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Invariant geodesic orbit metrics on homogeneous spaces with intermediate subgroups

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Abstract. In this paper, we study G-g.o. metrics on compact homogeneous spaces G/H with an intermediate subgroup K such that $H \subset K \subset G$. In the beginning, we prove that the restricted metrics of g on K/H and G/K are both g.o. metrics under certain conditions when g is a G-g.o. metric on G/H. Then we develop several methods to determine G-g.o. metrics on G/H by the representations of K/H and G/K. As an application, we study g.o. metrics on a class of homogeneous spaces and find that $\mathrm{SO}(11)/(\mathrm{Spin}(7) \times \mathrm{SO}(2))$ admits non-naturally reductive g.o. metrics.

Mathematics Subject Classification 2020: 53C25, 53C30

Key Words and Phrases: geodesic orbit manifold, representation of a compact Lie group, principal isotropy subgroup

1. Introduction

The concept of a geodesic orbit space was introduced by O. Kowalski and L. Vanhecke as a natural generalization of a Riemannian symmetric space ([KV91]). A homogeneous Riemannian manifold $(G/K, g)$ is called a geodesic orbit space if any geodesic of G/K is an orbit of a one-parameter subgroup of G, and g is known as a geodesic orbit metric. Naturally reductive Riemannian spaces and weakly symmetric spaces are subclasses of geodesic orbit spaces.

During the past few decades, research on g.o. spaces has been quite fruitful. There are many studies on the classification of g.o. metrics on homogeneous spaces, such as homogeneous spaces fibering over symmetric spaces ([Tam99]), flag manifolds ([AA07]), compact homogeneous spaces with positive Euler characteristics ([AN09]), homogeneous spaces with two isotropy summands ([CN19]), compact homogeneous spaces with simple isotropy subgroups ([CNN23]), compact homogeneous spaces with abelian isotropy subgroups ([Sou21]), compact Lie groups ([CCD18; CCW18; Nik19]) and others ([ASS21a; ASS21b; AW17; AWZ17; CCZ21; CNN23; Gor96; NN19]). A good survey on the development in this field can be found in [Arv17]. A recent book by V.N. Berestovskii and Yu.G. Nikonorov [BN20] provides detailed descriptions of various objects related to geodesic orbit Riemannian spaces. Besides, Yu.G. Nikonorov provides a systematic study of the structures of compact geodesic orbit spaces ([Nik17]).

Inspired by the work of C.S. Gordon ([Gor96]), H. Tamaru ([Tam99]), and A. Arvanitoyeorgos, N.P. Souris, and M. Statha ([ASS21a]), we carry out systematic research on G-invariant g.o. metrics on G/H with a subgroup K such that $H \subset K \subset$

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G. Homogeneous bundle G/H is over the base space G/K with fiber K/H. In this case, on the Lie algebra level, $\mathfrak{g}$ admits the following B-orthogonal decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}_2 = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 = \mathfrak{h} \oplus \mathfrak{m}, \tag{1}$$

where $\mathfrak{h}$ and $\mathfrak{k}$ are Lie algebras of H and K, $\mathfrak{m}_i (i = 1, 2)$ are tangent spaces of K/H and G/K, B is the minus Killing form of $\mathfrak{g}$. Besides, the decomposition is also orthogonal with respect to the G-invariant metric g (i.e., the $\mathrm{Ad}(H)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$). That is to say,

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathfrak{m}_1} + \langle \cdot, \cdot \rangle_{\mathfrak{m}_2}, \tag{2}$$

where $\langle \cdot, \cdot \rangle_{\mathfrak{m}_i}$ represents the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{m}_i (i = 1, 2)$.

We first prove that if the $\mathrm{Ad}(H)$-invariant inner product $\langle \cdot, \cdot \rangle$ of the form (2) induces a g.o. metric on G/H, then $\langle \cdot, \cdot \rangle_{\mathfrak{m}_1}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{m}_2}$ can induce a K-g.o. metric on K/H and a G-g.o. metric on G/K respectively (see Lemma 4.2). Based on this result and isotropy representations of K/H and G/K, we provide several methods to determine G-g.o. metrics on G/H (see Lemma 4.4 and Corollary 4.5).

Furthermore, we study a class of homogeneous spaces G/H satisfying the following conditions:

- there exists a subgroup K such that K is simple, $H \subset K \subset G$, and G/K is symmetric.
- K/H is one of compact homogeneous spaces listed in Table 1 in [Tam99].

Based on the above discussion, we prove that most of them admit only standard G-g.o. metrics and one of them has non-naturally reductive g.o. metrics. Up to our knowledge, it is the first time to give a non-naturally reductive g.o. metric on $\mathrm{SO}(11)/(\mathrm{Spin}(7) \times \mathrm{SO}(2))$.

Here a G-invariant metric induced by $\langle \cdot, \cdot \rangle$ is called standard if the inner product is a constant multiple of $B|_{\mathfrak{m}}$. A standard metric is naturally reductive and non-standard G-g.o. metrics are equivalent to non-naturally reductive G-g.o. metrics for G simple.

2. Geodesic orbit space and its algebraic properties

This section is to give a detailed introduction to Riemannian geodesic orbit spaces and discuss their algebraic properties. Let $(M = G/H, g)$ be a compact homogeneous Riemannian manifold with G compact semisimple. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of the compact Lie groups G and H, respectively. Let B be the minus Killing form of $\mathfrak{g}$. Then $\mathfrak{g}$ admits a B-orthogonal *reductive decomposition*:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{3}$$

where $\mathfrak{m}$ is isomorphic to the tangent space of M at $o = eH$ via $X \to \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \exp\, tX \cdot o$. Any G-invariant metric g on M is a one-to-one correspondence to an $\mathrm{Ad}(H)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$. And any $\mathrm{Ad}(H)$-invariant inner product is a one-to-one correspondence to an endomorphism $A : \mathfrak{m} \to \mathfrak{m}$ by

$$\langle X, Y \rangle = B(AX, Y), \ \forall\, X, Y \in \mathfrak{m}, \tag{4}$$

which is called the *metric endomorphism*. Obviously, A is $\mathrm{Ad}(H)$-equivariant (hence $\mathrm{ad}(H)$-equivariant), symmetric with respect to B and positively definite.

A geodesic $\gamma(t)$ on G/H through the origin $o = eH$ is called G-homogeneous if it is an orbit of a one-parameter subgroup of G, i.e.,

$$\gamma(t) = \exp(tX) \cdot o, \quad t \in \mathbb{R}, \tag{5}$$

where X is a non-zero vector in $\mathfrak{g}$ and we call X a *geodesic vector*. Then we have the following result (Geodesic Lemma).

Lemma 2.1 ([KV91, Proposition 2.1]). *A nonzero vector $X \in \mathfrak{g}$ is a geodesic vector if and only if*

$$\langle [X, Y]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = 0, \quad \forall\, Y \in \mathfrak{m}$$

where the subscript $\mathfrak{m}$ denotes the projection into $\mathfrak{m}$.

The terminology of geodesic orbit manifold was introduced by O. Kowalski and L. Vanhecke in [KV91].

Definition 2.2. A Riemannian homogeneous space $(M = G/H, g)$ is called a G-geodesic orbit space (G-g.o. space in short) if all geodesics on G/H are G-homogeneous. In this case, the metric g is called G-geodesic orbit (G-g.o. metric in short). If G is the full isometry group, then $(M = G/H, g)$ is called a geodesic orbit manifold (g.o. manifold in short).

A useful description of homogeneous geodesics is the following theorem.

Theorem 2.3 ([AA07, Proposition 1]). *Let $(M = G/H, g)$ be a compact homogeneous Riemannian manifold and A the corresponding metric endomorphism. Let $Z \in \mathfrak{h}$ and $X \in \mathfrak{m}$. Then the following are equivalent:*

1. *$Z + X$ is a geodesic vector in $\mathfrak{g}$.*
2. *$[Z + X, A(X)] \in \mathfrak{h}$.*
3. *$\langle [Z, X], Y \rangle = \langle X, [X, Y]_{\mathfrak{m}} \rangle$ for all $Y \in \mathfrak{m}$.*
4. *$\langle [Z + X, Y]_{\mathfrak{m}}, X \rangle = 0$ for all $Y \in \mathfrak{m}$.*

As a consequence, the authors obtained the following characterization of geodesic orbit spaces.

Corollary 2.4 ([AA07, Corollary 2]). *Let $(G/H, g)$ be a homogeneous Riemannian manifold. Then $(G/H, g)$ is a geodesic orbit space if and only if for every $X \in \mathfrak{m}$ there exists some $Z \in \mathfrak{h}$ such that*

$$[Z + X, A(X)] \in \mathfrak{h}, \tag{6}$$

where A is the corresponding metric endomorphism on $\mathfrak{m}$.

This corollary could be strengthened by the following lemma:

Lemma 2.5 ([Sou18, Proposition 2]). *A Riemannian homogenous space $(G/K, g)$ is a geodesic orbit space if and only if for any $X \in \mathfrak{m}$ there exists $Z \in \mathfrak{k}$ such that*

$$[Z + X, A(X)] = 0. \tag{7}$$

Proof. By Corollary 2.4, we only have to prove $[X, AX] \in \mathfrak{m}$ because $[Z, AX] \in \mathfrak{m}$ always holds for any $Z \in \mathfrak{k}$. Since the metric endomorphism A is $\mathrm{ad}(\mathfrak{k})$-equivariant and the Killing form B is $\mathrm{Ad}(H)$-invariant and A-symmetric, we have for any $\widetilde{Z} \in \mathfrak{k}$,

$$\begin{aligned} B([X, AX], \widetilde{Z}) &= -B(X, [\widetilde{Z}, AX]) = -B(X, A[\widetilde{Z}, X]) \\ &= -B(AX, [\widetilde{Z}, X]) = -B([X, AX], \widetilde{Z}), \end{aligned} \tag{8}$$

which means $B([X, AX], \widetilde{Z}) = 0$ holds for any $\widetilde{Z} \in \mathfrak{k}$. Therefore, $[X, AX] \in \mathfrak{m}$. □

A homogeneous Riemannian manifold $(G/H, g)$ is called naturally reductive if there is an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that for all $X, Y, Z \in \mathfrak{m}$,

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X, Z]_{\mathfrak{m}}\rangle = 0, \tag{9}$$

or, equivalently, for all $X, Y \in \mathfrak{m}$,

$$\langle [X, Y]_{\mathfrak{m}}, X\rangle = 0. \tag{10}$$

It is well-known that naturally reductivity is equivalent to the geometrical property that for each vector $X \in \mathfrak{m}$, the orbit $\gamma(t) = \exp tX \cdot o$ is a geodesic. Therefore, naturally reductive Riemannian homogeneous spaces are geodesic orbit spaces, which can be also deduced from the algebraic equivalent conditions. In particular, G-invariant standard metrics on G/H must be G-g.o. metrics.

3. Principal isotropy subgroups of representations of compact simple Lie groups

In this section, we first recall the study on principal isotropy subgroups of representations of connected compact simple Lie groups, which plays an important role in the next section.

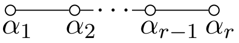
Let K be a compact linear Lie group acting linearly on a finite-dimensional vector space V. Then almost all points of V are situated on the orbits of K, which are pairwise isomorphic as K-manifolds. These orbits are called *orbits in general position*. The isotropy subgroups of all points in all orbits of general position are conjugated in K. Their conjugation class is called *a principal isotropy subgroup* for the linear Lie group K and the corresponding Lie algebra is called *a principal isotropy subalgebra* or *a stationary subalgebra* of points in general position (see [CN19; Èla72; HH70] and the reference therein).

Definition 3.1 ([HH70]). Let ρ be a real irreducible representation of a simple Lie group K. We say that a principal isotropy subgroup (H_ρ) (H_ρ is a representative in the conjugacy class (H_ρ)) is trivial if $H_\rho = \mathrm{Ker}(\rho) \subset Z(K)$, where $Z(K)$ is the center of K.

The classification of representations of connected compact simple Lie groups with non-trivial connected principal isotropy subgroups has been given in [Èla72; HH70] and [Krä66]. For readers' convenience, we will list the classification results in the following.

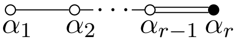
Theorem 3.2 ([HH70, Table A]). *All real irreducible representations of simple Lie groups with non-trivial principal isotropy subgroups* (H_ρ) *are listed in the following tables.*

I. $\mathfrak{g} = A_r$:



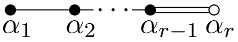
rank r	The representation ρ	The group G	(H_ρ)
$r=1$	$S^2\varphi_1$	$\mathrm{SO}(3)$	maximal tori
	$S^4\varphi_1$	$\mathrm{SO}(3)$	maximal $\mathbb{Z}_2$-tori$\approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$
$r \geq 2$	$\varphi_1 + \varphi_r$	$\mathrm{SU}(r+1)$	$\mathrm{SU}(r)$
	$S^2\varphi_1 + S^2\varphi_r$	$\mathrm{SU}(r+1)$	maximal $\mathbb{Z}_2$-tori
	$\varphi_1 \otimes \varphi_r$	$\mathrm{SU}(r+1)$	maximal tori
$r=3$	φ_2	$\mathrm{SO}(6)$	$\mathrm{SO}(5)$
	$S^2\varphi_2 - \theta$	$\mathrm{SO}(6)$	maximal $\mathbb{Z}_2$-tori
$r \geq 4$ even	$\varphi_2 + \varphi_{r-1}$	$\mathrm{SU}(r+1)$	$\mathrm{SU}(2) \times \cdots \times \mathrm{SU}(2)$ ($[\frac{r+1}{2}]$ copies)
$r > 4$ odd	$\varphi_2 + \varphi_{r-1}$	$\mathrm{SU}(r+1)/\mathbb{Z}_2$	$(\mathrm{SU}(2)\times\cdots\times\mathrm{SU}(2))/\mathbb{Z}_2$ ($\frac{r+1}{2}$ copies)
$r=5$	$2\varphi_3$	$\mathrm{SU}(6)$	$T^2 \subset \mathrm{SU}(3) \times \mathrm{SU}(3) \subset \mathrm{SU}(6)$
$r=7$	φ_4	$\mathrm{SU}(8)$	a finite group at least of order $128 \subset \mathrm{SU}(4) \times \mathrm{SU}(4)$

II. $\mathfrak{g} = B_r (r \geq 2)$:



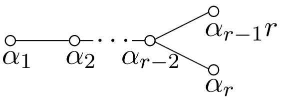
rank r	The representation ρ	The group G	(H_ρ)
$r \geq 2$	φ_1	$\mathrm{SO}(2r+1)$	$\mathrm{SO}(2r)$
	$S^2\varphi_1 - \theta$	$\mathrm{SO}(2r+1)$	maximal $\mathbb{Z}_2$-tori
	$\Lambda^2\varphi_1$	$\mathrm{SO}(2r+1)$	maximal tori
$r=2$	$2\varphi_2$	$\mathrm{Spin}(5) = \mathrm{Sp}(2)$	$\mathrm{Sp}(1)$
$r=3$	φ_3	$\mathrm{Spin}(7)$	G_2
$r=4$	φ_4	$\mathrm{Spin}(9)$	$\mathrm{Spin}(7)$ (with $\mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}$)

III. $\mathfrak{g} = C_r (r \geq 3)$:



rank r	The representation ρ	The group G	(H_ρ)
$r \geq 3$	$2\varphi_1$	$\mathrm{Sp}(r)$	$\mathrm{Sp}(r-1)$
	$S^2\varphi_1$	$\mathrm{Sp}(r)$	maximal tori
	$\Lambda^2\varphi_1 - \theta$	$\mathrm{Sp}(r)/\mathbb{Z}_2$	$(\mathrm{Sp}(1) \times \cdots \times \mathrm{Sp}(1))/\mathbb{Z}_2$ (r copies)
$r=3$	$2\varphi_3$	$\mathrm{Sp}(3)$	discrete subgroups isomorphic to $(\mathbb{Z}_2)^2$
$r=4$	φ_4	$\mathrm{Sp}(4)$	discrete subgroup of order 16

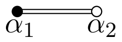
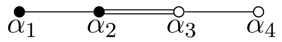
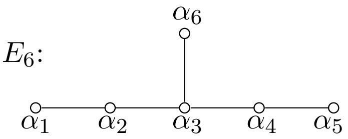
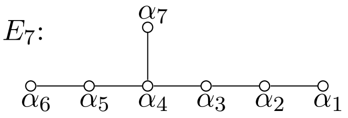
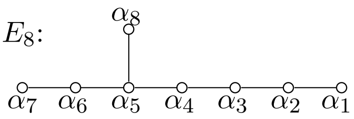
IV. $\mathfrak{g} = D_r (r \geq 4)$:



(There are automorphisms which interchange φ_{r-1}, φ_r or $\varphi_1, \varphi_3, \varphi_4$ in case $r = 4$)

rank r	The representation ρ	The group G	(H_ρ)
$r \geq 4$	φ_1	SO$(2r)$	SO$(2r-1)$
	φ_2	SO$(2r)$	maximal tori
	$S^2\varphi_1 - \theta$	SO$(2r)$	maximal $\mathbb{Z}_2$-tori
$r = 5$	$\varphi_4 + \varphi_5$	Spin(10)	SU(4)
$r = 6$	$2\varphi_5$ or $2\varphi_6$	Spin(12)	SU$(2) \times$ SU$(2) \times$ SU(2)
$r = 7$	$\varphi_6 + \varphi_7$	Spin(14)	**TRIVIAL**
$r = 8$	φ_7 or φ_8	Spin(16)	a finite group

V. Exceptional Lie groups

The Lie algebra	The representation ρ	(H_ρ)
G_2: 	φ_1	maximal tori
	φ_2	SU(3)(SU$(3)/G_2 \approx S^6$)
F_4: 	φ_1	maximal tori
	φ_4(dim $\varphi_4 = 26$)	Spin$(8) \subset$ Spin$(9) \subset F_4$
E_6: 	φ_6	maximal tori
	$\varphi_1 + \varphi_5$	Spin$(8) \subset F_4 \subset E_6$
E_7: 	φ_6	maximal tori
	$2\varphi_1$	Spin$(8) \subset E_6 \subset E_7$
E_8: 	Ad(E_8)	maximal tori

Remark 3.3. In Case IV, where $r = 7$, the principal isotropy subgroup (H_ρ) was previously identified as a finite group in [HH70]. However, we now establish that it is, in fact, trivial. This correction was made following our referee's valuable observation, which can be demonstrated through several approaches.

First, as demonstrated in Remark 3.2 of [Goz21]. Second, by complexifying the system to Spin$(14, \mathbb{C}), \mathbb{C}^{64} \oplus \mathbb{C}^{64}$, where it is established that the generic stabilizer on a single copy of $\mathbb{C}^{64}$ is reductive and isomorphic to $G_2(\mathbb{C}) \times G_2(\mathbb{C})$ ([GG17]). The action on the complementary copy, represented as $\mathbb{C}\oplus(\mathbb{C}^7\oplus\mathbb{C}^7)\oplus\mathbb{C}^7\otimes\mathbb{C}^7$, exhibits a trivial generic stabilizer. Additionally, this result is corroborated by the observation that the orbit space of $\mathrm{Spin}(14)$ on $\mathbb{C}^{64}$ has an empty boundary ([GKW24]).

4. Geodesic orbit metrics on homogeneous spaces with intermediate subgroups

In this section, we consider compact homogeneous spaces G/H with intermediate subgroups, i.e., there exists a subgroup K such that $H\subset K\subset G$. These homogeneous spaces can be viewed as total spaces G/H over homogeneous spaces G/K with fibers being homogeneous spaces K/H, i.e.,

$$K/H\to G/H\to G/K. \tag{11}$$

Denote by $\mathfrak{g},\mathfrak{k}$ and $\mathfrak{h}$ the corresponding Lie algebras of G, K and H respectively. Suppose G is semisimple, then B, the minus Killing form of $\mathfrak{g}$, is strictly positive definite. Besides, these structures can determine a B-orthogonal reductive decomposition of $\mathfrak{g}$:

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}_2=\underbrace{\mathfrak{h}\oplus\mathfrak{m}_1}_{\mathfrak{k}}\oplus\mathfrak{m}_2=\mathfrak{h}\oplus\mathfrak{m}, \tag{12}$$

where $\mathfrak{m}$ is isomorphic to the tangent space of G/H at eH, $\mathfrak{m}_1$ is isomorphic to the tangent space of K/H at eH and $\mathfrak{m}_2$ is isomorphic to the tangent space of G/K at eK.

In what follows, we will show how the GO-property affects the restrictions of G-g.o. metrics and K-g.o. metrics on homogeneous spaces G/K and K/H.

Lemma 4.1. *Assume that $(G/H,g)$ is a geodesic orbit space and $\langle\mathfrak{m}_1,\mathfrak{m}_2\rangle=0$. Then, $\langle\cdot,\cdot\rangle_{\mathfrak{m}_2}$, the restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{m}_2$, is not only $\mathrm{Ad}(H)$-invariant but also $\mathrm{Ad}(K)$-invariant.*

Proof. Since $\langle\cdot,\cdot\rangle_{\mathfrak{m}_2}$ is obviously $\mathrm{Ad}(H)$-invariant, it suffices to prove for any $Y\in\mathfrak{m}_1$, $\langle[Y,X],X\rangle_{\mathfrak{m}_2}=0$ holds for any $X\in\mathfrak{m}_2$. In fact, by the hypothesis, the $\mathrm{Ad}(H)$-invariant inner product $\langle\cdot,\cdot\rangle$ induces a G-g.o. metric on G/H, then by Theorem 2.3, for any $X\in\mathfrak{m}_2\subset\mathfrak{m}$, there exists some $Z\in\mathfrak{h}\subset\mathfrak{k}$ such that for any $Y\in\mathfrak{m}_1\subset\mathfrak{m}$, $0=\langle[Z+X,Y]_{\mathfrak{m}},X\rangle=\langle[Z,Y]_{\mathfrak{m}_1},X\rangle+\langle[X,Y]_{\mathfrak{m}_2},X\rangle=\langle[X,Y],X\rangle_{\mathfrak{m}_2}=0$. ☐

Lemma 4.2. *Suppose the $\mathrm{Ad}(H)$-invariant inner product $\langle\cdot,\cdot\rangle$ induces a G-g.o. metric on G/H. Then*

(1) *The restriction $\langle\cdot,\cdot\rangle_{\mathfrak{m}_1}$ of $\langle\cdot,\cdot\rangle$ to $\mathfrak{m}_1$ induces a K-g.o. metric on K/H;*
(2) *When $\langle\mathfrak{m}_1,\mathfrak{m}_2\rangle=0$, the restriction $\langle\cdot,\cdot\rangle_{\mathfrak{m}_2}$ of $\langle\cdot,\cdot\rangle$ to $\mathfrak{m}_2$ induces a G-g.o. metric on G/K.*

Proof. Since the $\mathrm{Ad}(H)$-invariant inner product $\langle\cdot,\cdot\rangle$ induces a G-g.o. metric on G/H, by Theorem 2.3, for any $X\in\mathfrak{m}_1\subset\mathfrak{m}$, there exists some $Z\in\mathfrak{h}$ such that for any $Y\in\mathfrak{m}_1\subset\mathfrak{m}$, $\langle[Z+X,Y]_{\mathfrak{m}},X\rangle=\langle[Z+X,Y]_{\mathfrak{m}_1},Y\rangle_{\mathfrak{m}_1}=0$. Again by Theorem 2.3, the $\mathrm{Ad}(H)$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{m}_1}$ induces a K-g.o. metric on K/H. Similarly, for any $X\in\mathfrak{m}_2$, there exists some $Z\in\mathfrak{h}\subset\mathfrak{k}$ such that for any $Y\in\mathfrak{m}_2$, $\langle[Z+X,Y]_{\mathfrak{m}},X\rangle=\langle[Z+X,Y]_{\mathfrak{m}_2},X\rangle_{\mathfrak{m}_2}=0$. By Lemma 4.1 and Theorem 2.3, the $\mathrm{Ad}(K)$-invariant $\langle\cdot,\cdot\rangle_{\mathfrak{m}_2}$ induces a G-g.o. metric on G/K. ☐

Assume that $\mathrm{Ad}(H)$ and $\mathrm{Ad}(K)$-modules $\mathfrak{m}_1$ and $\mathfrak{m}_2$ admit the following B-orthogonal decompositions:

$$\mathfrak{m}_1 = \mathfrak{m}_1^1 \oplus \mathfrak{m}_1^2 \oplus \cdots \oplus \mathfrak{m}_1^s, \quad \mathfrak{m}_2 = \mathfrak{m}_2^1 \oplus \mathfrak{m}_2^2 \oplus \cdots \oplus \mathfrak{m}_2^t, \tag{13}$$

where $\mathfrak{m}_1^i (i = 1, 2, \ldots, s)$ are irreducible $\mathrm{Ad}(H)$-modules, $\mathfrak{m}_2^j (j = 1, 2, \ldots, t)$ are irreducible $\mathrm{Ad}(K)$-modules.

Assume $\langle \mathfrak{m}_1, \mathfrak{m}_2 \rangle = 0$ and all G-g.o. metrics on G/K and K-g.o. metrics on K/H are diagonal with respect to the decompositions (13). Then, by Lemma 4.2, any G-g.o. metric on G/H must be of the following form:

$$\langle \cdot, \cdot \rangle = x_1 B|_{\mathfrak{m}_1^1} + x_2 B|_{\mathfrak{m}_1^2} + \cdots + x_s B|_{\mathfrak{m}_1^s} + y_1 B|_{\mathfrak{m}_2^1} + y_2 B|_{\mathfrak{m}_2^2} + \cdots + x_t B|_{\mathfrak{m}_2^t}, \tag{14}$$

where $x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_t \in \mathbb{R}^+$.

Remark 4.3. A G-g.o. metric on G/H with respect to the triple (G, K, H) is called *orthogonal* if its corresponding $\mathrm{Ad}(H)$-invariant inner product $\langle \cdot, \cdot \rangle$ satisfies $\langle \mathfrak{m}_1, \mathfrak{m}_2 \rangle = 0$. Throughout this paper, we only discuss orthogonal G-g.o. metrics on G/H.

In what follows, we will show how the properties of irreducible modules $\mathfrak{m}_1^i, \mathfrak{m}_2^j (i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, t)$ determine the form (14) of the G-g.o. metric on G/H.

Lemma 4.4. *Let the notation be as above. Assume that $(G/H, g)$ is a G-g.o. space.*

(1) *If the principal isotropy subalgebra of $\mathrm{Ad}(H)$-module $\mathfrak{m}_1^i$ is trivial and $[\mathfrak{m}_1^i, \mathfrak{m}_2^j] \neq \{0\}$ for some i, j, then $x_i = y_j$.*
(2) *If there exists a proper $\mathrm{Ad}(H)$-submodule $\mathfrak{n}_2^j$ contained in $\mathfrak{m}_2^j$ such that $[\mathfrak{m}_1^i, \mathfrak{n}_2^j] \nsubseteq \mathfrak{n}_2^j$ for some i, j, then $x_i = y_j$.*
(3) *If the principal isotropy subalgebra of $\mathrm{Ad}(K)$-module $\mathfrak{m}_2^j$ is trivial for some j, then $x_1 = x_2 = \cdots = x_t = y_j$.*

Proof. Since $(G/H, g)$ is a G-g.o. space, by Lemma 2.5, for any $X_i \in \mathfrak{m}_1^i, Y_j \in \mathfrak{m}_2^j$, there exists some $Z \in \mathfrak{h}$ such that $[Z + X_i + Y_j, x_i X_i + y_j Y_j] = 0$, which is equivalent to

(a) $[Z, X_i] = 0$;
(b) $y_j [Z, Y_j] = (x_i - y_j)[X_i, Y_j]$.

(1). There are $X_i \in \mathfrak{m}_1^i$ and $Y_j \in \mathfrak{m}_2^j$ such that $[X_i, Y_j] \neq 0$, and X_i is in the orbit in general position. This indicates that $Z = 0$ and $x_i = y_j$.

(2). There are $X_i \in \mathfrak{m}_1^i$ and $Y_j \in \mathfrak{n}_2^j$ such that $[X_i, Y_j]$ doesn't belong to $\mathfrak{n}_2^j$, while the left hand of the identity (b) belongs to $\mathfrak{n}_2^j$. This implies that $x_i = y_j$.

(3). If $x_i \neq y_j$, then the identities (a) and (b) are equivalent to

(A) $[Z, X_i] = 0$;
(B) $[\frac{y_j}{y_j - x_i} Z + X_i, Y_j] = 0$.

Since the principal isotropy subalgebra of $\mathrm{Ad}(K)$-module $\mathfrak{m}_2^j$ is trivial, for any $X_i \in \mathfrak{m}_1^i$ we can choose some $Y_j \in \mathfrak{m}_2^j$ in an orbit in general position. By the hypothesis, $\frac{y_j}{y_j - x_i} Z + X_i = 0$, which implies $Z = 0$ and $X_i = 0$, which is a contradiction. □

Corollary 4.5. *Assume that G/K and K/H only admit standard g.o. metrics and $\langle \mathfrak{m}_1, \mathfrak{m}_2 \rangle = 0$. If some $\mathrm{Ad}(K)$-irreducible module $\mathfrak{m}_2^j$ is $\mathrm{Ad}(H)$-reducible, then all G-g.o. metrics on G/H are standard.*

Proof. By Lemma 4.2, any G-g.o. metrics on G/H must be of the following form:

$$\langle \cdot, \cdot \rangle = xB|_{\mathfrak{m}_1} + yB|_{\mathfrak{m}_2}, \qquad x, y \in \mathbb{R}^+. \tag{15}$$

If $x \neq y$, then by Lemma 4.4 (2), $[\mathfrak{m}_1, \mathfrak{n}_2^j] \subset \mathfrak{n}_2^j$ for any proper $\mathrm{Ad}(H)$-submodule $\mathfrak{n}_2^j$ contained in $\mathfrak{m}_2^j$. This implies $\mathfrak{n}_2^j \subsetneq \mathfrak{m}_2^j$ is also $\mathrm{Ad}(K)$-irreducible, which is a contradiction. □

5. Some examples

In this section, we study g.o. metrics on some homogeneous spaces based on the above discussion.

5.1.

Here we consider the homogeneous spaces $\mathrm{SO}(n)/\mathrm{SO}(n_1) \times \cdots \times \mathrm{SO}(n_s)$ with $n_0 = n - (n_1 + n_2 + \cdots + n_s) > 1$ and $n_0, n \neq 4$, which can be viewed as a total space over a Stiefel manifold, with the fiber being a real flag manifold, e.g.,

$$\begin{aligned} &\mathrm{SO}(m)/\mathrm{SO}(n_1) \times \cdots \times \mathrm{SO}(n_s) \\ &\qquad \longrightarrow \mathrm{SO}(n)/\mathrm{SO}(n_1) \times \cdots \times \mathrm{SO}(n_s) \longrightarrow \mathrm{SO}(n)/\mathrm{SO}(m) \end{aligned} \tag{16}$$

with $m = n_1 + \cdots + n_s$. In this case, $G = \mathrm{SO}(n), K = \mathrm{SO}(n_1 + n_2 + \cdots + n_s)$ and $H = \mathrm{SO}(n_1) \times \cdots \times \mathrm{SO}(n_s)$. The corresponding decomposition of Lie algebra $\mathrm{so}(n)$ is as follows:

$$\mathrm{so}(n) = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 = \mathrm{so}(n_1) \oplus \cdots \oplus \mathrm{so}(n_s) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2. \tag{17}$$

Assume the $\mathrm{Ad}(\mathrm{SO}(n_1) \times \cdots \times \mathrm{SO}(n_s))$-invariant inner product $\langle \cdot, \cdot \rangle$ is G-geodesic orbit, by Lemma 4.2, its restriction to $\mathfrak{m}_1$ induces an $\mathrm{SO}(m)$-g.o. metric on the real flag manifold $\mathrm{SO}(m)/\mathrm{SO}(n_1) \times \cdots \times \mathrm{SO}(n_s)$, and the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{m}_2$ induces a $\mathrm{SO}(n)$-g.o. metric on the Stiefel manifold $\mathrm{SO}(n)/\mathrm{SO}(m)$. According to [AA07; ASS21a], any $\mathrm{SO}(m)$-g.o. metric on the real flag manifold $\mathrm{SO}(m)/\mathrm{SO}(n_1) \times \mathrm{SO}(n_2) \times \cdots \times \mathrm{SO}(n_s)$ is standard and any $\mathrm{SO}(n)$-g.o. metric on the Stiefel manifold $\mathrm{SO}(n)/\mathrm{SO}(m)$ is also standard. Clearly, we have $\langle \mathfrak{m}_1, \mathfrak{m}_2 \rangle = 0$. Hence,

$$\langle \cdot, \cdot \rangle = \lambda B|_{\mathfrak{k}} + xB|_{\mathfrak{m}}, \lambda, x \in \mathbb{R}^+. \tag{18}$$

If described in matrix form, the upper triangular part of $\mathrm{so}(n)$ is

$$\mathrm{so}(n) = \begin{bmatrix} \mathrm{so}(n_0) & \mathfrak{m}_{01} & \mathfrak{m}_{02} & \mathfrak{m}_{03} & \cdots & \mathfrak{m}_{0s} \\ & \mathrm{so}(n_1) & \mathfrak{m}_{12} & \mathfrak{m}_{13} & \cdots & \mathfrak{m}_{1s} \\ & & \mathrm{so}(n_2) & \mathfrak{m}_{23} & \cdots & \mathfrak{m}_{2s} \\ & * & & \ddots & \vdots & \vdots \\ & & & & & \mathrm{so}(n_s) \end{bmatrix}. \tag{19}$$

Consider the space $\mathfrak{n}_{02} = \mathrm{Span}_{\mathbb{R}}\{E_{1,n_0+n_1+1} - E_{n_0+n_1+1,1},\ E_{1,n_0+n_1+2} - E_{n_0+n_1+2,1}, \ldots,\ E_{1,n_0+n_1+n_2} - E_{n_0+n_1+n_2,1}\} \subsetneq \mathfrak{m}_{02}$, which is a proper $\mathrm{Ad}(H)$-submodule of $\mathfrak{m}_2$. A direct calculation shows that $[\mathfrak{m}_1, \mathfrak{n}_{02}] \not\subseteq \mathfrak{n}_{02}$. By Lemma 4.4, we can deduce $\lambda = x$. That is to say, any $\mathrm{SO}(n)$-invariant g.o. metric on $\mathrm{SO}(n)/\mathrm{SO}(n_1) \times \cdots \times \mathrm{SO}(n_s)$ with $n_0 = n - (n_1 + n_2 + \cdots + n_s) > 1$ and $n_0, n \neq 4$ is standard. In fact $\mathrm{SO}(n)$-g.o. metrics on $\mathrm{SO}(n)/\mathrm{SO}(n_1) \times \cdots \times \mathrm{SO}(n_s)$ were originally investigated in [ASS21b].

5.2.

Here we consider the compact homogeneous space G/H with $H \subset K \subset G$, where G/K is a non-symmetric strongly isotropy irreducible space of compact type with G simple and K/H is a strongly isotropy irreducible space of compact type. Assume $\langle \mathfrak{m}_1, \mathfrak{m}_2 \rangle = 0$, then by Lemma 4.2, any G-g.o. metric on G/H must be of the following form

$$\langle \cdot, \cdot \rangle = xB|_{\mathfrak{m}_1} + yB|_{\mathfrak{m}_2}, \qquad x, y \in \mathbb{R}^+. \tag{20}$$

If $\mathfrak{m}_2$ is $\mathrm{Ad}(H)$-reducible, then, by Corollary 4.5, all G-g.o. metrics on G/H must be standard. If $\mathfrak{m}_2$ is $\mathrm{Ad}(H)$-irreducible, then G/H is a compact homogeneous space with two irreducible submodules, where the classification of G-g.o. metrics has been obtained by Z. Chen and Yu. Nikonorov ([CN19, Theorem 2]). In particular, if G/H can admit non-standard g.o. metrics, then G/K is a symmetric space.

That is, let G/H be the compact homogeneous space with $H \subset K \subset G$ where G/K is a non-symmetric strongly isotropy irreducible space of compact type for G simple and K/H is a strongly isotropy irreducible space of compact type, then all the G-g.o. metrics of the form (20) is standard.

5.3.

H. Tamaru studied orthogonal G-g.o. metrics on a class of homogeneous spaces G/H fibered over irreducible symmetric spaces. He discovered 15 classes of such homogeneous spaces that admit non-standard G-g.o. metrics ([Tam99]) with the form $\langle \cdot, \cdot \rangle = xB|_{\mathfrak{m}_1} + yB|_{\mathfrak{m}_2}$ for $x, y \in \mathbb{R}^+$.

In this subsection, we will focus on the homogeneous spaces G/H where G/H is fibered over an irreducible symmetric space G/K with K simple, and K/H is one of the compact homogeneous spaces listed in Table 1 in [Tam99]. The complete classification appears below (see Table 1), and we will determine g.o. metrics except cases (1b), (7b) and (9b).

For any case, we have $H \subset K_1 \subset K \subset G$, where G/K is a symmetric space and K/H is a compact homogeneous space fibered over an irreducible symmetric space K/K_1. By Lemma 4.2, any orthogonal G-g.o. metric on G/H must be of the following form:

$$\langle \cdot, \cdot \rangle = x_{11}B|_{\mathfrak{m}_{11}} + x_{12}B|_{\mathfrak{m}_{12}} + x_2B|_{\mathfrak{m}_2} \tag{21}$$

The isotropy representation of the irreducible symmetric space $\mathrm{SU}(n)/\mathrm{SO}(n)$ is $\rho = 2\phi_1$ for $n \geq 5$, and that of $\mathrm{SU}(2n)/\mathrm{Sp}(n)$ is $\rho = \phi_1$ for $n \geq 2$. By Theorem 3.2, the principal isotropy subgroups of these isotropy representations are trivial. According to Lemma 4.4(3), we have $x_{11} = x_{12} = x_2$. This implies that homogeneous spaces G/H in Table 1 of Nos. (1a), (2a), (3), (4), (5a), (6a), (7a), (8a) and (9a) only admit standard G-g.o. metrics. For the cases (2b), (5b), and (6b), according to the result in [CNN23] for H simple, $\mathrm{SO}(4n+2)/\mathrm{SU}(2n)$ for $n \geq 1$

Table 1. Certain homogeneous spaces fiber over symmetric spaces

No.	H	K_1	K	G	Remark
(1a)	$\mathrm{U}(n)$	$\mathrm{SO}(2n)$	$\mathrm{SO}(2n+1)$	$\mathrm{SU}(2n+1)$	$n \geq 2$
(1b)	$\mathrm{U}(n)$	$\mathrm{SO}(2n)$	$\mathrm{SO}(2n+1)$	$\mathrm{SO}(2n+2)$	$n \geq 2$
(2a)	$\mathrm{SU}(2n)$	$\mathrm{SO}(4n)$	$\mathrm{SO}(4n+1)$	$\mathrm{SU}(4n+1)$	$n \geq 1$
(2b)	$\mathrm{SU}(2n)$	$\mathrm{SO}(4n)$	$\mathrm{SO}(4n+1)$	$\mathrm{SO}(4n+2)$	$n \geq 1$
(3)	$\mathrm{U}(1)\,\mathrm{Sp}(n)$	$\mathrm{Sp}(n) \times \mathrm{Sp}(1)$	$\mathrm{Sp}(n+1)$	$\mathrm{SU}(2n+2)$	$n \geq 1$
(4)	$\mathrm{Sp}(n)$	$\mathrm{Sp}(n) \times \mathrm{Sp}(1)$	$\mathrm{Sp}(n+1)$	$\mathrm{SU}(2n+2)$	$n \geq 1$
(5a)	$\mathrm{SU}(2n+1)$	$\mathrm{U}(2n+1)$	$\mathrm{SO}(4n+2)$	$\mathrm{SU}(4n+2)$	$n \geq 2$
(5b)	$\mathrm{SU}(2n+1)$	$\mathrm{U}(2n+1)$	$\mathrm{SO}(4n+2)$	$\mathrm{SO}(4n+3)$	$n \geq 2$
(6a)	$\mathrm{Spin}(7)$	$\mathrm{SO}(8)$	$\mathrm{SO}(9)$	$\mathrm{SU}(9)$	
(6b)	$\mathrm{Spin}(7)$	$\mathrm{SO}(8)$	$\mathrm{SO}(9)$	$\mathrm{SO}(10)$	
(7a)	$\mathrm{G}_2 \times \mathrm{SO}(2)$	$\mathrm{SO}(7) \times \mathrm{SO}(2)$	$\mathrm{SO}(9)$	$\mathrm{SU}(9)$	
(7b)	$\mathrm{G}_2 \times \mathrm{SO}(2)$	$\mathrm{SO}(7) \times \mathrm{SO}(2)$	$\mathrm{SO}(9)$	$\mathrm{SO}(10)$	
(8a)	$\mathrm{Spin}(7) \times \mathrm{SO}(2)$	$\mathrm{SO}(8) \times \mathrm{SO}(2)$	$\mathrm{SO}(10)$	$\mathrm{SU}(10)$	
(8b)	$\mathrm{Spin}(7) \times \mathrm{SO}(2)$	$\mathrm{SO}(8) \times \mathrm{SO}(2)$	$\mathrm{SO}(10)$	$\mathrm{SO}(11)$	
(9a)	$\mathrm{Spin}(7) \times \mathrm{SO}(3)$	$\mathrm{SO}(8) \times \mathrm{SO}(3)$	$\mathrm{SO}(11)$	$\mathrm{SU}(11)$	
(9b)	$\mathrm{Spin}(7) \times \mathrm{SO}(3)$	$\mathrm{SO}(8) \times \mathrm{SO}(3)$	$\mathrm{SO}(11)$	$\mathrm{SO}(12)$	

and $\mathrm{SO}(4n+3)/\mathrm{SU}(2n+1)$ for $n \geq 2$ only admit standard G-g.o. metrics, and $\mathrm{SO}(10)/\mathrm{Spin}(7)$ admits non-naturally reductive G-g.o. metrics.

For the Case (8b), we have $\mathrm{Spin}(7)\times\mathrm{SO}(2) \subset \mathrm{SO}(8)\times\mathrm{SO}(2) \subset \mathrm{SO}(10) \subset \mathrm{SO}(11)$. According to Table 1 in [CNN23], the compact homogeneous space $\mathrm{SO}(11)/\mathrm{Spin}(7)$ admits the following $\mathrm{Ad}(\mathrm{Spin}(7))$-invariant decomposition:

$$\mathrm{so}(11) = \mathrm{so}(7) \oplus \mathfrak{m}_1' \oplus \mathfrak{m}_2' = \mathrm{so}(7) \oplus \mathfrak{m}_{11} \oplus \underbrace{\mathrm{so}_2 \oplus \mathfrak{m}_{12} \oplus \mathfrak{m}_2}_{\mathfrak{m}_2'}, \tag{22}$$

and for any $x, y \in \mathbb{R}^+$, the $\mathrm{Ad}(\mathrm{Spin}(7))$-invariant metric $\langle\cdot,\cdot\rangle' = xB|_{\mathfrak{m}_1'} + yB|_{\mathfrak{m}_2'}$ is G-g.o. When considering $\mathrm{Spin}(7) \subset \mathrm{Spin}(7) \times \mathrm{SO}(2) \subset \mathrm{SO}(11)$, by Lemma 4.2, the restriction of $\langle\cdot,\cdot\rangle'$ to $\mathrm{SO}(11)/(\mathrm{Spin}(7) \times \mathrm{SO}(2))$, i.e. $\langle\cdot,\cdot\rangle = xB|_{\mathfrak{m}_{11}} + yB|_{\mathfrak{m}_{12}\oplus\mathfrak{m}_2}$ induces a non-naturally reductive G-g.o. metric. That is, $\mathrm{SO}(11)/(\mathrm{Spin}(7) \times \mathrm{SO}(2))$ admits non-naturally reductive G-g.o. metrics.

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