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Extrinsically Symmetric Spaces, Submanifolds of Clifford Type and a Theorem of Harish-Chandra

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Abstract. We prove that a compact, intrinsically symmetric submanifold of an Euclidean space is extrinsically symmetric if and only if its maximal tori are Clifford tori in the ambient space. Moreover, we show that this result can be used to give a geometric proof of a result of Harish-Chandra on strongly orthogonal roots in semisimple Lie algebras.

Mathematics Subject Classification 2020: 53C35, 53C40

Key Words and Phrases: extrinsically symmetric spaces, Clifford tori, Clifford type, strongly orthogonal roots

To the memory of Joseph A. Wolf

Introduction

A closed submanifold X of a Euclidean space E is called of *Clifford type* if every geodesic in X is contained in a totally geodesic submanifold of X which is a Clifford torus in E. For example, X is of Clifford type if all its geodesics are planar circles in E. It is easy to see (Proposition 2.2) that submanifolds of Clifford type are compact *extrinsically symmetric spaces*, that is they are compact submanifolds which are invariant under the orthogonal reflections along their affine normal spaces. By Theorem 2.5 also the converse holds. Surprisingly, this result is closely related to a theorem of Harish-Chandra on the existence of strongly orthogonal roots of noncompact type in semisimple Lie algebras. We show in fact how Harish-Chandra's result can be deduced from Theorem 2.5 by an explicit geometric construction and we indicate also a proof of the opposite direction.

1. Clifford tori

Let $r \geq 0$ be an integer and $\rho_1, \dots, \rho_r$ be positive real numbers. We call

$$C := C_{\rho_1 \dots \rho_r} = \{z = (z_1, \dots, z_r) \in \mathbb{C}^r \,:\, |z_j| = \rho_j \;\text{ for all }\; j = 1, \dots, r\}$$

a *standard Clifford torus*, generalizing the classical case $r = 2$ and $\rho_1 = \rho_2 = 1$. If $r = 0$ we have $C = \{0\}$. The inner product $\langle z, w\rangle = \mathrm{Re}(\sum_{j=1}^r z_j \overline{w_j})$ turns $\mathbb{C}^r$ into a Euclidean space and C becomes an extrinsically symmetric submanifold of $\mathbb{C}^r$ as round circles in planes are extrinsically symmetric and C in $\mathbb{C}^r$ is a product of such submanifolds. For each $w = (w_1, \dots, w_r) \in C$ the circles

$$C_k(w) = \{z \in \mathbb{C}^r \,:\, |z_k| = \rho_k \quad\text{and}\quad z_j = w_j \;\text{ for all }\; k \neq j\},$$

$k = 1, \dots r$, are called *generating circles* of C at w. They are contained in the pairwise orthogonal affine planes $\mathbb{C}_k(w) := \{z \in \mathbb{C}^r :\ z_j = w_j \text{ for all } k \neq j\}$ and C is isometric to their product.

More generally, a submanifold C of a Euclidean space E is called a *Clifford torus* if it is the image of a standard Clifford torus C' in $\mathbb{C}^r$ under an affine isometric (not necessarily surjective) map $\varphi : \mathbb{C}^r \to E$. Fixing φ we have for each $p = \varphi(w) \in C$ the generating circles $C_k(p) := \varphi(C'_k(w))$ for $k = 1, \dots, r$ lying in pairwise orthogonal affine planes $\varphi(\mathbb{C}_k(w))$. Up to order these generating circles of C are independent of φ as they only depend on the inner geometry of C. In fact, the unit lattice $\Gamma = \{v \in T_pC \, : \, \mathrm{Exp}_p(v) = p\}$, where Exp_p denotes the Riemannian exponential map of C at p, is rectangular, that is Γ has an orthogonal basis. Indeed, Γ is the direct sum of the unit lattices of the generating circles $C_k(p)$, $k = 1, \dots, r$. Thus uniqueness of the generating circles up to order follows from:

Lemma 1.1. *Let Γ be a rectangular lattice in a Euclidean vector space V. Then an orthogonal basis of Γ is unique up to signs and order.*

Proof. Let $B = \{b_1, \dots, b_r\}$ be an orthogonal basis of Γ, that is $\Gamma = \{\sum_{j=1}^r \alpha_j b_j \, : \, \alpha_j \in \mathbb{Z}$ for all $j = 1, \dots, r\}$. Assuming that $b_1 \dots, b_s,\ s \le r$, are the shortest elements in B then $\pm b_1, \dots, \pm b_s$ are the shortest element in Γ. Thus $b_1, \dots, b_s \in \Gamma$ are unique up to sign and order. Lemma 1.1 now follows by induction on the dimension of V. ☐

As the standard Clifford torus C' is extrinsically symmetric in $\mathbb{C}^r$, the Clifford torus $C = \varphi(C')$ is extrinsically symmetric in $\varphi(\mathbb{C}^r) \subset E$, which is the affine hull $\mathrm{aff}(C)$ of C, and thus extrinsically symmetric in E. Since at each $p \in C$ the affine hull $\mathrm{aff}(C)$ splits orthogonally as the direct sum $\bigoplus_{k=1}^r \mathrm{aff}(C_k(p))$ we have:

Lemma 1.2. *Let $p \in C$ and let ψ be an isometry of E with $\psi(p) = p$. If ψ induces a reflection on each generating circle $C_k(p)$ then ψ leaves C invariant.*

Lemma 1.3. *Each connected component of the fixed point set of an isometry f of C is a Clifford torus in E.*

Proof. Let $p \in C$ be fixed by f and let $\Gamma \subset T_pC$ be the unit lattice of C at p. By Lemma 1.1 there exists an orthogonal basis $b_1, \dots, b_r$ of T_pC unique up to sign and permutations with $\Gamma = \mathrm{span}_{\mathbb{Z}}\{b_1, \dots, b_r\}$. The differential f_* of f at p is an orthogonal map of T_pC which preserves Γ_p and therefore acts on $\{\mathbb{R}b_1, \dots, \mathbb{R}b_r\}$ by permutation. We decompose $I := \{1, \dots, r\}$ as $I = \bigsqcup_{j=1}^k I_j$ into non-empty subsets I_j such that f_* acts cyclically on the sets $\{\mathbb{R}b_s : s \in I_j\}$, $j \in \{1, \dots, k\}$. Then C is the product of the Clifford tori $T_j = \mathrm{Exp}_p(\mathrm{span}_{\mathbb{R}}\{b_s : s \in I_j\})$, $j = 1, \dots, k$, which are contained in pairwise orthogonal affine subspaces and are invariant under f. It therefore suffices to assume that f_* acts cyclically on $\{\mathbb{R}b_1, \dots, \mathbb{R}b_r\}$ and $f_*(b_j) = b_{j+1}$ for all $j \in \{1, \dots, r-1\}$. Thus $f_*(b_r) \in \{-b_1, b_1\}$. If $f_*(b_r) = -b_1$, then f_* has no fixed vector and p is an isolated fixed point of f. If $f_*(b_r) = b_1$, then the fixed point set of f_* is $\mathbb{R}(b_1 + \cdots + b_r)$ and the connected component of the fixed point set of f through p is the planar circle $\mathrm{Exp}_p(\mathbb{R}(b_1 + \cdots + b_r))$. ☐

2. Submanifolds of Clifford type

Definition 2.1. An (embedded) submanifold X of a Euclidean space E is called *of Clifford type* if every geodesic of X lies in a totally geodesic submanifold C of X which is a Clifford torus in E.

For brevity we call a totally geodesic submanifold in $X \subset E$ which is a Clifford torus in E a *totally geodesic Clifford torus* in X.

Proposition 2.2. *A connected submanifold $X \subset E$ of Clifford type is a compact extrinsically symmetric space.*

Proof. Since Clifford tori are compact X is geodesically complete. As there are no (distance minimizing) geodesic rays in a torus X is compact.

Let $p \in X$ and let $r_p : E \to E$ be the reflection along the affine normal space $p + N_pX$ of X at p. Since any point of X can be joined by a geodesic to p such a geodesic lies in a totally geodesic Clifford torus $C \subset X$ containing p. In order to prove $r_p(X) = X$ it suffices to show that $r_p(C) = C$. In view of Lemma 1.2 we have to see that r_p induces the geodesic symmetry at p on each generating circle $C_k(p)$ for $k = 1, \ldots, \dim(C)$. Now the second derivative of a geodesic parameterizing $C_k(p)$ is normal to X. Thus r_p leaves $\mathrm{aff}(C_k(p))$ invariant and induces on $C_k(p)$ the geodesic symmetry at p. □

As tori contain dense geodesics we conclude:

Corollary 2.3. *Maximal tori of connected submanifolds of Clifford type are Clifford tori.*

An advantage of the notion of submanifolds of Clifford type is its simple definition. It is sometimes possible to carry over results about Clifford tori to submanifolds of Clifford type. Proposition 2.4 illustrates this principle with Lemma 1.3. Recall that a non-empty connected component of an involutive isometry of a Riemannian manifold is called a *reflective subspace*.

Proposition 2.4. *A reflective subspace Y of a submanifold $X \subset E$ of Clifford type is itself of Clifford type.*

Proof. By Proposition 2.2, X is a compact extrinsically symmetric submanifold of E and Y is a compact symmetric space being a totally geodesic closed submanifold of X. It suffices to show that any maximal torus T_Y of Y is a Clifford torus. We may enlarge T_Y to a maximal torus T_X of X. Then T_X is a Clifford torus in E by Corollary 2.3. Let f be an involutive isometry of X having Y as a component of its fixed point set. It now suffices to show that T_X is f-invariant, since then T_Y is a connected component of the fixed point set of $f|_{T_X}$ and thus a Clifford torus by Lemma 1.3. Fixing $p \in T_Y$ it is actually sufficient to see that the differential f_* of f at p leaves the tangent space of T_X at p invariant.

Let G be the (full) isometry group of X and $\mathfrak{g}$ its Lie algebra. Every $g \in G$ induces an automorphism of G given by conjugation with g. Its differential $\mathrm{Ad}(g)$

at the identity is an automorphism of $\mathfrak{g}$. The involution $\mathrm{Ad}(s_p)$, where s_p is the geodesic symmetry of X at p, gives rise to a splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ and $\mathfrak{p}$ the (± 1)-eigenspace of $\mathrm{Ad}(s_p)$. The restriction to $\mathfrak{p}$ of the differential pr_* of $\mathrm{pr} : G \to X,\ g \mapsto g.p := g(p)$ at the identity identifies $\mathfrak{p}$ with T_pX. Under this identification $f_* : T_pX \to T_pX$ corresponds to $\mathrm{Ad}(f)|_{\mathfrak{p}}$.

Let $\mathfrak{g}_\pm$ be the (± 1)-eigenspace of the involution $\mathrm{Ad}(f)$. Then $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. As $s_p = s_{f(p)} = f s_p f^{-1}$, the involutions $\mathrm{Ad}(f)$ and $\mathrm{Ad}(s_p)$ commute and we get a finer decomposition $\mathfrak{g} = \mathfrak{k}_+ \oplus \mathfrak{k}_- \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$ where $\mathfrak{k}_\pm = \mathfrak{k} \cap \mathfrak{g}_\pm$ and $\mathfrak{p}_\pm = \mathfrak{p} \cap \mathfrak{g}_\pm$. Note that $\mathfrak{p}_+$ is identified with T_pY. Let $\mathfrak{a}_+ \subset \mathfrak{p}_+$ and $\mathfrak{a} \subset \mathfrak{p}$ be the maximal abelian subspaces corresponding to the tangent spaces of T_Y and T_X at p. Obviously $\mathfrak{a}_+ \subset \mathfrak{a}$. To show that $\mathfrak{a} = \mathfrak{a}_+ \oplus (\mathfrak{a} \cap \mathfrak{p}_-)$ let $\zeta \in \mathfrak{a}$ and $\zeta = \zeta_+ + \zeta_-$ with $\zeta_\pm \in \mathfrak{p}_\pm$. Then $0 = [\zeta, H] = [\zeta_+, H] + [\zeta_-, H]$ for any $H \in \mathfrak{a}_+$. From $[\mathfrak{p}_\pm, \mathfrak{p}_+] \subset \mathfrak{k}_\pm$ we obtain $[\zeta_\pm, \mathfrak{a}_+] = \{0\}$. By maximality we get $\zeta_+ \in \mathfrak{a}_+$ and hence $\zeta_- \in \mathfrak{a} \cap \mathfrak{p}_-$. Finally, since both $\mathfrak{a}_+$ and $\mathfrak{a} \cap \mathfrak{p}_-$ are $\mathrm{Ad}(f)$-invariant, $\mathfrak{a}$ is $\mathrm{Ad}(f)$-invariant and the tangent space of T_X at p is f_*-invariant. □

Next we show the converse of Proposition 2.2.

Theorem 2.5. *Every compact extrinsically symmetric submanifold X of a Euclidean space E is of Clifford type. More precisely, every maximal torus of X is a Clifford torus in E.*

To prepare the proof of Theorem 2.5 let $X \subset E$ be a compact, connected, extrinsically symmetric space. We may assume that the ambient Euclidean space is a (real) Euclidean vector space by choosing the barycenter of X as origin. We therefore will denote the ambient space by V instead of E. Then the identity component $\widehat{G}$ of the group generated by the reflections along the affine normal spaces of X is a subgroup $\widehat{G} \subset \mathrm{SO}(V)$. The restriction map $g \mapsto g|_X$ identifies $\widehat{G}$ with the identity component of the isometry group of X. Let $p \in X$. Since the reflection r_p along the affine normal space of X at p restricts to the geodesic symmetry s_p of X at p, conjugation with r_p gives rise to an involutive automorphism of $\widehat{G}$ whose differential at the identity coincides with $\mathrm{Ad}(s_p)$ on the Lie algebra of $\mathfrak{g}$ of the isometry group of X. Let T_X be a maximal torus of X that contains p and let $\mathfrak{a}$ be the maximal abelian subspace of $\mathrm{Fix}(-\mathrm{Ad}(s_p)) =: \mathfrak{p} \cong T_pX$ which corresponds to the tangent space of T_X at p. Then $T_X = \exp(\mathfrak{a}).p$.

We next show an auxiliary lemma needed in the proof of Theorem 2.5. Let γ be a geodesic in X. Then γ'' is a vector field along γ normal to X, more precisely $\gamma'' = \alpha(\gamma', \gamma')$, where α denotes the second fundamental form of X. Since X is extrinsically symmetric, α is parallel (as $D\alpha$ is equivariant with respect to the differentials of reflections along the affine normal spaces). Since γ' is parallel we get

$$\gamma''' = -S_{\alpha(\gamma', \gamma')}\gamma',$$

where S is the shape operator of X. In particular, γ''' is a vector field along γ which is tangent to X.

Lemma 2.6. *If γ is a geodesic in T_X with $\gamma(0) = p$, then $\gamma'''(0)$ is tangent to T_X.*

Proof. Let $\eta \in \mathfrak{k}$ and $k_t = \exp(t\eta)$, $t \in \mathbb{R}$, the corresponding one-parameter subgroup of K. Let $v \in \mathfrak{a}$. Since α is invariant under extrinsic isometries of X the map $t \mapsto \|\alpha(k_t.v, k_t.v)\|^2$ is constant. Differentiation at $t = 0$ yields $0 = \langle\alpha(v,v), \alpha(v,[\eta, v])\rangle$. If v is a regular vector in $\mathfrak{a} \cong T_pT_X$, then $[\mathfrak{k}, v] = \mathfrak{a}^\perp$, where $\mathfrak{a}^\perp$ denotes the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$ (polarity of the linear isotropy representation). Therefore $\langle S_{\alpha(v,v)}v, \mathfrak{a}^\perp\rangle = \langle\alpha(v,v), \alpha(v,\mathfrak{a}^\perp)\rangle = 0$ for all regular $v \in \mathfrak{a}$ and, by continuity, for all $v \in \mathfrak{a}$. This shows $\gamma_v'''(0) = -S_{\alpha(v,v)}v \in \mathfrak{a}$ for all $v \in \mathfrak{a}$. □

Proof of Theorem 2.5. Since all $H \in \mathfrak{a}$ are simultaneously diagonalizable over $\mathbb{C}$, they can be brought simultaneously into block diagonal form over $\mathbb{R}$. More precisely, $V = V_0 \oplus V_1 \oplus \cdots \oplus V_k$ such that each $H \in \mathfrak{a}$ acts trivially on V_0, and on each $V_j \cong \mathbb{C}$ $(j = 1,\ldots,k)$ it acts by the factor $i\alpha_j(H)$ for some real-valued linear form α_j on $\mathfrak{a}$. This shows that the maximal torus $T_X = \{\exp(H).p : H \in \mathfrak{a}\}$ is contained in a Clifford torus C in V, and each geodesic γ of T_X emanating from p, that is $\gamma(t) = \exp(tH).p$, $H \in \mathfrak{a}$, is a geodesic of C too. We may assume that C is a Clifford torus of minimal dimension with these properties and further that $C = \{(z_1,\ldots,z_k) : |z_j| = r_j\}$ is a standard torus with $r_j > 0$ and $p = (r_1,\ldots,r_k)$. Let γ be a dense geodesic in T_X with $\gamma(0) = p$. Since γ is also a geodesic in C, it has the form $\gamma(t) = (r_1e^{ia_1t},\ldots,r_ke^{ia_kt})$ for some $a_j \in \mathbb{R}$. As γ is dense in T_X and C is of minimal dimension, the numbers $(a_j)^2$ are pairwise different and do not vanish. In fact, $\{(re^{iat}, r'e^{iat}) : t \in \mathbb{R}\}$ and $\{(re^{iat}, r'e^{-iat}) : t \in \mathbb{R}\}$ are planar circles for any $r, r' > 0$ which would allow to reduce the dimension of C if $a_{j_1}^2 = a_{j_2}^2$ for some $j_1 \neq j_2$. Similarly, if $a_j = 0$, the corresponding S^1-factor of C could be deleted. Thus we may assume $0 < |a_1| < \cdots < |a_k|$.

By construction we have $T_X \subset C$. To show equality (and thus that T_X is a Clifford torus) let γ_ℓ be the geodesic of T_X emanating from p which is defined inductively by $\gamma_1 = \gamma$ and $\gamma_{\ell+1}$ with initial vector $\gamma_{\ell+1}'(0) := -\gamma_\ell'''(0) \in T_pT_X$ (see Lemma 2.6). From

$$\gamma_1(t) = \Big(r_1e^{ia_1t},\ldots,r_ke^{ia_kt}\Big),$$

we get

$$\gamma_\ell(t) = \Big(r_1e^{ita_1^{(3^\ell)}},\ldots,r_ke^{ita_k^{(3^\ell)}}\Big).$$

Now, if $w = i(w_1,\ldots,w_k) \in i\mathbb{R}^k = T_pC$ is perpendicular to $T_pT_X \subset T_pC$, we get $\langle\gamma_\ell'(0), w\rangle = -\sum_{j=1}^k r_ja_j^{(3^\ell)}w_j = 0$ for all $\ell \in \mathbb{N}$ and therefore $w_k = 0$ since otherwise the absolute value of the last summand would dominate the rest for sufficiently large ℓ. Similarly, $w_{k-1},\ldots,w_1 = 0$. This proves $T_pT_X = T_pC$ implying $T_X = C$. □

Corollary 2.7 (cf. [EQT15; Loo85; Tak65], see Remark 3.2)**.** *Maximal tori of compact extrinsically symmetric spaces have rectangular unit lattices.*

Combining Proposition 2.2 with Theorem 2.5 and observing that any flat torus contains a dense geodesic we get the next corollary.

Corollary 2.8. *Let X be a connected submanifold of a Euclidean space E. Then the following conditions are equivalent:*

(i) *X is a compact extrinsically symmetric space.*
(ii) *X is of Clifford type.*
(iii) *X is intrinsically a compact symmetric space and every maximal torus of X a Clifford torus in E.*
(iv) *X is intrinsically a compact symmetric space and every maximal torus of X is an extrinsically symmetric space in E.*

Remark 2.9. Corollary 2.8 is closely related to the results of Ferus and Schirrmacher in [FS82]. However, it seems that they take the implication "(i) $\Longrightarrow$ (iv)" for granted.

3. Harish-Chandra's theorem on strongly orthogonal roots

We give a new proof of Harish-Chandra's result on the existence of strongly orthogonal roots by means of Theorem 2.5 above.

Let G be a compact, connected, semisimple Lie group with a bi-invariant metric, $\mathfrak{g}$ its Lie algebra and $\xi \in \mathfrak{g}$ an non-zero element with $\operatorname{ad}(\xi)^3 = -\operatorname{ad}(\xi)$. Then $X := \operatorname{Ad}(G)\xi$ is a hermitian symmetric space of compact type and $X \subset \mathfrak{g}$ is extrinsically symmetric. In fact, the 1-parameter group $\varphi_t = e^{t\operatorname{ad}(\xi)} = \operatorname{Ad}(\exp(t\xi))$ of isometries of $\mathfrak{g}$ leaves X invariant and $\sigma := \varphi_\pi$ is the reflection along the affine normal space (in fact a linear subspace) of X at ξ. Indeed, $\mathfrak{k} := \mathfrak{z}(\xi) = \ker(\operatorname{ad}(\xi))$ and $\mathfrak{p} = [\mathfrak{g}, \xi]$ are the normal and the tangent spaces of X at ξ, and $\varphi_t = \mathrm{id}$ on $\mathfrak{k}$ and $\varphi_t = \cos(t)\cdot \mathrm{id} + \sin(t)\cdot \operatorname{ad}(\xi)$ on $\mathfrak{p}$ due to $\operatorname{ad}(\xi)^3 = -\operatorname{ad}(\xi)$. Thus σ is an involution with fixed point set $\mathfrak{k}$ and (-1)-eigenspace $\mathfrak{p}$. Hence $X \cong G/G_\xi$ with $G_\xi = \{g \in G : \operatorname{Ad}(g)\xi = \xi\}$ is a symmetric space of compact type which moreover is hermitian symmetric as $J := \varphi_{\frac{\pi}{2}}|_{\mathfrak{p}}$ defines an $\operatorname{Ad}(G_\xi)$-invariant almost complex structure on $\mathfrak{p}$. Conversely, every hermitian symmetric space of compact type is obtained in this way (cf. [Hel78, Chapter VIII, Theorem 4.5 (i)] or cf. [IT91, proof of Theorem 4.9, pp. 240–243]).

Let $\mathfrak{h}$ be a Cartan (that is maximal abelian) subalgebra of $\mathfrak{g}$ containing ξ and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha\in\mathcal{R}} \mathfrak{g}_\alpha$ the corresponding root space decomposition of the complexification of $\mathfrak{g}$ where an element $\alpha \in \mathfrak{h}^* \setminus \{0\}$ lies in $\mathcal{R}$ if and only if

$$\mathfrak{g}_\alpha := \Big\{X \in \mathfrak{g}^{\mathbb{C}} : \operatorname{ad}(H)X = i\alpha(H)X \quad \text{for all} \ \ H \in \mathfrak{h}\Big\} \neq \{0\}.$$

Note that $\operatorname{ad}(\xi)^3 = -\operatorname{ad}(\xi)$ implies $\alpha^3(\xi) = \alpha(\xi)$ for all $\alpha \in \mathcal{R}$. The roots $\alpha \in \mathbb{R}$ with $\alpha(\xi) \neq 0$ are called *of noncompact type* (this name has its origin in the dual case $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$) and two roots $\alpha, \beta \in \mathbb{R}$ are called *strongly orthogonal* if $\alpha \pm \beta \notin \mathcal{R} \cup \{0\}$.

Theorem 3.1 (Harish-Chandra [Har56, p. 582 f]). *There exists $r = \operatorname{rank}(X)$ pairwise strongly orthogonal roots of noncompact type.*

Proof. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{h}'$ a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $r = \dim(\mathfrak{a})$ and $\mathfrak{h}' = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{a}$ where $\mathfrak{h}_{\mathfrak{k}} = \mathfrak{h}' \cap \mathfrak{k}$. Note however that $\mathfrak{h}'$ does not contain ξ.

Since X is extrinsically symmetric and thus of Clifford type by Theorem 2.5, there exists pairwise orthogonal vectors $X_1, \ldots, X_r \in \mathfrak{a}$ such that the geodesics

$\gamma_j(t) = \mathrm{Ad}(\exp(tX_j)\xi$, $j = 1, \ldots, r$, in X are planar circles of smallest period 2π in $\mathfrak{g}$, contained in pairwise orthogonal planes. In particular, $\gamma_j'''(0) = -\gamma_j'(0)$. Let $Y_j = \gamma_j'(0)$ and $H_j = \gamma_j''(0)$. Then $Y_j = [X_j, \xi] \in \mathfrak{p}$, $H_j = [X_j, Y_j] \in \mathfrak{k} = \mathfrak{z}(\xi)$ and $Y_1, \ldots, Y_r, H_1, \ldots, H_r$ are pairwise orthogonal. Thus for each j we get a 3-dimensional subalgebra $\mathfrak{g}_j := \mathrm{span}\{X_j, Y_j, H_j\}$ of $\mathfrak{g}$ as $[X_j, H_j] = \gamma_j'''(0) = -\gamma_j'(0) = -Y_j$ and $[Y_j, H_j] = [[X_j, \xi], H_j] = [[X_j, H_j], \xi] = -[Y_j, \xi] = -\mathrm{ad}(\xi)^2 X_j = X_j$. Moreover, these subalgebras pairwise commute as $\mathfrak{a}$ (containing all X_j) and $[\mathfrak{a}, \xi] = \varphi_{\frac{\pi}{2}}(\mathfrak{a})$ (containing all Y_j) are abelian and $\|[X_j, Y_k]\|^2 = \langle [X_j, [X_k, \xi]], [X_j, Y_k] \rangle = \langle [X_k, [X_j, \xi]], [X_j, Y_k] \rangle = -\langle Y_j, [X_k, [X_j, Y_k]] \rangle = -\langle Y_j, [X_j, [X_k, Y_k]] \rangle = -\langle Y_j, [X_j, H_k] \rangle = \langle H_j, H_k \rangle = 0$ for $j \neq k$ implying the remaining relations by the Jacobi identity.

Let $\mathfrak{h} := \mathfrak{h}_{\mathfrak{k}} + \mathrm{Span}\{H_1, \ldots, H_r\}$. Since $\mathfrak{h}_{\mathfrak{k}}$ commutes with $\mathfrak{a}$ and ξ, and $H_j = [X_j, [X_j, \xi]]$, $\mathfrak{h}$ is abelian and $\mathfrak{h}_{\mathfrak{k}}$ is orthogonal to $H_1, \ldots, H_k$. In particular $\dim(\mathfrak{h}) = \dim(\mathfrak{h}')$. Therefore $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$ contained in $\mathfrak{k}$ which implies $\xi \in \mathfrak{h}$. For $j \in \{1, \ldots, r\}$ we define $\alpha_j := \frac{1}{\|H_j\|^2} \langle H_j, \ \ \rangle \in \mathfrak{h}^*$. Then $\alpha_1, \ldots, \alpha_r$ are pairwise strongly orthogonal roots of $\mathfrak{g}$ of noncompact type as

$$\alpha_j(\xi) = -\frac{\|Y_j\|^2}{\|H_j\|^2},$$

and $\mathfrak{g}_{\alpha_j}$ is spanned by $X_j + iY_j$, and $\mathfrak{g}_{\alpha_j \pm \alpha_k} = [\mathfrak{g}_{\alpha_j}, \mathfrak{g}_{\pm\alpha_k}] = 0$ for $j \neq k$. □

Remark 3.2. Here we indicate how conversely Theorem 2.5 may be deduced from Harish-Chandra's Theorem 3.1 using results of Ferus and Takeuchi. Let $X = \mathrm{Ad}(G)\xi$ be the extrinsically symmetric submanifold of $\mathfrak{g}$ as above. If $\alpha_1, \ldots, \alpha_r$ are $r = \mathrm{rank}(X)$ pairwise strongly orthogonal noncompact roots with respect to a Cartan subalgebra of $\mathfrak{g}$ containing ξ and if $\mathfrak{g}_{\alpha_j} = \mathbb{C}(X_j + iY_j)$, then the $\mathfrak{g}_j := \mathrm{span}\{X_j, Y_j, H_j := [X_j, Y_j]\}$, $j = 1, \ldots, r$, are pairwise commuting subalgebras of $\mathfrak{g}$ isomorphic to $\mathfrak{so}_3(\mathbb{R})$. The orbits through ξ of the corresponding subgroups of $\mathrm{Ad}(G)$ are round 2-spheres as they are complex and thus orientable submanifolds. Their product is called a *polysphere* and is not only a totally geodesic submanifold of X (yielding Harish-Chandra's polysphere theorem, see [Wol72]), but also extrinsically a product of round 2-spheres in pairwise orthogonal 3-spaces. Therefore the product of great circles passing through ξ in each of these 2-spheres gives a totally geodesic Clifford torus in X and thus proves Theorem 2.5 in this special case.

According to Ferus [Fer80] (cf. [EH95] for a different proof) every compact, connected, extrinsically symmetric space is congruent to an orbit $Y := \mathrm{Ad}_G(K')\xi$ in $\mathfrak{p}'$. Here (G, K') is a symmetric pair with G compact and semisimple, $\mathfrak{g} = \mathfrak{k}' \oplus \mathfrak{p}'$ is the corresponding decomposition of the Lie algebra of G and ξ is an element of $\mathfrak{p}'$ with $\mathrm{ad}(\xi)^3 = -\mathrm{ad}(\xi)$. In particular, Y is contained in $X = \mathrm{Ad}(G)\xi$. Takeuchi [Tak65] has noticed that Y is a connected component of the fixed point set of $-\tau|_X : X \to X$, where τ is the involution of $\mathfrak{g}$ with eigenspaces $\mathfrak{k}'$ and $\mathfrak{p}'$. Thus Theorem 2.5 follows for all compact extrinsically symmetric spaces by Proposition 2.4.

In [Tak65, Theorem 1, p. 158 and Theorem 5, p. 164] Takeuchi has used the ideas sketched above to prove that symmetric R-spaces have rectangular unit lattices, an essential step for the proof of Theorem 2.5.

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