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Discrete Series Representations Decomposing Discretely with Finite Multiplicity under Restriction to Symmetric Subgroups

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Abstract. For a semisimple Lie group G satisfying the equal rank condition, the most basic family of unitary irreducible representations is the Discrete Series found by Harish-Chandra. In this paper, we continue our study of the branching laws for Discrete Series when restricted to a subgroup H of the same type by use of integral and differential operators in combination with our previous duality principle. Many results are presented in generality, others are shown in detail for Holomorphic Discrete Series.

Mathematics Subject Classification: Primary 22E46; secondary 17B10.

Key Words: Intertwining operators, admissible restriction, duality theorem, reproducing kernel, discrete series.

Dedicated to Karl-Hermann Neeb on the occasion of his 60th birthday.

Contents

1. Introduction

For a unitary representation π of a Lie group G in a Hilbert space V we have for each one-parameter subgroup $exp(tX)$ a unitary one-parameter group of operators $U(t) = \pi(exp(tX))$, and this is given by the Fourier transform of a spectral measure on the real line with values in the projections in V. It is a fundamental question to know this spectral measure and in particular the spectrum, i.e. its support. When the generator of $U(t)$ corresponds to a physical interpretation such as energy, then one thinks of the notion of positive energy as related to the spectrum being bounded below. This notion has been studied in great detail by Karl-Hermann Neeb in connection with unitary highest weight representations, where also the connection

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to causality and field theory is central. In the works by Harish-Chandra this class of representations was introduced as holomorphic Discrete Series of a semi-simple Lie group G and related to analysis and holomorphic vector bundles on the Riemannian Symmetric space G/K. Now in general for a unitary representation of a Lie group G it is of interest to find the spectrum of its restriction to a subgroup H (just as for one-parameter subgroups) – the branching law – and this will reveal much about the nature of the representation. One important aspect is that one needs good models of the Hilbert space in order to carry out the restriction explicitly, and again typical models are in homogeneous vector bundles over G/K in the semi-simple situation, or equivalent versions (via parallelization of the bundles) in vector-valued function on G/K. Whereas the overall principles are simple, the computations involve the structures of G and H in suitable coordinates using the root systems. The coordinates are relevant since we want to represent the branching laws using integral kernel operators and also differential operators.

In this paper we continue our study of the Discrete Series of a connected linear semisimple Lie group G, namely the unitary irreducible representations π arising as closed subspaces of the left regular representation in $L^2(G)$. Here G and a maximal compact subgroup K have the same rank and Harish-Chandra gave a parametrization of such π, the so-called Discrete Series of G.

Our aim is to understand the restriction of the Discrete Series of G to a symmetric subgroup H of G in the admissible case, namely when a π restricted to H is a direct sum of irreducible subspaces, each of them Discrete Series ρ of H with finite multiplicities. There are a number of techniques that we use here, firstly the theory of reproducing kernels, corresponding to convenient models of the Discrete Series. There are two kinds of operators we analyze, namely

- *symmetry breaking operators*, i.e. H equivariant linear maps from π to the individual ρ
- *holographic operators,* i.e. H-equivariant linear maps from the individual ρ to π

As it turns out differential operators play a key role as symmetry breaking operators, and we shall explain this using homogeneous vector bundles over G/K. Another technique is that of pseudo-dual pair, a notion introduced in [28], and useful in our situation is that we associate to H another symmetric subgroup H_0 of G, and there is a corresponding duality Theorem, which in a sense reduces the branching law (the explicit decomposition of π) to a branching law for H_0 under its maximal compact subgroup.

We treat in special detail the case of G/K of Hermitian type and the holomorphic Discrete Series; here we give a simpler proof of the duality Theorem and also more information about the differential operators and kernel operators arising as holographic operators. This case has been treated earlier using the models specific to this case, namely seeing G/K as a bounded symmetric domain; we are particularly interested in the nature of the symmetry breaking differential operators, in particular whether they are purely tangential or contain normal derivatives.

There are a number of prerequisites of a technical nature that we assume in this paper; the topic of Discrete Series representations of semisimple Lie groups is by its nature (and history) quite demanding, and we ask the reader some patience and consultations of some of the literature. One important work here is the notion of lowest K-type, which has been introduced by D. Vogan [38, Definition 5.1, Proposition 5.8] as a way to characterize certain representations, in particular the Discrete Series. Also see the work by W. Schmid on realizations, i.e. concrete models of the Discrete Series [2][17][34]. Some of our results are for quite general Discrete Series – in fact as in our previous work [29][30][31] where the role of reproducing kernels in treated in detail, and also some criteria for admissible restrictions of Discrete Series are given. Finally we use the theory of the special case of holomorphic Discrete Series (we call this the holomorphic case), for the groups where the Riemannian symmetric space is a bounded symmetric domain; here we can be more explicit and make our duality quite concrete. The holomorphic case has been studied in great detail by R. Nakahama [25], and we have been much inspired by this work. For other technical prerequisites we refer to the works of N. Wallach [11] [17] [39] and obviously the fundamental works by T. Kobayashi on branching theory [19][21] [22].

In order to describe the main results we set up some notation.

From now on, G is a connected semisimple matrix Lie group. K is a maximal compact subgroup of G. θ denotes the Cartan involution associated to K. $Lie(G) = \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the associated Cartan decomposition. We fix (τ, W) a representation of K. dg denotes a Haar measure in G. We recall the space

$$L^2(G \times_\tau W) = \left\{ f : G \to W : \begin{array}{l} f(xk) = \tau(k^{-1})f(x), \ k \in K, \\ x \in G, \displaystyle\int_G \|f(x)\|_W^2 dx < \infty \end{array} \right\}.$$

The left action on functions is denoted by $L_\cdot^G, L_\cdot$.

A *reductive pair* (G, H) consists of G and a reductive, connected, closed subgroup H of G. We can arrange and we will do matters so that $L := K \cap H$ is a maximal compact subgroup of H. Similarly, for each representation (σ, Z) of L we have the space $L^2(H \times_\sigma Z)$. An example of reductive pair is a *symmetric pair* defined as follows: for each involution σ in G that commutes with the Cartan involution θ, we define $H := (G^\sigma)_0$ the connected component of the identity of G^σ. The *associated subgroup* to H is $H_0 := (G^{\sigma\theta})_0$. $L = K \cap H$ is a maximal compact subgroup of both H, H_0. We call these two groups a pseudo-dual pair.

An irreducible unitary representation of G is *square integrable*, equivalently a *Discrete Series representation,* if each of its matrix coefficient is square integrable with respect to Haar measure. It is a Theorem that square integrable representations are determined by their lowest K-type, in the sense of Vogan, and that the lowest K-type has multiplicity one when we restrict a given square integrable representation to the maximal compact subgroup K, [8][34][38]. Henceforth, we fix an irreducible representation (τ, W) of K which is the lowest K-type of a Discrete Series representation. Thus, Frobenius reciprocity lets us conclude that a Discrete Series representation of lowest K-type (τ, W) has multiplicity one in $L^2(G \times_\tau W)$.

Let $H^2(G, \tau) \subset L^2(G \times_\tau W)$ denote the unique closed linear subspace that realizes the square integrable representation of lowest K-type (τ, W). It can be shown that $H^2(G, \tau)$ is an eigenspace of the Casimir operator $\Omega_\mathfrak{g}$ [8]. We denote by λ_τ the value of such eigenvalue. A key property of $H^2(G, \tau)$ is being a reproducing kernel subspace [1]. We denote the corresponding reproducing kernel by $K_\tau : G \times G \to Hom_{\mathbb{C}}(W, W)$. We have $K_\tau(x, y) = K_\tau(y, x)^\star$, $K_\tau(x, \cdot)^\star w \in H^2(G, \tau), w \in W$. The orthogonal projector onto $H^2(G, \tau)$ is the integral operator $f \mapsto \int_G K_\tau(y, \cdot) f(y) dy$, and for every $f \in H^2(G, \tau), w \in W$ the identity $(f(x), w)_W = \int_G (f(y), K_\tau(y, x)^\star w)_W dy$ holds. Also, the map

$$W \ni w \mapsto K_\tau(e, \cdot)^\star w \in H^2(G, \tau)[W]$$

is a K-equivariant isomorphism. Similarly, for an irreducible representation (σ, Z) of L, we consider $H^2(H, \sigma) \subset L^2(H \times_\sigma Z)$.

To continue, we describe the main results and a resume of the paper. Before we proceed, we would like to point out that the style of this note is inspired by K.-H. Neeb in the sense of aiming first a presentation with a high degree of generality followed by particular (and interesting) more special cases. In Proposition 2.1, Proposition 2.3, we recall scattered results on basic properties of intertwining linear operators between two representations modeled on reproducing kernel spaces. In 2.4.1,2.4.2, we recall the general notion of differential operator in our context. With respect to the main results a description is as follows: In Theorem 2.5, we generalize a result of Helgason, we show whenever a symmetry breaking operator is equal to the restriction of a plain differential operator, we may replace a differential operator by a canonical differential operator between the vector bundles that describe the representations. In the same section, we carry out in thorough detail the problem of writing a symmetry breaking operator as a differential operator given the kernel and viceversa. In Theorem 2.8 we continue the study of the relation between symmetry breaking operators for different realizations of the same representation, among them, maximal globalization, Hilbert space realization, Harish-Chandra module realization.

In section 3, we consider the problem of representing a symmetry breaking operator as generalized gradient operator, the obtained result is valid for an arbitrary symmetric pair (G, H) and arbitrary symmetry breaking operator for an arbitrary H-admissible Discrete Series representation for G.

In section 4, we present a new proof of the duality Theorem for the holomorphic setting, Theorem 4.1, Theorem 4.3, Theorem 4.4, Theorem 4.7. Our proof is of algebraic nature. In order to carry out the proof, we introduce the subspace $\mathcal{L}^c_{W,H}$ to be the linear span of of subspaces of $H^2_{c_\tau}(G, W)$ that realizes the lowest L-type of each irreducible H-factor of $H^2_{c_\tau}(G, W)$. We also consider the subspace $\mathcal{U}(\mathfrak{h}_0)W :=$the $\mathcal{U}(\mathfrak{h}_0)$ submodule spanned by the subspace that realizes the lowest K-type of the representation. Then, in Proposition 4.2, we find an explicit isomorphism $\mathcal{L}^c_{W,H} \overset{D}{\cong} \mathcal{U}(\mathfrak{h}_0)W$. In Proposition 4.6 we show that D may be replaced by the orthogonal projector onto $\mathcal{U}(\mathfrak{h}_0)W$. In Theorem 4.12 we show a new formula for holographic operators based on the reproducing kernel for the initial space. As a consequence, we show a "separation of variable" formula for the kernel of holographic as well for symmetry breaking operators, we believe this formula is valid under more general hypothesis, see Corollary 4.16 and the previous corollaries.

In section 5 we present an overview of some of the results that will appear in a sequel to this paper. The results are built on a quite careful analysis of when all "first order" symmetry breaking operators are represented by normal derivative differential operators. Later on, we study, when the totality of symmetry breaking operators are represented by normal derivative operators. That is, following [31, Proposition 6.3], we analyze the equality $\mathcal{L}^c_{W,H} = \mathcal{U}(\mathfrak{h}_0)W$. Aspects of the notion of duality via dual pairs (also called pseudo dual pairs) has been considered in earlier papers by several authors, notably Jakobsen-Vergne, Kobayashi-Pevzner, Speh and Nakahama; we believe it is useful for branching theory in quite general situations. Some natural open problems arise from our study, for example we mention a few:

(1) the explicit nature of the branching laws in terms of the reproducing kernels for the representations; since these are essentially generalized hypergeometric functions with parameters coming from the large group (resp. the subgroup), there will be in the admissible case an explicit identity expressing the kernel for the large group as a sum over kernels for the subgroup.

(2) Since other families of unitary representations have aspects in common with the Discrete Series (such as reproducing kernels given in terms of certain matrix coefficients) we may have similar results for these.

(3) For the non-admissible cases of branching laws (with both a discrete and a continuous spectrum) it is still interesting to consider the symmetry-breaking operators for the discrete spectrum - here we have some conjectures, but the theory remains incomplete.

1.1. Notation

For unexplained concepts please consult the partial list below as well as the Section Partial list of symbols and definitions, or [30][31]. The complexification of a real vector space V is denoted $V_{\mathbb{C}}$. Quite often we are somewhat sloppy in writing the complex subindex $\mathbb{C}$. For a module M and a simple submodule N, $M[N]$ denotes the *isotypic component* of N in M. That is, $M[N]$ is the sum of all irreducible submodules isomorphic to N. If topology is involved, we define $M[N]$ to be the closure of $M[N]$. $Hom_H(V,U)$ denotes the linear space of continuous H-intertwining linear maps from V into U. For a finite dimensional representation Z and a representation U of a compact Lie group L, $Hom_L(Z,U)$ is equal to the set of linear maps from Z into U that intertwines the action of L. Given a representation (π, V) of G, and a linear subspace U of V, invariant under the set of linear operators $\pi(H)$, for short, we will refer as: U is *H-invariant*.

2. Analysis of symmetry breaking (holographic) operators in the symmetric space model

As we said in the introduction the Discrete Series has several different models on the Riemannian symmetric space G/K in concrete spaces of sections of vector bundles; thus the most natural linear operators to consider between such spaces are (a) integral kernel operators and (b) differential operators. For (a) we shall use both reproducing kernels (whose existence follows from the ellipticity of the equations defining the Discrete Series) and also kernel operators for both symmetry breaking operators and holographic operators. For (b) a crucial point is to find a canonical form of such differential operators, and this is most relevant for symmetry breaking in the admissible case. Since G/K is simply connected, bundles can be trivialized, and this will be relevant in our discussion. When we consider Discrete Series we tacitly assume the equal rank condition of Harish-Chandra. We also refer freely to some of our previous results on Discrete Series.

In the Introduction we have recalled the bundle model of a realization for Discrete Series representations, we begin this section by recalling realizations of Discrete Series representations in the symmetric space model. Next, in the realm of both realizations for Discrete Series, we recall the description of intertwining linear operators as integral operators and the description of intertwining linear operators as differential operators, whenever it is possible. In the following, we carry out a complete analysis on how to compute the kernel function that represents an intertwining linear operator knowing its representation as a differential operator and viceversa. To continue, we show a generalization of a Theorem of Helgason on differential operators to a very general setting. For the sake of completeness, we present (without proof) the general statement of our duality Theorem. We end up the section showing a generalization of results of T. Kobayashi on symmetry breaking operators for different models of the Harish-Chandra module attached to a Discrete Series representation. Some of the results in this section will be basic for the end of section 2, section 4, and the sequel to this paper.

2.1. Symmetric space model for Discrete Series

We recall G is connected, matrix, reductive Lie group, K is a maximal compact subgroup of G. We assume there exists a Discrete Series representation $H^2(G,\tau)$ of G of lowest K-type (τ, W). Let $c_\tau : G\times G/K \to GL(W)$ be a continuous cocycle. Thus, $c_\tau(ab,x)=c_\tau(a,bx)c_\tau(b,x)$ and $c_\tau(k,eK)=\tau(k), k\in K$. Let $dm_{G/K}$ denote a G-invariant measure on G/K adjusted so that

$$\int_G f(x)dx = \int_{G/K}\int_K f(xk)dk dm_{G/K}(xK).$$

We recall the space (for a reference See [14] [26], [30])

$$\begin{aligned} &L^2_c(G/K,W) := L^2_{c_\tau}(G/K,W) := \{f : G/K \to W : \\ &\|f\|^2_{L^2_c(G/K,W)} := \int_{G/K} \left(\left(c_\tau(x,eL)c_\tau(x,eL)^\star\right)^{-1} f(x), f(x)\right)_W dm_{G/K}(x) < \infty\}. \end{aligned} \tag{2.1}$$

and the unitary representation π_{c_τ} of G on $L^2_{c_\tau}(G/K,W)$ defined by means of the equality $\pi_{c_\tau}(x)(f)(yK) = c_\tau(x^{-1},yK)^{-1} f(x^{-1}yK)$.

As usual, $\dot{\pi}_{c_\tau}(X)(f)$ denotes the infinitesimal action of π_{c_τ}.

Roughly speaking, the space $L^2_{c_\tau}(G/K,W)$ may be thought of as the space of square integrable functions with respect to the "measure"

$$\mu_{c_\tau} := (c_\tau(x,e)c_\tau(x,e)^\star)^{-1} dm_{G/K}(x).$$

Henceforth, $H^2_{c_\tau}(G/K,W)$ denotes the λ_τ-eigenspace of $\pi_{c_\tau}(\Omega_\mathfrak{g})$ in $L^2_{c_\tau}(G/K,W)$.

The map
$$f \mapsto E_{c_\tau}(f)(\cdot) := c_\tau(\cdot,e)f(\cdot)$$

is a G-equivariant unitary equivalence between

$$(L^G_\cdot, L^2(G\times_\tau W)) \text{ and } (\pi_{c_\tau}, L^2_{c_\tau}(G/K,W)),$$

carrying $H^2(G,\tau)$ onto $H^2_{c_\tau}(G/K,W)$. Therefore, $H^2_{c_\tau}(G/K,W)$ is a *reproducing kernel space.* We denote the corresponding reproducing kernel by

$$K^c_\tau(xK,yK) = c_\tau(y,eK)K_\tau(x,y)c_\tau(x,eK)^\star. \tag{2.2}$$

We have, $K_\tau^c(hy,hx)=c_\tau(h,x)K_\tau^c(y,x)c_\tau(h,y)^\star$ for all $h\in G$, $x,y\in G/K$.
Since, $K_\tau(x,x)=d(\pi_{c_\tau})I_W$, for every $x\in G$ it holds that

$$K_\tau^c(xK,xK)=d(\pi_{c_\tau})\,c_\tau(x,e)c_\tau(x,e)^\star.$$

2.2. Symmetry breaking operators

Let (G,H) be an *arbitrary reductive pair*, $K, L := K\cap H$ respective maximal compact subgroups, (τ,W) is (resp. (σ,Z)) irreducible representations of K, L. We assume that there exists Discrete Series representations $H^2(G,\tau), H^2(H,\sigma)$. Let $c_\tau : G\times G/K\to GL(W), c_\sigma : H\times H/L\to GL(Z)$ be cocycles. Thus, $c(ab,x)=c(a,bx)c(b,x)$ and $c_\tau(k,eK)=\tau(k), c_\sigma(l,eL)=\sigma(l), k\in K, l\in L$.
Next, we fix a continuous symmetry breaking operator

$$S: H^2_{c_\tau}(G/K,W)\to H^2_{c_\sigma}(H/L,Z).$$

Since the target space is a reproducing kernel space, there exists

$$K_S^c: G/K\times H/L\to Hom_{\mathbb{C}}(W,Z)$$

so that for $z\in Z$, $h\in H$, $K_S^c(\cdot,hL)^\star z\in H^2_{c_\tau}(G/K,W)$ and for all $F\in H^2_{c_\tau}(G/K,W)$, $h\in H$, $z\in Z$, it holds

$$(S(F)(hL),z)_Z=(F,K_S^c(\cdot,hL)^\star z)_{L^2_{c_\tau}(G/K,W)}. \tag{2.3}$$

Similarly, for $S^\star$ and each $w\in W$, we have a kernel $K_{S^\star}^c(\cdot,xK)^\star w\in H^2_{c_\sigma}(H/L,Z)$ such that for all $g_1\in H^2_{c_\sigma}(H/L,Z), x\in G$, the equality

$$(S^\star(g_1)(xK),w)_W=(g_1,K_{S^\star}^c(\cdot,xK)^\star w)_{L^2_{c_\sigma}(H/L,Z)}$$

holds. In the following, we show a result analogous to some statements in [30, Proposition 3.7], and, in the proof of [30, Lemma 4.2].

Proposition 2.1. *Let (G,H) be a reductive pair of Lie groups and $H^2_{c_\tau}(G/K,W)$, $H^2_{c_\sigma}(H/L,Z)$ Discrete Series representations for G,H. Then, the following facts hold for an H-intertwining continuous linear map*

$$S: H^2_{c_\tau}(G/K,W)\to H^2_{c_\sigma}(H/L,Z):$$

Notation: $h,y\in H$, $x\in G$, $z\in Z$, $w\in W$, $F\in H^2_{c_\tau}(G/K,W)$, $g\in L^2_{c_\sigma}(H/L,Z)$.

(1) $S(F)(hL)=\int_{G/K}K_S^c(xK,hL)\big(c_\tau(x,e)c_\tau(x,e)^\star\big)^{-1}F(xK)dm_{G/K}(xK)$.

(2) $K_S^c(xK,hL)w=K_{S^\star}^c(hL,xK)^\star w=S(K_\tau^c(\cdot,xK)^\star w)(hL)$.

(3) $K_{S^\star}^c(xK,eL)w=S^\star(K_\sigma^c(\cdot,eL)^\star w)(xK)$.

(4) $K_S^c(hxK,hyL)=c_\sigma(h,yL)K_S^c(xK,yL)c_\tau(h,xK)^\star$.

(5) $K_{S^\star}^c(hL,xK)z=K_S^c(xK,hL)^\star z=S^\star(K_\sigma^c(\cdot,hL)^\star z)(xK)$.

(6) *The linear map,*
$Z\ni z\mapsto K_S^c(\cdot,eL)^\star z=K_{S^\star}^c(eL,\cdot)z\in H^2_c(G/K,W)$,
is an L-map. Moreover, for each $z\in Z$,
$K_S^c(\cdot,eL)^\star z=K_{S^\star}^c(eL,\cdot)z\in H^2_{c_\tau}(G/K,W)[H^2_{c_\sigma}(H/L,Z)][Z]$.

(7) $(S(K^c_\tau(\cdot, xK)^\star w)(hL), z)_Z = (w, S^\star(K^c_\sigma(\cdot, hL)^\star z)(xK))_W$.

(8) $(S^\star(g)(xK), w)_W = (g, S(K^c_\tau(\cdot, xK)^\star w)_{L^2_c(H/L,Z)}$.

If we replace the space $H^2(H,\sigma)$ by $L^2(H\times_\sigma Z)$, the fourth identity might not hold!

Proof. A justification of all the statements but the fifth, is in [26]. We carry out the necessary computation to justify the first and the sixth assertions.

$$\begin{aligned}
&(S(F)(hL), z)_Z \\
&= \int_{G/K} ((c_\tau(x,e)c_\tau(x,e)^\star)^{-1}F(xK), K^c_S(xK,hL)^\star z)_W dm(xK) \\
&= \int_{G/K} (K^c_S(xK,hL)(c_\tau(x,e)c_\tau(x,e)^\star)^{-1}F(xK), z)_Z dm(xK) \\
&= \Big(\int_{G/K} K^c_S(xK,hL)(c_\tau(x,e)c_\tau(x,e)^\star)^{-1}F(xK)dm(xK), z\Big)_Z.
\end{aligned}$$

For the sixth statement, we apply the fourth statement to $l \in L, x = eL = o = lo$, we obtain $K^c_S(ly,o)^\star = c_\tau(l,y)K^c_S(y,o)^\star c_\sigma(l,o)^\star$. Since, $c_\sigma(l,o)^\star = \sigma(l^{-1})$, we have $c_\tau(l,y)^{-1}K^c_S(ly,o)^\star = K^c_S(y,o)^\star\sigma(l^{-1})$. Thus, the map $z \mapsto K^c_S(\cdot,o)^\star z$ is an L-map. Now, for $G_1 \in (H^2_{c_\tau}(G/K,W)[H^2_{c_\sigma}(H/L,Z)])^\perp$, since $H^2_{c_\sigma}(H/L,Z)$ is H-irreducible, we have $S(G_1) = 0$. Hence, $0 = (S(G_1)(hL),z)_Z = (G_1, K^c_S(\cdot,hL)^\star z)_{H^2_c(G/K,W)}$ yields $K^c_S(\cdot,hL)^\star z \in H^2_{c_\tau}(G/K,W)[H^2_{c_\sigma}(H/L,Z)]$, the previous computation gives $K^c_S(\cdot,hL)^\star z \in H^2_{c_\tau}(G/K,W)[H^2_{c_\sigma}(H/L,Z)][Z]$ and we have verified (5). (6) and (7) readily follow. ∎

For Discrete Series $H^2(H,\sigma)$, $H^2(G,\tau)$ in bundle model and a symmetry breaking operator $S: H^2(G,\tau) \to H^2(H,\sigma)$, since both Hilbert spaces are reproducing kernel spaces, there exists a smooth function $K_S : G\times H \to Hom_{\mathbb{C}}(W,Z)$ so that

$$(S(f)(h), z)_Z = \int_G (f(x), K_S(x,h)^\star w)_Z dx$$

for all $f \in H^2(G,\tau)$, $z \in Z$. We have $K_S(\cdot,x)^\star w$ belongs to $H^2(G,\tau)$ for each $h \in H$.

The diagram below shows how to transfer symmetry breaking operators S for group models of representations, to symmetry breaking operators S^c in the symmetric space model. The corresponding relation between the function kernels K_S, $K^c_{S^c}$ is

$$K^c_{S^c}(xK,hL) = c_\sigma(h,e)K_S(x,h)c_\tau(x,e)^\star,\ \forall h \in H, x\in G. \tag{2.4}$$

[30, Proposition 3.7] yields $K^c_{S^c}$ is a smooth function.

$$\begin{array}{ccccccc}
H^2(G,\tau) & \xrightarrow{\cong} & H^2_{c_\tau}(G/K,W) & \qquad & f & \xrightarrow{E_{c_\tau}} & c_\tau(\cdot,e)f(\cdot) \\
\downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S^c} & & & & \\
H^2(H,\sigma) & \xrightarrow{\cong} & H^2_{c_\sigma}(H/L,Z) & \qquad & g & \xrightarrow{E_{c_\sigma}} & c_\sigma(\cdot,e)g(\cdot).
\end{array}$$

Remark 2.2. Let $r: H^2(G,\tau) \to C^\infty(H\times_\tau W)$ denote the restriction map. In [29] it is shown the image of r is contained in $L^2(H\times_\tau W)$ and r is $(2,2)$ continuous. It readily follows that the corresponding map $r^c : H^2_{c_\tau}(G/K,W) \to L^2_{c_\tau}(H/L,W)$ is again the restriction map. We will apply this observation for the restriction $r_0 : H^2(G,\tau) \to L^2(H_0\times_\tau W)$.

Note: Under the assumption that $res_H(H^2(G,\tau))$ is H-admissible, in [30, Proposition 6.8], we analyze the orthogonal projectors $P_{\tau,\sigma}$, (resp. $P^c_{\tau,\sigma}$) onto the isotypic components $H^2(G,\tau)[H^2(H\times_\sigma Z)]$ (resp. $H^2_{c_\tau}(G/K,W)[H^2_{c_\sigma}(H/L,Z)]$) we state without proofs the following relations among the respective kernels (see [30, Proposition 6.8]):

$$K^c_{\tau,\sigma}(xK,yK)=c_\tau(y,e)K_{\tau,\sigma}(x,y)c_\tau(x,e)^\star.$$
$$K_{\tau,\sigma}(x,y)=\Theta_{H^2(H\times_\sigma Z)}(h\mapsto K_\tau(h^{-1}x,y)).$$
$$K^c_{\tau,\sigma}(x,y)=\Theta_{H^2(H\times_\sigma Z)}(h\mapsto K^c_\tau(h^{-1}x,y)).$$

Here, $\Theta_{H^2(H\times_\sigma Z)}$ is the Harish-Chandra character of $H^2(H\times_\sigma Z)$.

2.3. Holographic operators

Let (G,H) be an *arbitrary reductive pair*, $K,L:=K\cap H$ respective maximal compact subgroups, (τ,W) is (resp. (σ,Z)) irreducible representations of K,L.

We assume that there exists Discrete Series representations $H^2(G,\tau),H^2(H,\sigma)$.

Let $c_\tau:G\times G/K\to GL(W)$ and $c_\sigma:H\times H/L\to GL(Z)$ be cocycles. Thus, $c(ab,x)=c(a,bx)c(b,x)$ and $c_\tau(k,eK)=\tau(k)$, $c_\sigma(l,eL)=\sigma(l)$, $k\in K$, $l\in L$. By definition, holographic operators are the continuous H-intertwining linear maps

$$T:H^2_{c_\sigma}(H/L,Z)\to H^2_{c_\tau}(G/K,W). \tag{2.5}$$

Holographic operators, have properties quite similar to the symmetry breaking operators, and they satisfy a Proposition quite similar to Proposition 2.1. In the following, we state without proof (the proof is quite similar to the proof of facts valid for symmetry breaking operators) facts about holographic operators necessary for further developments. A holographic operator T is represented by a smooth kernel

$$K^c_T:H/L\times G/K\to Hom_{\mathbb{C}}(Z,W)$$

so that for $z\in Z$, $h\in H$, $K^c_T(hL,\cdot)z\in H^2_{c_\tau}(G/K,W)$; for $w\in W, s\in G$, $K^c_T(\cdot,sK)^\star w\in H^2_{c_\sigma}(H/L,Z)$ and for $g_1\in H^2_{c_\sigma}(H/L,Z)$, $w\in W$, $x\in G$, we have $(T(g_1)(xK),w)_W=(g_1,K^c_T(\cdot,xK)^\star w)_{L^2_{c_\sigma}(H/L,Z)}$. The analogue to Proposition 2.1 is the following:

Proposition 2.3. *Let (G,H) be a reductive pair of Lie groups and $H^2_{c_\tau}(G/K,W)$, $H^2_{c_\sigma}(H/L,Z)$ Discrete Series representations for G,H. Then, the following facts hold for an intertwining continuous linear map*

$$T:H^2_{c_\sigma}(H/L,Z)\to H^2_{c_\tau}(G/K,W).$$

Notation: $h,y\in H$, $x\in G$, $z\in Z$, $w\in W$, $g\in H^2_{c_\sigma}(H/L,Z)$.

(1) $T(g)(xK)=\int_{H/L}K^c_T(hL,xK)\big(c_\sigma(h,e)c_\sigma(h,e)^\star\big)^{-1}g(hL)dm_{H/L}(hL)$.

(2) $K^c_T(hL,xK)z=K^c_{T^\star}(xK,hL)^\star z=T(K^c_\sigma(\cdot,hL)^\star z)(xK)$.

(3) $K^c_T(hyL,hxK)=c_\tau(h,xK)K^c_T(yL,xK)c_\sigma(h,yL)^\star$.

(4) $K^c_{T^\star}(xK,hL)w=K^c_T(hL,xK)^\star w=T^\star(K^c_\tau(\cdot,xK)^\star w)(hL)$.

(5) *The linear map*
$Z \ni z \mapsto K_T^c(eL,\cdot)z = K_{T^\star}^c(\cdot,eL)^\star z \in H{c_\tau}^2(G/K,W)$
is an L*-map. Moreover, for each* $z \in Z$*,*
$K_T^c(\cdot,eL)z = K_{T^\star}^c(\cdot,eL)^\star z \in H^2_{c_\tau}(G/K,W)[H^2_{c_\sigma}(H/L,Z)][Z]$.

(6) $(T^\star(K_\tau^c(\cdot,xK)^\star w)(hL),z)_Z = (w, T(K_\sigma^c(\cdot,hL)^\star z)(xK))_W$.

(7) $\forall g,\ (T(g)(xK),w)_W = (g, T^\star(K_\tau^c(\cdot,xK)^\star w)_{L^2_{c_\sigma}(H/L,Z)}$.

For Discrete Series $H^2(H,\sigma)$, $H^2(G,\tau)$ in bundle model and a holographic operator $T : H^2(H,\sigma) \to H^2(G,\tau)$, since both Hilbert spaces are reproducing kernel spaces, there exists a smooth function $K_T : H \times G \to Hom_{\mathbb{C}}(Z,W)$ so that

$$(T(g)(x),w)_W = \int_H (g(h), K_T(h,g)^\star w)_Z dh \quad \text{for all } g \in H^2(H,\sigma),\ w \in W.$$

We have that $K_T(\cdot,x)^\star w$ belongs to $H^2(H,\sigma)$ for each $x \in G$.

The diagram below shows how to transfer holographic operators T on bundle model to holographic operators T^c in the symmetric space model. The corresponding relation between the function kernels K_T, $K_{T^c}^c$ is

$$K_{T^c}^c(hL,xK) = c_\tau(x,e)K_T(h,x)c_\sigma(h,e)^\star,\ \forall h \in H,\ x \in G. \tag{2.6}$$

[30, Proposition 3.7] yields that $K_{T^c}^c$ is a smooth function.

$$\begin{array}{ccc} H^2(G,\tau) & \xrightarrow{\cong} & H^2_{c_\tau}(G/K,W) \\ \uparrow T & & T^c \uparrow \\ H^2(H,\sigma) & \xrightarrow{\cong} & H^2_{c_\sigma}(H/L,Z) \end{array} \qquad \begin{array}{c} f \xrightarrow{E_{c_\tau}} c_\tau(\cdot,e)f(\cdot) \\ \\ g \xrightarrow{E_{c_\sigma}} c_\sigma(\cdot,e)g(\cdot). \end{array}$$

2.4. Differential operators

2.4.1. Generalities on differential operators

Since Discrete Series are realized in spaces of functions, sometimes, symmetry breaking operators are equal to the restriction of differential operators. For example, Rankin-Cohen brackets. The objective of this subsection is to study symmetry breaking operators represented by differential operators. As a result we obtain a way to find the integral kernel of a symmetry breaking operator knowing its representation as differential operator and vice-versa. We also obtain a generalization of a Theorem of Helgason.

For $X \in \mathcal{U}(\mathfrak{g})$, let R_X denote infinitesimal right derivative by X. Let (G,H) be an *arbitrary reductive pair*, $K, L := K \cap H$ respective maximal compact subgroups, (τ,W) is (resp. (σ,Z)) irreducible representations of K, L. We assume there exists Discrete Series representations $H^2(G,\tau), H^2(H,\sigma)$. Then, a definition of differential operator is based on the fact (for a proof, [40, chap V]): A linear map $D : \Gamma^\infty(G \times_\tau W) \to \Gamma^\infty(G \times_\sigma Z)$ is a *differential operator* if and only if there exists finitely many smooth functions $c_\alpha : G \to Hom_{\mathbb{C}}(W,Z)$ so that $D(f) = \sum c_\alpha R_{X_1^{\alpha_1} \cdots X_n^{\alpha_n}}(f)$ or a similar expression by means of left derivatives.

Such a D is invariant by left translations by G if and only if, for every α, c_α is a constant function and $\sum_\alpha c_a X_1^{\alpha_1} \cdots X_n^{\alpha_n} \in (Hom_{\mathbb{C}}(W,Z) \otimes \mathcal{U}(\mathfrak{g}))^L$.

As in [31, 4.0.1] $S : H^2(G,\tau) \to H^2(H,\sigma)$ is *represented by a differential operator* if there exists a G-invariant differential operator $D : \Gamma^\infty(G \times_\tau W) \to \Gamma^\infty(G \times_\sigma Z)$ so that $\forall f \in H^2(G,\tau)$ we have $S(f) = res(D(f))$. Here, res is the restriction map $res : \Gamma^\infty(G \times_\sigma Z) \to \Gamma^\infty(H \times_\sigma Z)$. According to Theorem 2.5, in order to show that S is the restriction of a differential operator, as in Fact 2.4, it suffices to show S is the restriction of a global differential operator. We also recall that in [30, Lemma 4.2] it is shown that S is the restriction of a differential operator if and only if $K_S(\cdot, e)^\star z$ is a K-finite vector for each $z \in Z$.

2.4.2. A theorem shown by Kobayashi-Pevzner

Let (τ, W) and (η, F) be two finite dimensional representations of K. Then, from $C^\infty(G) \otimes W$ into $C^\infty(G) \otimes F$ we have two families of operators, namely the one constructed by means of the left infinitesimal translation

$$L_D, D \in \mathcal{U}(\mathfrak{g}) \otimes Hom_{\mathbb{C}}(W, F),$$

and the one constructed via right infinitesimal translation

$$R_D, D \in \mathcal{U}(\mathfrak{g}) \otimes Hom_{\mathbb{C}}(W, F).$$

We know [40], when $D \in (\mathcal{U}(\mathfrak{g}) \otimes Hom_{\mathbb{C}}(W,F))^{(Ad \otimes \tau^\vee \otimes \eta)(K)}$, the operator R_D transforms the subspace $(C^\infty(G) \otimes W)^{(R.\otimes\tau)(K)}$ into the subspace $(C^\infty(G) \otimes F)^{(R.\otimes\eta)(K)}$, whence, R_D defines a left invariant differential operator from $\Gamma^\infty(G \times_\tau W)$ into $\Gamma^\infty(G \times_\eta F)$. In [40] it is shown that every G-left invariant differential operator from $\Gamma^\infty(G \times_\tau W)$ into $\Gamma^\infty(G \times_\eta F)$ is obtained in this way.

Kobayashi-Pevzner have generalized the previous result; we present a version according to our future needs.

For this, we fix $H \subset G$ connected reductive subgroups, H closed in G. We fix respective maximal compact subgroups $L \subset H, K \subset G$ so that $L = K \cap H$.

We fix (τ, W) (resp. (η, F)) a finite dimensional representation of K (resp. of L). Then, we have the natural inclusion $H/L \hookrightarrow G/K$ and the spaces

$$\Gamma^\infty(G \times_\tau W) \simeq C^\infty(G, W)^{(R\otimes W)(K)}, \;\; \Gamma^\infty(H \times_\eta F) \simeq C^\infty(H, F)^{(R\otimes\sigma)(L)}.$$

In [23, Theorem 3.1], they have shown the space of H-invariant differential operators from $\Gamma^\infty(G \times_\tau W)$ into $\Gamma^\infty(H \times_\eta F)$ is isomorphic to the space

$$Hom_L(F^\vee, \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} W^\vee).$$

An isomorphism is given by the map: $\phi \in Hom_L(F^\vee, \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} W^\vee)$ maps to the differential operator D_ϕ, defined, for $f \in C^\infty(G) \otimes W$, by the equality,

$$\langle D_\phi(f), z^\vee \rangle = \sum_j \langle < R_{u_j}(f), w_j^\vee \rangle|_H \text{ for } z^\vee \in F^\vee, \tag{2.7}$$

where $\phi(z^\vee) = \sum_j u_j w_j^\vee$ $(u_j \in \mathcal{U}(\mathfrak{g}),\, w_j^\vee \in W^\vee)$.

An equivalent expression for the H-invariant differential operator is

Fact 2.4. $D_\phi(f) = \sum_j T_j R_{D_j}(f), \quad T_j \in Hom_{\mathbb{C}}(W, Z), \quad D_j \in \mathcal{U}(\mathfrak{g})$

which corresponds to $\phi \in Hom_L(F^\vee, \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} W^\vee)$ *defined by*

$$\phi(z^\vee) = \textstyle\sum_j D_j \otimes T_j^t(z^\vee).$$

The L-equivariance of ϕ is represented by the equality

$$\sum_j D_j \otimes T_j = \sum_j Ad(l)(D_j) \otimes \sigma(l) T_j \tau(l^{-1}), \ \forall\, l \in L. \tag{2.8}$$

Next, we fix cocycles $c_\tau : G \times G/K \to Gl(W)$, $c_\eta : H \times H/L \to Gl(F)$. The cocycles c_τ, c_η carries respective parallelization for the respective vector bundles $G \to G \times_\tau W$, $H \to H \times_\eta F$ and respective isomorphisms as the following picture shows

$$\begin{array}{ccccc} \Gamma(G \times_\tau W) & \xrightarrow{\cong} & \Gamma_{c_\tau}(G/K, W) & \qquad & f \xrightarrow{E_{c_\tau}} c_\tau(\cdot, e) f(\cdot) \\ T \uparrow \cdot \downarrow S & & T^c \uparrow \downarrow S^c & & \\ \Gamma(H \times_\sigma F) & \xrightarrow{\cong} & \Gamma_{c_\sigma}(H/L, F) & \qquad & g \xrightarrow{E_{c_\sigma}} c_\sigma(\cdot, e) g(\cdot). \end{array}$$

Here, Γ, Γ_c represents $\Gamma^\infty, \Gamma^\infty_{c_\tau}, L^2, L^2_{c_\tau}$ and so on.

In particular, when $H = G$, $(\tau, W) = (\eta, F)$ we obtain three families of differential operators on $\Gamma^\infty(G/K, W)$, namely,

$$\text{for } D \in \mathcal{U}(\mathfrak{g}) \rightsquigarrow \dot{\pi}_{c_\tau}(D), \quad L^c_D := E_{c_\tau} L_D E^{-1}_{c_\tau}; \tag{2.9}$$
$$\text{for } D \in (\mathcal{U}(\mathfrak{g}) \otimes Hom_{\mathbb{C}}(W, W))^{(Ad \otimes \tau^\vee \otimes \tau)(K)} \rightsquigarrow R^c_D := E_{c_\tau} R_D E^{-1}_{c_\tau}.$$

Moreover, the action of G on G/K by left translation gives rise to another family of differential operators $L^{G/K}_D$, $D \in \mathcal{U}(\mathfrak{g})$, $f \in C^\infty(G/K) \otimes W$ by the formula for $X \in \mathfrak{g}$, $L^{G/K}_X(f)(x) := \{\frac{d}{dt} f(exp(-tX)x)\}_{t=0}$. We would like to recall the following identities

$$\text{for } x \in G,\ X \in \mathfrak{g},\ f \in C^\infty(G/K, W) = C^\infty(G/K) \otimes W),$$
$$\dot{\pi}_{c_\tau}(X)(f)(x) = L^{G/K}_X(f)(x) + \big\{\frac{d}{dt} c_\tau(exp(tX), exp(-tX)x)\big\}_{t=0} f(x),$$
$$L^c_D = \dot{\pi}_{c_\tau}(D),\ D \in \mathcal{U}(\mathfrak{g}), \tag{2.10}$$
$$L_D(f)(e) = R_{\check{D}}(f)(e), \quad D \in \mathcal{U}(\mathfrak{g}),\ f \in C^\infty(G, W). \tag{2.11}$$

In these formulas $D \mapsto \check{D}$ is the antiautomorphism of $\mathcal{U}(\mathfrak{g})$ associated to the map $\mathfrak{g} \ni X \mapsto -X \in \mathfrak{g}$.

The equality $L^c_D(f)(e) = R^c_{\check{D}}(f)(e), D \in \mathcal{U}(\mathfrak{g}), f \in C^\infty(G/K, W)$, is not true, as it readily follows from the formula for L^c_X.

However, it holds that $[L^c_D(f)](e) = [R_{\check{D}}(c_\tau(\cdot, o)^{-1} f(\cdot))](e)$.

We could define R^{cnv}_D for any $D \in \mathcal{U}(\mathfrak{g}) \otimes Hom_{\mathbb{C}}(W, W)$ as follows: Let P_0 be the projector of $C^\infty(G) \otimes W$ onto $(C^\infty(G) \otimes W)^{R \otimes \tau(K)}$; the projector is given by integration along K. For any $D \in \mathcal{U}(\mathfrak{g})$ we set $R^{cnv}_D = E_{c_\tau} P_0 R_D E^{-1}_{c_\tau}$.

Certainly, the definition of differential operator in the language of supports, yields R^{cnv}_D is a G-invariant differential operator on $\Gamma^\infty(G \times_\tau W)$. Whence, the result of Kobayashi-Pevzner just quoted, implies the existence of

$$D_0 \in (\mathcal{U}(\mathfrak{g}) \otimes Hom_{\mathbb{C}}(W, W))^{(Ad \otimes \tau^\vee \otimes \tau)(K)}$$

so that $R^{cnv}_D = R^c_{D_0}$. In Theorem 2.5, we show that $D_0 = Q_0(D)$. Here, Q_0 is usual projector from $\mathcal{U}(\mathfrak{g}) \otimes Hom_{\mathbb{C}}(W, W)$ onto

$$(\mathcal{U}(\mathfrak{g}) \otimes Hom_{\mathbb{C}}(W, W))^{(Ad \otimes \tau^\vee \otimes \tau)(K)}.$$

2.4.3. Generalization of a theorem of Helgason

The technique developed in the next result has several applications. We would like to point out that the result generalizes [13, Ch. II Theorem 4.6].

Theorem 2.5. *We assume $H \subset G$ is an arbitrary reductive pair, and there exists Discrete Series representations $H^2(G,\tau), H^2(H,\sigma)$. Let $S : H^2(G,\tau) \to H^2(H,\sigma)$ be a continuous H-intertwining map. We suppose there exists a differential operator $D : C^\infty(G) \otimes W \to C^\infty(H) \otimes Z$ so that D restricted to $H^2(G,\tau)$ is equal to S and $D = \sum_j T_j R_{D_j}, T_j \in Hom_{\mathbb{C}}(W,Z), D_j \in \mathcal{U}(\mathfrak{g})$. Then, there exists*

$$D_0 = \textstyle\sum_j P_j \otimes L_j \in (\mathcal{U}(\mathfrak{g}) \otimes Hom_{\mathbb{C}}(W,Z))^{(Ad\otimes\tau^\vee\otimes\sigma)(L)},$$

so that $\sum P_j R_{L_j}$ restricted to $H^2(G,\tau)$ is equal to S.

In Kobayashi-Pevzner notation,

$$\begin{aligned} D_0 &\in \mathrm{Diff}_H\big((C^\infty(G)\otimes W)^{(Ad\otimes\tau)(K)}(C^\infty(H)\otimes Z)^{(Ad\otimes\sigma)(L)}\big) \\ &\cong \mathrm{Diff}_H(\Gamma^\infty(G\times_\tau W),\, \Gamma^\infty(H\times_\sigma Z)) \end{aligned}$$

represents S. That is, in vector bundle language, and, according to definition 2.4, S is equal to the restriction of a H-invariant differential operator from $\Gamma^\infty(G\times_\tau W)$ into $\Gamma^\infty(H\times_\sigma Z)$.

Proof. For $\nu \in \widehat{K}$, let χ_ν (resp. d_ν) be the character of ν (resp. the dimension of ν). Then, for the representation $R\otimes\tau$, the isotypic component corresponding to ν is

$$(C^\infty(G)\otimes W)[\nu] := \{f : d_\nu \int_K \bar{\chi}_\nu(k)\tau(k)(f(xk))dk = f(x)\},$$

then
$$C^\infty(G)\otimes W = \mathrm{Cl}(\oplus_{\nu\in\hat{K}} C^\infty(G)\otimes W[\nu])$$

and
$$C^\infty(G)\otimes W[Triv] := \{f : \tau(k)(f(xk)) = f(x)\}.$$

Similarly, for $\eta \in \widehat{L}$ and the action $R\otimes\sigma$ the isotypic component is

$$(C^\infty(H)\otimes Z)[\eta] := \{f : d_\eta \int_L \bar{\chi}_\eta(l)\sigma(l)(f(xl))dl = f(x)\},$$

then
$$C^\infty(H)\otimes Z = \mathrm{Cl}(\oplus_{\eta\in\hat{L}}(C^\infty(H)\otimes Z)[\eta]).$$

We recall that

$$\begin{aligned} Hom_L(Z^\vee, \mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{k})} W^\vee) &= \big((\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{k})} W^\vee)\otimes Z\big)[Triv] \\ &\cong (S(\mathfrak{p})\otimes W^\vee\otimes Z)^{(Ad\otimes\tau^\vee\otimes\sigma)(L)}. \end{aligned}$$

The equality is the expression for the isotypic component associated the trivial character for the representation $Ad\otimes\tau^\vee\otimes\sigma$ of L. The equivalence is given by means of the symmetrization map. For the representation $Ad\otimes\tau^\vee\otimes\sigma$ of L, we write the decomposition $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{k})} W^\vee\otimes Z = \oplus_{\eta\in\hat{L}}(\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{k})} W^\vee\otimes Z)[\eta]$.

We write
$$D = D_0 + \sum_{\eta\neq Triv\in\hat{L}} D_\eta, \quad D_0 \in (\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{k})} W^\vee\otimes Z)^{(Ad\otimes\tau^\vee\otimes\sigma)(L)},$$

and
$$D_\eta \in (\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{k})} W^\vee\otimes Z)[\eta].$$

We notice that $D=\sum_s D_s\otimes T_s \in (\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{k})} W^\vee \otimes Z)[\eta]$ if and only if

$$\sum_s \int_L d_\eta \bar{\chi}_\eta(l)\sigma(l)T_s\tau(l^{-1})R_{Ad(l)D_s}dl = \sum_s T_s R_{D_s}. \tag{2.12}$$

In the following, we verify for $D_\eta=\sum_s D_s\otimes T_s \in (\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{k})} W^\vee \otimes Z)[\eta]$, that $D_\eta(C^\infty(G)\otimes W)[Triv]) \subset (C^\infty(H)\otimes Z)[\eta]$. For this purpose we must verify for $f\in (C^\infty(G)\otimes W)[Triv]$ that

$$\int_L d_\eta \bar{\chi}_\eta(l)\sigma(l)D_\eta(f)(hl)dl = D_\eta(f)(h), \quad h\in H,$$

and $$\int_L d_\eta \bar{\chi}_\eta(l)\sigma(l)D_\eta(f)(hl)dl = \int_L d_\eta \bar{\chi}_\eta(l)\sum_s \sigma(l)T_s(R_{D_s}(f)(hl))dl.$$

The equality $(R_{D_s}(f))(hl)=\tau(l^{-1}))(R_{Ad(l)D_s}(f))(h)$, applied to the last term yields that the first term is equal to

$$\int_L d_\eta \bar{\chi}_\eta(l)\sum_s \sigma(l)T_s\tau(l^{-1})(R_{Ad(l)D_s}(f))(h)dl = D_\eta(f)(h).$$

We now finish the proof of the Theorem. For this purpose we notice that

$$H^2(G,\tau)\subset C^\infty(G)\otimes W[Triv]$$

and $$S(H^2(G,\tau)) = D(H^2(G,\tau)) = D_0(H^2(G,\tau)) \oplus \oplus_{\eta\neq Triv\in\hat{L}} D_\eta(H^2(G,\tau)).$$

Since our hypothesis is $S(H^2(G,\tau))\subset H^2(H,\sigma)\subset C^\infty(H)\otimes Z[Triv]$, and the sum is direct we have $D_\eta(H^2(G,\tau))=0$ for $\eta\neq Triv\in\hat{L}$, so D restricted to $H^2(G,\tau)$ is equal to D_0 restricted to $H^2(G,\tau)$. ∎

Actually, the proof applies to the subspaces

$$(C^\infty(G)\otimes W)[Triv], \text{ and } (C^\infty(H)\otimes Z)[Triv],$$

to obtain that any $D=\sum_s T_s R_{D_s}$ so that

$$D((C^\infty(G)\otimes W)[Triv]) \subset (C^\infty(H)\otimes Z)[Triv],$$

can be replaced by the component D_0 as above.

2.5. Differential symmetry breaking operators

In the following, we apply the previous definitions and results to the different realization of Discrete Series representations.

2.5.1. Symmetry breaking op's in case G-bundle model

We assume that $H\subset G$ is an arbitrary reductive pair, and there exists Discrete Series representations $H^2(G,\tau), H^2(H,\sigma)$.

Next, we recall and sketch a proof of a result in [30, Thm 4.3].

Theorem 2.6. *We assume the restriction to H of $(L^G_\cdot, H^2(G,\tau))$ is an admissible representation. Then, any intertwining map $S: H^2(G,\tau)\to H^2(H,\sigma)$ is the restriction of a differential operator.*

In the following, we outline a verification of Theorem 2.6, in turn, the proof yields how to represent S as a differential operator from K_S and viceversa. We recall in [30, Remark 3.8], we have shown that the function $K_S(\cdot,e)^\star z$ is L-finite. Thus, owing to [19, Proposition 1.6], "In an H-admissible representation, L-finite vectors are K-finite vectors", our hypothesis forces that $K_S(\cdot,e)^\star z$ is a K-finite vector in $H^2(G,\tau)$. We fix $0 \neq w_0 \in W$. In consequence the irreducibility of the representation $L^G_\cdot$ implies that there exists a linear function $Z \ni z \to D_z \in \mathcal{U}(\mathfrak{g})$, so that $K_S(\cdot,e)^\star z = L^c_{D_z} K_\tau(\cdot,e)^\star w_0$. Therefore,

$$\begin{aligned}(S(f)(e),z)_Z &= \int_G (f(x), K_S(x,e)^\star z)_W dx \\ &= \int_G (f(x), L_{D_z}(K_\tau)(x,e)^\star w_0)_W dx \\ &= \int_G (L_{D_z^\star} f(x), K_\tau(x,e)^\star w_0)_W dx = (L_{D_z^\star} f(e), w_0)_W.\end{aligned}$$

Thus, $(S(f)(h),z)_Z = (S(L_{h^{-1}}(f))(e),z)_Z = (L_{D_z^\star}(L_{h^{-1}}(f))(e),w_0)_W$.
We recall the equality $L_D(f)(e) = R_{\check{D}}(f)(e)$. Hence,

$$(S(f)(h),z)_Z = (R_{\check{D}_z^\star}(L_{h^{-1}}(f))(e),w_0)_W = (R_{\check{D}_z^\star}(f)(h),w_0)_W.$$

Whence, after we fix an orthonormal basis $\{z_j, 1 \leq j \leq \dim Z\}$ for Z, an expression of S by means of left invariant differential operators is

$$S(f)(h) = \sum_{1\leq j\leq \dim Z} (R_{\check{D}_{z_j}^\star}(f)(h), w_0)_W \, z_j.$$

Next, we apply Theorem 2.5 and obtain a differential operator D_0, according to Fact 2.4, that represents S. We define $T_j : W \to Z$ by the equality $T_j(w) = (w,w_0)_W \, z_j$. Hence, $S(f) = \sum_j T_j R_{\check{D}_{z_j}^\star}(f)$. The component "$D_0$" of the right hand side, according to 2.12, is $\int_L \sum_j \sigma(l) T_j \tau(l^{-1}) R_{Ad(l)(\check{D}_{z_j}^\star)} dl$. Next, we fix a ordered linear basis $\{D_t\}_t$ for $\mathcal{U}(\mathfrak{g})$. Thus, $Ad(l)(\check{D}_{z_j}^\star) = \sum_t a_{j,t}(l) D_t$ and we obtain

$$S(f)(h) = \textstyle\sum_{j,t} \int_L \sigma(l) T_j \tau(l^{-1}) a_{j,t}(l)\, dl \; R_{D_t}(f)(h),$$

an expression of S as differential operator in the realm of Fact 2.4.
An expression for S by means of invariant differential operators is

$$S(f)(h) = \sum_{1\leq j\leq \dim Z} (L_{Ad(h)(D_{z_j}^\star)}(f)(h), w_0)_W \, z_j.$$

In the ordered linear basis $\{D_t\}_t$, we write $Ad(h)(D_{z_j}^\star) = \sum_t \phi_{t,j}(h) D_t$ where $\phi_{t,j}(\cdot) \in C^\infty(H)$, and we obtain the expression

$$S(f)(h) = \sum_{t,j} \phi_{t,j}(h)(L_{D_t} f(h), w_0)_W \, z_j.$$

This concludes the outline for the direct implication in the proof of Theorem 2.6.
Next, we study the converse statement to Theorem 2.6, that is, we sketch a way to recover the kernel K_S when we know a representation of S as a differential operator.

A tool is [19, Proposition 16] and the identity $S(K_\tau(\cdot,x)^\star w)(h) = K_S(x,h)w$, [30, Proposition 3.7]. The hypothesis yields the existence of $D_j \in \mathcal{U}(\mathfrak{g})$, and ϕ_j, ψ_j, $Hom_{\mathbb{C}}(W,Z)$-valued smooth functions, so that for $f \in H^2(G,\tau)^\infty$,

$$S(f)(h) = \sum_j \phi_j(h) L_{D_j}(f)(h) = \sum_j \psi_j(h) R_{D_j}(f)(h), h \in H.$$

Both sums are finite sums!

In [30] it is verified that $K_\tau(\cdot,h)^\star w$ is a smooth vector for $H^2(G,\tau)$, hence, we have

$$\begin{aligned} K_S(x,h)w = S(K_\tau(\cdot,x)^\star w)(h) &= \sum_j \phi_j(h) L^{(1)}_{D_j}\big(K_\tau(\cdot,x)^\star(w)\big)(h) \\ &= \sum_j \psi_j(h) R^{(1)}_{D_j}\big(K_\tau(\cdot,x)^\star(w)\big)(h). \end{aligned}$$

Therefore, knowing an expression of S as differential operator, we just follow the above recipe to obtain the kernel K_S.

2.5.2. Symmetry breaking op's for symmetric space G/K-model

We recall our hypothesis, (G,H) is an arbitrary reductive pair, and there exists Discrete Series representations $H^2(G,\tau), H^2(H,\sigma)$. We assume the representation $res_H((\pi^c_{c_\tau}, H^2_{c_\tau}(G/K,W))$ is admissible. We fix $S^c : H^2_{c_\tau}(G/K,W) \to H^2_{c_\tau}(H/L,Z)$ a symmetry breaking operator.

Under these hypotheses it follows S^c is represented by a differential operator.

Note that we already have an isomorphism between the bundle model and the symmetric space model; but it is convenient to work entirely within the symmetry space model.

In the following we show how to compute a differential operator that represents the operator S^c knowing the kernel $K^c_{S^c}$ and viceversa.

We recall that Proposition 2.1 shows

$$K^c_{S^c}(\cdot,eL)^\star z \in H^2_{c_\tau}(G/K,W)[H^2_{c_\sigma}(H/L,Z)][Z].$$

We fix a nonzero $w_0 \in W$. Then, due to the hypothesis that the representation $H^2_{c_\tau}(G/K,W)$ is H-admissible, we have $K^c_{S^c}(\cdot,eL)^\star z \in H^2_{c_\tau}(G/K,W)_{K-fin}$ [18], also we have that infinitesimal representation in $H^2_{c_\tau}(G/K,W)_{K-fin}$ is algebraically irreducible. Thus, there exists a linear function $Z \ni z \mapsto D_z \in \mathcal{U}(\mathfrak{g})$ so that

$$\pi^c_{c_\tau}(D_z)\big(K^c_\tau(\cdot,e)^\star w_0\big)(\cdot) = K^c_{S^c}(\cdot,eL)^\star z, z \in Z.$$

Next, we point out equalities that will justify a later development:

$$\pi^c_{c_\tau}(D_z)\big(K^c_\tau(\cdot,e)^\star w_0\big)(\cdot) = L^c_{D_z}\big(K^c_\tau(\cdot,e)^\star w_0\big)(\cdot).$$

$$L^c_D(F)(x) = c_\tau(x,o) L^G_D\big((c_\tau(\cdot,e))^{-1}F(\cdot)\big)(x); \;\; c_\tau(e,yK) = I_W$$

$$K^c_\tau(xK,eK) = c_\tau(e,eK)K_\tau(x,e)c_\tau(x,eK)^\star = K_\tau(x,e)c_\tau(x,eK)^\star.$$

Equation 2.3 justifies the first equality, the others are justified by means of the previous equalities

$$(S^c(f)(o),z)_Z = \int_{G/K} ((c_\tau(x,o)c_\tau(x,o)^\star)^{-1} f(x), K^c_{S^c}(x,eL)^\star z)_W dm_{G/K}(x)$$

$$
\begin{aligned}
&= \int_{G/K} ((c_\tau(x,o)c_\tau(x,o)^\star)^{-1} f(x), L^c_{D_z}\big(K^c_\tau(\cdot,e)^\star w_0\big)(x))_W dm_{G/K}(x) \\
&= \int_{G/K} (c_\tau(x,o)^\star (c_\tau(x,o)^\star)^{-1} c_\tau(x,o)^{-1} f(x), \\
&\qquad L_{D_z}\big(c_\tau(\cdot,e)^{-1} K^c_\tau(\cdot,e)^\star w_0\big)(x))_W dm_{G/K}(x) \\
&= \int_{G/K} (L_{D_z^\star}\big(c_\tau(\cdot,o)^{-1} f(\cdot)\big)(x),\ c_\tau(x,o)^{-1} K^c_\tau(x,e)^\star w_0)_W dm_{G/K}(x) \\
&= \int_{G/K} (L_{D_z^\star}\big(c_\tau(\cdot,o)^{-1} f(\cdot)\big)(x),\ c_\tau(x,o)^{-1} c_\tau(x,o) K_\tau(x,e)^\star w_0)_W dm_{G/K}(x) \\
&= (L_{D_z^\star}\big(c_\tau(\cdot,o)^{-1} f(\cdot)\big)(e), w_0)_W = (R_{\check{D}_z^\star}\big(c_\tau(\cdot,o)^{-1} f(\cdot)\big)(e), w_0)_W.
\end{aligned}
$$

We compute,

$$
\begin{aligned}
(S^c(f)(h), z)_Z &= (c_\sigma(h,o) S^c(L^c_{h^{-1}}(f))(o), z)_Z = (S^c(L^c_{h^{-1}}(f))(o), c_\sigma(h,o)^\star z)_Z \\
&= (R_{\check{D}^\star_{c_\sigma(h,o)^\star z}} (L_{h^{-1}}(c_\tau(\cdot,o)^{-1} f(\cdot)))(o), w_0)_W \\
&= (R_{\check{D}^\star_{c_\sigma(h,o)^\star z}} ((c_\tau(\cdot,o)^{-1} f(\cdot)))(h), w_0)_W.
\end{aligned}
$$

We fix a linear basis $\{D_i\}$ for $\mathcal{U}(\mathfrak{g})$. Then, it readily follows from the fact, for $X \in \mathfrak{g}$, R_X is a derivation, that there exists a finite family ϕ_i of $Hom_{\mathbb{C}}(W,Z)$-valued smooth functions on H, giving,

$$
S^c(f)(h \cdot o) = \sum_{1 \le j \le \dim Z} \Big(\sum_{i=1}^N \phi_i(h) R_{D_i}(f)(h \cdot o), z_j \Big)_Z z_j. \tag{2.13}
$$

whence, S^c is represented by a differential operator.

Next, in a similar path to the development of the G-model, we apply Theorem 2.5 and obtain a differential operator D_0, according to Fact 2.4, that represents S^c.

In the following, we describe an algorithm to compute the kernel $K^c_{S^c}$ when we know a representation of S^c by means of differential operators.

The hypothesis is that π_{c_τ} is H-admissible, the tools are 2.13 and the equality $K^c_S(xK, hL)w = K^c_{S^\star}(hL, xK)^\star w = S(K^c_\tau(\cdot, xK)^\star w)(hL)$. Therefore

$$
\begin{aligned}
K^c_S(xK, hL)w &= S(K^c_\tau(\cdot, xK)^\star w)(hL) \\
&= \sum_{1 \le j \le \dim Z} (\sum_{i=1}^N \phi_i(h) R_{D_i}(K^c_\tau(\cdot, xK)^\star w)(h \cdot o), z_j)_Z\, z_j.
\end{aligned}
$$

and we have computed the kernel K_{S^c}.

Remark 2.7. We would like to point out, the following fact shown in [30, Lemma 4.2]. S^c is represented by a differential operator if and only if the function $K_{S^c}(\cdot, e)^\star z$ is a K-finite vector for each $z \in Z$.

2.6. Some properties of symmetry breaking operators

Let (G,H) be an arbitrary reductive pair, $K, L := K \cap H$ respective maximal compact subgroups, (τ, W) is (resp. (σ, Z)) irreducible representations of K (resp. L). We assume there exists Discrete Series representations $H^2(G,\tau), H^2(H,\sigma)$. In order to begin with, we recall the Schmid operator $D_{G,\tau} : \Gamma^\infty(G \times_\tau W) \to \Gamma^\infty(G \times_\tau W_\Psi)$.

For this, let $q_\Psi : W \otimes \mathfrak{p}_\mathbb{C} \twoheadrightarrow W_\Psi$ be the quotient of $W \otimes \mathfrak{p}_\mathbb{C}$ associated to the Harish-Chandra parameter for $H^2(G,\tau)$. Then, we have $D_{G,\tau}(f) = q_\Psi(\sum_j R_{X_j}(f))$, $f \in \Gamma^\infty(G \times_\tau W)$, where $\{X_j\}$ is an orthonormal basis for $\mathfrak{p}$ (see [17], [29]). The Schmid operator is elliptic and $Ker(D_{G,\tau}) \cap L^2(G \times_\tau W) = H^2(G,\tau)$. In [34] and further work of [41], it is shown that $Ker(D_{G,\tau})$ is a maximal model for the representation $H^2(G,\tau)$. Under the assumption $res_H((L_\cdot^G, H^2(G,\tau)))$ is H-admissible in [30, Theorem 4.11] we find a proof that any continuous symmetry breaking operator from $H^2(G,\tau)$ into $H^2(H,\sigma)$ extends to continuous symmetry breaking operator from the maximal model of the first representation given by the kernel of the Schmid operator, into the maximal model of the second representation constructed by means of the Schmid operator, and that any continuous symmetry breaking operator from the maximal model of $(L_\cdot^G, H^2(G,\tau))$ into the maximal model of $(L_\cdot^H, H^2(H,\sigma))$ is represented by a differential operator. In [23, Theorem 5.13], under the assumption of $res_H((L_\cdot^G, H^2(H,\sigma))$ being H-admissible, and, in the realm of holomorphic Discrete Series, they show that any symmetry breaking continuous operator from the maximal model given by holomorphic functions on the bounded domain $\mathcal{D}_G$ with values into the lowest K-type, into the maximal model given by holomorphic functions on the bounded domain $\mathcal{D}_H$ into the lowest L-type, carries the unitary model inside the maximal model onto the unitary model contained in the maximal model. We would like to point out, that the result, as well as the proof, yields the following generalization.

Let $Ker(D_{G,\tau})$ denote the kernel of the Schmid operator, [30], owing to $D_{G,\tau}$ being elliptic, $Ker(D_{G,\tau})$ is contained in $\Gamma^\infty(G \times_\tau W)$. We endow $\Gamma^\infty(G \times_\tau W)$ as usual with the smooth topology. That is, the topology of uniform convergence on compact subsets of the sequence as well as any of its derivatives. Then we have that $H^2(G,\tau) \subset Ker(D_{G,\tau})$ is a dense subspace, the subspace of K-finite vectors of each of them are identical [1][34][35]. A similar results holds for H. Let us recall that when $H^2(G,\tau)$ is a holomorphic Discrete Series, the Schmid operator is equal to the $\bar{\partial}$-operator.

Theorem 2.8. *Let (G,H) a general reductive pair. $H^2(G,\tau)$, $H^2(H,\sigma)$ Discrete Series representations. We assume $res_H(H^2(G,\tau))$ is an H-admissible representation. Then, the following four statements hold:*

(a) *Any continuous intertwining linear map from $Ker(D_{G,\tau})$ into $Ker(D_{H,\sigma})$ is the restriction of a differential operator.*

(b) *Any continuous H-intertwining map from $H^2(G,\tau)$ into $H^2(H,\sigma)$ extends to a continuous linear map from $Ker(D_{G,\tau})$ into $Ker(D_{H,\sigma})$.*

(c) *Any nonzero continuous H-intertwining linear map from $Ker(D_{G,\tau})$ into $Ker(D_{H,\sigma})$ maps continuously $H^2(G,\tau)$ onto $H^2(H,\sigma)$.*

(d) *Any $(\mathfrak{h}, L)$ morphism from $H^2(G,\tau)_{K-fin}$ into $H^2(H,\sigma)_{L-fin}$ extends to a continuous intertwining map from $H^2(G,\tau)$ into $H^2(H,\sigma)$.*

In [25, Theorem 3.6], for the holomorphic setting, we find a proof of some of the statements in the Theorem.

Proof of Theorem 2.8. (a) and (b) are shown in [30]. For (c) we follow [23, Theorem 5.13] quite closely. Let $S : Ker(D_{G,\tau}) \to Ker(D_{H,\sigma})$ be a continuous H-map.

$Ker(D_{G,\tau})$ is a maximal model [34] [41], which yields

$$S(Ker(D_{G,\tau})_{K-fin}) = S(H^2(G,\tau)_{K-fin}) = Ker(D_{H,\sigma})_{L-fin} = H^2(H,\sigma)_{L-fin}.$$

The hypothesis is that $res_H((L^G_\cdot, H^2(G,\tau))$ is H-admissible. We conclude by [6] that $res_L((L^G_\cdot, H^2(G,\tau))$ is L-admissible. Then the work of Kobayashi [18] [19] yields, [we write, for short, $V^G_\lambda := H^2(G,\tau), V^H_{\mu_j} := H^2(H,\sigma_j)$]

$$H^2(G,\tau) = \oplus_{1\leq j<\infty} V^G_\lambda[V^H_{\mu_j}] \cong \oplus_{1\leq j<\infty} m_j V^H_{\mu_j} \text{ (Hilbert sum), and}$$

$$H^2(G,\tau)_{K-fin} = \oplus_{1\leq j<\infty} V^G_\lambda[V^H_{\mu_j}]_{L-fin} \cong \oplus_{1\leq j<\infty}[m_j(V^H_{\mu_j})]_{L-fin} \text{ (algebraic sum).}$$

Here, $\forall j, 1 \leq m_j < \infty$; for $i \neq j, V^H_{\mu_i} \ncong V^H_{\mu_j}$.

Thus, for all k but one, S carries the isotypic component corresponding to V_{μ_k} in $H^2(G,\tau)_{K-fin}$ to the zero subspace. Thus, one $V^H_{\mu_k}$ is equal to $H^2(H,\sigma)$, say $\mu_k = \mu_1$, and $S(V^G_\lambda[V^H_{\mu_1}]_{L-fin}) \neq \{0\}$. Let B_j linear, invariant $\mathcal{U}(\mathfrak{h})$-irreducible subspaces so that we write $m_1(V^H_{\mu_1})_{L-fin} \cong (V_\lambda)_{K-fin}[V^H_\mu] = \oplus_{1\leq j\leq m_1} B_j$, and let's say $S(B_j) \neq \{0\}$ for exactly $1 \leq j \leq m \leq m_1$. Then, H-irreducibility implies S restricted to each $B_j, 1 \leq j \leq m$ is a equivalence with $H^2(H,\sigma)_{L-fin}$.

By classical theory of Harish-Chandra, [40, Lemma 8.6.7], for $1 \leq j \leq m$, S extends to a continuous linear, injective (S_j is injective in the subspace of L-finite vectors), H-map S_j from the closure $\mathrm{Cl}(B_j)$ of B_j in $H^2(G,\tau)$ into $H^2(H,\sigma)$.

Considering the polar decomposition $S_j = U_j\sqrt{S_j^\star S_j}$ we obtain that $\sqrt{S_j^\star S_j}$ is a continuous intertwining linear operator on $\mathrm{Cl}(B_j)$. The H-irreducibility of $\mathrm{Cl}(B_j)$ forces $\sqrt{S_j^\star S_j}$ is equal to a multiple of the identity mapping. Hence, S_j is up to a constant a partial isometry with dense image (the image contains the L-finite vectors). Thus, S_j is a *surjective* map. We define $\tilde{S} = \sum_{1\leq j\leq m} S_j$ on $\oplus \mathrm{Cl}(B_j)_{1\leq j\leq m}$ and $\tilde{S} = 0$ in either $\mathrm{Cl}(\sum_{2\leq j<\infty} m_j \mathrm{Cl}(V^H_{\mu_j})_{L-fin}))$ or $\oplus_{m+1\leq j\leq m_1}\mathrm{Cl}(B_j)$.

Thus, $\tilde{S}$ is a continuous H-map from $H^2(G,\tau)$ *onto* $H^2(H,\sigma)$. Owing to (b), the map $\tilde{S}$ extends to a continuous linear map from $Ker(D_G)$ into $Ker(D_{H,\sigma})$ and the map $\tilde{S}$ restricted to $H^2(G,\tau)_{K-fin}$ agrees with the restriction of S. Since the subspace of K-finite vectors is dense in $Ker(D_G)$, we have they agree everywhere. Thus, S maps $H^2(G,\tau)$ onto $H^2(H,\sigma)$, and we have shown (c). Finally, (d) is shown in [21]. ■

Remark 2.9. *Summary on automatic continuity.*

If we assume that $res_H((L^G_\cdot, H^2(G,\tau))$ is an admissible representation, then, in [21], we find a proof of the following statements:

(a) Any $(\mathfrak{h}, L)$ morphism T from $H^2(G,\tau)_{K-fin}$ into $H^2(H,\sigma)_{L-fin}$ extends to a continuous intertwining map from $H^2(G,\tau)$ into $H^2(H,\sigma)$.

(b) Any continuous intertwining linear operator from the space of smooth vectors in $H^2(G,\tau)$ into the space of smooth vectors in $H^2(H,\sigma)$ extends to a continuous morphism between the corresponding maximal model representations.

We do not know whether or not (a) and (b) are true after we drop the H-admissibility assumption.

2.7. Duality theorem in G-bundle space model.

In this subsection we assume (G,H) is a *symmetric pair.* More precisely, G is a connected reductive group so that H is the identity connected component of the fixed points of an involution σ of G. Let $K := G^\theta$ a maximal compact subgroup for G so that σ commutes with θ. Hence $L := H \cap K$ is a maximal compact subgroup of H. Let $H_0 := (G^{\sigma\theta})_0$ the associated subgroup to H, then L is a maximal compact subgroup of H_0. Next, we fix $H^2(G,\tau)$ a Discrete Series representation for G so that the $res_H(H^2(G,\tau))$ is H-admissible. Let $\mathcal{U}(\mathfrak{h}_0)W := \mathcal{U}(\mathfrak{h}_0)\big(H^2(G,\tau)[W]\big)$ denote the subspace of $H^2(G,\tau)$ spanned by the left translates by $\mathcal{U}(\mathfrak{h}_0)$ of the lowest K-type $H^2(G,\tau)[W]$ of $H^2(G,\tau)$. Finally, we fix a Discrete Series $H^2(H,\sigma)$ for H. Then, the *duality Theorem* [31, Theorem 1] claims that

$$Hom_H(H^2(H,\sigma), H^2(G,\tau)) \cong Hom_L(Z, \mathcal{U}(\mathfrak{h}_0)W).$$

In the following, we describe the isomorphism $\cong$ and some structural facts for $\mathcal{U}(\mathfrak{h}_0)W$. Later on, we will present an analogous statement in the realm of symmetric space model realization of Discrete Series representations. In Section 4 we will present an independent proof of the duality Theorem under the extra hypothesis: the inclusion $H/L \to G/K$ is holomorphic and $H^2(G,\tau)$ is a holomorphic Discrete Series representation.

The following statements are verified in [31, 4.1.5].

(a) We write $res_L(\tau)$ as a sum of irreducible representations. That is, we write $res_L(\tau) = \sigma_1 \oplus \cdots \oplus \sigma_R$ with (σ_j, Z_{σ_j}) an irreducible representation of L. Then, each σ_j is the lowest L-type of a Discrete Series $H^2(H_0,\sigma_j)$ for H_0, and, $\mathcal{U}(\mathfrak{h}_0)W$ is $(\mathfrak{h}_0, L)$-equivalent to the direct sum over $1 \le j \le R$ of the underlying Harish-Chandra modules for $H^2(H_0,\sigma_j)$. Formally,

$$\mathcal{U}(\mathfrak{h}_0)W \cong \oplus_{1\le j\le R} H^2(H_0,\sigma_j)_{L-fin}$$

(b) Let $\mathcal{L}_{W,H}$ the linear span of subspaces of $H^2(G,\tau)$ that realizes the lowest L-type of each irreducible H-factor of $H^2(G,\tau)$. That is,

$$\mathcal{L}_{W,H} = \oplus_{H^2(H,\mu)\in Spec(res_H(H^2(G,\tau))} H^2(G,\tau)[H^2(H,\mu)][Z_\mu]$$

Here, (μ, Z_μ) is the lowest L-type of $H^2(H,\mu)$. Then, there exists

$$\text{a bijective, linear}, L-\text{map}, \ D \text{ from } \mathcal{L}_{W,H} \text{ onto } \mathcal{U}(\mathfrak{h}_0)W. \tag{2.14}$$

(c) Each $T \in H^2(H,\mu) \to H^2(G,\tau)$ is represented by a smooth kernel

$$K_T : H \times G \to Hom_{\mathbb{C}}(Z_\mu, W).$$

Then, for $z \in Z_\mu$, $K_T(e,\cdot)z \in H^2(G,\tau)[H^2(H,\mu)][Z_\mu]$, whence we obtain a map from Z_μ into $\mathcal{U}(\mathfrak{h}_0)W$ via D. This is the map

$$Z_\mu \ni z \mapsto D\big(G \ni x \mapsto K_T(e,x)z)\big)(\cdot) \in \mathcal{U}(\mathfrak{h}_0)W.$$

The resulting map belongs to $Hom_L(Z_\mu, \mathcal{U}(\mathfrak{h}_0)W)$. In [31, Theorem 1] it is shown that the map $T \mapsto \big(z \mapsto D(K_T(e,\cdot)z)(\cdot)\big)$ is *bijective.*

(d) For each j we may and will realize Z_{σ_j} as a linear subspace of W. Thus, we may write $W=\oplus_{1\leq j\leq R} Z_{\sigma_j}$, and we have the inclusions $H^2(H_0,\sigma_j)\subset L^2(H_0\times_{\sigma_j} Z_{\sigma_j})$ and the equality $L^2(H_0\times_\tau W)=\oplus_{1\leq j\leq R} L^2(H_0\times_{\sigma_j} Z_{\sigma_j})$. We define the subspace

$$\mathbf{H}^2(H_0,\tau):=\oplus_{1\leq j\leq R} H^2(H_0,\sigma_j)\subset L^2(H_0\times_\tau W).$$

Now, since $H^2(G,\tau)$ consists of smooth functions, let $r_0: H^2(G,\tau)\to C^\infty(H_0\times_\tau W)$ denotes the restriction map. It is shown in [31, Proposition 1] that r_0 yields a $(\mathfrak{h}_0,L)$-*equivalence* from $\mathcal{U}(\mathfrak{h}_0)W$ onto $\mathbf{H}^2(H_0,\tau)_{L-fin}$.

(e) The previous comments provide another version of the duality theorem, namely:

Theorem 2.10. $Hom_H(H^2(H,\sigma),H^2(G,\tau))\cong Hom_L(Z,\mathbf{H}^2(H_0,\tau))$ *via the map*

$$\begin{aligned}&Hom_H(H^2(H,\sigma),H^2(G,\tau))\ni T\\ &\mapsto\Big(Z\ni z\mapsto r_0(D(K_T(e,\cdot)z))(\cdot)\in Hom_L(Z,\mathbf{H}^2(H_0,\tau))\Big).\end{aligned}$$

(f) An expression for the inverse to the map defined in e) has been computed in [31, Section 4]. A formula is:

$$K_T(h,x)z=\Big(D^{-1}\Big[\int_{H_0}K_\tau(h_0,\cdot)C(r_0(D(K_T(e,\cdot)z))(\cdot))(h_0)dh_0\Big]\Big)(h^{-1}x).$$

Here, $C=(r_0r_0^\star)^{-1}$ is a bijective endomorphism of $\mathbf{H}^2(H_0,\tau)$.

When, D can be chosen equal to the identity map, the formula reads

$$K_T(h,x)z=\int_{H_0}K_\tau(h_0,h^{-1}x)z)(C(r_0((K_T(e,\cdot))(\cdot))(h_0))dh_0.$$

Remark 2.11. In the holomorphic setting, in the formulas in (e), (f) we are able to replace the linear map r_0D by r_0. (See 4.7).

2.8. Duality theorem in symmetric space model

The statement in this case is:

$$Hom_H(H^2_{c_\sigma}(H/L,Z),H^2_{c_\tau}(G/K,W))\cong Hom_L(Z,\mathbf{H}^2_{c_\tau}(H_0/L,W)).$$

An explicit isomorphism is:

$$\begin{aligned}&Hom_H(H^2_{c_\sigma}(H/L,Z),H^2_{c_\tau}(G/K,W))\ni T^c\\ &\mapsto\Big(Z\ni z\mapsto r_0^c(D^c(K^c_{T^c}(e,\cdot)z))(\cdot)\in Hom_L(Z,\mathbf{H}^2_{c_\tau}(H_0/L,W))\Big).\end{aligned}$$

Here, $\quad res_L(\tau)=\oplus_j\sigma_j,\quad \mathbf{H}^2_{c_\tau}(H_0/L,W):=\oplus_j H^2_{c_\tau}(H_0,\sigma_j),$

$$\mathcal{U}(\mathfrak{h}_0)W:=\mathcal{U}(\mathfrak{h}_0)\big(\mathbf{H}^2_{c_\tau}(H_0/L,W)[res_L(W)]\big)\subset\mathbf{H}^2_{c_\tau}(H_0/L,W),$$

D as in 2.14, $D^c:=E_{c_\tau}DE^{-1}_{c_\tau}:E_{c_\tau}(\mathcal{L}_{W,H})=:\mathcal{L}^c_{W,H}\to\mathcal{U}(\mathfrak{h}_0)W$.

2.9. Symmetry breaking operators in case of a holomorphic imbedding $H/L\to G/K$

The concepts and notation for this subsection is recalled in Section 4. For this subsection, we assume, both symmetric spaces $H/L,G/K$ are Hermitian symmetric and the inclusion $H/L \to G/K$ is holomorphic. We consider two holomorphic Discrete Series representations realized, respectively on space of functions defined over the corresponding bounded symmetric domains obtained by Harish-Chandra. An aim of this subsection is to present a result on symmetry braking operators under this setting.

We follow the notation of Section 4. We quote the necessary notation. $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{h} = \mathfrak{l} + \mathfrak{p}_{\mathfrak{h}} = \mathfrak{k} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{h}$, $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ + \mathfrak{p}^-$, $\mathfrak{p}_{\mathfrak{h}}^+ = \mathfrak{p}^+ \cap \mathfrak{h}$, $\mathcal{D} \subset \mathfrak{p}^+$ the Harish-Chandra realization of the Hermitian symmetric space G/K, $\mathcal{D}_{\mathfrak{h}} \subset \mathfrak{p}_{\mathfrak{h}}^+$ the realization of H/L. Thus, we have the holomorphic inclusions $\mathcal{D}_{\mathfrak{h}} \subset \mathcal{D}$. Let (τ, W) (resp. (σ, Z)) irreducible representations of K (resp. L) so that the respective spaces of holomorphic functions $(L^\tau_\cdot, V_\tau)$ (resp. $(L^\sigma_\cdot, V_\sigma)$ on $L^2_\tau(\mathcal{D}, W)$ (resp. $L^2_\tau(\mathcal{D}_{\mathfrak{h}}, Z)$) are nonzero. From now on, $(L^\sigma_\cdot, V_\sigma)$ is an irreducible H-subrepresentation of $res_H(L^\tau_\cdot, V_\tau)$. The action of $\mathfrak{g}$ in V_τ has been computed by [15], (See 4.0.1), the formulas they obtained let us state.

For every $D \in \mathcal{U}(\mathfrak{g})$ (resp. $D \in \mathcal{U}(\mathfrak{h})$), L^τ_D (resp. L^σ_D) is a *holomorphic differential operator* on V_τ (V_σ).

Next, we fix a symmetry breaking operator $S : V_\tau \to V_\sigma$. Then, due that V_σ is a reproducing kernel space, as in 2.1, it follows there exists $K^c_S : \mathcal{D} \times \mathcal{D}_{\mathfrak{h}} \to Hom_{\mathbb{C}}(W, Z)$ a smooth function, anti holomorphic in the first variable and holomorphic in the second variable, so that

$$S(f)(z) = \int_{\mathcal{D}} K^c_S(w, z) K^c_\tau(\bar{w}, w)^{-1} f(w)\, dm_{G/K}(w), \ \forall f \in V_\tau, z \in \mathcal{D}_{\mathfrak{h}}.$$

Here, we have appealed to equation 2.2 and its consequences.
Our hypothesis implies V_τ is H*-admissible*, next, reproducing word by word the computation in 2.5.2 and recalling L^τ_D is a holomorphic differential operator, we obtain S is represented by a holomorphic differential operator.

[23] have strengthed this observation to:

S is represented by a constant coefficient holomorphic differential operator.

In 2.6, 2.5.2, in order to obtain a representation of S as differential operator from the kernel K^c_S, we construct a linear map $Z \ni z \to D_z \in \mathcal{U}(\mathfrak{g})$ so that $L^\tau_{D_z} K^c_\tau(\cdot, e)^\star w_0 = K^c_S(\cdot, e)^\star z$, and compute.

In either the group model or symmetric space model, the function $(z \to D_z)$ is in $Hom_{\mathbb{C}}(Z, \mathcal{U}(\mathfrak{g}))$ so that $L^\tau_{D_z} K^c_\tau(\cdot, e)^\star w_0 = K^c_S(\cdot, e)^\star z$ is far from unique. In fact, if we consider the annihilator $\mathcal{A}$ in $\mathcal{U}(\mathfrak{g})$ of the copy of W in each model, then we may replace the function D by $D + \tilde{D}$, where $\tilde{D}$ is an arbitrary element of $Hom_{\mathbb{C}}(Z, \mathcal{A})$. However, when we consider the case $H/L \to G/K$ is a holomorphic embedding and both V_τ, V_σ are holomorphic Discrete Series representations, we may replace the function $Z \ni z \mapsto D_z \in \mathcal{U}(\mathfrak{g})$ by a function from Z that takes on values in the space $\mathcal{U}(\mathfrak{p}^-) \otimes W$, and, then, the new function $z \mapsto D_z$ is unique.

Proposition 2.12. *We assume that $H/L \to G/K$ is a holomorphic embedding and that both V_τ and V_σ are holomorphic Discrete Series representations. Let $S : V_\tau \to V_\sigma$ be a symmetry breaking operator. Then, there exists an unique linear function $Z \ni z \mapsto D_z := \sum_i D^i_z \otimes w_i \in \mathcal{U}(\mathfrak{p}^-) \otimes W$ so that we have for all $z \in Z$,*

$$K^c_S(\cdot, e)^\star z = \textstyle\sum_i L^\tau_{D^i_z}(K^c_\tau(\cdot, e)^\star w_i).$$

Proof. Under our hypothesis, we may and use the notation in Section 4. Thus, $(V_\tau)_{K-fin}$ is *K-isomorphic* to $\mathcal{U}(\mathfrak{p}^-)\otimes W$ by means of the map

$$\mathcal{U}(\mathfrak{p}^-)\otimes W \ni D\otimes w \mapsto L_D^\tau(K_\tau^c(\cdot,e)^\star w)(\cdot),$$

hence, after we fixed an ordered linear basis $\{w_i,\ i=1\cdots\dim W\}$ for W, for each $v\in(V_\tau)_{K-fin}$, there exists a unique family $(D_i)_{1\leq i\leq \dim W}\in\mathcal{U}(\mathfrak{p}^-)$ so that $v=\sum_i L_{D_i}^\tau K_\tau^c(e,\cdot)^\star w_i$. Therefore, for each $z\in Z$ there is unique family $D_z^i\in\mathcal{U}(\mathfrak{p}^-)$ so that $K_S^c(\cdot,e)^\star z=\sum_i L_{D_z^i}^\tau(K_\tau^c(\cdot,e)^\star w_i)$. We define the new function D_z to be function $Z\ni z\mapsto\sum_i D_z^i\otimes w_i$. This function does not depend on the choice of basis and is unique. ∎

A consequence of the previous discussion is that whenever τ is one dimensional, that is, V_τ is a *scalar Discrete Series representation*, there is a *unique*, up to a constant that depends only on the choice of w_0, map $Z\ni z\to D_z\in\mathcal{U}(\mathfrak{p}^-)$ so that $K_S^c(\cdot,e)^\star z=L_{D_z}^\tau(K_\tau^c(\cdot,e)^\star w_0)$.

Remark 2.13. We like to point out, that Nakahama in [25, Section 3.3] has considered the problem of representing intertwining operators via differential operators under the hypothesis of Proposition 2.12. His solution is constructive and quite different to the one we found.

2.10. Normal derivative symmetry breaking operators

For this section (G,H) is a symmetric pair and $H^2(G,\tau),H^2(H,\sigma)$ are respective Discrete Series representations. We keep the hypothesis and notation in 2.7, 2.4.1. Generalizing the concepts in 2.4.1 and as in [31, 4.0.1] we say a symmetry breaking operator $S:H^2(G,\tau)\to H^2(H,\sigma)$ is *normal derivative differential operator* if there exist a G-invariant differential operator $D:\Gamma^\infty(G\times_\tau W)\to\Gamma^\infty(G\times_\sigma Z)$ so that $\forall f\in H^2(G,\tau)$ we have $S(f)=res(D(f))$ and

$$D=\textstyle\sum_\alpha c_\alpha R_{X_1^{\alpha_1}\cdots X_n^{\alpha_n}},\quad\text{with}\quad X_1^{\alpha_1}\cdots X_n^{\alpha_n}\in\mathcal{U}(\mathfrak{h}_0).$$

Here, res is the restriction map $res:\Gamma^\infty(G\times_\sigma Z)\to\Gamma^\infty(H\times_\sigma Z)$.

In [31, Proposition 5, 6.1], we have shown in the group model the following result: *Let (G,H) denote a symmetric pair and $H^2(G,\tau)$, $H^2(H,\sigma)$ respective Discrete Series representations. Then, $\mathcal{L}_{W,H}=\mathcal{U}(\mathfrak{h}_0)W$ if and only if for every (σ,Z) we holds that every symmetry breaking operator is represented by a normal derivative differential operator.* Actually, it follows from the proof of the previous statement that if $S:H^2(G,\tau)\to H^2(H,\sigma)$ is a continuous H-map with the property $\forall z\in Z$, $K_{S^\star}(e,\cdot)z=K_S^c(\cdot,e)^\star z\in\mathcal{U}(\mathfrak{h}_0)W$, then S is represented by a normal derivative differential operator, and conversely.

3. Symmetry breaking operators as generalized gradients

It is natural to find inspiration in the holomorphic case, i.e. in the theory of the holomorphic Discrete Series, to find more general facts in connection with branching laws, symmetry breaking and holographic operators, at least in the admissible case where we consider quite general Discrete Series representations for both G and H. In [4] and references therein, we find a proof that intertwining operators between representations realized as spaces of holomorphic functions are given by covariant differential operators. We may reinterpret their results in the language of symmetric spaces as follows, we assume G/K is Hermitian symmetric space and consider the "symmetric pair" (G, G), then, a special case of a symmetry breaking operator is that of an intertwining operator between two respective holomorphic Discrete Series representations of both G, G. Then, for this setting, they have shown that each symmetry breaking operator is represented by a covariant differential operator. In this section, *we generalize their result to an arbitrary symmetric pair (G, H) and an H-admissible Discrete Series representation for G.* As side remark, we recall that in [30, Theorem 4.4] we find a proof that the existence of a nonzero H-symmetry breaking operator represented by a differential operator forces the Discrete Series of G that we are dealing with is H-discretely decomposable. In the same paper, we find a proof that if every H-symmetry breaking operator is represented by differential operators, then the representation is H-admissible. Thus, symmetry breaking operators can be represented as "generalized gradients" only under the hypothesis of H-admissibility.

For this section, G is an arbitrary connected semisimple Lie group, $K = G^\theta$ is a maximal compact subgroup of G. $(G, H) := (G^\sigma)_0)$ is a symmetric pair, and $H_0 := (G^{\sigma\theta})_0$ the associated subgroup to H.

We fix $(L^G_\cdot, H^2(G, \tau))$, an H-admissible Discrete Series for H of lowest K-type (τ, W).

Let $(L^H_\cdot, H^2(H, \sigma))$ be an irreducible factor of $res_H(\pi)$. The lowest L-type of $H^2(H, \sigma)$ is (σ, Z). We note that for each representative of the equivalence class associated to (σ, Z), we obtain a different space of functions $H^2(H, \sigma)$. We will show that the expression of each symmetry breaking operator $S : H^2(G, \tau) \to H^2(H, \sigma)$ as "generalized gradient operator" is obtained from a particular choice of a representative of the class of (σ, Z).

In order to present the definition of "generalized gradient operator" we recall definitions and facts from [29]. For this section the symbol $\bullet$ denotes the natural representation of L in the vector space $S^n(\mathfrak{p}_{\mathfrak{h}_0}) \otimes W$.

The *normal derivative operator* of order n, r_n, is a map from

$$H^2(G, \tau) \quad \text{into} \quad C^\infty(H \times_\bullet Hom_{\mathbb{C}}(S^n(\mathfrak{p}_{\mathfrak{h}_0}), W)) \cong C^\infty(H \times_\bullet S^n(\mathfrak{p}_{\mathfrak{h}_0}) \otimes W).$$

The formal definition of r_n is: for $X_1, \cdots, X_n \in \mathfrak{p}_{\mathfrak{h}_0}$,

$$r_n(f)(X_1 \cdots X_n)(h) = \textstyle\sum_{\iota \in \mathfrak{S}_n} R_{X_{\iota(1)}} \cdots R_{X_{\iota(n)}}(f)(h).$$

The first order normal derivative map is a gradient in the direction orthogonal to the tangent space to $H/L \hookrightarrow G/K$. The n^{th}-order normal derivative may be thought as an iteration of first order normal derivative, whence, we call a n^{th}-order normal derivative a "generalized gradient", actually for $H = \{e\}$, r_n is the iteration of gradients. In [29] we find a proof for the statement:

$$r_n(H^2(G, \tau)) \subset L^2(H \times_\bullet Hom_{\mathbb{C}}(S^n(\mathfrak{p}_{\mathfrak{h}_0}), W)) \text{ and } r_n \text{ is (2,2)-continuous.}$$

A *generalized gradient* representation of a symmetry breaking operator

$$S : H^2(G, \tau) \to H^2(H, \sigma)$$

is a composition of the form $S = \tilde{R} r_n$, where $R : S^n(\mathfrak{p}_{\mathfrak{h}_0}) \otimes W \to Z$ is an L-map, and $\tilde{R} : C^\infty(H \times_\bullet S^n(\mathfrak{p}_{\mathfrak{h}_0}) \otimes W) \to C^\infty(H \times_\sigma Z)$ is the map $\tilde{R}(g)(h) := R(g(h))$, $g \in C^\infty(H \times_\bullet S^n(\mathfrak{p}_{\mathfrak{h}_0}) \otimes W)$, $h \in H$.

It readily follows that the uniqueness of the lowest L-type of a Discrete Series representation yields that any nonzero intertwining operator from $H^2(H,\sigma) \subset L^2(H\times_\sigma Z)$ into $H^2(H,\sigma_0) \subset L^2(H \times_{\sigma_0} Z_0)$ is the restriction of an operator $\tilde{R}$. R is an equivalence from Z onto Z_0.

Theorem 3.1. *Let (G,H) denote a symmetric pair of reductive groups. We assume $H^2(G,\tau)$ is an H-admissible Discrete Series representation for G. Let $S : H^2(G,\tau) \to H^2(H,\sigma)$ be a symmetry breaking operator. Then, for a convenient choice of a representative of the equivalence class of (σ, Z), S is represented as a generalized gradient operator.*

Proof. We define $T := S^\star$. In [31, 3.1], we find a proof for the inclusion

$$T(H^2(H,\sigma)[Z]) \subseteq H^2(G,\tau)[H^2(H,\sigma)][Z].$$

We claim the subspace $T(H^2(H,\sigma)[Z])$ is intrinsic, that is, does not depend on the choice of representative of the equivalence class of (σ, Z). In fact, we fix an intertwining linear operator, which we may assume is unitary, R from (σ, Z) onto (σ_1, Z_1). Hence, the map $L^2(H\times_\sigma Z) \ni g \overset{\tilde{R}}{\mapsto} \big(h \mapsto R(g(h))\big) \in L^2(H\times_{\sigma_1} Z_1)$ is a unitary, H-equivalence that maps $H^2(H,\sigma)$ onto $H^2(H,\sigma_1)$, as well as $\tilde{R}$ maps $H^2(H,\sigma)[Z]$ onto $H^2(H,\sigma_1)][Z]$. Finally, $T\tilde{R}^{-1}$ is equivalent to T. Whence, we have shown the claim.

Next, we fix some $n \geq 0$ so that for $j < n$,

$$r_j(T(H^2(H,\sigma)[Z])) = \{0\} \text{ and } r_n(T(H^2(H,\sigma)[Z])) \neq \{0\}.$$

We claim that

– $H^2(H,\sigma)$ does not occur as a subrepresentation for $L^2(H\times_\bullet Hom_{\mathbb{C}}(S^j(\mathfrak{p}_{\mathfrak{h}_0}), W))$ for $j < n$, and that

– (σ, Z) occurs in $S^n(\mathfrak{p}_{\mathfrak{h}_0}) \otimes W$.

In fact, to begin with, we assume that the rank of K is equal to the rank of L and the system of positive roots Ψ determined by the Harish-Chandra parameter of $H^2(G,\tau)$ has the Borel de Siebenthal property, for unexplained notation or facts we refer to [31, Section 4]. The duality Theorem 2.7, [31, Theorem 1] together with the elements in the proof of [31, Lemma 3, Lemma 4] implies the Harish-Chandra parameter of (σ, Z) is equal to $\lambda + \rho_n + B$, here, B stands for a sum of noncompact roots in $\Psi_{\mathfrak{h}_0}$, we observe that the coefficient, of the unique noncompact simple β for $\Psi_{\mathfrak{h}_0}$ in the expression of B as a linear combination of simple roots, is equal to n. The general case follows from a similar line of computation.

An argument based on the fact, that any noncompact root in $\Psi_{\mathfrak{h}_0}$ is equal to β plus a linear combination of compact simple roots, as well as that the highest weight of any irreducible component of a tensor product is equal to the highest weight of one factor plus weight of the other factor, shows that $\lambda + \rho_n - \rho_L + B$ is not an L-type for $S^j(\mathfrak{p}_{\mathfrak{h}_0}) \otimes W$ and $j < n$.

We note that since $r_n(T(H^2(H,\sigma))) \neq 0$, Frobenius reciprocity yields:

$$r_n(T(H^2(H,\sigma))) \text{ contains an } L\text{-type in } S^n(\mathfrak{p}_{\mathfrak{h}_0}) \otimes W.$$

Now, a result of Schmid, See [31, Proof of Lemma 3, Lemma 4] gives that any L-type of $H^2(H,\sigma)$ is equal $\lambda + \rho_n - \rho_L + B + B_h$ where B_h stands for a sum of roots in $\Psi_{\mathfrak{h}}^n$. Next, comparing the coefficient of the noncompact the simple root of Ψ in $\lambda+\rho_n-\rho_L+B+B_h$ and the weight structure of $S^n(\mathfrak{p}_{\mathfrak{h}_0})\otimes W$ we obtain $B_h=0$, and, hence Z is an L-type of $S^n(\mathfrak{p}_{\mathfrak{h}_0})\otimes W$. Thus, $H^2(H,\sigma)$ does not occur as a subrepresentation for $L^2(H\times_\bullet Hom_{\mathbb{C}}(S^j(\mathfrak{p}_{\mathfrak{h}_0}),W))$ for $j<n$. Thus, we have shown the claim.

We set $Z_0 := r_n(T(H^2(H,\sigma)[Z])))$ and we conclude the proof of Theorem 3.1. The subspace Z_0 of $S^n(\mathfrak{p}_{\mathfrak{h}_0})\otimes W$ gives a representation of L equivalent to (σ,Z). Let P denote the orthogonal projector in $S^n(\mathfrak{p}_{\mathfrak{h}_0})\otimes W$ onto Z_0. In consequence, $g\mapsto \tilde{P}(g)(\cdot):=P(g(\cdot))$ maps $L^2(H\times_\bullet Hom_{\mathbb{C}}(S^n(\mathfrak{p}_{\mathfrak{h}_0}),W))$ into $L^2(H\times_\bullet Z_0)$, and $R:=\tilde{P}r_nT$ is a nonzero H-intertwining continuous linear map from $H^2(H,\sigma)$ onto $H^2(H,\bullet)$. Hence, $R=\tilde{R}_0$, with R_0 an L-map from Z onto Z_0. $T=S^\star$ yields the equality $\tilde{R}_0=\tilde{P}r_nS^\star$. Now $SS^\star=T^\star T=cI_{H^2(H,\sigma)}$, $c>0$. Thus, $\tilde{R}_0S=\frac{1}{c}\tilde{P}r_nS^\star S$, and $S=\tilde{R}_0^{-1}\tilde{P}r_n$. Whence, we have verified Theorem 3.1. ■

Comment. In Theorem 3.1, we have obtained a generalization of [4], we would like to point that the covariant differential operators in their setting are also generalized gradients (covariant derivative followed by projection).

4. Holomorphic embedding, duality theorem

In this section we present a new and self contained proof of the duality Theorem for the holomorphic setting. We actually, present four different versions of the duality Theorem in the context of a holomorphic embedding. The precise statements are in: 4.1, 4.3, 4.4, 4.7. The original result for this section is Theorem 4.12 on the structure of holographic operators. Later on, we derive consequences on the structure of symmetry breaking operators. For specific applications of the duality Theorem to branching laws, see [31].

A statement for the duality Theorem is:

Theorem 4.1. *Let (G,H) be a symmetric pair so that the inclusion $H/L\to G/K$ is a holomorphic map. Let V_σ, V_τ be holomorphic Discrete Series representations for respectively H,G. Then, there exists a linear isomorphism between $Hom_H(V_\sigma,V_\tau)$ and $Hom_L(Z,\mathcal{U}(\mathfrak{h}_0)W)$.*

In order to present the remaining elements in the formulation of the Theorem, in the following, we recall the standard description of a symmetric complex domain realizartion of the holomorphic Discrete Series (details in [5], [27, XII.5]):

- G a semisimple Lie group and a maximal compact subgroup K of G.
- $\big(G,H=(G^\sigma)_0\big)$ is a symmetric pair.
- $H_0:=(G^{\sigma\theta})_0$ associated subgroup to H.
- $L=K\cap H$ a maximal compact subgroup of H.
- $H/L\to G/K$ a holomorphic embedding.
- $(L^\tau_\cdot, V_\tau)$ is a holomorphic Discrete Series for G of lowest K-type (τ,W), which is realized as holomorphic W-valued functions in the bounded symmetric domain $\mathcal{D}\cong G/K$. The action, $L^\tau_\cdot$ of $\mathfrak{g}$, is recalled bellow.

- $(L^{\sigma}_{\cdot}, V_{\sigma})$ is a holomorphic Discrete Series for H of lowest L-type (σ, Z), which is realized as holomorphic Z-valued functions in the bounded symmetric domain $\mathcal{D}_{\mathfrak{h}} \cong H/L$.

Next, we recall the definition of *holomorphic embedding*. First of all, the symmetric space G/K (G simple) admits a G-invariant holomorphic structure if and only if the center of K is one dimensional torus. Similarly, for H/L, H simple. The natural inclusion $H/L \to G/K$ is a *holomorphic embedding* if both symmetric spaces admits an (H-invariant) G-invariant complex structure and the natural inclusion is holomorphic. The inclusion is holomorphic is equivalent to: the holomorphic tangent bundle of H/L is a subset of the holomorphic tangent bundle for G/K. The inclusion $H/L \to G/K$ being holomorphic, implies the center of K is contained in the center of L and viceversa.

Further notation: T maximal torus of K so that $U = T \cap H$ is a maximal torus of L. $o :=$the coset eK. See Section 6 for holomorphic system. We fix Ψ holomorphic system in $\Phi(\mathfrak{g}, \mathfrak{t})$, Ψ_H, Ψ_{H_0} holomorphic systems in $\Phi(\mathfrak{h}, \mathfrak{u}), \Phi(\mathfrak{h}_0, \mathfrak{u})$ so that

$$\textstyle\sum_{\beta \in \Psi^n} \mathfrak{g}_\beta := \mathfrak{p}^+ = \mathfrak{p}^+_{\mathfrak{g}} = \mathfrak{p}^+_{\mathfrak{h}} + \mathfrak{p}^+_{\mathfrak{h}_0}$$

are isomorphic to the respective holomorphic tangent spaces at o. Therefore, $G \subset P_+ K_{\mathbb{C}} P_-$, and we have the Harish-Chandra realization as bounded symmetric domain $G/K \cong \mathcal{D} \subset \mathfrak{p}^+$, $H/L \cong \mathcal{D}_{\mathfrak{h}} \subset \mathfrak{p}^+_{\mathfrak{h}}$, $H_0/L \cong \mathcal{D}_{\mathfrak{h}_0} \subset \mathfrak{p}^+_{\mathfrak{h}_0}$. For this, we recall the triangular decomposition

$$P_+ K_{\mathbb{C}} P_- \ni x = exp(x_+) x_0 exp(x_-), x_\pm \in \mathfrak{p}^\pm, x_0 \in K_{\mathbb{C}}.$$

Then, the Harish-Chandra embeddings are: $G/K \ni xK \mapsto x_+ \in \mathcal{D} \subset \mathfrak{p}^+$, $H/L \ni hL \mapsto h_+ \in \mathcal{D}_{\mathfrak{h}} \subset \mathcal{D} \subset \mathfrak{p}^+$, $H_0/L \ni hL \mapsto h_+ \in \mathcal{D}_{\mathfrak{h}_0} \subset \mathcal{D}$.

The cocycle c_τ and the reproducing kernel K^c_τ that defines the Hilbert structure and the representation on V_τ are: for $g \in G, z, w \in \mathcal{D}$,

$$c_\tau(g, z) = \tau((gexpz)_0),\ K^c_\tau(w, z) = \tau(((exp\bar{w})^{-1} expz)_0)^{-1}. \tag{4.1}$$

Here, we take the conjugation with respect to the real form $\mathfrak{g}$.

We consider the space $L^2_\tau(\mathcal{D}, W) := L^2_{c_\tau}(\mathcal{D}, W)$ and the subspace

$$V_\tau := V^G_\tau = \{f \in \mathcal{O}(\mathcal{D}, W) : \int_{\mathcal{D}} (K^c_\tau(w, w) f(w), f(w))_W dm_{G/K}(w) < \infty\}. \tag{4.2}$$

The action $L^\tau_g := \pi_{c_\tau}(g), g \in G$ in $\mathcal{O}(\mathcal{D}, W)$ is defined by the formula

$$L^\tau_g(f)(z) = \tau((g^{-1} expz)_0)^{-1} f((g^{-1} expz)_+),\ g \in G,\ z \in \mathfrak{p}^+.$$

Then, [27], it can be shown that $(L^\tau_{\cdot}, V_\tau)$ is a unitary, irreducible representation for G. Due that any matrix coefficient of $(L^\tau_{\cdot}, V_\tau)$ is square integrable for the Haar measure on G, henceforth, we refer to:

$(L^\tau_{\cdot}, V_\tau)$ is *a holomorphic Discrete Series* for G.

We recall the subspace V_{K-fin} of K-finite vectors in V_τ is

$$V_{K-fin} = \mathcal{P}(\mathfrak{p}^+, W) \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k}+\mathfrak{p}^+)} W \cong S(\mathfrak{p}^-) \otimes W \tag{4.3}$$

Here, we have identified W with the subspace of constant functions.

The isomorphisms in 4.3 are $L^\tau_D(w) \leftarrow [D\otimes w] \leftarrow D\otimes w$.
Here, we use that $\mathfrak{p}^\pm$ is an abelian Lie algebra, and hence, the symmetrization map may be thought as the identity map.

4.0.1. In [15] Jacobsen and Vergne have computed the action of $\mathfrak{g}$ (resp. K) in $\mathcal{O}(\mathcal{D},W)$. For this

we define $(\delta(x)f)(v) = \lim_{t\to 0} \dfrac{f(v+tx)-f(v)}{t}$,

then, for $p\in\mathcal{O}(\mathcal{D},W)$, and for

$x\in\mathfrak{k}$: $\quad L^\tau_x(p)(v) = \tau(x)(p(v)) - (\delta([x,v])p)(v)$,

$x\in\mathfrak{p}^+$: $\quad L^\tau_x(p)(v) = -(\delta(x)p)(v)$,

$x\in\mathfrak{p}^-$: $\quad L^\tau_x(p)(v) = \tau([x,v])(p(v)) - \frac{1}{2}(\delta([[x,v],v])p)(v)$,

$k\in K$: $\quad L^\tau_k(p)(v) = \tau(k)(p(k^{-1}v))$.

It readily follows that

$$V_\tau^{\mathfrak{p}^+} = W,\ \ V_\tau^{\mathfrak{p}^+_{\mathfrak{h}}} = \{p\in V_{K-fin} : \delta(x)p = 0, \forall x\in \mathfrak{p}^+_{\mathfrak{h}}\} = \mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}, W).$$

The hypothesis, the inclusion $H/L\to G/K$ is a holomorphic embedding and V_τ is a holomorphic Discrete Series representation yields that the restriction of $(L^\tau_\cdot, V_\tau)$ to H is an H-admissible representation and the totality of irreducible factors are holomorphic Discrete Series for H for a reference [22]. Therefore, after we write

$$res_H(V_\tau) = \oplus_{\mu\in Spec(res_H(V_\tau))} V_\tau[V^H_\mu],$$

the subspace $\mathcal{L}^c_{W,H} := E_{c_\tau}(\mathcal{L}_\lambda)$, defined in [37][31], equal to the sum of the lowest L-type's of the totality of irreducible H-factors of $res_H(V_\tau)$, is equal to $\mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}, W)$. That is,

$$\begin{aligned}\mathcal{L}^c_{W,H} = E_{c_\tau}(\mathcal{L}_{W,H}) &= \oplus_{\mu\in Spec(res_H(V_\tau))} V_\tau[V^H_\mu][V^L_{\mu^H+\rho^H_n}]\\ &= \{p\in V_{K-fin} : \delta(x)p = 0, \forall x\in\mathfrak{p}^+_{\mathfrak{h}}\} = \mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}, W).\end{aligned}$$

Via the Killing form, B, $\mathfrak{p}^-_{\mathfrak{h}_0}$ is in duality with $\mathfrak{p}^+_{\mathfrak{h}_0}$, which provides an isomorphism between $\mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}, W)$ and $S(\mathfrak{p}^-_{\mathfrak{h}_0})\otimes W$. That is, the inverse map to

$$\mathfrak{p}^-_{\mathfrak{h}_0}\otimes W \ni Y\otimes w \mapsto (\mathfrak{p}^+_{\mathfrak{h}_0}\ni X \xrightarrow{p_Y} B(X,Y)w) \in \mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}, W),$$

extends to an L-equivariant isomorphism

$$D_0 : \mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}, W) \to S(\mathfrak{p}^-_{\mathfrak{h}_0})\otimes W.$$

The action of L in $S(\mathfrak{p}^-_{\mathfrak{h}_0})\otimes W$ is the tensor product action.
We recall that W is identified with the constant functions, we have the equality $\mathfrak{h}_0 = \mathfrak{p}^-_{\mathfrak{h}_0} + \mathfrak{l} + \mathfrak{p}^+_{\mathfrak{h}_0}$, and V_τ is a holomorphic Discrete Series representation. Next, as in [31][37], we consider the subspace

$$\mathcal{U}(\mathfrak{h}_0)W := \{L^\tau_D w, D\in\mathcal{U}(\mathfrak{h}_0), w\in W\} = \{L^\tau_D w, D\in \mathcal{U}(\mathfrak{p}^-_{\mathfrak{h}_0}), w\in W\}$$

and the map D_1 defined by $\mathcal{U}(\mathfrak{p}^-_{\mathfrak{h}_0})\otimes W\ni D\otimes w \xmapsto{D_1} L^\tau_D w\in\mathcal{U}(\mathfrak{h}_0)W$.

4.0.2. Then, 4.3 yields that D_1 is a L-equivariant isomorphism.

Finally, we recall that $\mathfrak{p}^-_{\mathfrak{h}_0}$ is an abelian Lie algebra, hence, the symmetrization map from $S(\mathfrak{p}^-_{\mathfrak{h}_0})$ onto $\mathcal{U}(\mathfrak{p}^-_{\mathfrak{h}_0})$ may be thought of as an identification. Thus, we have given a new proof of [31, Proposition 1] [37] in the holomorphic setting.

Proposition 4.2. *The map $D := D_1 D_0$ is a L-equivariant, degree preserving, linear isomorphism between $\mathcal{L}^c_{W,H}$ and $\mathcal{U}(\mathfrak{h}_0)W$.*

Obviously, D_0 preserves the degree of polynomials. D_1 preserves the degree of polynomials owing to the explicit formula for $L^\tau_x, x \in \mathfrak{p}^-$, hence $L_R(w)$ is a polynomial of the same degree as the degree of $R \in \mathcal{U}(\mathfrak{p}^-)$.
The map D in Proposition 4.2 is an example of the map D^c in 2.8.

4.0.3. Statement and proof of the first version of duality

We are ready to provide a statement and proof of the first version of the duality Theorem for holomorphic setting.

For this, we assume the inclusion $H/L \to G/K$ is holomorphic. Let $(L^\tau_\cdot, V_\tau)$ be a holomorphic Discrete Series representation for G realized in a Hilbert subspace of $\mathcal{O}(\mathcal{D}, W)$. Let V_σ a holomorphic Discrete Series for H of lowest L-type (σ, Z), realized as a Hilbert space of holomorphic Z-valued functions on the bounded domain $\mathcal{D}_\mathfrak{h} \subset \mathfrak{p}^+_\mathfrak{h}$ associated to H/L. Thus, $res_H(V_\tau)$ is an H-admissible representation. Each continuous linear, H-map, T, from V_σ into V_τ is represented by a kernel $K^c_T : \mathcal{D}_\mathfrak{h} \times \mathcal{D} \to Hom_\mathbb{C}(Z, W)$. Among the properties of K^c_T we recall for

$$x \in \mathcal{D},\ w \in W,\ K^c_T(\cdot, x)^* w \in V_\sigma,\ z \in Z,\ K^c_T(o, \cdot)z \in V_\tau[V_\sigma][Z] \subset \mathcal{L}_{W,H} \subset V_{K-fin}:$$

$$\text{the map } Z \ni z \to K^c_T(o, \cdot)z \in V_\tau \text{ is an } L\text{-map.}$$

Theorem 4.3. *We suppose that (G, H) is a symmetric pair and that the inclusion $H/L \to G/K$ is a holomorphic map. We fix V_σ, V_τ holomorphic Discrete Series representations for respectively H, G. Then, we have a linear isomorphism between $Hom_H(V_\sigma, V_\tau)$ and $Hom_L(Z, \mathcal{U}(\mathfrak{h}_0)W)$. An explicit isomorphism is the map*

$$Hom_H(V_\sigma, V_\tau) \ni T \mapsto \big(Z \ni z \mapsto D(K^c_T(o, \cdot)z)(\cdot) \in Hom_L(Z, \mathcal{U}(\mathfrak{h}_0)W)\big).$$

Proof. We recall a Discrete Series representation for H is determined by its lowest L-type. The following computation shows that the two involved spaces have the same dimension

$$\dim Hom_H(V_\sigma, V_\tau) = \dim \mathcal{L}^c_{W,H}[Z] \overset{D}{=} \dim \mathcal{U}(\mathfrak{h}_0)W[Z] = \dim Hom_L(Z, \mathcal{U}(\mathfrak{h}_0)W).$$

The map $T \mapsto D(K^c_T(e, \cdot))$ being injective, owing to D is injective and the equality $K^c_T(h \cdot o, w) = c_\sigma(.., ..) K^c_T(o, h^{-1} \cdot w) c_\tau(.., ..)$. The equality of dimensions shows the surjectivity. ∎

We suggest to compare Theorem 4.3 with [25, Theorem 3.10][31, Theorem 1] and Subsection 2.7.

4.0.4. Multiplicity formulae

In the following, we present a brief summary of [31, Subsection 3.4]. The duality Theorem provides a formula for $\dim Hom_H(H^2(H, \sigma),\ H^2(G, \tau))$ in terms of a partition function based on the noncompact roots for $(\mathfrak{h}_0, \mathfrak{u})$ and a Weyl group, see [6], [31] and references therein. In [11] we also find a multiplicity formula based on a partition function and Weyl groups. In [20, Theorem 8.3] the author computes the Harish-Chandra parameters of all the irreducible factors in $res_H(V_\tau)$. In [12] is shown a multiplicity formula for either weak factors or factors based on the map r_n considered by [29]. On the other hand, Paradan, [32], Duflo and Vergne, [7] have obtained $\dim Hom_H(V_\sigma, V_\tau)$ as the volume of an orbifold.

4.1. Structure of $\mathcal{U}(\mathfrak{h}_0)W$ and another versions of the duality theorem

We keep the hypotheses and notation of the previous subsection. We fix a decomposition $W = Z_1 + \cdots + Z_r$, with Z_j a L-invariant, L-irreducible linear subspace. Then, since for $x \in \mathfrak{p}^+_{\mathfrak{h}_0}, w \in W$ we have $L^\tau_x w = 0$. It readily follows that $\mathcal{U}(\mathfrak{h}_0)Z_j$ is the underlying Harish-Chandra module of the irreducible square integrable representation of lowest L-type Z_j. Therefore,

$$\mathcal{U}(\mathfrak{h}_0)W = \oplus_{1\leq j\leq r}\mathcal{U}(\mathfrak{h}_0)Z_j \cong \oplus_{1\leq j\leq r}\mathcal{U}(\mathfrak{h}_0) \otimes_{\mathcal{U}(\mathfrak{l}_\mathbb{C}+\mathfrak{p}^+_{\mathfrak{h}_0})} Z_j. \tag{4.4}$$

Whence, we have obtained the decomposition of $\mathcal{U}(\mathfrak{h}_0)W$ as the sum of underlying Harish-Chandra modules of Discrete Series representations for H_0. The conclusion of Theorem 4.3 can be written as,

Theorem 4.4. *For $H/L \to G/K$ a holomorphic embedding and Discrete Series representations $V_\sigma = \mathcal{O}(\mathcal{D}_\mathfrak{h}, Z) \cap L^2_\sigma(\mathcal{D}_\mathfrak{h}, Z)$, $V_\tau = \mathcal{O}(\mathcal{D}, W) \cap L^2_\tau(\mathcal{D}, W)$, we have a linear isomorphism*

$$Hom_H(V_\sigma, V_\tau) \cong \oplus_{1\leq j\leq r} Hom_L(Z, \mathcal{U}(\mathfrak{h}_0) \otimes_{\mathcal{U}(\mathfrak{l}_\mathbb{C}+\mathfrak{p}^+_{\mathfrak{h}_0})} Z_j).$$

We would like to point out that Theorem 4.4 follows from facts proven in Kobayashi-Pevzner [24, Theorem 2.7], Nakahama [25][Lemma 3.4, Theorem 3.6]. Our proof is of algebraic nature.

Owing to our hypothesis $\mathcal{D}_{\mathfrak{h}_0}$ is a subset of $\mathcal{D}$ and the inclusion is holomorphic, we may restrict holomorphic function on $\mathcal{D}$ and we obtain holomorphic functions on $\mathcal{D}_{\mathfrak{h}_0}$. Let $r_0 : \mathcal{O}(\mathcal{D}, W) \to \mathcal{O}(\mathcal{D}_{\mathfrak{h}_0}, W)$ denote the restriction map. In [29] it is shown that r_0 maps L^2-continuously the space V_τ into $L^2_{c_\tau}(\mathcal{D}_{\mathfrak{h}_0}, W)$. In [31, Lemma 1] it is shown that the kernel of r_0 is equal to the orthogonal to the subspace $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$. Whence, $r_0 : \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W) \to L^2_{c_\tau}(\mathcal{D}_{\mathfrak{h}_0}, W)$ is injective. Now, $L^2_{c_\tau}(\mathcal{D}_{\mathfrak{h}_0}, Z_j)$ contains just once the holomorphic Discrete Series of lowest L-type Z_j. (See Introduction) Let's denote by $H^2_{c_\tau}(\mathcal{D}_{\mathfrak{h}_0}, Z_j)$ such a subspace. Actually, $H^2_{c_\tau}(\mathcal{D}_{\mathfrak{h}_0}, Z_j)$ is the kernel of the $\bar\partial$ operator in $L^2_{c_\tau}(\mathcal{D}_{\mathfrak{h}_0}, Z_j))$. As in [31], we define

$$\mathbf{H}^2_{c_\tau}(\mathcal{D}_{\mathfrak{h}_0}, W) := \oplus_{1\leq j\leq r} H^2_{c_\tau}(\mathcal{D}_{\mathfrak{h}_0}, Z_j) \subset L^2_{c_\tau}(\mathcal{D}_{\mathfrak{h}_0}, W)$$

Then, from the previous calculations, it readily follows that the image of r_0 is $\mathbf{H}^2_{c_\tau}(\mathcal{D}_{\mathfrak{h}_0}, W)$. Finally, we define

$$r_0^D : Hom_H(V_\sigma, V_\tau) \to Hom_L(Z, \mathbf{H}^2_{c_\tau}(\mathcal{D}_{\mathfrak{h}_0}, W)) \tag{4.5}$$

by the rule $$r_0^D(T)(z) = r_0(D(K^c_T(o, \cdot)z)).$$

Thus we have:

Theorem 4.5. *Hypothesis as in Theorem 4.4, r_0^D is a linear bijection.*

The isomorphism r_0^D is our third version of the duality Theorem.

4.2. Another equivalence map for $\mathcal{L}^c_{W,H} \cong \mathcal{U}(\mathfrak{h}_0)W$ and fourth version of the duality theorem

To begin with, we show that we can replace the map D defined in Proposition 4.2 by the orthogonal projector Q onto $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$. That is,

Proposition 4.6. *We assume $H/L \to G/K$ is a holomorphic embedding. Let $V_\tau = \mathcal{O}(\mathcal{D}, W) \cap L^2_{c_\tau}(\mathcal{D}, W)$ be a holomorphic Discrete Series for G. Let $Q : V_\tau \to V_\tau$ denote the orthogonal projector onto $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$. Then,*

$$Q : \mathcal{L}^c_{W,H} \to \mathcal{U}(\mathfrak{h}_0)W \text{ is an } L\text{-equivalence.}$$

Proof. Indeed, to begin with we recall $\mathcal{P}(\mathfrak{p}^+, W) \subset V_\tau = \mathcal{O}(\mathcal{D}, W) \cap L^2_\tau(\mathcal{D}, W)$. Here, $\mathcal{L}^c_{W,H} = \mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}, W)$. Thus we have $r_0(p) = p$, $\forall p \in \mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}, W)$. We write $V_\tau \ni p = p_1 + Q(p)$ with $p_1 \in \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)^\perp, Q(p) \in \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$. Since, [31, Lemma 1] $Ker(r_0) = \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)^\perp$, we have

$$r_0(p) = r_0(Q(p)) \quad \forall p \in V_\tau,$$

hence, Q restricted to $\mathcal{L}^c_{W,H} = \mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}, W)$ is injective. A proof of the surjectivity of Q is: Q is an L-map, hence Q maps each isotypic component in $\mathcal{L}^c_{W,H}$ into the correspond isotypic component in $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)_{L-fin} = \mathcal{U}(\mathfrak{h}_0)W$. The L-admissibility of V_τ gives that each L-isotypic component is a finite dimensional vector space. The bijective map D shows that each L-isotypic component in $\mathcal{L}^c_{W,H}$ has the same dimension that the corresponding L-isotypic component in Q. Thus, Q from $\mathcal{L}^c_{W,H} = \mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}, W)$ into $\mathcal{U}(\mathfrak{h}_0)W$ is surjective. Hence, we obtain Q restricted to $\mathcal{L}^c_{W,H}$ is a bijection onto $\mathcal{U}(\mathfrak{h}_0)W$. Since the equivalences between the different models are unitary intertwining operators, it readily follows the general statement. ∎

We believe that Proposition 4.6 holds for arbitrary symmetric pairs (G, H) under the hypothesis of H-admissibility. A consequence of this proposition is an improvement of the duality theorem.

Theorem 4.7. *We assume $\mathcal{D}_\mathfrak{h} \cong H/L \to G/K \cong \mathcal{D}$ is a holomorphic embedding. Let $V_\sigma = \mathcal{O}(\mathcal{D}_\mathfrak{h}, Z) \cap L^2_{c_\sigma}(\mathcal{D}_\mathfrak{h}, Z)$, V_τ be holomorphic Discrete Series representations for H, G. Then,*

(a) *For $T_1, T_2 \in Hom_H(V_\sigma, V_\tau)$, we have $K^c_{T_1} = K^c_{T_2}$ if and only if*
$K_{T_1}(e, h_0)z = K_{T_2}(e, h_0)z \;\; \forall z \in Z, h_0 \in H_0$.

(b) *The following linear map is bijective,*
$Hom_H(V_\sigma, V_\tau) \ni T \mapsto$
$(Z \ni z \mapsto r_0(K^c_T(e, \cdot)z)(\cdot) \in \mathbf{H}^2(\mathcal{D}_{\mathfrak{h}_0}, W))) \in Hom_L(Z, \mathbf{H}^2(\mathcal{D}_{\mathfrak{h}_0}, W))$.

Actually, (a) can be extracted from [25, Lemma 3.4]. Statement (b) can be deduced from [25, Lemma 3.4] or the work of Kobayashi [20, Proof of Lemma 8.8]. Their proofs apply quite different techniques.

Note: (a) In [31, Proposition 4] we have computed the orthogonal projector Q in terms of the operator $r_0^\star r_0$, for example, when τ restricted to L is irreducible, Q is up to a constant the operator $r_0^\star r_0$, whence, applying the results in [31, 4.2.4] we may compute the inverse to the map defined in part b) of the Theorem. In general, under the hypothesis of H-admissibility, in [31, Proposition 1] it is shown that $Im(r_0)$ is a finite sum of irreducible Discrete Series representations for H_0, and that $r_0 r_0^\star$ is a bijective linear operator for $Im(r_0)$. From the computation in [31, 4.2.4] it can be shown that $Q = r_0^\star (r_0 r_0^\star)^{-1} r_0$.

(b) Due to the equality $r_0 = r_0 Q$, we obtain a simpler description of the inverse to the map in Theorem 4.7 b). In fact, formula 2.7 (f) simplifies to

$$K_T^c(h,x)z = \big(Q^{-1}\big[\int_{H_0} K_\tau(h_0,\cdot)(C(r_0(K_T^c(e,\cdot)z))(\cdot))(h_0)dh_0\big]\big)(h^{-1}x).$$

Here, $C := (r_0 r_0^\star)^{-1}$ is a bijective endomorphism of $\mathbf{H}^2(H_0,\tau)$, and, Q^{-1} is the inverse map to $Q : \mathrm{Cl}(\mathcal{L}_{W,H}) \to \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$.

4.3. Other versions of the inversion formula

In Theorem 4.7 we have presented a new proof that the map

$$Hom_H(V_\sigma, V_\tau) \ni T \mapsto (z \to r_0(K_T^c(e,\cdot)z)(\cdot)) \in Hom_L(Z, \mathbf{H}^2(\mathcal{D}_{\mathfrak{h}_0}, W)).$$

is bijective. Actually, we have shown injectivity and that both spaces are equidimensional. In this manner, we have obtained the surjectivity of the map. Nakahama [25](See Proposition 4.8) has written an explicit inverse map based on the reproducing kernel of the target space V_τ and the realization of holomorphic Discrete Series in spaces of functions on the bounded symmetric domain attached to the group G. In the next paragraph we present Nakahama's formula, and, in Theorem 4.12, we show another expression for the inverse map, based on the reproducing kernel of the initial factor V_σ and a different realization of the Discrete Series representation on space of functions on G.

4.3.1. Nakahama's formula in bounded symmetric realization's

The hypothesis for Proposition 4.8 is:

The inclusion $\mathcal{D}_{\mathfrak{h}} \cong H/L \to G/K \cong \mathcal{D}$ is a holomorphic map, and V_τ, V_σ are holomorphic Discrete Series representations for G, H realized in a space of holomorphic functions. Thus, $V_\tau = \mathcal{O}(\mathcal{D}, W) \cap L^2_\tau(\mathcal{D}, W)$ is an H-admissible holomorphic Discrete Series representation for G.

We consider the triangular decomposition,

$$P_+ K_{\mathbb{C}} P_- \ni x = exp(x_+)\,(x)_0\, exp(x_-), x_\pm \in Lie(P_\pm) = \mathfrak{p}^\pm,\ (x)_0 \in K_{\mathbb{C}}.$$

Let $P_2 : \mathfrak{p}^+ \to \mathfrak{p}^+_{\mathfrak{h}_0}$ be the projection onto $\mathfrak{p}^+_{\mathfrak{h}_0}$ along $\mathfrak{p}^+_{\mathfrak{h}}$. Let $x \mapsto \bar{x}$ denote the conjugation associated to the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g})$.

In [25, Proposition 3.3] we find a proof of

Proposition 4.8. (Nakahama's formula) *Let $T : V_\sigma \to V_\tau$ a continuous H-map. Then, for $w \in \mathcal{D}_{\mathfrak{h}}, z \in \mathcal{D}$, the function kernel K_T^c is given by,*

$$K_T^c(w,z) = K_\tau^c(w,z)\ K_T^c(o, P_2\big(\big(exp(-\bar{w})exp(z)\big)_+\big)). \tag{4.6}$$

Thus, Nakahama provides an explicit inversion formula to the map in Theorem 4.7(b).

Note. Nakahama's statement as well as his proof is written in the language of Jordan algebras. We are able to obtain a quite long proof of the Proposition without the terminology of Jordan algebras, however, it is just a copy, without Jordan algebras language, of the techniques of Nakahama. Our proof replaces, the result Nakahama needs from [9, Part V, Thorem III.5.1(ii)], by facts in [27, Lemma XII.1.8] [33, Chapter II, § 5].

4.3.2. Nakahama´s formula for the kernel of a holographic operator together with Theorem 2.1 let us compute the kernel of a symmetry breaking operator. The final result is:

Proposition 4.9. *We assume $H/L \to G/K$ is a holomorphic embedding, and $V_\tau = \mathcal{O}(\mathcal{D}, W) \cap L^2_\tau(\mathcal{D}, W)$ (resp. $V_\sigma = \mathcal{O}(\mathcal{D}_\mathfrak{h}, Z) \cap L^2_\tau(\mathcal{D}_\mathfrak{h}, Z)$) is a holomorphic Discrete Series representation of G (resp.is a holomorphic Discrete Series for H). Then, for each $S \in Hom_H(V_\tau, V_\sigma)$, and the function*

$$\Phi_S = K^c_{S^\star}(o, \cdot) \in Hom_L(Z, \mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}, W)) = (\mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}) \otimes Hom_\mathbb{C}(Z, W))^{(L\cdot\otimes\sigma^\star\otimes\tau)(L)}$$

we have $K^c_S(z, w)(\cdot) = \big(\Phi_S(P_2\big((exp(-\bar{w})exp(z)\big)_+)\big)^\star(K^c_\tau(z, w)(\cdot))$, $z \in \mathcal{D}$, $w \in \mathcal{D}_\mathfrak{h}$. Actually, for each $\Phi \in Hom_L(Z, \mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}, W))$, there is a unique symmetry breaking operator S so that $\Phi_S = \Phi$.

When we write $\Phi_S(v)(z_0) = \sum_r C_r(v) z_0^r, v \in Z$, $C_r \in Hom_\mathbb{C}(Z, W)$, $z_0 \in \mathfrak{p}^+_{\mathfrak{h}_0}$, then, $\big(\Phi_S(z)\big)^\star = \sum_r C_r^\star \, \bar{z}_0{}^r$.

4.3.3. Nakahama's formula in group model

Proposition 4.10. *We assume $H/L \to G/K$ is a holomorphic embedding, $H^2(H, \sigma), H^2(G, \tau)$ are respective holomorphic Discrete Series representations. Let $T : H^2(H, \sigma) \to H^2(G, \tau)$ be a holographic operator. Then,*

$$K_T(h, x) = K_\tau(h, x) \; K_T(e, (h)_0^\star \, (g_a \, (g_a)_0^{-1}) \, (h)_0^{\star-1}).$$

Proof. To begin with we recall that for $z \in \mathcal{D}$, the unique $g_z \in exp(\mathfrak{p}) \cap G$ so that $g_z \cdot o = z$ is $g_z = exp(z) K^c_\tau(z, z)^{1/2} exp(\bar{z})$. We note $K^c_\tau(z, z)^{1/2}$ is well defined due to the identity $K^c_\tau(w, z) = \overline{K^c_\tau(z, w)}^{-1}$, and in consequence, $K^c_\tau(z, z)$ belongs to the $exp(\mathfrak{p})$-part of $K_\mathbb{C} = Kexp(iLie(K))$ so $log(K^c_\tau(z, z))$ is defined [33, Chap II, § 5, Exercise 2][27].

For $w \in \mathcal{D}_\mathfrak{h}, z \in \mathcal{D}$, we write (at least for small w, z it is true)

$$P_2((exp(-\bar{w})exp(z))_+) = g_a \cdot o, \; a \in \mathcal{D}_{\mathfrak{h}_0}, \; g_a \in exp(\mathfrak{p} \cap \mathfrak{h}_0).$$

Now, we apply 2.6 and Nakahama's formula 4.6 to obtain

$$\begin{aligned} K_T(g_w, g_z) &= K_\tau(g_w, g_z) \, \tau((g_w)_0)^\star \tau((g_a)_0) \, K_T(e, g_a) \, \sigma((g_w)_0)^{\star-1} \\ &= K_\tau(g_w, g_z) \; K_T(e, ((g_w)_0)^\star \, g_a \, \big(((g_w)_0)^\star (g_a)_0\big)^{-1}) \\ &= K_\tau(g_w, g_z) \; K_T(e, ((g_w)_0)^\star \, (g_a \, (g_a)_0^{-1}) \, ((g_w)_0)^{\star-1}). \end{aligned}$$

The second equality is due to $K_T(e, lxk) = \tau(k^{-1}) K_T(e, x) \sigma(l^{-1})$. Owing to real analyticity of both sides in the formula, the equality holds for every w, z. Next, we write $h = g_{h\cdot o} \, l, x = g_{x\cdot o} \, k$, and compute

$$\begin{aligned} K_T(h, x) &= \tau(k^{-1}) K_T(g_{h\cdot o}, g_{x\cdot o}) \sigma(l) \\ &= \tau(k^{-1}) K_\tau(g_{h\cdot o}, g_{x\cdot o})) \, \tau((g_{h\cdot o})_0)^\star \tau((g_a)_0) \, K_T(e, g_a) \, \sigma((g_{h\cdot o}))_0)^{\star-1} \sigma(l) \end{aligned}$$

$$= K_\tau(g_{h\cdot o}l, g_{x\cdot o}k)\,\sigma(l^{-1})\,K_T(e,\sigma(l^{-1})((g_{h\cdot o})_0)^\star\,g_a\,\big(((g_{h\cdot o})_0)^\star(g_a)_0\big)^{-1})$$
$$= K_\tau(h,x)\;K_T(e,\sigma(l^{-1})(g_{h\cdot o})_0)^\star\,(g_a\,(g_a)_0^{-1})\,((g_{h\cdot o}))_0)^{\star-1}\sigma(l))$$
$$= K_\tau(h,x)\;K_T(e,(h)_0^\star\,(g_a\,(g_a)_0^{-1})\,(h)_0^{\star-1}). \qquad \blacksquare$$

We note that $((g_w)_0)^\star\,(g_a\,(g_a)_0^{-1})\,((g_w)_0)^{\star-1} = e^r e^s,\ r\in\mathfrak{p}^+_{\mathfrak{h}_0},\ s\in\mathfrak{p}^-_{\mathfrak{h}_0}$.
When both representations are scalar the formula turns into

$$K_T(g_w,g_z)=\tau((g_w)_0)^\star\tau((g_a)_0)\;\sigma((g_w)_0)^{\star-1}\,K_\tau(g_w,g_z)\,K_T(e,g_a).$$

4.4. A formula for holographic operator's in the H^2_{hol}-model

In Theorem 4.7 we have presented a new proof that the map

$$Hom_H(V_\sigma,V_\tau)\ni T\mapsto (z\to r_0(K_T(e,\cdot)z)(\cdot))\in Hom_L(Z,\mathbf{H}^2(H_0\times_\tau W)).$$

is bijective. Actually, we have shown injectivity and that both spaces are equidimensional. Nakahama (See Proposition 4.8) have written an explicit inverse map based on the reproducing kernel of the target space V_τ and the realization of holomorphic Discrete Series in spaces of functions on the bounded symmetric domain attached to the group G. In the next paragraph we will present another expression for the inverse map, based on the reproducing kernel of the initial factor V_σ that we are considering and a realization of the Discrete Series representation on space of functions on G.

In order to present the formula we recall a fact shown by Bailey-Borel and another realization for the holomorphic Discrete Series. We assume G/K is isomorphic to a bounded symmetric domain $\mathcal{D}\subset\mathfrak{p}^+$. Let (τ,W) denote the lowest K-type of a holomorphic Discrete Series representation realized in $V_\tau=\mathcal{O}(\mathcal{D},W)\cap L^2_\tau(\mathcal{D},W)$.

Then, we have triangular decomposition $G\subset P^+K_{\mathbb{C}}P^-$, $x=exp(x_+)(x)_0exp(x_-)$, the cocycle $c_\tau(x,w)=\tau((xexp(w))_0)$, $x\in G, w\in\mathcal{D}$, the reproducing kernel for V_τ is $K_\tau(w,z)=\tau((exp(-\bar{w})exp(z))_0^{-1})$. We also have the linear isomorphism $E_{c_\tau}: C^\infty(G\times_\tau W)\to C^\infty(\mathcal{D},W)$ defined by $E_{c_\tau}(f)(xK)=c_\tau(x,o)f(x)$ (see 2.9). Then, Bailey-Borel [3] have shown the pre-image of the space of holomorphic functions $\mathcal{O}(\mathcal{D},W)$, via the map E_{c_τ}, is equal to the subspace

$$\mathcal{O}(G\times_\tau W):=\begin{Bmatrix} f:G\to W:smooth,\ f(xk)=\tau(k^{-1})f(x),\\ k\in K,\ x\in G,\ R_X(f)=0\ \ \forall X\in\mathfrak{p}^-\end{Bmatrix}. \tag{4.7}$$

We also consider similar isomorphisms for the domains $H/L, H_0/L$. Now, we may realize holomorphic Discrete Series representations in spaces of "holomorphic" sections in $\Gamma^\infty(G\times_\tau W)$. That is, given (τ,W) the lowest K-type of a *Holomorphic* Discrete Series representation, we consider the realization

$$H^2_{hol}(G\times_\tau W)=\begin{Bmatrix} f:G\to W:C^\infty,\ f(xk)=\tau(k^{-1})f(x),\ k\in K,\ x\in G,\\ \int_G |f(x)|^2dx<\infty,\ \text{and}\ R_X(f)=0,\ \forall X\in\mathfrak{p}^-\end{Bmatrix}. \tag{4.8}$$

In a similar way, we realize the holomorphic Discrete Series representations for H or H_0. In order to justify this realization of Holomorphic Discrete Series we recall that the equivalence E_{c_τ} is an isometry from $L^2(G\times_\tau W)$ onto $L^c_\tau(\mathcal{D},W)$, see section 2 and references therein.

Let (G,H) be a reductive pair. We suppose there exists Discrete Series representations $H^2_{hol}(G\times_\tau W)$, $H^2_{hol}(H\times_\sigma Z)$, for the respective groups. Since both spaces are reproducing kernel spaces, for each continuous linear map,

$$T: H^2_{hol}(H\times_\sigma Z)\to H^2_{hol}(G\times_\tau W),$$

there exists a smooth function $K^{hol}_T : H\times G\to Hom_{\mathbb{C}}(Z,W)$ with the properties $K^{hol}_T(\cdot,x)\in H^2_{hol}(H\times_\sigma Z)$, $x\in G$, and $(T(g)(x),w)_W=\int_H (g(h),K^{hol}_T(h,x)^\star w)_Z dh$, $x\in G$, $w\in W$, $g\in H^2_{hol}(H\times_\sigma Z)$ (see [30]).

The hypotheses for the next result, Theorem 4.11*, are*:

(G,H) is a symmetric pair, $H_0=(G^{\sigma\theta})_0$, and the generalized Cartan decomposition, [10], is the smooth decomposition

$$G=exp(\mathfrak{h}\cap\mathfrak{p})exp(\mathfrak{h}_0\cap\mathfrak{p})K.$$

For $x\in G$, we write the corresponding unique decomposition $x=x_1x_2k$ [10]. The respective Discrete Series representations are $H^2_{hol}(G\times_\tau W)$ and $H^2_{hol}(H\times_\sigma Z)$.

Theorem 4.11. *The function kernel K^{hol}_T of a holographic operator*

$$T: H^2_{hol}(H\times_\sigma Z)\to H^2_{hol}(G\times_\tau W),$$

decomposes as the composition "separation of variables formula"

$$K^{hol}_T(h,x_1x_2k)(z)=\tau(k^{-1})K^{hol}_T(e,x_2)(K_\sigma(h,x_1)z),$$

where $h\in H$, $x_1\in exp(\mathfrak{h}\cap\mathfrak{p})$, $x_2\in exp(\mathfrak{h}_0\cap\mathfrak{p})$, $k\in K$, $z\in Z$.

To begin with, we consider a holomorphic Discrete Series $H^2_{hol}(H\times_\sigma Z)$ for H and we fix $\Phi: Z\to\mathcal{O}(H_0\times_\tau W)$. We define $T_\Phi:\mathcal{O}(H\times_\sigma Z)\to C^\infty(G\times_\tau W)$ by the rule: For $x=x_1x_2k$, $x_1\in exp(\mathfrak{h}\cap\mathfrak{p})$, $x_2\in exp(\mathfrak{h}_0\cap\mathfrak{p})$, $k\in K$

$$T_\Phi(g)(x):=\tau(k^{-1})\Phi(g(x_1))(x_2). \tag{4.9}$$

We claim: *T_Φ is H-intertwining map for the respective left translation actions if and only if*

$$\Phi(\cdot)(lyl^{-1})=\tau(l)\Phi(\sigma(l^{-1})(\cdot))(y),\forall\, l\in L, y\in exp(\mathfrak{h}_0\cap\mathfrak{p}). \tag{4.10}$$

In fact, we fix $h\in H$. We write $hx_1=x_3l_3$ with $x_3\in exp(\mathfrak{h}\cap\mathfrak{p})$, $l_3\in L$, $hx_1x_2k=x_3l_3x_2l_3^{-1}l_3k$. Then, the equality $T_\Phi(g)(hx)=T_\Phi(L_{h^{-1}}g)(x)$ is equivalent to the equalities

$$\begin{aligned}\tau(k^{-1})\tau(l_3^{-1})\Phi(g(x_3))(l_3x_2l_3^{-1}) &= \tau(k^{-1})\tau(l_3^{-1})\Phi(g(hx_1l_3^{-1}))(l_3x_2l_3^{-1})\\ &= \tau(k^{-1})\tau(l_3^{-1})\Phi(\sigma(l_3)g(hx_1))(l_3x_2l_3^{-1})\\ &\overset{?}{=} \tau(k^{-1})\Phi(g(hx_1))(x_2).\end{aligned}$$

Since the set $linspan_{\mathbb{C}}\{g(hx_1): g\in H^2_{hol}(H,\sigma)[Z]\}$ is equal to Z we get the equivalence.

We would like to point out that condition (4.10) on Φ is equivalent to Φ belongs to $Hom_L(Z, C^\infty(H_0\times_\tau W))$.

4.4.1. Injectivity of r_0 restricted to $\mathcal{L}_{W,H} = \mathcal{L}_\lambda$

In 4.4 we have shown the restriction map $r_0^c : C^\infty(\mathcal{D}, W) \to C^\infty(\mathcal{D}_{\mathfrak{h}_0}, W)$ when restricted to $\mathcal{L}^c_{W,H}$ is injective. The commutativity of the diagram below shows that the restriction map $r_0 : \Gamma^\infty(G \times_\tau W) \to \Gamma^\infty(H_0 \times_\tau W)$ restricted to $\mathcal{L}_{W,H} = \mathcal{L}_\lambda$ is one to one.

$$\begin{array}{ccccc}
\mathcal{L}_{W,H} & \xrightarrow{=} & \mathcal{L}_{W,H} & \xrightarrow{\subset} & H^2_{hol}(G \times_\tau W) \subset L^2(G \times_\tau W) \\
\downarrow{\scriptstyle \subset} & & \downarrow{\scriptstyle E_{c_\tau}} & & \downarrow{\scriptstyle E_{c_\tau}} \\
H^2_{hol}(G \times_\tau W) & & \mathcal{L}^c_{W,H} & \xrightarrow{\subset} & V_\tau \subset L^2_\tau(\mathcal{D}, W) \\
\downarrow{\scriptstyle r_0} & & \downarrow{\scriptstyle r_0^c} & & \downarrow{\scriptstyle r_0^c} \\
L^2(H_0 \times_\tau W) & \xrightarrow{E_{c_\tau}} L^2_\tau(\mathcal{D}_{h_0}, W) & & \xrightarrow{=} & L^2_\tau(\mathcal{D}_{h_0}, W).
\end{array}$$

Here, $\mathcal{L}_{W,H} := \mathcal{L}_\lambda$ is the linear span of the totality of lowest L-type subspaces on each irreducible factor for $res_H(H^2_{hol}(G \times_\tau W))$, that is,

$$\mathcal{L}_{W,H} = \oplus_{H^2_{hol}(H \times_\sigma Z) \in Spec_H(H^2_{hol}(G \times_\tau W))} H^2_{hol}(G \times_\tau W)[H^2_{hol}(H \times_\sigma Z)][Z].$$

$V_\tau = \mathcal{O}(\mathcal{D}, W) \cap L^2_\tau(\mathcal{D}, W)$. $\mathcal{L}_{W,H} = \mathcal{P}(\mathfrak{p}^+_{\mathfrak{h}_0}, W)$ is the linear span of the lowest L-type subspaces on each H-isotypic component for $res_H(V_\tau)$. Further, E_{c_τ} is the map $f \mapsto c_\tau(\cdot, e) f(\cdot)$. E_{c_τ} establishes a H-bijection between $H^2_{hol}(G \times_\tau W)$ (resp. $L^2(G \times_\tau W)$) and V_τ (resp. $L^2_\tau(\mathcal{D}, W)$). $\mathcal{L}^c_{W,H} = E_{c_\tau}(\mathcal{L}_{W,H})$.

4.4.2. In the following Theorem 4.12 we present a new expression for holographic operators and its kernel.

The hypotheses are:

(G, H) is a symmetric pair. We recall $H_0 = (G^{\sigma\theta})_0$ and the "generalized Cartan decomposition", that is, the smooth decomposition $G = exp(\mathfrak{h} \cap \mathfrak{p}) exp(\mathfrak{h}_0 \cap \mathfrak{p}) K$ [10]. We also assume $H/L \to G/K$ is a holomorphic embedding, and $H^2_{hol}(G \times_\tau W)$ (resp. $H^2_{hol}(H \times_\sigma Z)$) is a holomorphic Discrete Series representation of G (resp. is a holomorphic Discrete Series representation for H). Then,

Theorem 4.12. *For each holographic operator*

$$T : Hom_H(H^2_{hol}(H \times_\sigma Z) \to H^2_{hol}(G \times_\tau W),$$

there exists $\Phi \in Hom_L(Z, H^2_{hol}(H_0 \times_\tau W))$ *so that for every* $g \in H^2_{hol}(H \times_\sigma Z)$, $G \ni x = x_1 x_2 k$, $x_1 \in exp(\mathfrak{h} \cap \mathfrak{p})$, $x_2 \in exp(\mathfrak{h}_0 \cap \mathfrak{p})$, $k \in K$, *we have*

$$T(g)(x) = T_\Phi(g)(x) = \tau(k^{-1}) \Phi(g(x_1))(x_2).$$

Whence, its kernel function K_T^{hol} *is:*

$$K_T^{hol}(h, x) z = \tau(k^{-1}) \Phi(K_\sigma(x_1, h)^\star z)(x_2) = \tau(k^{-1}) \Phi(K_\sigma(h, x_1) z)(x_2).$$

Here, $K_\sigma(h, x_1) z$ is the reproducing kernel for $H^2_{hol}(H \times_\sigma Z)$.

Theorem 4.12 generalizes [25, Theorem 5.1].

Proof. For $H_0 \ni h_0 = x_2 l$, $x_2 \in exp(\mathfrak{h}_0 \cap \mathfrak{p})$, $l \in L$, $z \in Z$, we define

$$\Phi(z)(x_2 l) := K_T^{hol}(e, x_2 l)(z) = \tau(l^{-1}) K_T^{hol}(e, x_2)(z).$$

Hence, $\Phi(z)(\cdot) \in \Gamma^\infty(H_0 \times_\tau W)$. Next, the H-invariance ($K_T^{hol}(hx, hy) = K_T^{hol}(x, y)$)

for the kernel function K_T^{hol} yields that Φ satisfies condition 4.10, and, owing to $R_X(K_T^{hol}(h,\cdot)z)=0$ for each $h\in H, X\in\mathfrak{p}^-$ we obtain Φ is "holomorphic", that is, $R_X(\Phi(z)(\cdot))=0$ for each $X\in\mathfrak{p}_{\mathfrak{h}_0}^-$. Therefore, $\Phi\in Hom_L(Z,\mathcal{O}(H_0\times_\tau W))$. In the following, we verify $\Phi(z)(\cdot)$ belongs to $L^2(H_0\times_\tau W)$. Owing to the representation $H^2_{hol}(G\times_\tau W)$ is H-admissible, we may apply the result of T. Kobayashi [19] "In an H-admissible representation, every L-finite vector is K-finite" and we obtain in consequence $K_T^{hol}(e,\cdot)z\in H^2_{hol}(G\times_\tau W)[H^2_{hol}(H\times_\sigma Z)][Z]\subset H^2_{hol}(G\times_\tau W)_{K-fin}$. A result due to Schmid, Knapp-Wallach [34] [17, Cor. 9.6] shows

$$H^2_{hol}(G\times_\tau W)_{K-fin}=H^2(G\times_\tau W)_{K-fin}\quad\text{and}\quad H^2_{hol}(G\times_\tau W)\subset H^2(G\times_\tau W).$$

Finally, r_0 maps $H^2(G\times_\tau W)$ into $L^2(H_0\times_\tau W)$ (See [29]), and, $\Phi(z)(x_2l)=\tau(l^{-1})r_0(K_T^{hol}(e,\cdot)z)(x_2)$, whence we have verified $\Phi(z)(\cdot)\in H^2_{hol}(H_0\times_\tau W)$.

It readily follows: $T_\Phi(g)$ is "holomorphic" for each $g\in H^2_{hol}(H\times_\sigma Z)$.

Since, via E_{c_σ}, $H^2_{hol}(H\times_\sigma Z)$ is isomorphic to the kernel of an elliptic operator, we have that L^2-convergence in $H^2_{hol}(H\times_\sigma Z)$ implies convergence in the natural topology in $\mathcal{O}(H\times_\sigma Z)$. We conclude that $T_\Phi : H^2_{hol}(H\times_\sigma Z)\to\mathcal{O}(G\times_\tau W)$ is a continuous linear map. Moreover, since for $g\in H^2_{hol}(H\times_\sigma Z)$ the equality $g(h)=\int_H K_\sigma(y,h)g(y)dy$, $h\in H$ holds, we have T_Φ is a kernel linear map represented by the kernel

$$K_{T_\Phi}(h,x_1x_2k)(\cdot):=\tau(k^{-1})\Phi(K_\sigma(h,x_1)(\cdot))(x_2).$$

Owing to $T_\Phi(K_\sigma(\cdot,e)^\star)(z))=K_{T_\Phi}(e,\cdot)(z)$, as in Proposition 2.3(5), we obtain $K_{T_\Phi}(e,\cdot)(z)$ is a L-finite vector. Thus, $E_{c_\tau}(K_{T_\Phi}(e,\cdot)z)$ is a L-finite vector in the H-admissible representation $(L^\tau_\cdot,\mathcal{O}(\mathcal{D},W))$, [19] implies $E_{c_\tau}(K_{T_\Phi}(e,\cdot))$ is a K-finite vector, now, we recall that a K-finite holomorphic function is polynomial, thus $E_{c_\tau}(K_{T_\Phi}(e,\cdot))\in\mathcal{O}(\mathcal{D},W)\cap L^2_\tau(\mathcal{D},W)$ and hence, $K_{T_\Phi}(e,\cdot)z$ as well as $L_h(K_{T_\Phi}(e,\cdot))=K_{T_\Phi}(h,\cdot)$ belongs to $H^2_{hol}(G\times_\tau W)$. Finally, the equality $K_{T_\Phi}(e,\cdot)z=T_\Phi(K_\sigma(\cdot,e)^\star z)$ yields

$$K_{T_\Phi}(e,\cdot)\in H^2_{hol}(G\times_\tau W)[H^2_{hol}(H\times_\sigma Z)[Z]\subset\mathcal{L}_{W,H}.$$

Also, $K_T^{hol}(e,\cdot)(z)\in\mathcal{L}_{W,H}$. By construction, $r_0(K_T^{hol}(e,\cdot)z)=r_0(K_{T_\Phi}(e,\cdot)z)$, whence, due that r_0 is injective in $\mathcal{L}_{W,H}$ (see 4.4.1) we obtain that both kernels are equal and $T=T_\Phi$. ■

Remark 4.13. The converse to Theorem 4.12 holds too. In fact, if we have $\Phi\in Hom_L(Z,\ H^2_{hol}(H_0\times_\tau W)$, the duality Theorem 4.7 shows that there exists $T_1\in Hom_H(H^2_{hol}(H\times_\sigma Z),H^2_{hol}(G\times_\tau W))$ so that $r_0(K_{T_1}(e,\cdot)z)=\Phi(z)$.

Corollary 4.14. *The kernel K_T^{hol} of each holographic operator $T\in Hom_H$ $(H^2_{hol}(H\times_\sigma Z),H^2_{hol}(G\times_\tau W))$, decomposes as the composition "separation of variables formula"*

$$K_T^{hol}(h,x_1x_2k)(z)=\tau(k^{-1})K_T^{hol}(e,x_2)(K_\sigma(h,x_1)z),$$

where $h\in H$, $x_1\in exp(\mathfrak{h}\cap\mathfrak{p})$, $x_2\in exp(\mathfrak{h}_0\cap\mathfrak{p})$, $k\in K$.

Theorem 4.12 above and its corollary 4.14 generalize [30, example 10.1] for the case $G=SU(n,1)$, $H=S(U(n-1,1)\times U(1))$, $H_0=S(U(n-1)\times U(1,1))$.

In the following, we rewrite Theorem 4.7 in the symmetric space model. Later on, after a change of the coordinates (x_1, x_2) to holomorphic coordinates we obtain another formula for the kernel of a holographic operator.

We recall that to each holographic operator $T : H^2_{hol}(H \times_\sigma Z) \to H^2_{hol}(G \times_\tau W)$ we have associated an unique H-operator $T^c : H^c_{c_\sigma}(H/L, Z) \to H^c_{c_\tau}(G/K, W)$ via the composition $T_c = E_{c_\tau} T E^{-1}_{c_\sigma}$ and the respective kernels are related by the equality 2.6

$$K^c_{T^c}(hL, xK) = c_\tau(x, e) K^{hol}_T(h, x) c_\sigma(h, e)^\star, \; \forall h \in H, x \in G. \qquad (\natural)$$

Hence, the formula for K^{hol}_T obtained in Corollary 4.14 and 2.3 (5) yields

Corollary 4.15. $K^c_{T^c}(hL, x_1x_2K) = \tau((x_1 exp(x_2)_+)_0) K^c_{T^c}(o, (x_1)_0 x_2 K) K^c_\sigma(hL, x_1K)$.

Here, $x_1 \in exp(\mathfrak{h} \cap \mathfrak{p})$, $x_2 \in exp(\mathfrak{h}_0 \cap \mathfrak{p})$, $G \ni a = exp(a_+)\,(a)_0\, exp(a_-)$, $a_\pm \in \mathfrak{p}^\pm$ and $(a)_0 \in K_\mathbb{C}$. In fact, the equality $(\natural)$, $a \cdot o = a_+$,

$$c_\tau(x_1x_2, o) = c_\tau(x_1, x_2 \cdot o) c_\tau(x_2, o) = \tau((x_1 exp((x_2)_+)))_0) \tau((x_2)_0),$$

$c_\sigma(b, o) = \sigma((b)_0)$ and 2.2 imply

$$\begin{aligned} K^c_{T^c}(hL, x_1x_2K)z &= \tau((x_1 exp((x_2)_+)))_0) \tau((x_2)_0) K_T(e, x_2)(K_\sigma(h, x_1)\sigma((h)_0)^\star z \\ &= \tau((x_1 exp((x_2)_+)))_0) K^c_{T^c}(e, x_2K) \sigma((x_1)_0)^{-1} K^c_\sigma(hL, x_1K)z. \end{aligned}$$

[30, Example 10.1] for $(SU(n,1), S(U(n-1,1) \times U(1))$ and holomorphic scalar representations in the bounded symmetric space realization, shows that the factor

$$\tau((x_1 exp(x_2)_+)_0)\, K^c_{T^c}(o, x_2K)\, \sigma((x_1)_0^{-1})$$

does not depends on $x_1 \in \mathfrak{p}^+_\mathfrak{h}$. We do not know, in the bounded symmetric realization of holomorphic Discrete Series, when "separation of variables via K^c_τ" does hold. On the positive side, in the H_{hol} realization, Corollary 4.14 gives:

Corollary 4.16. *Let $S : H^2_{hol}(G \times_\tau W) \to H^2_{hol}(H \times_\sigma Z)$ be a symmetry breaking operator. Then,* $K_S(x_1x_2k, h) = K_S(x_2, e) K_\sigma(x_1, h) \tau(k)$.

The next example, shows the complexity of computations for splitting variables.

Example 4.17. $G = SU(1,1) \times SU(1,1)$, $\sigma(x,y) = (y,x)$, $H = \{(x,x) : x \in SU(1,1)\}$, $K = T = \{(diag(e^{i\phi}, e^{-i\phi}), diag(e^{i\psi}, e^{-i\psi}))\}$, $G^{\sigma\theta} = H_0 = \{(x, x^{*-1}) : x \in SU(1,1)\}$, $\mathfrak{h}_0 = Lie H_0 = \{(x, -x^*) : x \in \mathfrak{su}(1,1)\} = (1 \times \theta_{\mathfrak{su}(1,1)})H$.

$$exp(\mathfrak{p} \cap \mathfrak{h}) = \Big\{ x_1(a) := \Big(cosh(|a|)I + \frac{sinh(|a|)}{|a|} \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix}, cosh(|a|)I + \frac{sinh(|a|)}{|a|} \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix} \Big),\ a \in \mathbb{C} \Big\}.$$

$$exp(\mathfrak{p} \cap \mathfrak{h}_0) = \{ x_2(a) := \Big(cosh(|a|)I + \frac{sinh(|a|)}{|a|} \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix}, cosh(|a|)I + \frac{sinh(|a|)}{|a|} \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix} \Big), a \in \mathbb{C} \}.$$

The generalized Cartan decomposition for G reads: $G \ni x = x_1(a) x_2(b) k$, $a, b \in \mathbb{C}$, $k \in K$. We set $\tau_\lambda(exp(\phi \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right))) = e^{i\lambda\phi}$, $\lambda \geq 2$.

Let $v_s(\left(\begin{smallmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{smallmatrix}\right)) := (\lambda)_s \bar{\alpha}^{-\lambda-s} \beta^s$, then, $v_s \in H^2_{hol}(SU(1,1), \tau_\lambda)[\tau_{\lambda+2s}]$. An orthonormal basis for $H^2_{hol}(SU(1,1), \tau_\lambda)$ is $\{v_s, s = 0, 1, \dots\}$. For $H^2_{hol}(SU(1,1), \tau_{\lambda'})$ we denote the corresponding basis for $v'_t, t = 0, 1, \dots$.

A computation in the unit disc, for $\lambda, \lambda' \geq 2$, yields

$$H^2_{hol}(SU(1,1),\tau_\lambda)\otimes H^2_{hol}(SU(1,1),\tau_{\lambda'})[\tau_{\lambda+\lambda'+2n}]\cap \mathcal{L}_{\mathbb{C},SU(1,1)}=\mathbb{C}\,\Phi(1)(\cdot),$$

where, for $z \in \mathbb{C}$,

$$\Phi(z)(\cdot)=z\sum_{0\leq s\leq n}(-1)^s\binom{n}{s}\frac{(\lambda-1)!}{(\lambda-1+s)!}\ \frac{(\lambda'-1)!}{(\lambda'-1+n-s)!}v_s\otimes v'_{n-s}(\cdot).$$

Therefore, the associated holographic operator

$$T_\Phi : H^2_{hol}(SU(1,1),\tau_{\lambda+\lambda'+2n})\to H^2_{hol}(SU(1,1),\tau_\lambda)\otimes H^2_{hol}(SU(1,1),\tau_{\lambda'})$$

is the operator that maps $g \in H^2_{hol}(SU(1,1),\tau_{\lambda+\lambda'+2n})$ to

$$\begin{aligned}&T_\Phi(g)(x_1(a)x_2(b)(k_1,k_2)) = \tau_\lambda\otimes\tau_{\lambda'}(k_1,k_2)\big(\Phi(g(x_1(a)))\big)(x_2(b))\\ &= \tau_\lambda(k_1^{-1})\tau_{\lambda'}(k_2^{-1})\ g(x_1(a))\\ &\quad\times\sum_{0\leq s\leq n}(-1)^s\binom{n}{s}\frac{(\lambda-1)!}{(\lambda-1+s)!}\ \frac{(\lambda'-1)!}{(\lambda'-1+n-s)!}v_s(x_2(b))v'_{n-s}(x_2(b)^{*-1}).\end{aligned}$$

Remark 4.18. For an arbitrary symmetric pair (G,H), a pair $H^2(H,\sigma)$, $H^2(G,\tau)$, and, for $\Phi \in Hom_L(Z,\mathbf{H}^2(H_0\times_\tau W))$, the linear map T_Φ defines a H-map, $T_\Phi \in Hom_H(H^2(H\times_\sigma Z), C^\infty(G\times_\tau W))$. If we could show $D_{Schmid}(K_{T_\Phi}(h,\cdot))=0$ and $\int_G \|T_\Phi(g)\|^2 dg < \infty$ for all $g \in H^2_{hol}(H,\sigma)$ we would obtain holographic operators. By means of the analysis of leading exponents of a representation (See [39, Chap IV]) we are able to show that $T_\Phi(g)$ is square integrable for Harish-Chandra parameters far away from the walls. The equality $D_{Schmid}(K_{T_\Phi}(h,\cdot))=0$ would follows if we knew a Hartog's Theorem for the Schmid operator. We refer to Hartog's Theorem as the Theorem that shows: a function is holomorphic if and only if it is holomorphic in each variable.

4.4.3. A result of Kitagawa on holographic operators

Recently, some remarkable new results have been obtained by Kitagawa, as follows: Let $(L^G_\cdot, V^G := H^2(G,\tau))$ be an H-admissible Discrete Series representation. Let $V^H := H^2(H,\sigma)$ be an irreducible factor of $res_H(V^G)$.

The H-admissibility hypothesis yields:

For any intertwining map $T:(V^G)^\infty[V^H]\to(V^G)^\infty$, in particular, for each $R\in\mathcal{U}(\mathfrak{g})^H$, either T or $L^G_R : (V^G)^\infty[V^H]\to(V^G)^\infty$ extents to a continuous linear endomorphism for $V^G[V^H]$.

In fact, we write $V^G[V^H]=M_1\oplus\cdots\oplus M_k$, where M_i are irreducible H-factors, thus L^G_R maps the $\mathcal{U}(\mathfrak{h})$-irreducible representation $(M_i)_{L-fin}$ into the unitary representation $V^G[V^H]$, we now apply [40, Lemma 8.6.7] and obtain T or L^G_R extents to a continuous linear map from M_i into $V^G[V^H]$, whence the claim follows.

A consequence of this is that for any $T \in Hom_H(V^H,V^G)$ and for any $R\in\mathcal{U}(\mathfrak{g})^H$, the composition map $L^G_R T$ from $(V^H)_{L-fin}\to V^G$ extends to a holographic operator. Whence, $\mathcal{U}(\mathfrak{g})^H$ also acts in $Hom_H(V^H,V^G)$ by the rule $R\cdot T = L^G_R T$. We note, we have verified the inclusion $Hom_H(V^H,V^G)\subseteq Hom_H((V^H)^\infty,(V^G)^{H-\infty})$ becomes an equality. In [16, Theorem 5.26], appealing to a particular Zuckerman functor's realization of the space of K-finite vectors of a Discrete Series representation, it is shown that the action of $\mathcal{U}(\mathfrak{g})^H$ in $Hom_H(V^H,V^G)$ is irreducible.

Thus, we obtained the observation: once we know one nonzero holographic operator T from V^H into V^G, all the others have the shape $L_R^G T, R \in \mathcal{U}(\mathfrak{g})^H$.

5. Further perspectives

In a sequel to this paper, we shall continue the more detailed analysis of the nature of differential operators that are symmetry breaking (holographic). Below, we mention some of those results to be proven in the sequel to this paper, and in the final Section 5.1 below we give two final concrete examples of the case of admissible restriction, our primary focus, this time outside the holomorphic case. A major goal of branching laws is to understand the structure of symmetry breaking (holographic) operators for a general pair (G,H) and (π,V) an H-admissible Discrete Series. For this, we present some considerations on the structure of symmetry breaking operators. Whence, we fix a symmetry breaking operator $S : H^2(G,\tau) \mapsto H^2(H,\sigma)$, represented by the kernel $K_S : G\times H \to Hom_{\mathbb{C}}(W,Z)$ and recall that the subspace

$$Z_S := Image(Z \ni z \mapsto K_{S^*}(e,\cdot)(z) = K_S(\cdot,e)^\star z \in H^2(G,\tau))$$

is a L-irreducible subspace contained in $\mathcal{L}_{W,H}$. Thus, either $Z_S \cap \mathcal{U}(\mathfrak{h}_0)W = \{0\}$ or $Z_S \subset \mathcal{U}(\mathfrak{h}_0)W$. Our hypothesis is that (π,V) is an H-admissible representation, hence, every symmetry breaking operator is represented by differential operator's 2.4.1, 2.6, [30, Proposition 4.4]. We recall in [31, Proposition 6.1], it is shown that in the case $Z_S \subset \mathcal{U}(\mathfrak{h}_0)W$, S is represented by a normal derivative differential operator, See 2.10, whereas in the case $Z_S \cap \mathcal{U}(\mathfrak{h}_0)W = \{0\}$, S is represented by a differential operator that never will be a normal derivative differential operator. To be more precise, we write

$$Hom_H(V,V_\sigma) = \{S : Z_S \subset \mathcal{U}(\mathfrak{h}_0)W\} \cup \{S : Z_S \cap \mathcal{U}(\mathfrak{h}_0)W = \{0\}\}. \qquad (\ddagger)$$

After ignoring the zero operator, this is a disjoint union, the first subset is the one that contains the symmetry breaking operators that are represented by normal derivatives operators and the second subset is its complement. Roughly speaking, $\mathcal{L}_{W,H} \cap \mathcal{U}(\mathfrak{h}_0)W$ measures the "quantity" of symmetry breaking operators represented by normal derivatives operators, whereas, $\mathcal{L}_{W,H} \backslash (\mathcal{L}_{W,H} \cap \mathcal{U}(\mathfrak{h}_0)W)$, "measures" the totality of "non normal" derivative symmetry breaking operators. We do not know, if for given σ,τ, both subsets in $(\ddagger)$ might be nonempty.

Obviously, if $H^2(H,\sigma)$ has multiplicity one in $H^2(G,\tau)$, the nonzero symmetry breaking operators are either normal derivatives differential operators or not.

Henceforth, our hypotheses are: $H/L \to G/K$ is a holomorphic immersion and $(L^\tau_\cdot, V_\tau)$ is H-admissible holomorphic Discrete Series representation of lowest K-type (τ,W).

Since the Lie algebra $\mathfrak{p}^-$ is an Abelian Lie algebra, the symmetrization from $S(\mathfrak{p}^-)$ onto $\mathcal{U}(\mathfrak{p}^-)$ becomes an associative algebra isomorphism. Whence, after we recall 4.3, $\mathcal{U}(\mathfrak{p}^-)\otimes W$ may be written as a graded vector space, the n^{th} subspace being

$$\mathcal{V}^{(n)} := lin.span_{\mathbb{C}}\{L^\tau_{x_1\cdots x_n}(w),\ x_j \in \mathfrak{p}^-\backslash\{0\}, w \in W\}.$$

It follows from the computation in 2.5.2 that whenever $Z_S \subset \mathcal{V}^{(n)}$, then, S is represented by an n^{th}-order differential operator.

The meaning of next Proposition 5.1 is: when we represent symmetry breaking operators via differential operators, the totality of "first order" symmetry breaking operators are given by means of normal derivatives if and only if τ is a one dimensional representation unless G is not locally isomorphic to $SU(m,n)$, $m \geq 2$, $n \geq 2$.

Proposition 5.1. *$H/L \to G/K$ is a holomorphic embedding. V_σ, V_τ holomorphic Discrete Series representations for H, G. We assume $\mathfrak{g}$ is simple and is not isomorphic to $\mathfrak{su}(m,n)$, $m \geq 2, n \geq 2$, then, $\mathcal{U}(\mathfrak{h}_0)W \cap \mathcal{L}_{W,H} \cap \mathcal{V}^{(1)} = \mathcal{L}_{W,H} \cap \mathcal{V}^{(1)}$ if and only if τ is a one dimensional representation.*

We also have a similar equivalence for $\mathfrak{g} \cong \mathfrak{su}(m,n)$, $\quad m \geq 2, n \geq 2$.

We have computed examples, for τ not unidimensional, so that among the symmetry "first order" operators, we find nonzero "first order normal differential operators", as well as, nonzero "first order non normal differential operators". This statement, is equivalent to that the inclusions $\{0\} \subset \mathcal{L}_{W,H} \cap \mathcal{V}^{(1)} \cap \mathcal{U}(\mathfrak{h}_0)W \subset \mathcal{L}_{W,H} \cap \mathcal{V}^{(1)}$ are proper. A part of the result is,

Proposition 5.2. *Same hypothesis as in* 5.1. *For a pair $(\mathfrak{g}, \mathfrak{h})$ so that both $\mathfrak{h}, \mathfrak{h}_0$ are isomorphic to the product of one noncompact simple Lie algebra times a compact Lie algebra, and, τ is not a scalar representation. Then, the representation of L in $\mathcal{U}(\mathfrak{h}_0)W \cap \mathcal{L}_{W,H} \cap \mathcal{V}^{(1)}$ is irreducible and not a scalar representation. Moreover, $\mathcal{U}(\mathfrak{h}_0)W \cap \mathcal{L}_{W,H} \cap \mathcal{V}^{(1)}$ is a proper subspace of $\mathcal{L}_{W,H} \cap \mathcal{V}^{(1)}$.*

The following fact shows that, sometimes, every "second order" symmetry breaking operator is represented via normal derivative operators.

Proposition 5.3. *Same hypothesis as in* 5.1. *For a scalar representation (τ, W), we have: $\mathcal{U}(\mathfrak{h}_0)W \cap \mathcal{L}_{W,H} \cap \mathcal{V}^{(2)} = \mathcal{L}_{W,H} \cap \mathcal{V}^{(2)}$ if and only if $[[\mathfrak{p}^+_{\mathfrak{h}_0}, \mathfrak{p}^-_{\mathfrak{h}}], \mathfrak{p}^+_{\mathfrak{h}_0}] = \{0\}$.*

It follows, via the Poincaré-Birkoff-Witt Theorem, that

Proposition 5.4. *Same hypothesis as in* 5.1. *For a scalar representation (τ, W), whenever every "first order" as well as every "second order" symmetry breaking operator is represented by normal derivative differential operator. Then, every symmetry breaking operator is represented by normal derivatives.*

This is due that the equalities $\mathcal{U}(\mathfrak{h}_0)W \cap \mathcal{V}^{(i)} = \mathcal{L}_{W,H} \cap \mathcal{V}^{(i)}, i = 1, 2$ yields the equality $\mathcal{L}_{W,H} = \mathcal{U}(\mathfrak{h}_0)W$.

5.1. Examples and comments on branching laws

Many open problems on branching laws remain, for example, in [30], for a quaternionic Discrete Series representation $\pi^{Sp(1,p+1)}$, we derive an abstract decomposition for $res_{Sp(1)\times Sp(1,p)}(\pi^{Sp(1,p+1)})$, however, we do not explicitly present such a decomposition. That is, we do not provide equations for the isotypic component $\pi^{Sp(1,p+1)}[\pi^{Sp(1)\times Sp(1,p)}]$, as well as, for the symmetry (holographic) operators. Other open problem is to explicit the continuous spectrum for the restriction of a Discrete Series representation, important results on the subject are found in the work Harris-He-Olafsson [12], see also [29].

The final part of this section is to make explicit two examples of admissible restriction. See [31]. The first symmetric pair (G, H) is so that the corresponding pair of Lie algebras is $(\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})$. This is not a holomorphic pair. We have, there exists a Borel-de Siebenthal system Ψ_{BS} of positive roots in $\Phi(\mathfrak{e}_6, \mathfrak{t})$ so that a Discrete Series for $E_{6(2)}$ whose Harish-Chandra parameter λ is dominant with respect to Ψ_{BS} has an admissible restriction to a subgroup $H \cong F_{4(4)}$. We fix a compact Cartan subgroup $T \subset K \cong SU_2(\alpha_{max}) \times SU(6)$ so that $U := T \cap H$ is a compact Cartan subgroup of $L = K \cap H \cong SU_2(\alpha_{max}) \times Sp(3)$. Let α_{max} denote the maximal root for Ψ_{BS}. Among the H-admissible representations are the so-called quaternionic representations, they are the representations $H^2(E_{6(2)}, \pi^{SU_2(\alpha_{max})\times SU(6)}_{n\frac{\alpha_{max}}{2}+\rho_{SU(6)}}), n = 1, 2, \cdots$. The associated pair is $(\mathfrak{e}_{6(2)}, \mathfrak{sp}(1,3))$. Let $q_{\mathfrak{u}}$ denote the restriction map from $\mathfrak{t}^\star$ into $\mathfrak{u}^\star$. Then, λ dominant with respect to Ψ_{BS}, determines systems of positive roots, $\Psi_{\mathfrak{f}_{4(4)},\lambda}$, $\Psi_{\mathfrak{sp}(1,3),\lambda}$ that satisfies the Borel-de Siebenthal property and they are quaternionic. We denote by β_{max} the maximal root for $\Psi_{\mathfrak{f}_{4(4)},\lambda}$. The fundamental weight $\tilde{\Lambda}_1$ associated to external short root in the Dynkin diagram is $\frac{1}{2}\beta_{max}$.

The highest weight of the lowest K-type for Discrete Series of Harish-Chandra parameter $n\frac{\alpha_{max}}{2} + \rho_{SU(6)}$ for $E_{6(2)}, n \geq 1$ is

$$\nu_n := n\frac{\alpha_{max}}{2} + \rho_{SU(6)} + \rho_n^{\Psi_{BS}} - \rho_c = (n+10)\frac{\alpha_{max}}{2}.$$

Thus, the duality Theorem, applied to the restriction of the quaternionic representation $H^2(E_{6(2)}, \pi_{\nu_n}^{SU_2(\alpha_{max})\times SU(6)})$ to $F_{4(4)}$, yields

$$res_{F_{4(4)}}\big(H^2(E_{6(2)}, \pi_{\nu_n}^{SU_2(\alpha_{max})\times SU(6)})\big) = \bigoplus_{m\geq 0} H^2(F_{4(4)}, \pi_{\sigma_{n,m}}^{SU_2(\alpha_{max})\times Sp(3)}).$$

Here, the highest weight $\sigma_{n,m}$ is: $\sigma_{n,m} = (n+11+m)\frac{\alpha_{max}}{2} + m\tilde{\Lambda}_1$.

However, this is in some sense an abstract decomposition, owing to we do not provide either the equations or a description of each isotypic component

$$H^2(E_{6(2)}, \pi^{SU_2(\alpha_{max})\times SU(6)}_{n\frac{\alpha_{max}}{2}+\rho_{SU(6)}})\big[H^2(F_{4(4)}, \pi^{SU_2(\alpha_{max})\times Sp(3)}_{(n+11+m)\frac{\alpha_{max}}{2}+m\tilde{\Lambda}_1})\big].$$

Nevertheless, in [30, Proposition 6.8] we have obtained a formula for the kernel that represents the orthogonal projector onto a given isotypic component, for the precise statement, see Remark 2.2. We note that the Harish-Chandra parameter of the irreducible $F_{4(4)}$-factors are dominant with respect to a Borel-de Siebenthal system of positive roots. Moreover, each $F_{4(4)}$-irreducible factor is a generalized quaternionic representation. This ends the first example.

The second example is on the pair $(\mathfrak{so}(2m,2), \mathfrak{so}(2m,1))$. From $Spin(2m,2)$, $m \geq 2$, we restrict to $Spin(2m,1)$. We notice the isomorphism between the pair $(Spin(4,2),\ Spin(4,1))$ with $(SU(2,2), Sp(1,1))$. In this setting,

$$K = Spin(2m) \times Z_K, \quad L = Spin(2m), \quad Z_K \cong \mathbb{T}.$$

Obviously, we may conclude that any irreducible representation of K is irreducible when restricted to L. In this case $H_0 \cong Spin(2m,1)$ (for $m = 2,\ H_0 \cong Sp(1,1)$). We always have: the representation $\mathbf{H}^2(H_0, \tau)$ is irreducible. Therefore, the duality theorem together with the fact that any irreducible representation for $Spin(2m,1)$ is $L = Spin(2m)$-multiplicity free [36, page 11], implies:

Any Discrete Series representation for $Spin(2m,2)$, with an admissible restriction to $Spin(2m,1)$, is a multiplicity free representation.

We fix a maximal torus T for K, so that $U := L \cap T$ is a maximal torus for L. Then, there exists a orthogonal basis $\epsilon_1, \ldots, \epsilon_m, \delta$ for $i\mathfrak{t}^\star$ so that $\mathfrak{z}_K^\star = \mathbb{C}\delta$, and $\Phi(\mathfrak{so}(2m,2), \mathfrak{t}) := \{\pm(\epsilon_k \pm \epsilon_s), 1 \leq k < s \leq m\} \cup \{\pm(\epsilon_j \pm \delta), 1 \leq j \leq m\}$.

We consider the systems of positive roots in $\Phi(\mathfrak{so}(2m,2),\mathfrak{t})$ defined as follows: $\Psi_- = S_{\epsilon_m-\delta}S_{\epsilon_m+\delta}\Psi_+$ and $\Psi_+ = \{\epsilon_k \pm \epsilon_s,\ 1\leq k<s\leq m,\ (\epsilon_j\pm\delta),\ 1\leq j\leq m\}$.

The systems $\Psi_\pm$ are not Borel-de Siebenthal.

For $Spin(2m,2), m\geq 3$ in [37, Table 2], [22] it is verified that any Discrete Series of Harish-Chandra parameter λ dominant with respect to one of the systems $\Psi_\pm$ has admissible restriction to $Spin(2m,1)$ and no other Discrete Series representation has admissible restriction to $Spin(2m,1)$.

Let $q_\mathfrak{u}$ denote the restriction map from $\mathfrak{t}^\star$ onto $\mathfrak{u}^\star$. For $\lambda\in i\mathfrak{t}^\star$ dominant integral for one of the systems $\Psi_\pm$, the highest weight Harish-Chandra parameter (infinitesimal character) of the lowest K-type for the Discrete Series of $Spin(2m,1)$ attached to λ is $\lambda+\rho_n^{\Psi_\pm}$. Thus,

$$(\tau,W) = (\tau^{Spin(2m)\times SO(2)}_{\lambda+\rho_n^{\Psi_\pm}}, V^{Spin(2m)\times SO(2)}_{\lambda+\rho_n^{\Psi_\pm}}).$$

The restriction of (τ,W) to L is the irreducible the representation

$$(\tau^{Spin(2m)}_{q_\mathfrak{u}(\lambda+\rho_n^{\Psi_\pm})}, (V^{Spin(2m)}_{q_\mathfrak{u}(\lambda+\rho_n^{\Psi_\pm})}).$$

Let $Spec_L(H^2(Spin(2m,1),res_L((\tau,W))))\subset i\mathfrak{u}^\star$ denote the totality of the infinitesimal character, dominant with respect to $\Psi_\pm\cap\Phi_c\cap i\mathfrak{u}^\star$, of the irreducible L-factors of $H^2(Spin(2m,1),res_L((\tau,W)))$. In [36], we find an algorithm to compute the set $Spec_L(H^2(Spin(2m,1),\ res_L((\tau,W))))$. Then, the duality theorem yields

$$\begin{aligned} &res_{Spin(2m,1)}\big(H^2(Spin(2m,2),\tau^{Spin(2m)\times SO(2)}_{\lambda+\rho_n^{\Psi_\pm}})\big) \\ &= \bigoplus_{\sigma\in Spec_L(H^2(Spin(2m,1),res_L((\tau,W))))} H^2(Spin(2m,1),\tau_\sigma^{Spin(2m)}). \end{aligned}$$

The above formula provides a description of the restriction, however, we have not been able to compute the isotypic component,

$$H^2(Spin(2m,2),\tau^{Spin(2m)\times SO(2)}_{\lambda+\rho_n^{\Psi_\pm}})[H^2(Spin(2m,1),\tau_\sigma^{Spin(2m)})].$$

Also we did not make explicit, either holographic or symmetry breaking operators. Nevertheless, in [30, Proposition 6.8] we have obtained a formula for the kernel that represents the orthogonal projector onto a given isotypic component, for the precise statement see Remark 2.2 in this paper.

6. Partial list of symbols and definitions

- (τ,W), (σ,Z), $(L^G_\cdot, L^2(G\times_\tau W))$, $(L^H_\cdot, L^2(H\times_\sigma Z))$, K_τ, K_σ, $H^2(G,\tau)=V_\lambda=V^G_\lambda$, $H^2(H,\sigma)=V^H_\mu$, $(L^H_\cdot)_{|_{H^2(H,\sigma)}}=\pi^H_\mu, \tau=\pi^K_\nu$. (Section 1).
- $H^2_{c_\tau}(G/K,W)$, π_{c_τ}, $\dot{\pi}_{c_\tau}(X)$, $X\in\mathcal{U}(\mathfrak{g})$, E_{c_τ}, $d(\pi^G_\lambda)$, K^c_τ, K^c_σ, K^c_S, K_S, K^c_T, K_T (Section 2).
- $(L^\tau_\cdot, V_\tau)$, $(L^\sigma_\cdot, V_\sigma)$, K_τ, $\mathcal{P}(\mathfrak{p}^+,W)$=$W$-valued holomorphic polynomial functions on $\mathfrak{p}^+$, $L^2_\tau(\mathcal{D},W)$ (Section 4).
- P_X orthogonal projector onto subspace X. I_X identity map on the set X.
- For a closed linear map R between Hilbert spaces, $R^\star$ is its adjoint.

- $M_{K-fin}(resp\, M^{\infty}, M^{H-\infty})$ $K-$finite vectors in M (smooth vectors in M, H-smooth vectors in M).
- dg, dh Haar measures on G, H.
- A unitary representation is *square integrable*, equivalently a *Discrete Series* representation, (resp. *integrable*) if some nonzero matrix coefficient is square integrable (resp. integrable) with respect to Haar measure on the group in question.
- $\Theta_{\pi_\mu^H}(...)$ Harish-Chandra character of the representation π_μ^H.
- M_{H-disc} is the closure of the linear subspace spanned by the totality of irreducible H-submodules. $M_{disc} := M_{G-disc}$.
- A representation M is H-*discretely decomposable* if $M_{H-disc} = M$.
- A representation is H-*admissible* if it is H-discretely decomposable and each isotypic component is equal to a finite sum of H-irreducible representations.
- $\mathcal{U}(\mathfrak{g})$ (resp. $\mathfrak{z}(\mathcal{U}(\mathfrak{g})) = \mathfrak{z}_{\mathfrak{g}}$) universal enveloping algebra of the Lie algebra $\mathfrak{g}$ (resp. center of universal enveloping algebra).
- $\mathrm{Cl}(X)$ =closure of the set X.
- $\mathbb{T} = S^1 = SO(2)$ one dimensional torus.
- $S^{(r)}(V)$ the r^{th}-symmetric power of the vector space V.
- $\cong$, $\equiv$ isomorphic.
- Cartan decomposition $Lie(G) = \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, θ Cartan involution associated to $\mathfrak{k}$, $\mathfrak{t}$ maximal abelian subalgebra for $\mathfrak{k}$.
- $\Phi(\mathfrak{g}, \mathfrak{t}) = \Phi(\mathfrak{g}) = \Phi$ root system attached to the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then, either $\mathfrak{g}_\alpha \subset \mathfrak{k}_{\mathbb{C}}$ (α is compact) or $\mathfrak{g}_\alpha \subset \mathfrak{p}_{\mathbb{C}}$ (α is noncompact).
- $\Phi_c = \Phi(\mathfrak{k}, \mathfrak{t})$ set of compact roots.
- $\Phi_n = \Phi(\mathfrak{p}, \mathfrak{t}) = \Phi_n(\mathfrak{g}) = \Phi^n(\mathfrak{g}) = \Phi_n^{\mathfrak{g}} = \Phi_{\mathfrak{g}}^n$ set of noncompact roots.
- For a system of positive roots $\Psi = \Psi_{\mathfrak{g}} = \Psi(\mathfrak{g})$ in $\Phi(\mathfrak{g}, \mathfrak{t})$:
  $\Psi_c := \Psi(\mathfrak{k}, \mathfrak{t}) := \Psi \cap \Phi_c = \Psi \cap \Phi(\mathfrak{k}, \mathfrak{t})$,
  $\Psi_n := \Psi_n(\mathfrak{g}) := \Psi_n^{\mathfrak{g}} := \Psi \cap \Phi_n = \Psi \cap \Phi(\mathfrak{p}, \mathfrak{t}) = \Psi \cap \Phi_n(\mathfrak{g}) = \Psi \cap \Phi^n(\mathfrak{g})$.
- A system of positive roots $\Psi \subset \Phi(\mathfrak{g}, \mathfrak{t})$ is *holomorphic* if on every irreducible component there is at most one noncompact simple root and each noncompact simple root has multiplicity at most one in maximal root.
- σ involution in $\mathfrak{g}$ that commutes with θ. $\mathfrak{h} = \{X \in \mathfrak{g} : \sigma(X) = X\}$, $\mathfrak{q} = \{X \in \mathfrak{g} : \sigma(X) = -X\}$, $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$, $\mathfrak{l} = \mathfrak{h} \cap \mathfrak{k}$, $\mathfrak{h}_0 = \mathfrak{l} + \mathfrak{q} \cap \mathfrak{p}$, $\mathfrak{u} = \mathfrak{t} \cap \mathfrak{k}$.
- Similar notation for the pairs $(\mathfrak{h}, \mathfrak{u}), (\mathfrak{h}_0, \mathfrak{u})$.

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