as $(\alpha' u', \beta' v')$ for

$$v' = \pi^{\chi}_{V/W}(g)v, \quad \text{and} \quad u' = \pi_W^{\chi^{-1}}(g)u + \frac{\beta}{\alpha'}P_W\pi(g)v.$$

Note that since $v \in V^F$, we have $v' \in V$. To see that $u' \in U$, note that since $u \in (U_r)^F$ we have $\pi_W^{\chi^{-1}}(g)u \in U_r$. Further, $\|P_W\pi(g)w\| \leq \rho$. Since U is balanced, U_r is balanced as well. By Lemma 4.4, we may assume that $|\alpha^{d_1}\beta^{d_2}| = \varepsilon$. Then

$$\frac{|\beta|}{|\alpha|} \leq (1/\varepsilon)^{1/d_1} R_2^{1+\frac{d_2}{d_1}},$$

and $\frac{|\beta|}{|\alpha'|} \leq (D/\varepsilon)^{1/d_1} R_2^{1+\frac{d_2}{d_1}}$. Putting this together, we find $\|\frac{\beta}{\alpha'}P_W\pi(g)v\| \leq r$ for r as above, and we conclude that $u' \in U$. Since $|\alpha'^{d_1}\beta'^{d_2}| = |\alpha^{d_1}\beta^{d_2}| = \varepsilon$ and $|\alpha'| \leq D^{1/d_1}R_1 =: R_1'$, and $|\beta'| \leq D^{1/d_2}R_2 =: R_2'$, we conclude that

$$\pi(g)\big((U_r)^F \times_{R_1,R_2,\varepsilon} V^F\big) \subseteq U \times_{R_1',R_2',\varepsilon} V$$

for all $g \in F$, and hence, since F is symmetric,

$$(U \times_{R_1',R_2',\varepsilon} V)^F \supseteq (U_r)^F \times_{R_1,R_2,\varepsilon} V^F.$$

By Lemma 4.5, it follows that

$$\begin{aligned}\delta_F^V(U \times_{R_1',R_2',\varepsilon} V) &\geq \frac{\Lambda\Big((U_r)^F \times_{R_1,R_2,\varepsilon} V^F\Big)}{\Lambda\Big(U \times_{R_1',R_2',\varepsilon} V\Big)} \\ &= \frac{\Lambda\big((U_r)^F\big)}{\Lambda(U)} \cdot \frac{\Lambda(V^F)}{\Lambda(V)} \cdot \frac{\varepsilon\left(1+\log\left(\frac{R_1^{d_1}R_2^{d_2}}{\varepsilon}\right)\right)}{\varepsilon\left(1+\log\left(\frac{R_1'^{d_1}R_2'^{d_2}}{\varepsilon}\right)\right)}.\end{aligned} \tag{27}$$

We choose $R_1(\varepsilon)$ and $R_2(\varepsilon)$ such that $R_1(\varepsilon) \downarrow 0$ and $R_2(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$, for example by setting $R_1(\varepsilon) = \varepsilon^a$ and $R_2(\varepsilon) = \varepsilon^b$ with $a, b > 0$. Further, we can achieve $r(\varepsilon) \downarrow 0$ for

$$r(\varepsilon) = \rho D^{1/d_1}\frac{R_2(\varepsilon)^{1+\frac{d_2}{d_1}}}{\varepsilon^{1/d_1}},$$

by choosing $b > 1/(d_1+d_2)$. The first factor in (27) then converges to the constant $\delta_F^{\pi_W^{\chi^{-1}}}(U) = \Lambda(U^F)/\Lambda(U)$. The second factor is $\delta_F^{\pi^{\chi}_{V/W}}(V)$.

For the third factor, we can achieve

$$\frac{R_1^{d_1}(\varepsilon)R_2(\varepsilon)^{d_2}}{\varepsilon} \to \infty$$

for $\varepsilon \downarrow 0$ by choosing $a > 0$ such that $d_1 a + d_2 b < 1$. The third factor in (27) then converges to 1, because ${R'_1}^{d_1} {R'_2}^{d_2}$ differs from ${R_1}^{d_1} {R_2}^{d_2}$ by a multiplicative constant D^2. Note that the inequalities $a > 0$, $d_1 a + d_2 b < 1$ and $b > \frac{1}{d_1+d_2}$ do admit solutions. ■

5. Connected Lie groups

If G is a connected Lie group and (π, V) is a finite dimensional real G-representation, then the character $\chi(g) = \det(\pi(g))$ takes values in $\mathbb{R}^{>0}$.

Definition 5.1 (Rescaled representations)**.** We denote by $\overline{\pi}(g) := \det\big(\pi(g)\big)^{-\frac{1}{d}}\pi(g)$ the representation on V that is rescaled to have determinant 1. ■

Definition 5.2. We denote by $\overline{\delta}_F^V$ the supremum of $\delta_F^{(\overline{\pi},V)}(\mathcal{V})$ over all *balanced* open neighbourhood bases of $0 \in V$. ■

Note that if (π, V) has determinant 1, then $\delta_F^V \geq \overline{\delta}_F^V$. From Lemma 4.6, we immediately obtain the following splitting theorem:

Lemma 5.3 (Rescaled splitting)**.** *Let G be a connected Lie group, and $W \subseteq V$ a subrepresentation of the (real) representation (π, V). Then*

$$\overline{\delta}_F^V \geq \overline{\delta}_F^W \cdot \overline{\delta}_F^{V/W}. \tag{28}$$

5.1. Connected solvable Lie groups

Recall that a connected Lie group R is solvable if and only if its Lie algebra $\mathfrak{r}$ is solvable [17, Theorem 3.18.8]. From the above rescaled splitting lemma, it is not hard to see that $\overline{\delta}_F^V = 1$ for connected solvable Lie groups.

Theorem 5.4. *Let R be a connected Lie group with solvable Lie algebra $\mathfrak{r}$, and let $F \subseteq R$ be a relatively compact, symmetric subset. Then $\overline{\delta}_F^V = 1$ for any real representation (π, V).*

Proof. We proceed by induction on $d = \dim_{\mathbb{R}}(V)$. For $d = 1$, the statement holds because $\overline{\pi}$ is the trivial representation. Suppose the statement holds for all representations of dimension less than d. By Lie's Theorem, the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ admits a nonzero vector v_λ and a weight $\lambda\colon \mathfrak{r} \to \mathbb{C}$ such that $d\pi(x)v_\lambda = \lambda(x)v_\lambda$ for all $x \in \mathfrak{r}$. Since R is connected and since the infinitesimal action of $\mathfrak{r}$ on $\mathbb{C}\mathrm{P}(V)$ has $[v_\lambda] \in \mathbb{C}\mathrm{P}(V)$ as a fixed point, the action of R on $\mathbb{C}\mathrm{P}(V)$ fixes the ray $[v_\lambda] = \{zv_\lambda\,;\, z \in \mathbb{C}^\times\}$. The homomorphism λ therefore integrates to a character $e_\lambda\colon R \to \mathbb{C}^\times$, and $\pi(g)v_\lambda = e_\lambda(g)v_\lambda$ for all $g \in R$. Let $U_\lambda \subseteq V$ be the real subrepresentation spanned by the real vectors $\frac{1}{2}(v_\lambda + \overline{v}_\lambda)$ and $\frac{1}{2i}(v_\lambda - \overline{v}_\lambda)$, and let $\chi(g) := |e_\lambda(g)|$. If v_λ is real, then U_λ is 1-dimensional, and the representation (e_λ, V_λ) becomes trivial when rescaled to have determinant 1. If v_λ is not real, then U_λ is 2-dimensional, and the real representation $\overline{e}_\lambda(g) = \chi(g)^{-1}e_\lambda(g)$ (rescaled to have determinant 1) acts by rotations. Either way, U_λ admits an invariant neighbourhood basis for the rescaled representation, so $\overline{\delta}_F^{U_\lambda} = 1$.

Since the quotient representation $(\pi_{V/W}, V/U_\lambda)$ is of dimension less than d, it too has $\overline{\delta}_F^{V/U_\lambda} = 1$. So $\overline{\delta}_F^V \geq 1$ by Lemma 5.3. ■

Since a connected Lie group is unimodular if and only if the adjoint representation has determinant one, we have $1 \geq \delta_F^{\mathfrak{r}} \geq \overline{\delta}_F^{\mathfrak{r}}$, and the following result is an immediate consequence.

Corollary 5.5. *If R is a connected, unimodular Lie group with solvable Lie algebra $\mathfrak{r}$, then $\delta_{\mathfrak{r}}^F = 1$ for any relatively compact, symmetric subset $F \subseteq R$.*

From Theorem 1.2, it then follows that

$$\|T_{m|_\Gamma} \colon L_p(\widehat{\Gamma}) \to L_p(\widehat{\Gamma})\| \leq \|T_m \colon L_p(\widehat{R}) \to L_p(\widehat{R})\| \tag{29}$$

for any discrete subgroup Γ of a connected, solvable Lie group R.

This is in agreement with results obtained earlier in [2]; since every solvable Lie group is amenable [5], and since every discrete subgroup Γ of an amenable group is amenable, it follows from [2, Theorem 8.7] that R has small almost-invariant neighbourhoods with respect to Γ, and (29) follows from [2, Theorem A].

5.2. Connected Lie groups

We end with some remarks on $\delta_F^{\mathfrak{g}}$ for a connected unimodular Lie group G with Lie algebra $\mathfrak{g}$. Let $\mathfrak{r} \subseteq \mathfrak{g}$ be the radical of $\mathfrak{g}$, i.e. the maximal solvable ideal. Then $\mathfrak{r}$ integrates to a closed, connected, normal Lie subgroup $R \subseteq G$, and the semisimple quotient $S := G/R$ has Lie algebra $\mathfrak{s} := \mathfrak{g}/\mathfrak{r}$ [17, Thm. 3.18.13].

Let $F \subseteq G$ be a relatively compact, symmetric subset. Using the Reduction Lemma 3.3 on the G-subrepresentation $\mathfrak{r} \subseteq \mathfrak{g}$, we obtain a bound

$$\delta_F^{\mathfrak{g}} \geq a_{\mathrm{ss}} a_{\mathrm{rad}} \tag{30}$$

with two types of contributions: the 'semisimple' contribution $a_{\mathrm{ss}} = \delta_F^{\mathfrak{g}/\mathfrak{r}}$ and the contribution $a_{\mathrm{rad}} = \delta_F^{\mathfrak{r}}$ from the radical.

5.2.1. The semisimple contribution a_{ss} from $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$. The factor $a_{\mathrm{ss}} := \delta_F^{\mathfrak{g}/\mathfrak{r}}$ depends only on the semisimple quotient S. Indeed, since the adjoint action of $\mathfrak{r}$ on $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$ is trivial, the action of G on $\mathfrak{s}$ is trivial on the connected subgroup R, and the representation of G on $\mathfrak{g}/\mathfrak{r}$ factors through the adjoint representation of S. It follows that a_{ss} is equal to $\delta_{[F]}^{\mathfrak{s}}$ for $[F] \subseteq S$. Let B be the Killing form on $\mathfrak{s}$, and θ a Cartan involution. By [3, Theorem 8.1], we have

$$a_{\mathrm{ss}} \geq \rho_S^{-d/2}, \tag{31}$$

where $\rho_S = \sup_{s\in[F]} \|\mathrm{Ad}_s\|$ is the maximal norm of $s \in [F] \subseteq S$ in the adjoint representation, with respect to the inner product $B_\theta(x, y) = -B(x, \theta(y))$. The exponent d is the maximal dimension of a nilpotent orbit in $\mathfrak{s}$. Note that d is an even integer, since the adjoint orbit in a semisimple Lie algebra is a symplectic manifold. In terms of the Lie algebra $\mathfrak{s}$, it can be expressed as $d = \dim_{\mathbb{R}}(\mathfrak{s}) - \min_{x\in\mathcal{N}} \dim_{\mathbb{R}}(\mathfrak{s}_x)$, where the second term is the minimal dimension of the stabilizer $\mathfrak{s}_x$, where x runs over the nilpotent cone $\mathcal{N} \subseteq \mathfrak{s}$ [3, Remark 8.2].

5.2.2. The contribution a_{rad} from the radical. Since G is unimodular, the adjoint representation $(\mathrm{Ad}_G, \mathfrak{g})$ has determinant 1. As the semisimple quotient $S = G/R$ is unimodular as well, the adjoint representation $(\mathrm{Ad}_S, \mathfrak{s})$ of S also has determinant 1. It follows that the restriction $(\pi, \mathfrak{r})$ of the adjoint G-representation to the radical $\mathfrak{r} \subseteq \mathfrak{g}$ has determinant 1 as well, so that $a_{\text{rad}} = \delta_F^{\mathfrak{r}} \geq \overline{\delta}_F^{\mathfrak{r}}$.

Recall from §4.1.3 that the real G-representation $V^k(\mathfrak{r}) = \mathfrak{r}^k/\mathfrak{r}^{k+1}$ decomposes as

$$V^k(\mathfrak{r}) = \bigoplus_{\{\lambda,\overline{\lambda}\}\subseteq\Phi^k(\mathfrak{r}_{\mathbb{C}})} U_\lambda^k, \tag{32}$$

where U_λ^k is a real form of the complex representation V_λ^k if λ is real, and $U_\lambda^k \simeq V_\lambda^k$ as a real representation otherwise. From Lemma 5.3, we immediately obtain the following result.

Lemma 5.6. *Let $F \subseteq G$ be a relatively compact, symmetric subset. Then*

$$\delta_F^{\mathfrak{r}} \geq \prod_{k=0}^{r-1} \prod_{\{\lambda,\overline{\lambda}\}\subseteq\Phi^k(\mathfrak{r}_{\mathbb{C}})} \overline{\delta}_F^{U_\lambda^k}. \tag{33}$$

Proof. Since $\mathfrak{r}^k/\mathfrak{r}^{k+1} = \bigoplus_{\{\lambda,\overline{\lambda}\}\subseteq\Phi^k(\mathfrak{r}_{\mathbb{C}})} U_\lambda^k$, Lemma 5.3 yields

$$\overline{\delta}_F^{\mathfrak{r}^k/\mathfrak{r}^{k+1}} \geq \prod_{\{\lambda,\overline{\lambda}\}\subseteq\Phi^k(\mathfrak{r}_{\mathbb{C}})} \overline{\delta}_F^{U_\lambda^k}. \tag{34}$$

The result then follows by applying Lemma 5.3 to the subrepresentations $\mathfrak{r}^k/\mathfrak{r}^{k+1}$ of $\mathfrak{r}/\mathfrak{r}^{k+1}$ repeatedly for $k = 2, \ldots, r-1$. ∎

If λ is real, then the maximal solvable subgroup $R \subseteq G$ acts on the real subspace $U_\lambda^k = V_\lambda^k \cap V^k(\mathfrak{r})$ by the real character $e_\lambda \colon R \to \mathbb{R}^{>0}$. The rescaled representation $\overline{\pi}_\lambda^k$ of G is therefore trivial on $R \subseteq G$, and factors through a real representation of the semisimple quotient $S = G/R$.

If λ is not real, then R acts on $U_\lambda^k = V_\lambda^k$ by the (complex) character $e_\lambda \colon R \to \mathbb{C}^\times$, and the restriction of $\overline{\pi}_\lambda^k$ to $R \subseteq G$ factors through a homomorphism $\chi_\lambda \colon R \to \mathrm{U}(1)$. In fact, χ_λ is trivial on the commutator subgroup $[R, R]$ because λ vanishes on $[\mathfrak{r}, \mathfrak{r}]$, so it factors through a compact quotient of the abelianization $R/[R, R]$ of R. Since G is connected, it acts trivially on the weight λ, and hence on the character χ_λ. It follows that $\chi_\lambda(grg^{-1}) = \chi_\lambda(r)$ for all $r \in R$ and $g \in G$. The representation $\overline{\pi}_\lambda$ of G on U_λ^k therefore factors through the central extension

$$S_\lambda^\sharp := G \times_{\chi_\lambda} \mathrm{U}(1), \tag{35}$$

the quotient of $G \times \mathrm{U}(1)$ by the closed, normal subgroup $\{(r^{-1}, \chi_\lambda(r))\,;\, r \in R\}$ isomorphic to R. Note that

$$1 \to \mathrm{U}(1) \to S_\lambda^\sharp \to S \to 1 \tag{36}$$

is a central extension of the semisimple Lie group S by the abelian Lie group $\mathrm{U}(1)$, and hence a real reductive Lie group. The relevant subset of $S_\lambda^\sharp$ is the image $[F] = \{[g, 1]\,;\, g \in F\}$ of $F \times \{1\}$ under the quotient map, and the factors $\overline{\delta}_F^{U_\lambda^k}$ in (33) come from the representation of $S_\lambda^\sharp$ on U_λ^k.

Remark 5.7. In fact, the central extension (36) splits at the Lie algebra level by the second Whitehead Lemma. It follows that $S_\lambda^\sharp$ can be expressed in terms of the simply connected cover $\widetilde{S}$ and a character $\phi_\lambda \colon \pi_1(S) \to \mathrm{U}(1)$ by

$$S_\lambda^\sharp = \widetilde{S} \times_{\phi_\lambda} \mathrm{U}(1),$$

the quotient of $\widetilde{S} \times \mathrm{U}(1)$ by the normal subgroup $\{([\gamma]^{-1}, \phi_\lambda([\gamma]))\,;\, [\gamma] \in \pi_1(S)\}$.

The inclusion $\mathfrak{s} \hookrightarrow \mathfrak{g}$ of Lie algebras integrates to a Lie group homomorphism $\phi\colon \widetilde{S} \to G$, and the restriction of ϕ to $\pi_1(S)$ takes values in R. In terms of χ_λ and ϕ, the character ϕ_λ is given by $\phi_\lambda = \chi_\lambda \circ \phi|_{\pi_1(S)}$. ■

Remark 5.8. If we consider $U^k_\lambda = V^k_\lambda$ as a *complex* representation for $\lambda \neq \overline{\lambda}$, and consider only neighbourhood bases that consist of *complex* balanced open subsets (so $\alpha U \subseteq U$ for all *complex* numbers α with $|\alpha| \leq 1$), then the U(1)-action is immaterial. So at the cost of restricting attention from $\mathbb{R}$-balanced neighbourhoods to $\mathbb{C}$-balanced neighbourhoods, the real reductive Lie groups $S^\sharp_\lambda$ can be replaced by the connected semisimple Lie group $\widetilde{S}$. ■

Note that $\dim_{\mathbb{R}}(U^k_\lambda) = \dim_{\mathbb{C}}(V^k_\lambda)$ if λ is real, and $\dim_{\mathbb{R}}(U^k_\lambda) = \dim_{\mathbb{C}}(V^k_\lambda) + \dim_{\mathbb{C}}(V^k_{\overline{\lambda}})$ if λ is not real. Applying the trivial bound (7) to each of the factors $\delta^{U^k_\lambda}_{[F]}$ in (33) therefore yields the explicit bound

$$a_{\mathrm{rad}} \geq \prod_{\lambda \in \Phi(\mathfrak{r}_{\mathbb{C}})} (\rho^{[F]}_\lambda)^{-\dim_{\mathbb{C}}(V^k_\lambda)}, \tag{37}$$

where $\rho^{[F]}_\lambda$ is the maximal value of $\|\overline{\pi}_\lambda(g)\colon V^k_\lambda \to V^k_\lambda\|$ for $[g] \in [F] \subseteq S^\sharp_\lambda$. If $\mathfrak{g}$ is solvable, then $G = R$, S is trivial, and $S^\sharp_\lambda$ is a compact quotient of $R/[R,R]$, acting by operators with norm 1. The bound (37) then reduces to Theorem 5.5, which is of course sharp.

In general, however, the trivial bound on $\delta^{U^k_\lambda}_{[F]}$ will not be sharp. A case in point is the tangent group $TS = A \rtimes S$ of a semisimple Lie group S, the semidirect product of S with its Lie algebra $A = \mathfrak{s}$, considered as an abelian Lie group with the adjoint action of S. The trivial bound then yields $\delta^{\mathfrak{a}\rtimes\mathfrak{s}}_F \geq \rho^{-d/2}\rho^{-\dim_{\mathbb{R}}(\mathfrak{s})}$ for $\rho = \sup_{s\in F}\|\mathrm{Ad}_s\|$. But since $\mathfrak{r} = \mathfrak{s}$ as an S-representation, [3, Theorem 8.1] yields the superior bound $\delta^{\mathfrak{a}\rtimes\mathfrak{s}}_F \geq \rho^{-d/2}\rho^{-d/2}$.

5.3. Open problems and further directions of research

This leads one to consider the following problem:

Problem 5.9. *Determine nontrivial bounds on $\overline{\delta}^V_F$, where F is a compact subset of a connected real reductive Lie group H, and (π, V) is a finite-dimensional, irreducible, real representation of H with* $\det(\pi(h)) = 1$ *for all $h \in H$.*

By 'nontrivial', we mean superior to the trivial bound from Proposition 3.1. Except for the case where V is the adjoint representation [3, Theorem 8.1], this is an open problem to the best of our knowledge.

By the above considerations, a solution to Problem 5.9 would imply nontrivial bounds on $\delta^{\mathfrak{g}}_F$ for a general connected unimodular Lie group G. Indeed, if one can find nontrivial bounds in the case where $H = S^\sharp_\lambda$ and V is an irreducible real subrepresentation of U^k_λ, then repeated application of the Splitting Lemma 5.3 will result in nontrivial bounds for $\overline{\delta}^{U^k_\lambda}_{[F]}$, and hence for $\delta^{\mathfrak{g}}_F$.

Conversely, suppose that H is a connected real reductive Lie group and (π, V) is an irreducible, finite-dimensional, real representation of H with determinant one. Then for the connected unimodular Lie group $G = V \rtimes H$, it seems unlikely that one would be able to arrive at nontrivial bounds on $\delta^{\mathfrak{g}}_F$ for G without first considering $\overline{\delta}^V_{[F]}$ for H.

Rather than considering neighbourhood bases of balls $B_r(0) \subseteq V$, which lead to the trivial bound (7), it would be a natural direction of research (following [3, §9]) to consider neighbourhood bases $\mathcal{V}$ consisting of sets $U_{r,R} \subseteq V$ of the form

$$U_{r,R} := \big(\pi(H)B_r(0)\big) \cap B_R(0)$$

for $0 < r \ll R < 1$. Note that $\bigcap_{r>0} \bigcup_{R>0} U_{r,R}$ is the union of the H-orbits in V whose closure contains the origin. Although this would probably require detailed information about the orbit structure of the H-action on V, we believe that determining $\delta_F^V(\mathcal{V})$ for such neighbourhood bases has the potential to yield nontrivial bounds on a_{rad}. This has been substantiated in the special case of the adjoint representation, where this is precisely the Ansatz used in [3, §9].

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