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Matsumoto theorem for skeleta

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Abstract. We present a proof of a generalization of the theorem of H. Matsumoto on Coxeter groups. Our generalized version is applicable to "graphs admitting geometric realization". The original version of the theorem for Coxeter groups is a special case when applied to the Cayley graph and the geometric representation of a Coxeter group. Our version of Matsumoto theorem is also applicable to *skeleta*, graphs that were defined in the recent paper [GHS24] on root Lie superalgebras.

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Key Words and Phrases: Coxeter groups, Weyl groups, Lie superalgebras, skeleta, Coxeter graphs

In memory of Joe Wolf

1. Introduction

Let (G, S) be a Coxeter group. Recall that this means that

$$G = \mathrm{Free}_S /((st)^{m_{st}}),$$

the factor of the free group generated by S, where $M = (m_{st})$ is a symmetric matrix over $\mathbb{Z}$ satisfying the conditions $m_{ss} = 1,\ m_{st} > 1$ for $s \neq t$. A reduced decomposition of $g \in G$ is a shortest expression

$$g = s_1 \dots s_n,\ s_i \in S$$

and we denote by $\ell(g)$ the length of a reduced decomposition of g.

It is convenient to rewrite the relations as follows

$$\begin{cases} s^2 = 1, & s \in S \\ sts\dots = tst\dots \ \text{(both expressions of length } m_{st}), & s \neq t \in S. \end{cases}$$

The second equality is called the generalized braid (also Artin–Tits) relation. We define an equivalence relation (called *the braid relation*) on the set of words in the alphabet S as the minimal relation closed under concatenation and containing the Artin–Tits relation.

The following result is proven by Matsumoto [Mat64] in 1964.

Theorem 1.1. *Let $g = s_1 \dots s_n = t_1 \dots t_n$ be two reduced decompositions of g, with s_i and t_j in S. Then the sequences $(s_1, \dots, s_n)$ and $(t_1, \dots, t_n)$ are braid equivalent.*

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The original proof of Matsumoto theorem is based on the following *replacement property*.

Lemma 1.2. *Let $g = s_1 \dots s_n$ be a reduced decomposition and let $s \in S$ satisfy the condition*

$$\ell(sg) < n.$$

Then there exists j such that

$$ss_1 \dots s_n = s_1 \dots s_{j-1}s_{j+1} \dots s_n.$$

In a recent paper [GHS24] the authors introduced a family of Lie superalgebras (root Lie superalgebras) generalizing Kac–Moody superalgebras and Borcherds Lie algebras. The set of Borel subalgebras of a root Lie superalgebra can be described in terms of a certain graph called a skeleton. The skeleton plays the role of the Cayley graph of a Weyl group. A version of Matsumoto theorem can be formulated for these graphs but the replacement property does not seem to hold. This led us to an attempt to generalize Matsumoto theorem so that it would cover the case of skeleta.

Our solution turns out to be very simple: it gives a proof that is, in our opinion, easier even for the Coxeter groups. Note that our proof does not use replacement property. Our approach is somewhat similar (though not completely equivalent) to that of Heckenberger–Yamane [HY08]. It seems that validity of the Matsumoto theorem for skeleta in the symmetrizable fully reflectable case could have been deduced from [HY08] and the Coxeter property of skeleta proven in [GHS24, Section 6].

Matsumoto graphs have a dual presentation in terms of convex geometry of hyperplanes arrangements, see Section 6. In this dual form our notion of Matsumoto graph turns out to be equivalent to the notion of thin simplicial arrangement, see [CMW17].

In the case of finite crystallographic arrangements a connection to Weyl groupoids was studied by M. Cuntz [Cun11] (for the detailed treatment of Weyl groupoids see the book [HS20]).

2. Matsumoto graphs

2.1. Rays and cones

Our basic geometric object will consist of closed rays in a real vector space $\mathbb{V}$.

For a collection of rays $\underline{\alpha} = (\alpha_1, \dots, \alpha_k)$ their sum $C(\underline{\alpha}) = C(\alpha_1, \dots, \alpha_k)$ is the convex cone generated by α_i. For a collection $\underline{\alpha} = (\alpha_i,\ i = 1, \dots, \dim \mathbb{V})$ of linearly independent rays the cone $C(\underline{\alpha})$ will be called the *simplicial cone* spanned by α_i.

Note that the rays spanning a simplicial cone C are uniquely determined by C.

2.2. Graphs and their realization

Our graphs $\Gamma = (V, E)$ are connected. Their edges may be compact (having two ends) or noncompact (infinite, connected to only one vertex), and we write $E = E_c \sqcup E_\infty$.

We think of compact edges as the ones allowing both orientations, whereas infinite edges are always incoming. We define the set of oriented edges $\overrightarrow{E} = \overrightarrow{E_c} \sqcup E_\infty$

where $\overrightarrow{E_c}$ is a two-fold covering of E_c that we identify with $E_c \times \{\pm 1\}$. Thus,

$$\overrightarrow{E} = \{(e, \epsilon) \in E \times \{\pm 1\} \,|\, \epsilon = 1 \ \text{ if } \ e \in E_\infty\}.$$

A geometric realization of a graph Γ is an assignment of a ray $\alpha_{e,\epsilon}$ to each oriented edge (e, ϵ) so that $\alpha_{e,-\epsilon} = -\alpha_{e,\epsilon}$ for $e \in E_c$ so that the axioms (1)–(4) below hold. If $e \in E_\infty$, we will write α_e instead of $\alpha_{e,1}$.

We define the set of roots $R = \{\alpha_{e,\epsilon}\}$ and we require the following axioms:

(1) If $e \in E_\infty$ then $-\alpha_e \notin R$.
(2) For any $v \in V$ the set of roots $\alpha_1, \ldots, \alpha_d$ assigned to the edges ingoing to v, form a basis of $\mathbb{V}$. In particular, the graph Γ is regular of degree $d = \dim \mathbb{V}$. We denote $R_v^+ = \{\alpha \in R \mid \alpha \in C(\alpha_1, \ldots, \alpha_d)\}$.
(3) For each $(e, \epsilon) : v \to v'$ one has

$$R_{v'}^+ \setminus R_v^+ = \{\alpha_{e,\epsilon}\}.$$

Note that this condition implies that a graph having a geometric realization has no loops $e : v \to v$ and no multiple edges.
(4) If $R_v^+ = R_{v'}^+$ then $v = v'$.

Definition 2.1. A Matsumoto graph is a connected graph endowed with a geometric realization.

Remark 2.2. The notion of Matsumoto graph can be seen as a generalization of Yamane's notion of *(set-theoretical) generalized root system*, see [Yam16, p. 2.1]. These generalized root systems are defined as certain subsets of a lattice in a real vector space, with a chosen subset in the set of bases. They correspond to Matsumoto graphs in the crystallographic fully reflectable case.

Note that a root α corresponds to a compact edge iff $-\alpha \notin R$. Having this in mind, the roots corresponding to compact edges will be called *invertible* and the roots of infinite edges will be called *non-invertible*.

Our generalization of Matsumoto theorem is formulated in terms of Matsumoto graphs. It claims that any two shortest paths from one vertex to another in a Matsumoto graph are braid equivalent. The result is formulated only in Theorem 2.8 as the braid equivalence requires a certain effort to define.

Basic properties. The property below immediately follows from the axioms.

Lemma 2.3.

(1) *For any $e \in E_\infty$ and any $v \in V$ one has $\alpha_e \in R_v^+$.*
(2) $R \subset R_v^+ \sqcup (-R_v^+)$.

We define $R_v^- := R \setminus R_v^+ \subset -R_v^+$.

Lemma 2.4. *If $R_v^+ \subset R_{v'}^+$ then $v = v'$.*

Proof. Let $v_0 = v, v_1, \ldots, v_s = v'$ be a path from v to v'. Set $A_i := R_{v_i}^+ \setminus \bigcap_{i=0}^s R_{v_i}^+$. The sets A_i are finite and A_i, A_{i+1} have the same cardinality for $i = 0, \ldots, s-1$. Thus A_0 and A_s have the same finite cardinality. The embedding $R_v^+ \subset R_{v'}^+$ implies $A_0 \subset A_s$, so $A_0 = A_s$. Hence $R_v^+ = R_{v'}^+$ and, by Axiom 4 $v = v'$. ∎

For vertices v, v' we define the distance $d(v, v')$ as the length of a shortest path connecting v with v'.

Lemma 2.5. *Any path from v to v' has length equal to $d(v, v')$ modulo 2. $d(v, v')$ is the cardinality of $R^+_{v'} \setminus R^+_v$. In particular, any closed path has an even cardinality; that is, any graph admitting a geometric realization, is bipartite.*

Proof. The first claim follows from Axiom (3). The inequality $d(v, v') \geq |R^+_{v'} \setminus R^+_v|$ is obvious. The converse is deduced by induction, as by Lemma 2.4 for $v' \neq v$ there exists an incoming edge (e, ϵ) to v which does not belong to $R^+_{v'}$. □

Matsumoto graphs are regular of degree $d = \dim \mathbb{V}$. The lemma below shows that any edge e connecting two vertices v and w establishes a one-to-one correspondence between the edges adjacent to v and to w so that e corresponds to itself. In Proposition 5.1 we will show that this correspondence leads to a consistent coloring of the set of edges E with the elements of a set X of cardinality d, so that the correspondence above assigns to an edge adjacent to v the edge adjacent to w of the same color.

Lemma 2.6. *Let $(e, \epsilon) : v \to v'$, $\alpha = \alpha_{e,\epsilon}$ and $R^+_v = C(\alpha_1, \ldots, \alpha_d)$ with $\alpha_1 = -\alpha$. Then there is a unique numbering of the basic roots of $R^+_{v'} = C(\alpha'_1, \ldots, \alpha'_d)$ such that $\alpha'_1 = -\alpha_1 = \alpha$ and for each $i > 1$ α'_i lies in the cone spanned by α_i and α_1.*

Proof. Let $\mathbb{V}' = \mathbb{V}/\mathbb{R}\alpha$ and $\pi : \mathbb{V} \to \mathbb{V}'$ denote the natural projection. Then $C' = \pi(C(\alpha_2, \ldots, \alpha_d)) = \pi(C(\alpha'_2, \ldots \alpha'_d))$ coincides with the image under π of the convex hull of the set $R^+_v \cap R^+_{v'}$. On the other hand, C' is a simplicial cone in $\mathbb{V}'$ with generators $\pi(\alpha_2), \ldots, \pi(\alpha_d)$ and similarly with generators $\pi(\alpha'_2), \ldots, \pi(\alpha'_d)$. We choose the enumeration so that $\pi(\alpha_i) = \pi(\alpha'_i)$. □

Lemma 2.7. *Let v be a vertex of a Matsumoto graph with basis $\{\alpha_1, \ldots, \alpha_d\}$. Let $\{\alpha_1, \ldots, \alpha_k\}$ be a subset and $\mathbb{W}$ be the span of $\{\alpha_1, \ldots, \alpha_k\}$. Then $R \cap \mathbb{W}$ gives a geometric realization of a full subgraph Γ' of degree k.*

Proof. We define Γ' as the connected component of v of the subgraph Γ'' of Γ having the same vertices as Γ and the edges $e \in E$ such that $\alpha_{e,1} \in \mathbb{W}$.

The vertex $v \in \Gamma'$ has k incoming edges marked by $\{\alpha_1, \ldots, \alpha_k\}$. Lemma 2.6 implies that Γ' is the Matsumoto graph of degree k with the set of roots $R' \subset R \cap \mathbb{W}$. We will show that $R' = R \cap \mathbb{W}$.

For any vertex $v' \in \Gamma'$ we denote by $\alpha'_1, \ldots, \alpha'_k$ the basis at v'. Note first that for any $v' \in \Gamma'$ we have $R^+_{v'} \cap \mathbb{W} \subset C(\alpha'_1, \ldots, \alpha'_k)$ again by Lemma 2.6. Next by induction on the cardinality of $(R^+_{v'} \cap \mathbb{W}) \setminus (R^+_u \cap \mathbb{W})$ we can prove that for any $u \in \Gamma$ there exists $v' \in \Gamma'$ such that $(R^+_{v'} \cap \mathbb{W}) \subset (R^+_u \cap \mathbb{W})$.

Now let $\beta \in R \cap \mathbb{W}$. There exists $u \in \Gamma$ such that β is an element of the basis at u. Then either β or $-\beta$ lies in $R^+_{v'} \cap \mathbb{W} \subset C(\alpha'_1, \ldots, \alpha'_k)$ where $v' \in \Gamma'$ is as above. Suppose that $\beta \in R^+_{v'} \cap \mathbb{W}$. We claim that $\beta = \alpha'_i$ for some $i \leq k$. Indeed, consider a linear function ψ such that $\psi(\beta) = 0$ and $\psi(\gamma) > 0$ for all other $\gamma \in R^+_u$. If $\beta \neq \alpha'_i$ for all $i \leq k$ then $\psi(\beta) > 0$ and we get a contradiction. Similarly, if $-\beta \in R^+_{v'} \cap \mathbb{W}$ we can prove that $-\beta = \alpha'_i$ by the same method. The proof is complete. □

Notation

The graph Γ' in the above construction will be denoted $\Gamma(v, \alpha_1, \ldots, \alpha_k)$.

2.3. The case $d = 2$

Here are all connected graphs of degree two.

(1) A polygon. Here all edges are compact. If an n-gon admits a geometric realization, n has to be even by Lemma 2.5. A polygon with an even number of edges is the Cayley graph of a dihedral group, so it admits a realization, see Section 3.
(2) A graph with an infinite number of compact edges. This is the Cayley graph of the infinite dihedral group, so it admits a realization, see Section 3.
(3) A graph with one infinite edge and an infinite number of compact edges.
(4) A graph with two infinite edges and a finite number of compact edges.

A geometric realization for the cases (3), (4) can be easily found. Note that in all Matsumoto cases there is a canonical coloring of the edges so that any two adjacent edges have different colors.

2.4. Braid relation and Matsumoto theorem

2.4.1. Braid relation. Fix a vertex $v \in \Gamma$. Choose two edges adjacent to v, one incoming and one outgoing. Let α and β be the corresponding roots. There is a unique continuation of the path (α, β), potentially in both directions, so that the roots assigned to the new edges belong to $\mathrm{Span}_{\mathbb{R}}(\alpha, \beta)$. This yields a rank 2 Matsumoto subgraph $\Gamma(v, \alpha, \beta)$ of Γ spanned by α and β.

We denote by $\mathcal{P} = \mathcal{P}(\Gamma)$ the set of paths in Γ. We define the braid relation on $\mathcal{P}$ as the minimal equivalence relation satisfying the following two properties.

(1) For any rank two Matsumoto subgraph $\Gamma(v, \alpha, \beta)$ presented by a $2m$-gon and colored by $X = \{x, y\}$, the length m paths

$$xyx, \ldots, \quad \text{and} \quad yxy, \ldots$$

starting at v (and having the same end) are equivalent.
(2) The equivalence relation is closed under the concatenation of paths.

Note that equivalent paths have the same ends and the same length.

The main result of this note is the following generalization of Matsumoto theorem [Mat64].

Theorem 2.8. *Let Γ be a Matsumoto graph. Then any two shortest paths in Γ from v to v' are braid equivalent.*

The proof is given in Section 4.

3. Examples

In this section we present two examples of Matsumoto graphs: the Cayley graphs of Coxeter groups and the skeleta of admissible components of the root groupoid, see [GHS24, Section 5].

3.1. Coxeter groups

The Matsumoto structure on the Cayley graph of a general Coxeter group (G, X), based on its geometric representation [Hum90, p. 5.3], is described below. We start presenting an even more classical construction that makes sense for Weyl groups.

3.1.1. Weyl group. Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h}$, the root system Δ and the set of simple roots $\alpha_1, \ldots, \alpha_n$. The Weyl group W is generated by the simple reflections $s_i = s_{\alpha_i}$. We define Γ as the Cayley graph of W with respect to the set S of simple reflections, so that the edges (they are all compact) are of the form $w \to ws$, $s \in S$. The arrow $w \to ws_i$ is marked with the root $\mathbb{R}_{\geq 0} \cdot w(\alpha_i)$.

The same construction of Matsumoto structure generalizes to Weyl groups of Kac–Moody Lie algebras, as well as for root Lie superalgebras [GHS24]. Note that the Matsumoto structures corresponding to the classical root systems B_n and C_n are isomorphic, as in our formalism roots are rays and not vectors.

3.1.2. General Coxeter group. Let (G, X) be a Coxeter group with the generating set X. We define Γ as the Cayley graph of G with respect to the set X of generators. All edges in Γ are compact and are colored by the set of generators X. The geometric representation $\sigma : G \to GL(\mathbb{V})$, [Hum90, p. 5.3], where $\mathbb{V} = \mathrm{Span}_{\mathbb{R}}\{\alpha_x, x \in X\}$, yields a geometric realization of Γ as follows. It assigns to an arrow $x : g \to gx$ the root $\mathbb{R}_{\geq 0} \cdot \sigma_g(\alpha_x) \in \mathbb{V}$. The set R_g^+ is spanned by $-\sigma_g(\alpha_x)$, $x \in X$. The axioms (1)–(3) of the geometric realization are verified immediately. The Axiom (4) follows from [Hum90, p. 5.6].

3.2. Skeleta

The application to skeleta is the *raison-d'être* of the present note. Let X be a finite set and let $v = (\mathfrak{h}, a : X \to \mathfrak{h},\ b : X \to \mathfrak{h}^*,\ p : X \to \mathbb{Z}_2)$ be an admissible root datum, see [GHS24, Section 2]. Let $\mathtt{Sk}(v)$ be the connected component of v in the skeleton of the root groupoid $\mathcal{R}$, see [GHS24, Sections 3–4]. The graph Γ assigned to it is a regular graph with the edges colored by X having the same vertices as $\mathtt{Sk}(v)$. The compact edges of Γ are the edges of $\mathtt{Sk}(v)$. The infinite edges of Γ are described by the nonreflectable pairs (v', x), with $v' \in \mathtt{Sk}(v)$ and $x \in X$. The geometric realization of Γ assigns to each arrow $r_x : v \to v'$ the real root $b_{v'}(x)$ and to an infinite edge defined by the nonreflectable pair (v', x) the real nonreflectable root $b_{v'}(x)$. In this realization the roots of Γ are precisely the real roots of the component of $v \in \mathcal{R}$.

It is proven in [GHS24] that $\mathtt{Sk}(v)$ is a Coxeter graph. Now Theorem 2.8 implies that the analog of Matsumoto theorem holds of $\mathtt{Sk}(v)$. Note that Theorem 2.8 gives another proof of coxeterity of $\mathtt{Sk}(v)$.

4. Proof of Theorem 2.8

We will prove the theorem by induction in the length n of the shortest paths.

Step 1. The cases $n = 0$ are obvious as a Matsumoto graph does not have loops and multiple edges.

Step 2. Assume the result is proven for shortest paths of length $\leq n-1$. Let

$$\underline{\alpha} : v = v_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} v_n = v'$$

and

$$\underline{\beta} : v = v'_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} v'_n = v'$$

be a pair of length n shortest paths leading from v to v'. By Lemma 2.5 the sets of α_i and of β_j coincide. Thus, $\beta_1 = \alpha_i$ for some i.

Step 3. Assume that $i < n$. Consider the path

$$v'_1 \xrightarrow{-\beta_1} v'_0 = v \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_i} v_i.$$

Obviously $d(v'_1, v_i) = i-1$. Choose a path γ of length $i-1$ connecting v'_1 with v_i. Then, by the inductive hypothesis, the path α is braid equivalent to the concatenation of β_1, γ, and the segment of α connecting v_i with $v_n = v'$. In the same manner the above concatenation is equivalent to the path β. This proves that in this case the paths α and β are braid equivalent. Therefore, we may assume that $i = n$, that is $\beta_1 = \alpha_n$. In the same way, we can assume that $\alpha_1 = \beta_n$.

Step 4. It remains to deal with the case when $\alpha_1 = \beta_n$ and $\alpha_n = \beta_1$. One has $\alpha_1, \beta_1 \in R^+_{v'} \cap R^-_v$. This means that

$$\Gamma(v, -\beta_1, \alpha_1) = C(\alpha_1, \beta_1) \cap R \subset R^+_{v'} \cap R^-_v.$$

The roots assigned to infinite edges are everywhere positive, the intersection $R^+_{v'} \cap R^-_v$ is finite, so $\Gamma(v, -\beta_1, \alpha_1)$ is a polygon with an even number of edges. Denote by v'' the vertex opposite to v. One has

$$d(v, v') = d(v, v'') + d(v'', v').$$

Denote by γ a shortest path connecting v'' with v'. Denote by γ', γ'' two equivalent halves of $\Gamma(v, -\beta_1, \alpha_1)$ connecting v to v''. Then the concatenation of γ' with γ is equivalent to α by the inductive hypothesis, as they start with the same root α_1, and similarly the concatenation of γ'' with γ is equivalent to β.

This proves Theorem 2.8. □

5. Complements

5.1. Edge coloring

We will now prove the existence of a consistent coloring of the set of edges E of a Matsumoto graph Γ.

Proposition 5.1. *Let Γ be a graph of degree d endowed with a geometric realization. Let $|X| = d$. There exists a coloring $E \to X$ of the edges of Γ consistent with the isomorphisms defined by Lemma 2.6 for each arrow $v \to v'$. The coloring is unique up to automorphism of X.*

Proof. Let us identify X with the set of generators of R^+_v. Any closed path $\gamma : v \to v$ in Γ defines an automorphism $\theta_\gamma : X \to X$. It is enough to prove that $\theta_\gamma = \mathrm{id}_X$ for any γ. Theorem 2.8 implies that if $\gamma', \gamma'' : v \to v'$ are two shortest paths, $\theta_{\gamma' \circ \gamma''^{-1}} = \mathrm{id}_X$.

Let us assume, to the contrary, that there exists a closed path $\gamma : v \to v$ for which $\theta_\gamma \neq \mathrm{id}_X$. Choose a shortest such γ, of length $2m$. For any root α in γ the root $-\alpha$ should also appear. Let us choose a closest pair of edges is γ with assigned roots α and $-\alpha$.

It is easy to see that if α and $-\alpha$ are not precisely opposite to each other, one can produce a shorter counterexample.

In the remaining case when all pairs α and $-\alpha$ are opposite to each other, the loop γ can necessarily be presented as $\gamma' \circ \gamma''^{-1}$ where γ' and γ'' are two shortest paths with the same ends. This proves the claim. □

Remark 5.2. Note that in the examples of graphs with a geometric realization presented in Section 3, the coloring provided by Proposition 5.1 coincides with the apriori coloring given, in the first case by the set S of generator, and in the case of skeleta (by the set X).

5.2. Exchange condition

In this subsection we discuss a partial exchange condition that holds in the context of skeleta [GHS24, p. 4.2.5]. The edges of a skeleton $\mathtt{Sk}(v)$ are *reflexions* that can be *anisotropic* or *isotropic*, see definitions in [GHS24, p. 4.1.1]. The following result shows that, in the context of skeleta, the exchange condition holds for anisotropic reflexions.

Proposition 5.3. *Let*

$$v_0 \xrightarrow{r_{x_1}} \cdots \xrightarrow{r_{x_n}} v_n$$

be a sequence of reflexions that is a shortest path from v_0 to v_n and let $r_y : v_0' \to v_0$ be an anisotropic reflexion such that the path $v_0' \to v_0 \to \cdots \to v_n$ is not shortest. Then there exists j so that the composition

$$v_0' \xrightarrow{r_y} v_0 \longrightarrow \cdots \xrightarrow{r_{x_i}} v_j$$

coincides with the composition

$$v_0' \xrightarrow{r_{x_1}} v_1' \longrightarrow \cdots \xrightarrow{r_{x_{j-1}}} v_{j-1}' = v_j.$$

Proof. By the assumptions, $\alpha = b_{v_0}(y) = b_{v_j}(x_j)$ for some j. The path $v_0' \to \cdots \to v_{j-1}'$ exists as the namesake of the path $v_0 \to \cdots \to v_{j-1}$, see [GHS24, pp. 4.3.6, 4.3.7]. The equality $v_{j-1}' = v_j$ then follows from ibid 4.3.7. □

5.3. Geometric properties of roots.

Lemma 5.4. *Let Γ be a Matsumoto graph with set of roots R. Consider R as a set of points on the unit sphere in $\mathbb{V}$. If $\alpha, -\alpha \in R$ (that is, if α is the root of a compact edge) then α is an isolated point of R in the usual topology on the unit sphere.*

Proof. Let α mark an arrow $v' \to v$, so that $\alpha = \alpha_1, \ldots, \alpha_d$ form the basis of roots at v. If there is a sequence of roots $\beta_k \in R_v^+$ converging to α, then α belongs to the closure of $R_v^+ \setminus \alpha = R_v^+ \cap R_{v'}^+ \subset C(\alpha_2, \ldots, \alpha_d)$ which is impossible. For the same reason a sequence of $\beta_k \in R_v^-$ cannot possibly converge to α: then $-\beta_k \in C(\alpha_2, \ldots, \alpha_d)$ would converge to $-\alpha$. This proves the result. □

Note that a noninvertible root α may be a limit point in R as we will see in Example 6.2 below.

6. Dual picture

The notion of Matsumoto graph can be equivalently expressed in the dual picture, in terms of convex geometry. We present below this equivalent language.

Let Γ be a Matsumoto graph with the set of roots R in a vector space $\mathbb{V}$. For every $\alpha \in R$ define a hyperplane H_α and a half-space U_α in the dual space $\mathbb{V}^*$ by

$$H_\alpha = \{\xi \in \mathbb{V}^* \mid \langle \xi, \alpha \rangle = 0\}, \qquad U_\alpha = \{\xi \in \mathbb{V}^* \mid \langle \xi, \alpha \rangle \geq 0\}.$$

For a vertex $v \in \Gamma$ with assigned basis $\alpha_1, \ldots, \alpha_d$ we define a chamber $C_v^\vee$ to be the intersection $\cap U_{\alpha_i}$. Then $C_v^\vee$ is a simplicial cone in $\mathbb{V}^*$.

Lemma 6.1. *Let $D = \bigcup_{v \in \Gamma} C_v^\vee$ and D' be the set of all $\xi \in \mathbb{V}^*$ such that $\langle \xi, \alpha \rangle \geq 0$ for all but finitely many invertible $\alpha \in R_v^+$. Then $D = D'$. In particular, D is a conical convex set.*

Proof. Obviously $D \subset D'$. Let us show that $D = D'$. Suppose that $\xi \in D'$ and M be the set of all invertible $\alpha \in R_v^+$ such that $\langle \xi, \alpha \rangle < 0$. If $M \neq \emptyset$ then there is an invertible root α in the basis at v such that $\alpha \in M$. Consider the arrow $v' \to v$ marked by α. Then $\langle \xi, \beta \rangle \geq 0$ for all $\beta \in R_{v'}^+ \setminus M'$ with $M' = M \setminus \{\alpha\}$. Using induction on $|M|$ we can prove existence of a vertex u such that ξ is nonnegative on all roots of R_u^+. This shows that $\xi \in C_u^\vee \subset D$. ☐

Vice versa, here is a way to reconstruct a Matsumoto graph from a convex conical set in the dual space $\mathbb{V}^*$.

Let $\{H_\alpha\}_{\alpha \in A}$ be a collection of hyperplanes in $\mathbb{V}^*$, V be the collection of chambers $C_v^\vee$, that is connected components of $\mathbb{V}^* \setminus \bigcup_{\alpha \in A} H_\alpha$, and let $\Gamma \subset V$ satisfy the following properties:

(1) All $C_v^\vee$ for $v \in \Gamma$ are simplicial, that is have the minimal possible ($\dim(V)$) number of faces.
(2) $D = \bigcup_{v \in \Gamma} C_v^\vee$ is convex.

We claim that the data $\{H_\alpha\}, \Gamma$ as above uniquely define a Matsumoto graph. Its vertices correspond to the set Γ of chambers of D. The edges ingoing to a vertex $v \in \Gamma$ are the pairs $(C_v^\vee, H_\alpha)$ where H_α contains a boundary component of $C_v^\vee$. The edge defined by a pair $(\mathcal{C}_c, H_\alpha)$ is non-compact when D belongs to one of halfspaces defined by H_α. The invertible edge corresponds to a pair of chambers having a common boundary component. Convexity of D implies that if D does not belong to one of the halfspaces defined by H_α then any edge $(C_v^\vee, H_\alpha)$ is invertible. In fact, let $(C_w^\vee, H_\alpha)$ be an edge with $C_w^\vee$ and $C_v^\vee$ being the chambers belonging to the different components of $\mathbb{V}^* \setminus H_\alpha$. Choose $x \in C_v^\vee \cap H_\alpha$ and $y \in C_w^\vee \setminus H_\alpha$. Since $x, y \in D$, the segment connecting x, y lies in D. This segment has a nontrivial intersection with the H_α-reflection of $C_v^\vee$. Since D is a union of chambers, this implies that the H_α-reflection of $C_v^\vee$ lies in D.

The geometric realization for this graph is defined as follows. We assign to an edge $(\mathcal{C}_c, H_\alpha)$ the ray defining the hyperplane H_α and the halfspace containing $\mathcal{C}_c$.

Example 6.2. The convex conic set D is uniquely determined by its intersection with the unit sphere in $\mathbb{V}^*$. Moreover, if $D \setminus \{0\}$ lies in some open half-space

$$\{\xi \in \mathbb{V}^* \mid \langle \xi, \alpha \rangle > 0\},$$

one can use instead the intersection of D with the affine hyperplane $H = \{\xi \in \mathbb{V}^* \mid \langle \xi, \alpha \rangle = 1\}$.

Let us give one example in the case $\dim H = 2$. Consider a triangle A, B, C in H, let M_0 be the midpoint BC, M_1 the midpoint of AC, M_2 the midpoint of M_1C, and then inductively M_n is defined to be the midpoint of $M_{n-1}C$. This configuration gives a dual realization of a Matsumoto graph with chambers $ABM_0, AM_0M_1, \ldots,$ $AM_nM_{n+1} (n > 0), \ldots$ The roots are dual to the lines AB, BM_0, M_0C, M_0M_i for all $i > 0$, AM_1, M_nM_{n+1}. The noninvertible roots are AB, BM_0, M_0C, AM_1 and M_nM_{n+1} for $n > 0$. Note that M_0C is the limit of the set $\{M_0M_n\}_{n \geq 1}$.

Warning

The numbering of sections, formulas and theorems in this version of the paper was changed according to the guidelines of the journal. We retained the original numbering in the arXiv version of the paper.

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References

[Cun11] Michael Cuntz, *Crystallographic arrangements: Weyl groupoids and simplicial arrangements*, Bull. Lond. Math. Soc. **43** (2011), no. 4, pp. 734–744.

[CMW17] Michael Cuntz, Bernhard Mühlherr, and Christian J. Weigel, *Simplicial arrangements on convex cones*, Rend. Semin. Mat. Univ. Padova **138** (2017), pp. 147–191.

[GHS24] Maria Gorelik, Vladimir Hinich, and Vera Serganova, *Root groupoid and related Lie superalgebras*, Ann. Represent. Theory **1** (2024), no. 4, pp. 465–516.

[HS20] István Heckenberger and Hans-Jürgen Schneider, *Hopf algebras and root systems*, Mathematical Surveys and Monographs, no. 247, American Mathematical Society, 2020.

[HY08] István Heckenberger and Hiroyuki Yamane, *A generalization of Coxeter groups, roots systems and Matsumoto's theorem*, Math. Z. **259** (2008), no. 2, pp. 255–276.

[Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, no. 29, Cambridge University Press, 1990.

[Mat64] Hideya Matsumoto, *Générateurs et relations des groupes de Weyl généralisés*, C. R. Acad. Sci. Paris **258** (1964), pp. 3419–3422.

[Yam16] Hiroyuki Yamane, *Generalized root systems and the affine Lie superalgebra* $\mathrm{G}^{(1)}(3)$, São Paulo J. Math. Sci. **10** (2016), no. 1, pp. 9–19.

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