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The affine root system of an isoparametric submanifold in Hilbert space

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Abstract. An alternative construction of the affine root system of an isoparametric submanifold in Hilbert space to that in [GH12] is provided, without invoking Dadok's theorem.

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Key Words and Phrases: Coxeter groups, affine root systems, isoparametric submanifolds

Dedicated to the memory of Joseph A. Wolf

Introduction

An open conjecture states that every connected complete full irreducible isoparametric submanifold M of rank at least two in an infinite-dimensional separable Hilbert space V arises as a principal orbit of the isotropy representations of a symmetric space of affine Kac–Moody type (which is obtained from an involution of the second kind of an affine Kac–Moody group, compare [Hei06]). Such submanifolds are already known to be extrinsically homogeneous in V by the main result of [HL99]. The paper [GH12] contributed to this conjecture in that, for fixed $x \in M$, we associated to M a certain bilinear map $\Gamma_x : T_xM \times T_xM \to T_xM$, called a homogeneous structure, and proved that this invariant together with the second fundamental form at x determines M completely. Although Terng [Ter89] had already constructed the affine Weyl group of M, in [GH12] it was necessary to refine that construction and associate an affine root system to M, in order to prove the continuity of Γ and obtain several of its properties.

This note is a small addition to [GH12]. Our purpose herein is to provide an alternative, more direct construction of the affine root system, without invoking Dadok's theorem [Dad85] and several abstract identifications as was done in [GH12, Section 7] (cf. Theorem 2.5). Instead we use the finite root systems of the involutive Lie algebras attached to slices of rank one (curvature spheres) and two, and assemble them into the affine root system of M.

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1. Some notation and terminology

Let E be an affine Euclidean space and denote by T its group of translations (a finite dimensional real vector space). Let $\mathcal{H}$ be a given set of affine hyperplanes in E which is invariant under the group $\mathcal{W}$ generated by all the orthogonal reflections in the elements of $\mathcal{H}$. It is assumed that the normal vectors to the $H \in \mathcal{H}$ span T and that $\mathcal{W}$ is a finite or an affine Weyl group. In the first case $\mathcal{W}$ has a fixed point that necessarily is contained in all $H \in \mathcal{H}$; taking this point as the origin in E, we can identify E with T and view the hyperplanes as linear subspaces. The second case may be characterized as $\mathcal{H}$ consisting of finitely many families of equidistant hyperplanes. It is well known [Bou68, Chapter VI, Section 2.5, Proposition 8] that these actually are described by a unique reduced root system Δ in T as the set of hyperplanes

$$L_{\alpha,k} = \{x \in T \mid \alpha(x) = k\}, \quad \alpha \in \Delta, \quad k \in \mathbb{Z} \tag{1}$$

after choosing a special point in E as origin, i. e. a point through which there passes one hyperplane from each family of parallel hyperplanes, to identify E with T. We also recall that $\mathcal{W}$ acts simply transitively on the set of Weyl chambers, which are the connected components of the complement of the union of the hyperplanes in $\mathcal{H}$. Weyl chambers are also called alcoves in case $\mathcal{W}$ is an affine Weyl group. The Coxeter graph of $\mathcal{W}$ (or $\mathcal{H}$) is obtained from a Weyl chamber $\mathcal{C}$ by taking as vertices the walls of $\mathcal{C}$ (hyperplanes bounding $\mathcal{C}$) and linking two vertices by 0, 1, 2, 3 or infinitely many edges according to whether the corresponding walls make an angle $\pi/2$, $\pi/3$, $\pi/4$, $\pi/6$ or are parallel (other cases cannot occur). $\mathcal{W}$ is called irreducible if its Coxeter graph is connected. In this case $\mathcal{C}$ is a simplicial cone (resp. simplex) if $\mathcal{W}$ is finite (resp. affine) and hence the Coxeter graph has n (resp. $n+1$) vertices, where $n = \dim E$ is called the rank of $\mathcal{W}$. The isomorphism type of the Coxeter graph is independent of the chosen Weyl chamber and determines $\mathcal{H}$ and $\mathcal{W}$ up to isomorphism. Here an isomorphism between two sets of hyperplanes $\mathcal{H} \subset E$, $\mathcal{H}' \subset E'$ as above with irreducible Weyl groups is a map $f : E \to E'$ that is the composition of an isometry with a homothety and takes $\mathcal{H}$ onto $\mathcal{H}'$. It turns out that the isomorphism classes of irreducible finite (resp. affine) Weyl groups correspond bijectively to the irreducible reduced root systems Δ in T and are correspondingly denoted by A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, E_n $(n = 6, 7, 8)$, F_4 and G_2 in case $\mathcal{W}$ is finite, and $\widetilde{A}_n$ $(n \geq 1)$, $\widetilde{B}_n$ $(n \geq 3)$, $\widetilde{C}_n$ $(n \geq 2)$, $\widetilde{D}_n$ $(n \geq 4)$, $\widetilde{E}_n$ $(n = 6, 7, 8)$, $\widetilde{F}_4$, $\widetilde{G}_2$ in case $\mathcal{W}$ is affine.

Definition 1.1. A *root system* associated to $\mathcal{H}$ is a subset R of $T \times \mathcal{H}$ such that:

(i) $v \neq 0$ and $v \perp H$ for all $(v, H) \in R$.
(ii) $2\langle v, v'\rangle/\|v\|^2 \in \mathbf{Z}$ for all (v, H), $(v', H') \in R$.
(iii) The projection $T \times \mathcal{H} \to \mathcal{H}$ maps R onto $\mathcal{H}$.
(iv) R is invariant under $\mathcal{W}$, that is, $(w_* v, wH) \in R$ for all $(v, H) \in R$ and $w \in \mathcal{W}$.

The *rank* of R is defined to be the rank of $\mathcal{W}$. Each $(v, H) \in R$ may be identified with the nonconstant affine mapping $E \to \mathbb{R}$ whose gradient is v and whose zero set is H. In case $\mathcal{W}$ is finite and E is identified with T by taking the point $x \in \bigcap_{H \in \mathcal{H}} H$ as origin, these affine mappings are linear functionals and one gets a root system in the ordinary sense. In case $\mathcal{W}$ is affine, one gets an affine root system in the sense of

Macdonald [Mac71]. In this case we thus call R an *affine root system* associated to $\mathcal{H}$. An equivalent definition, under the name *échelonnage*, has been given by Bruhat and Tits [BT72]. We refer to [GH12, Section 2] for the definition of isoparametric submanifolds and other related notations and concepts, and to [GH12, Section 7] for an extensive discussion of affine root systems. We now recall some of that terminology and notation.

- $TM = \bigoplus_{\mathbf{i}\in\mathbf{I}} E_{\mathbf{i}}$: Hilbert direct sum decomposition into curvature distributions (common eigenspaces of shape operators);
- $v_{\mathbf{i}}$: curvature normal;
- $\mathbf{I}$: countable index set containing $\mathbf{0}$;
- $\mathbf{I}^* = \mathbf{I} \setminus \{\mathbf{0}\}$;
- $S_{\mathbf{i}}(x)$: curvature sphere through x (leaf of $E_{\mathbf{i}}$ through x);
- $H_{\mathbf{i}}(x) = x + \{\xi \in \nu_x M \mid \langle \xi, v_{\mathbf{i}} \rangle = 1\}$: (affine) focal hyperplane with respect to x;
- P: affine subspace of $x + \nu_x M$;
- $\mathcal{D}_P$: $\bigoplus_{v_{\mathbf{i}}(x)\in P} E_{\mathbf{i}}(x)$ (closure of algebraic sum; by Codazzi, it is an integrable distribution with totally geodesic leaves);
- $W_P(x) = x + \mathcal{D}_P(x) + \mathrm{span}\{v_{\mathbf{i}}(x) \mid v_{\mathbf{i}}(x) \in P\}$;
- $L_P(x)$: leaf of $\mathcal{D}_P$ through x, it is called the *slice* through x associated to P; it is a complete full isoparametric submanifold of $W_P(x)$ of rank $\dim P$ if $0 \in P$ and $\dim P + 1$ if $0 \notin P$;
- ξ_P: parallel normal vector field ξ_P such that $\langle \xi_P, v_{\mathbf{i}} \rangle = 1$ if and only if $v_{\mathbf{i}}(x) \in P$ (in case $0 \notin P$);
- $c_P(x) = x + \xi_P(x)$: center of round hypersphere in $W_P(x)$ containing $L_P(x)$ (in case $0 \notin P$);
- $\Phi^*_P(x)$: identity component of normal holonomy group of focal manifold obtained from M by focalizing $\mathcal{D}_P(x) = \bigoplus_{v_{\mathbf{i}}(x)\in P} E_{\mathbf{i}}(x)$.

2. The construction

Lemma 2.1. *Let $\mathfrak{k}$ be a (finite-dimensional) Lie algebra and let $\rho : \mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ be a faithfull irreducible representation where $\dim \mathfrak{p} \geq 2$. Assume there exist bilinear maps*

$$[\ ,\ ]_i : \mathfrak{p} \times \mathfrak{p} \longrightarrow \mathfrak{k}, \quad i = 1,\, 2$$

each one defining a Lie algebra structure on $\mathfrak{g}_i = \mathfrak{k} + \mathfrak{p}$ which: extends the Lie algebra structure on $\mathfrak{k}$, the map given by $[A, x] = \rho(A)x$ for $A \in \mathfrak{k}$ and $x \in \mathfrak{p}$, and has $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Then there exists $c \neq 0$ such that

$$[\ ,\ ]_2 = c\,[\ ,\ ]_1.$$

Moreover, for the corresponding Killing forms we have $\beta_2 = c\,\beta_1$ on $\mathfrak{p} \times \mathfrak{p}$.

Proof. The assumptions imply that $\mathfrak{g}_i = \mathfrak{k} + \mathfrak{p}$ is an effective irreducible involutive Lie algebra, $i = 1, 2$. By irreducibility, $\beta_2 = c\,\beta_1$ on $\mathfrak{p} \times \mathfrak{p}$ for some $c \neq 0$. The rest follows from

$$\begin{aligned}\beta_2(A, [x, y]_2) &= \beta_2(\rho(A)x, y)\\ &= c\,\beta_1(\rho(A)x, y)\\ &= c\,\beta_1(A, [x, y]_1)\\ &= c\,\beta_2(A, [x, y]_1)\end{aligned}$$

for all $A \in \mathfrak{k}$, x, $y \in \mathfrak{p}$, where we used $\beta_1 = \beta_2$ on $\mathfrak{k}$ in the last equality as

$$\beta_1(A, B) = \beta_{\mathfrak{k}}(A, B) + \mathrm{trace}_{\mathfrak{p}}(\rho(A)\rho(B))$$

and similarly for β_2, where A, $B \in \mathfrak{k}$ and $\beta_{\mathfrak{k}}$ denotes the Killing form of $\mathfrak{k}$. □

Remark 2.2. For an effective involutive Lie algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of compact type and a fixed Cartan subspace $\mathfrak{a}$ of $\mathfrak{p}$, the associated root system Δ is defined by

$$\Delta = \{\alpha \in \mathfrak{a}^* \setminus \{0\} \,|\, \mathfrak{p}_\alpha \neq 0\}\,,$$

where

$$\mathfrak{p}_\alpha = \left\{X \in \mathfrak{p} \,\middle|\, [A, [A, X]] = -\alpha(A)^2 X \ \text{ for all } \ A \in \mathfrak{a}\right\}.$$

Under the assumptions of Lemma 2.1, we further suppose that $(\mathfrak{g}, [\ ,\ ]_1)$ and $(\mathfrak{g}, [\ ,\ ]_2)$ are compact Lie algebras so that $c > 0$. Then it is clear that the corresponding root systems satisfy $\Delta_2 = \sqrt{c}\Delta_1$.

Lemma 2.3. *For any irreducible finite dimensional slice through x, there exists a naturally defined irreducible finite root system associated to it.*

Proof. Let $P \subset \nu_x M$ be an affine subspace such that the correspondind slice $L_P(x)$ is irreducible and finite dimensional. By [HL99, Proposition 2.2], the compact connected Lie group $\Phi^*_P(x)$ acts on the affine span $W_P(x)$ of $L_P(x)$ as an irreducible s-representation, and has $L_P(x)$ as the orbit through x. Now there is a natural way of putting a Lie algebra structure on

$$\mathfrak{g}_P(x) := \mathrm{Lie}(\Phi^*_P(x)) + W_P(x)$$

extending the Lie algebra structure of $\mathfrak{k} := \mathrm{Lie}(\Phi^*_P(x))$ and the action ρ of $\mathfrak{k}$ on $\mathfrak{p} := W_P(x)$, namely, we put $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and

$$\langle A, [u, v]\rangle_{\mathfrak{k}} = \langle Au, v\rangle_V \tag{2}$$

for all $A \in \mathfrak{k}$ and u, $v \in \mathfrak{p}$, where

$$\langle A, B\rangle_{\mathfrak{k}} = -\beta_{\mathfrak{k}}(A, B) - \mathrm{trace}_{\mathfrak{p}}(\rho(A)\rho(B)),$$

for all A, $B \in \mathfrak{k}$, and $\beta_{\mathfrak{k}}$ denotes the Killing form of $\mathfrak{k}$ (cf. [EH99, Lemma 4.1]). This turns $\mathfrak{g}_P(x)$ into an effective irreducible involutive Lie algebra. By Lemma 2.1, this structure is unique if we impose that minus the Killing form of $\mathfrak{g}_P(x)$ coincides with the Hilbert inner product on $W_P(x)$; in other words, we suitably renormalize the Lie bracket (2) on $\mathfrak{p} \times \mathfrak{p}$. Moreover, $\mathfrak{g}_P(x)$ is a compact Lie algebra.

Now the normal space to $L_P(x)$ at x in $W_P(x)$ is spanned by $\{v_i(x) \mid v_i \in P\}$ and it is a maximal Abelian subspace $\mathfrak{a}_P$ of $W_P(x)$. There is a (finite) Weyl group $\mathcal{W}_P$ acting on $\mathfrak{a}_P$ and an associated irreducible finite root system Δ_P. To the slice $L_P(x)$

we have associated the finite root system Δ_P. Note that the kernels of the roots in Δ_P coincide with the focal hyperplanes $H_i(x)$ ($v_i \in P$) in $\mathfrak{a}_P$. □

Recall that a *root subsystem* of a root system is just a subset which is itself a root system.

Lemma 2.4. *Let P_1 and P_2 be two affine subspaces of $\nu_x M$ such that $P_1 \subsetneq P_2$ and $0 \notin P_2$. If $L_{P_1}(x)$, $L_{P_2}(x)$ are irreducible then Δ_{P_1} is isometric to a root subsystem of Δ_{P_2}.*

Proof. Since P_1, P_2 do not contain 0, the slices $L_{P_1}(x)$, $L_{P_2}(x)$ are finite dimensional. For the sake of simplicity, we denote $\mathfrak{g}_{P_i}(x) = \mathfrak{g}_i$, $\Phi^*_{P_i}(x) = K_i$, $\mathrm{Lie}(\Phi^*_{P_i}(x)) = \mathfrak{k}_i$, $W_{P_i(x)} = \mathfrak{p}_i$ and the bracket in $\mathfrak{g}_i$ by $[\,,\,]_i$. Of course $\mathfrak{p}_1$ is a proper affine subspace of $\mathfrak{p}_2$ and $c_1 := c_{P_1}(x) \neq c_2 := c_{P_2}(x)$. The subgroup of K_2 that preserves $\mathfrak{p}_1$ is the isotropy subgroup at c_1. By [HL99, Proposition 2.3], K_1 is the effectivization of $(K_2)_{c_1}$ acting on $\mathfrak{p}_1$, so we can view K_1 as a subgroup of K_2 by embedding it as the connected subgroup whose Lie algebra is the orthogonal complement in $\mathfrak{k}_2$ to the kernel of the action of $(\mathfrak{k}_2)_{c_1}$ on $\mathfrak{p}_1$ with respect to the Killing form of $\mathfrak{g}_2$.

Next we check that $\mathfrak{g}_1 = \mathfrak{k}_1 + \mathfrak{p}_1$ is closed under the bracket operation of $\mathfrak{g}_2 = \mathfrak{k}_2 + \mathfrak{p}_2$, the only case which is not immediate being $[\mathfrak{p}_1, \mathfrak{p}_1]_2 \subset \mathfrak{k}_1$. Let u, $v \in \mathfrak{p}_1$ and let $A \in (\mathfrak{k}_2)_{c_1}$ be such that $[A, \mathfrak{p}_1]_2 = 0$. Then

$$\beta_2(A, [u, v]_2) = \beta_2(Au, v) = 0$$

implies that $[u, v]_2 \in \mathfrak{k}_1$. It follows from Lemma 2.1 that $[\,,\,]_2 = c\,[\,,\,]_1$ on $\mathfrak{p}_1 \times \mathfrak{p}_1$ for some $c > 0$.

Now $\mathfrak{a}_1 := \mathrm{span}\{v_i(x) \mid v_i \in P_1\}$ is a Cartan subspace of $\mathfrak{p}_1$ in $\mathfrak{g}_1$ but also in $(\mathfrak{g}_1, [\,,\,]_2)$, and it is contained in the Cartan subspace $\mathfrak{a}_2 := \mathrm{span}\{v_i(x) \mid v_i \in P_2\}$ of $\mathfrak{p}_2$ in $\mathfrak{g}_2$. If $A \in \mathfrak{a}_1$, then ad^2_A preserves the decomposition $\mathfrak{p}_2 = \mathfrak{p}_1 + \mathfrak{p}_1^\perp$ (orthogonal with respect to β_2). Denote by Δ_1, $\Delta_{1,2}$, Δ_2 the set of roots of $(\mathfrak{g}_1, [\,,\,]_1)$, $(\mathfrak{g}_1, [\,,\,]_2)$, $\mathfrak{g}_2$ with regard to $\mathfrak{a}_1$, $\mathfrak{a}_1$, $\mathfrak{a}_2$, respectively. Recall that the curvature normal of L_2 corresponding to $\alpha \in \Delta_2$ is specified by $\langle v_\alpha, \cdot\rangle = -\alpha(\cdot)/\alpha(x)$. Therefore

$$\begin{aligned} v_\alpha \in P_1 &\iff \langle v_\alpha, \xi_1\rangle = 1 \\ &\iff -\frac{\alpha(\xi_1)}{\alpha(x)} = 1 \\ &\iff \alpha(c_1) = \alpha(x + \xi_1) = 0, \end{aligned}$$

namely (compare [HO92, p. 873]),

$$\Delta_{1,2} = \{\alpha|_{\mathfrak{a}_1} \mid \alpha \in \Delta_2 \quad \text{and} \quad \alpha(c_1) = 0\}\,.$$

Note also that $\mathfrak{a}_2 = \mathfrak{a}_1 \oplus \mathfrak{a}_1^\perp$, where $\mathfrak{a}_1^\perp$ is killed by all the $\alpha \in \Delta_2$ that restrict to an element of $\Delta_{1,2}$ (for in this case $v_\alpha \perp \mathfrak{a}_1^\perp$), namely, such a α "lives" in $\mathfrak{a}_1$. This shows that $\Delta_{1,2}$ is isometric to a closed subsystem of Δ_2. Since minus the Killing forms of $(\mathfrak{g}_1, [\,,\,]_1)$ and $(\mathfrak{g}_2, [\,,\,]_2)$ both coincide with the Hilbert inner product on $\mathfrak{p}_1$, by Lemma 2.1 we have $[\,,\,]_2 = [\,,\,]_1$ on $\mathfrak{p}_1$ and hence $\Delta_{1,2} = \Delta_1$. This proves the desired result, as $\Delta_{P_i} = \Delta_i$ for $i = 1$, 2. □

We can now describe the construction of the affine root system Δ of M on the affine normal space $E = x + \nu_x M$. This affine root system is associated to the family

$\mathcal{H} = \{H_{\mathbf{i}}(x)\}_{\mathbf{i}\in\mathbf{I}^*}$ of focal hyperplanes of M in E. We consider the curvature sphere $S_{\mathbf{i}}(x)$, which equals the slice $L_{\{v_{\mathbf{i}}\}}(x)$, and we construct the rank one root system $\Delta_{\{v_{\mathbf{i}}\}}$ according to Lemma 2.3. Each $\alpha \in \Delta_{\{v_{\mathbf{i}}\}}$ is a linear form on $\mathfrak{a}_{\{v_{\mathbf{i}}\}} = \mathrm{span}\{v_{\mathbf{i}}(x)\}$, but we view it as an affine mapping $x + \nu_x M \to \mathbf{R}$ vanishing on $H_{\mathbf{i}}(x)$. Now Δ is defined to be the disjoint union of the $\Delta_{\{v_{\mathbf{i}}\}}$ over $\mathbf{i} \in \mathbf{I}^*$. This yields a subset Δ of $T \times \mathcal{H}$ satisfying conditions (i) and (iii) in the definition of an affine root system. We finish this discussion by checking that Δ also satisfies (ii) and (iv).

Let (v, H), $(v', H') \in \Delta$. If v, v' are not parallel, there is a rank two slice L of M through x whose finite root system contains (v, H), (v', H'), due to Lemma 2.4. By finite root theory $2\langle v, v'\rangle/\|v\|^2 \in \mathbf{Z}$.

If v, v' are parallel, we need to see that

$$q := 2\|v'\|/\|v\| \in \mathbf{Z}.$$

In this case, using that M is irreducible and has rank at least two, we can find $(v'', H'') \in \Delta$ such that v'' is neither parallel nor orthogonal to v, v'. Let L, L' be rank two slices of M through x whose finite root systems contain (v, H), (v'', H'') and (v'', H''), (v', H'), respectively. By finite root theory, H and H'' make an angle of π/m, where $m = 3$, 4 or 6, and this is also the angle that H'' and H' make. If $m = 3$ then we must have $\|v\| = \|v''\|$ and $\|v''\| = \|v'\|$, and we deduce that $q = 1$. In case $m = 6$, we can replace (v'', H'') by another element of Δ and assume that $m = 3$, so this case reduces to the first one. Finally, suppose that $m = 4$. Here the ratios $\|v'\|/\|v''\|$, $\|v''\|/\|v\| \in \{\sqrt{2}, 1/\sqrt{2}\}$; note that this is true even if some of the finite root systems involved are not reduced. It follows that $q \in \{1, 2, 4\}$. This shows that Δ satisfies (ii).

Recall that the affine Weyl group $\mathcal{W}$ of M at x is generated by the reflections on the focal hyperplanes in E. Let $(v, H_{\mathbf{i}}(x)) \in \Delta$ and $w \in \mathcal{W}$. We already know that $wH_{\mathbf{i}}(x) = H_{\mathbf{k}}(x) \in \mathcal{H}$ for some $\mathbf{k} \in \mathbf{I}$ by [Ter89]. In order to show that $(w_*v, H_{\mathbf{k}}(x)) \in \Delta$, we may assume that w is a single reflection in a hyperplane $H_{\mathbf{j}}(x) \in \mathcal{H}$. If $H_{\mathbf{i}}(x)$ and $H_{\mathbf{j}}(x)$ are not parallel, the result follows as above by considering the rank two slice L_P, where $P = \text{affine span}\{v_{\mathbf{i}}(x), v_{\mathbf{j}}(x)\}$. Suppose $H_{\mathbf{i}}(x)$ and $H_{\mathbf{j}}(x)$ are parallel; the following argument can also be applied in case they are not parallel. By the main result of [HL99, Section 4] (see also [GH12, pp. 100–101]), there is an isometry F of V that preserves M, maps x to its antipodal point $\varphi_{\mathbf{j}}(x)$ in $S_{\mathbf{j}}(x)$, and induces the reflection across the affine focal hyperplane $H_{\mathbf{j}}(x)$ along $x + \nu_x M = \varphi_j(x) + \nu_{\varphi_j(x)} M$, namely, F induces w along $x + \nu_x M$. Now F conjugates the holonomy group $\Phi^*_{\{v_{\mathbf{i}}\}}(x)$ to the holonomy group $\Phi^*_{\{v_{\mathbf{i}}\}}(\varphi_{\mathbf{j}}(x))$, and the latter equals $\Phi^*_{\{v_{\mathbf{k}}\}}(x)$, as $S_{\mathbf{i}}(\varphi_{\mathbf{j}}(x))$ and $S_{\mathbf{k}}(x)$ are parallel submanifolds. Also, F maps

$$W_{\{v_{\mathbf{i}}\}}(x) = c_{\mathbf{i}}(x) + E_{\mathbf{i}}(x) + \mathrm{span}\{v_{\mathbf{i}}(x)\}$$

to

$$W_{\{v_{\mathbf{i}}\}}(\varphi_{\mathbf{j}}(x)) = c_{\mathbf{i}}(\varphi_{\mathbf{j}}(x)) + \underbrace{E_{\mathbf{i}}(\varphi_{\mathbf{j}}(x))}_{=E_{\mathbf{k}}(x)} + \mathrm{span}\Big\{\underbrace{v_{\mathbf{i}}(\varphi_{\mathbf{j}}(x))}_{=v_{\mathbf{k}}(x)}\Big\},$$

so F conjugates the involutive Lie algebra $\mathfrak{g}_{\{v_{\mathbf{i}}\}}(x)$ to the involutive Lie algebra $\mathfrak{g}_{\{v_{\mathbf{k}}\}}(x)$. It follows that F^* pulls-back the root system $\Delta_{\{v_{\mathbf{k}}\}}$ to $\Delta_{\{v_{\mathbf{i}}\}}$. Therefore we see that $(w_*v, H_{\mathbf{k}}(x)) \in \Delta$ and this shows that Δ satisfies (iv).

Hence we have proved:

Theorem 2.5. *For each infinite dimensional connected complete full irreducible isoparametric submanifold M of rank at least two in a separable Hilbert space V and $x \in M$, there exists a naturally defined affine root system associated to the family of focal hyperplanes in $x + \nu_x M$.*

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