JoLT
Journal
of Lie
Theory

A structure theorem along fibers of extreme points of the momentum polytope

Peter Heinzner and Christian Zöller

Journal of Lie Theory
Volume 36 (2026) 51–76
https://doi.org/10.5802/jolt.1419

CENTRE MERSENNE
JoLT is a member of Centre Mersenne
https://www.centre-mersenne.org/
e-ISSN: 2981-2917

Journal of Lie Theory
Volume **36** (2026) 51–76

A structure theorem along fibers of extreme points of the momentum polytope

Peter Heinzner and Christian Zöller

Communicated by Ivan Penkov

Abstract. Let G be a complex reductive Lie group acting on a compact Kähler manifold X. Assume that the action of a maximal compact subgroup K of G is Hamiltonian. For each extreme point of the convex hull of the momentum map image, there exists an associated open dense subset of X, that is invariant under the action of a parabolic subgroup Q of G. We prove a Q-equivariant product decomposition for the Q-action on this subset and discuss some applications of this result. Additionaly, we establish a similar statement for real reductive subgroups of G and the restricted momentum map.

In memory of Joseph A. Wolf

Introduction

For a finite dimensional representation V of a complex reductive group G and $X = \mathbb{P}(V)$, the following local structure theorem holds for any point $x \in X$ with a compact G-orbit (see [BLV86] for the precise statement or Section 9). There exists a Zariski open neighborhood Ω of x, a parabolic subgroup Q of G with Levi decomposition $Q = R \rtimes L$, an L-representation F such that the associated bundle $Q \times^L F$ to the L-principal bundle $Q \to Q/L$ is Q-equivariantly and algebraically isomorphic to Ω. The Q orbit through x is open in the compact orbit. Note that the map $R \times F \to Q \times^L F$ induced by the inclusion $R \times F \subset Q \times F$ is an isomorphism. In particular, Ω is an affine neighborhood of x. When restricting this situation to G-invariant subvarieties of $\mathbb{P}(V)$, the same result holds, with the L-representation F replaced by an L-invariant affine subvariety of F.

In this paper we prove a version of the local structure theorem for an invariant compact submanifold of a Kähler manifold. More precisely, we consider a holomorphic action of a connected complex reductive group $U^{\mathbb{C}}$ (with maximal compact subgroup U) on a Kähler manifold Z such that the restriction of the action to U on Z is Hamiltonian. This means, that the Kähler form ω on Z is U-invariant and there exists a U-equivariant momentum map $\mu \colon Z \to \mathfrak{u}^*$. We adoped the notations introduced in [HSS08]. Specifically, we fix a U-invariant inner product $\langle\, , \rangle$ on $\mathfrak{u}$ and use it to identify $\mathfrak{u}$ with its dual. Additionally, we identify $\mathfrak{u}$ with $i\mathfrak{u}$. With these identifications, a momentum map μ on Z is a smooth U-equivariant map $\mu \colon Z \to \mathrm{i}\mathfrak{u}$ such that for every function $\mu^\alpha \colon Z \to \mathbb{R}$, defined by $\mu^\alpha(x) = \langle \mu(x), \alpha \rangle$ for $\alpha \in \mathrm{i}\mathfrak{u}$, the gradient condition

$$\operatorname{grad} \mu^\alpha = \alpha_Z$$

is satisfied. Here, the gradient is computed with respect to the Riemannian structure induced by the Kähler form ω, and α_Z denotes the vector field on Z given by the one-parameter subgroup $t \mapsto \exp t\alpha$ of $U^{\mathbb{C}}$ and the $U^{\mathbb{C}}$-action on Z.

In the following, we fix a closed real subgroup G of $U^{\mathbb{C}}$ that is compatible with the Cartan decomposition

$$U^{\mathbb{C}} = U \exp(i\mathfrak{u})$$

in the sense that for $K := U \cap G$ (a compact group) and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$, the mapping

$$K \times \mathfrak{p} \longrightarrow G, \quad (k, \alpha) \longmapsto k \exp \alpha,$$

is an isomorphism.

The gradient remains unchanged when computed with respect to the induced Riemannian structure on a G-invariant real submanifold X of Z. For $\alpha \in \mathfrak{p}$, the flow of $\operatorname{grad} \mu^\alpha | X$ is given by

$$(t, y) \longmapsto \exp(t\alpha) \cdot y.$$

On X, we consider restricted momentum map (or gradient map)

$$\mu_{\mathfrak{p}} \colon X \longrightarrow \mathfrak{p}$$

defined by $\mu_{\mathfrak{p}}^\alpha = \mu^\alpha | X$ for $\alpha \in \mathfrak{p}$. Note that for $X = Z$, the restricted momentum map $\mu_{\mathfrak{p}}$ coincides with the map $\pi_{\mathfrak{p}} \circ \mu$, where $\pi_{\mathfrak{p}} \colon i\mathfrak{u} \to \mathfrak{p}$ denotes the orthogonal projection.

There are two important special cases to consider:

- *Complex reductive groups:* for example, we may assume $X = Z$ and consider the action of $G = U^{\mathbb{C}}$ on X.
- *Real forms:* here, G is a real form of $U^{\mathbb{C}}$, and X is a G-stable real or complex submanifold of Z.

In the following, we always assume that G is compatible with the Cartan decomposition of $U^{\mathbb{C}}$ and that X is a G-invariant compact submanifold of Z. For simplicity, we refer to the case where $G = U^{\mathbb{C}}$ and X is a compact complex submanifold of Z as the complex case.

In general, we are interested in points $x \in X$ with compact G-orbit such that the local structure theorem holds. In contrast to the complex case, very little is known in the general real case. Under the quite strong assumption that X is a real spherical variety (see [MS10]) a local structure theorem is established in [KKS15].

In Section 6, we will prove the local structure theorem for points lying in special G-orbits. To be precise, let E denote the convex envelope of the compact set $\mu_{\mathfrak{p}}(X)$. Since E is a convex body in $\mathfrak{p}$, it is the convex envelope of the set of its extreme points. The orbits of interest are the G-orbits through points in the $\mu_{\mathfrak{p}}$-fibers of extreme points of E.

The set of extreme points of E has a rather simple description in terms of the so called momentum polytope P which is as follows:

Let $\mathfrak{a}$ be a maximal Lie subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. The map

$$\mu_{\mathfrak{a}} := \pi_{\mathfrak{a}} \circ \mu,$$

where $\pi_{\mathfrak{a}} \colon \mathfrak{p} \to \mathfrak{a}$ denotes the orthogonal projection of $\mathfrak{p}$ onto $\mathfrak{a}$, is a restricted momentum map for the action of $A = \exp \mathfrak{a} \subset G$ on X. Note that A is compatible with the Cartan decomposition of $U^{\mathbb{C}}$. The convex envelope P of $\mu_{\mathfrak{a}}(X)$ is a convex polytope. In fact, P is the convex hull the finite set $\mu_{\mathfrak{a}}(X^A)$, where X^A denotes the set of A-fixed points in X (see [BGH16, Proposition 3.1]).

In the complex case we have $\mathfrak{p} = i\mathfrak{k}$, $\mathfrak{a} = i\mathfrak{t}$, where $\mathfrak{t}$ is the Lie algebra of a maximal torus T of K. Additionally, for a connected complex manifold X we have $P = \mu_{\mathfrak{a}}(X)$ ([Ati82; GS82]). In the general non-complex case, however, no connected compact manifold X is known for which $\mu_{\mathfrak{a}}(X) \neq P$.

Let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathfrak{p}$. For any $\beta \in \mathfrak{p}$ we define the face

$$F_\beta(E) := \Big\{ \alpha \in E : \langle \beta, \alpha \rangle = \max\{ \langle \beta, \gamma \rangle : \gamma \in E \} \Big\}.$$

A face F of E is called exposed by β if $F = F_\beta(E)$. In this case, we also say that β exposes F. If F is exposed then the set

$$C_F := \{ \beta \in \mathfrak{p} : F = F_\beta(E) \}$$

of β which exposes F is a cone in $\mathfrak{p}$. It is known that every face F of E is exposed. Moreover, the set of extreme points of E coincides with the union of the K-orbits of the extreme points of P with respect to the coadjoint action of K on $\mathfrak{p}$ (see [BGH16, Theorems 0.1, 0.2 and 0.3]).

Now let σ be an extreme point of E. The cone C_σ is invariant under the group

$$K^\sigma := \{ g \in K : g \cdot \sigma = \sigma \}$$

which is the centralizer of σ in K. Since C_σ is K^σ-invariant and convex, the set

$$C_\sigma^{K^\sigma} := \{ \beta \in C_\sigma : g \cdot \beta = \beta \ \text{ for all } \ g \in K^\sigma \}$$

of K^σ-fixed points in C_σ is non empty, and σ is exposed by some $\beta \in \mathfrak{p}^{K^\sigma}$.

For any $\beta \in \mathfrak{p}$ we define the parabolic subgroup

$$G^{\beta-} := \Big\{ g \in G : \lim_{t \to +\infty} \exp(t\beta) g \exp(-t\beta) \text{ exists} \Big\}$$

and its opposite

$$G^{\beta+} := \Big\{ g \in G : \lim_{t \to -\infty} \exp(t\beta) g \exp(-t\beta) \text{ exists} \Big\}.$$

The Levi factor of $G^{\beta-}$ is given by the centralizer

$$G^{\beta-} \cap G^{\beta+} = \{ g \in G : \mathrm{Ad}(g) \cdot \beta = \beta \},$$

and the unipotent radical of $G^{\beta-}$ is given by

$$R^{\beta-} := \Big\{ g \in G : \lim_{t \to +\infty} \exp(t\beta) g \exp(-t\beta) = e \Big\}.$$

We also consider the following G^β-stable subsets of X:

$$X^\beta_{\max} := \Big\{ x \in X : \mu^\beta_P(x) = \max \mu^\beta_P(X) \Big\} \subset X^\beta := \{ x \in X : \beta_X(x) = 0 \},$$

and

$$X^{\beta-}_{\max} := \Big\{ x \in X : \lim_{t \to +\infty} \exp(t\beta) \cdot x \text{ exists in } X^{\beta-}_{\max} \Big\}.$$

The set $X^{\beta-}_{\max}$ is open and $G^{\beta-}$-stable, while $X^\beta_{\max}$ is a smooth, G^β-stable, compact submanifold of X. There exists a surjective, smooth, and G^β-equivariant map

$$p^{\beta-} : X^{\beta-}_{\max} \longrightarrow X^\beta_{\max}, \quad p^{\beta-}(z) = \lim_{t \to +\infty} \exp(t\beta) \cdot z,$$

which defines a smooth G^β-equivariant fibration (see Section 4 for further details).

Our main result is the following theorem, whose proof is presented in Section 6:

Theorem A (Local structure Theorem). *Let X be a G-invariant compact submanifold of Z, and let σ be an extreme point of the convex envelope E of $\mu_{\mathfrak{p}}(X)$. Then, for any $\beta \in C_\sigma^{K^\sigma}$, there exists a normal subgroup I^β of G^β that is compatible with the Cartan decomposition and contains $\exp(\mathbb{R}\beta)$. Moreover, the quotient G^β/I^β is compact, and for every $x \in X^\beta_{\max}$, the following holds:*

There exists

(i) *an open neighborhood $U \subseteq X^\beta_{\max}$ of x,*
(ii) *an I^β-representation F, with all weights of β strictly negative, and*
(iii) *a smooth $R^{\beta-} \rtimes I^\beta$-equivariant isomorphism*

$$\Phi : \left(\left(R^{\beta-} \rtimes I^\beta\right) \times^{I^\beta} F\right) \times U \longrightarrow (p^{\beta-})^{-1}(U)$$

such that $p^{\beta-} \circ \Phi = p_U$, where p_U is the projection onto U.

For $x \in \mu_{\mathfrak{p}}^{-1}(\sigma)$, the orbits $G^\beta \cdot x$ and $G \cdot x$ are compact. It should be emphasized that $\mu_{\mathfrak{p}}^{-1}(\sigma) = X^\beta_{\max}$ for every β in C_σ ([BGH16, Lemma 3.1]). Furthermore, for $\beta \in C_\sigma^{K^\sigma}$ the open subset $X^{\beta-}_{\max}$ and the smooth fibration $p^{\beta-}$ depend only on the face σ and not on the choice of $\beta \in C_\sigma^{K^\sigma}$ (see [BGH16, Lemma 3.4]).

In Section 7, we apply the above theorem to obtain the following result.

Theorem B (Structure Theorem). *Let σ be an extreme point of the convex envelope E of $\mu_{\mathfrak{p}}(X)$, and let $\beta \in C_\sigma$. Then*

$$\mu^{-1}(\sigma) = X^\beta_{\max} \subset X^{G^\beta}.$$

If $\beta \in C_\sigma^{K^\sigma}$, then $K^\beta = K^\sigma$ and the $R^{\beta-}$-action on $X^{\beta-}_{\max}$ is proper and free. The quotient

$$X^{\beta-}_{\max}/R^{\beta-}$$

is a complex manifold with an induced holomorphic G^β-action. The map

$$p^{\beta-} : X^{\beta-}_{\max} \longrightarrow X^\beta_{\max}$$

induces a I^β-invariant holomorphic bundle map

$$q \colon X^{\beta-}_{\max}/R^{\beta-} \longrightarrow X^\beta_{\max}$$

with typical fiber F.

The same theorem holds in the complex case. In this case, if X is connected, then $X^\beta_{\max}$ and I^β contains the connected component of the identity of G^β. If G is connected, then $G^\beta = I^\beta$. In particular, the holomorphic bundle map

$$q \colon X^{\beta-}_{\max}/R^{\beta-} \longrightarrow X^\beta_{\max}$$

is a G^β-invariant Stein map (see [HMP98]). The same is true for the holomorphic map

$$p^{\beta-} : X^{\beta-}_{\max} \longrightarrow X^\beta_{\max}.$$

Let G be a connected complex reductive group. and let σ be an extreme point σ of E. For every $\beta \in C_\sigma$ we have $X^\beta_{\max} \subset X^{G^\beta}$. If $\beta \in C_\sigma^{K^\sigma}$, we obtain the following complex version of the above theorem:

Theorem C (Local Structure Theorem, complex case). *For any* $\beta \in C_\sigma^{K^\sigma}$ *and every* $x \in X^\beta_{\max}$, *there exists*

(i) *an open neighborhood* $U \subset X^\beta_{\max}$ *of* x,
(ii) *a* G^β*-representation* F, *with all weights of* β *being strictly negative, and*
(iii) *a* $G^{\beta-}$*-equivariant biholomorphic map*

$$\Phi : \left(G^{\beta-} \times^{G^\beta} F\right) \times U \longrightarrow (p^{\beta-})^{-1}(U),$$

such that $p^{\beta-} \circ \Phi = p_\Omega$, *where* p_Ω *is the projection onto* Ω.

For examples of non-algebraic Kähler G-manifold with Hamiltonian K-action, we refer to [GM18; GM23].

In contrast to the algebraic local structure theorem of Brion–Luna–Vust in [BLV86], our result does not imply the local structure theorem for all compact G-orbits in X. Nevertheless, we show in Section 9 that the result in [BLV86] is implied by the above theorem.

The proof of the local structure theorem in [BLV86] does not rely on momentum maps. Instead, given a point $x \in \mathbb{P}(V)$ with a compact G-orbit, Brion–Luna–Vust explicitly construct an open cone in $V \setminus \{0\}$ by utilizing the complete reducibility of G-representations. The image of this cone in $\mathbb{P}(V)$ yields the desired affine neighborhood Ω of x. This neighborhood naturally contains candidates for F and U. The main challenge in this approach lies in proving the surjectivity of the map Φ. We provide a proof in Section 9, Corollary 9.2.

Personal note of the first author

I recall with great joy the many visits Joe made to Ruhr University Bochum. Our connection with him was not merely mathematical, it was deeply personal. Joe was always a welcome guest in our family, and he became a true friend to us all.

Mathematically, our contact was intense and fruitful. The Bochum complex analysis group benefited immensely from Joe's visits. It is no coincidence that the work submitted in his memory aligns closely with Joe's central interests: the actions of real Lie groups on complex manifolds, with the fundamental example being the flag manifolds he so cherished.

1. Basic properties of the momentum map

Let Z be a complex manifold with a holomorphic action of the complex reductive group $U^{\mathbb{C}}$, where $U^{\mathbb{C}}$ is the complexification of the compact Lie group U. The Cartan involution associated with this setup is denoted by Θ.

We assume that Z admits a smooth, U-invariant Kähler structure and a U-equivariant momentum map

$$\mu \colon Z \longrightarrow \mathfrak{u}^*,$$

where $\mathfrak{u}$ is the Lie algebra of U and $\mathfrak{u}^*$ its dual. As in the introduction, we choose a U-invariant inner product on $\mathfrak{u}$ and use it to identify $\mathfrak{u}^*$ with $\mathfrak{u}$. To simplify notation, we isometrically identify $\mathfrak{u}$ as a U-representation with $\mathrm{i}\mathfrak{u}$ by multiplying with i.

We also assume that $G \subset U^{\mathbb{C}}$ is a closed subgroup such that the Cartan decomposition $U^{\mathbb{C}} = U \exp(i\mathfrak{u}) \simeq U \times i\mathfrak{u}$ induces a Cartan decomposition $G = K \exp(\mathfrak{p}) \simeq K \times \mathfrak{p}$, where $K = U \cap G$ and $\mathfrak{p} \subset i\mathfrak{u}$ is an $(\mathrm{Ad}\, K)$-stable linear subspace.

In this case, we also say that G is compatible with Θ or compatible with the Cartan decomposition of $U^{\mathbb{C}}$.

Given this setup, we have the subspace $\mathfrak{p} \subset \mathrm{i}\mathfrak{u}$ and, by restriction, an induced restricted momentum map

$$\mu_{\mathfrak{p}} \colon Z \longrightarrow \mathfrak{p},$$

which is the composition of the orthogonal projection of $\mathrm{i}\mathfrak{u}$ onto $\mathfrak{p}$ with μ.

For $\beta \in \mathfrak{p}$ and $x \in Z$, let

$$\mu_{\mathfrak{p}}^{\beta}(x) := \langle \mu(x), \beta \rangle.$$

Note that $\mu_{\mathfrak{p}}^{\beta} = \mu^{\beta}$ and that with our convention, the momentum map condition means that

$$\operatorname{grad} \mu_{\mathfrak{p}}^{\beta} = \beta_Z,$$

where β_Z is the vector field on Z corresponding to β and grad is computed with respect to the Riemannian metric induced by the Kähler structure. We call $\mu_{\mathfrak{p}}$ the G-gradient map associated with μ or the restricted momentum map. Note that the condition on μ means that for every $\xi \in \mathfrak{p}$, the vector field $(\mathrm{i}\xi)_Z$ is a Hamiltonian vector field on Z with Hamiltonian function μ_P^{ξ}.

For a G-stable subset X of Z, we may consider $\mu_{\mathfrak{p}}$ as a K-equivariant map

$$\mu_{\mathfrak{p}} \colon X \longrightarrow \mathfrak{p},$$

such that

$$\operatorname{grad} \mu_{\mathfrak{p}}^{\beta} = \beta_X$$

holds orbit-wise. The gradient can be computed on each G-orbit with respect to the induced Riemannian metric. The vector field β_X is given by the flow $(t, x) \mapsto \exp(t\beta) \cdot x$. Since X is G-stable, we have $\beta_Z(z) = \beta_X(z)$ for $z \in X$, and the map $\mu_{\mathfrak{p}}$ on X is the restriction of the map $\mu_{\mathfrak{p}} \colon Z \to \mathfrak{p}$. We are primarily interested in the case where X is a compact G-invariant subset of Z.

Remark 1.1. If X in the above setup is smooth, we refer to this as the smooth case. If X is a smooth complex submanifold of Z and G is a complex reductive subgroup of $U^{\mathbb{C}}$ that is compatible with the Cartan decomposition of $U^{\mathbb{C}}$, then the Kähler form ω on Z as well as the momentum map $\mu_{\mathfrak{p}}$ restrict to X, and we have $\mathfrak{p} = \mathrm{i}\mathfrak{k}$. We refer to this setup as the smooth complex case. This case must be distinguished from the case where X is smooth and complex and G is a non-complex, Θ-compatible Lie subgroup of $U^{\mathbb{C}}$.

We collect a few elementary properties of the restricted momentum map $\mu_{\mathfrak{p}} \colon X \to \mathfrak{p}$. For more details, see [HS07a].

For a subspace $\mathfrak{m} \subset \mathfrak{g}$ and $x \in X$, we set

$$\mathfrak{m} \cdot x = \{\xi_X(x) : \xi \in \mathfrak{m}\}, \qquad \mathfrak{m}_x := \{\xi \in \mathfrak{m} : \xi_X(x) = 0\}.$$

If x is a smooth point of X and V is a subspace of T_xX, let $V^{\perp}$ denote the orthogonal complement of V with respect to the Riemannian metric $(\,\cdot\,,\cdot\,)_x$ on the smooth part of X. Note that any point x is a smooth point in its G-orbit.

Lemma 1.2. *Let* $\mathfrak{m} \subset \mathfrak{p}$ *be a subspace. Then*

$$\ker \mathrm{d}\,\mu_{\mathfrak{m}}(x) = (\mathfrak{m} \cdot x)^{\perp}$$

on any smooth G*-invariant submanifold* X *of* Z*.*

Proof. This follows from $\operatorname{grad} \mu^{\beta} = \beta_Z$ for $\beta \in \mathfrak{p}$. □

Lemma 1.3. *For all* $\beta \in \mathfrak{p}$ *and* $x \in X$*, consider the curve* $\gamma\colon \mathbb{R} \to X$ *defined by* $\gamma(t) = \exp(t\beta) \cdot x$*. Then, one of the following holds:*

- (i) $x \in X^{\beta}$, *or*
- (ii) *the function* $t \mapsto \mu_{\mathfrak{p}}^{\beta}(\gamma(t))$ *is strictly increasing.*

Proof. This follows from the computation

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Big(\mu_{\mathfrak{p}}^{\beta} \circ \gamma\Big)(t) = \Big(\beta_X(\gamma(t)),\ \beta_X(\gamma(t))\Big)_{\gamma(t)}. \qquad \square$$

For $\beta \in \mathfrak{p}$, let $\mathcal{M}_{\mathfrak{p}}(\beta) := \mu_{\mathfrak{p}}^{-1}(\beta)$. Since $\beta = 0$ plays a prominent role, we set $\mathcal{M}_{\mathfrak{p}} := \mathcal{M}_{\mathfrak{p}}(0)$. Using Lemma 1.3 and the K-equivariance of $\mu_{\mathfrak{p}}$, we obtain the following (see [HS07a, Lemma 5.5]):

Corollary 1.4. *Let* β *be a* K*-fixed point in* $\mathfrak{p}$ *and* $x \in \mathcal{M}_{\mathfrak{p}}(\beta)$*. Then:*

1. $G \cdot x \cap \mathcal{M}_{\mathfrak{p}}(\beta) = K \cdot x$.
2. *The* G*-isotropy group* G_x *of* x *is compatible with the Cartan decomposition of* $U^{\mathbb{C}}$*. The decomposition is given by* $G_x = K_x \exp(\mathfrak{p}_x)$*.*

Remark 1.5. If $\mu_{\mathfrak{p}}$ is a restricted momentum map and $\alpha_0 \in \mathfrak{p}$ is a K-fixed point, then $\widetilde{\mu_{\mathfrak{p}}}\colon Z \to \mathfrak{p}$, defined by $\widetilde{\mu_{\mathfrak{p}}}(x) = \mu_{\mathfrak{p}}(x) + \alpha_0$, is also a restricted momentum map for the G-action on Z (see Remark 2.2 (3)).

If we assume Z to be connected, then up to an additive K-invariant constant in $\mathfrak{p}$, a restricted momentum map $\mu_{\mathfrak{p}}$ is uniquely defined by the Kähler structure on Z.

In Section 6, we need the following:

Proposition 1.6. *Let* X *be a* G*-invariant subset of* Z *and assume that* $\mu_{\mathfrak{p}}$ *is constant on* X*. Then, for every* $x \in X$*, the Lie algebra* $\mathfrak{g}_x$ *of the isotropy group* G_x *contains the ideal* $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ *of* $\mathfrak{g}$*.*

Proof. First, we note that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ show that $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ is an ideal of $\mathfrak{g}$. For the remaining part, it is sufficient to show the statement orbit-wise. Hence, we may assume that X is smooth.

For all $x \in X$, we have $\ker \mathrm{d}\,\mu_{\mathfrak{p}}(x) = (\mathfrak{p} \cdot x)^{\perp}$. Since $\mu_{\mathfrak{p}}$ is constant, this shows that $\mathfrak{p} \cdot x = \{0\}$. This implies $\mathfrak{p} \subset \mathfrak{g}_x$. Since $\mathfrak{g}_x$ is a Lie algebra, it also contains the ideal $[\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$. □

The Lie algebra $\mathfrak{g}$ of a Θ-compatible subgroup G is real reductive. It admits a direct sum decomposition:

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m,$$

where $\mathfrak{z}(\mathfrak{g})$ denotes the center of $\mathfrak{g}$ and the $\mathfrak{g}_j$ are the simple ideals of the semisimple part of $\mathfrak{g}$.

Since Cartan involutions of semisimple Lie algebras are unique up to conjugation, we obtain Cartan decompositions of the Lie algebras:

$$\mathfrak{g}_j = \mathfrak{k}_j \oplus \mathfrak{p}_j$$

for each j. Consequently, $\mathfrak{k}$ and $\mathfrak{p}$ decompose as:

$$\mathfrak{k} = \mathfrak{z}_{\mathfrak{k}} \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_m, \qquad \mathfrak{p} = \mathfrak{z}_{\mathfrak{p}} \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_m,$$

where $\mathfrak{z}_{\mathfrak{k}} = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k}$ and $\mathfrak{z}_{\mathfrak{p}} = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{p}$.

An ideal $\mathfrak{g}_j$ is called compact if $\mathfrak{g}_j = \mathfrak{k}_j$. Without loss of generality, we may assume that:

$$\mathfrak{g}_j = \mathfrak{k}_j \quad \text{for} \quad j = 1, \ldots, l.$$

The connected Lie subgroup G_{nc} of G with Lie algebra

$$\mathfrak{g}_{nc} = \mathfrak{z}_{\mathfrak{p}} \oplus \mathfrak{g}_{l+1} \oplus \cdots \oplus \mathfrak{g}_m$$

is called the non-compact factor of G. The subgroup G_{nc} of G is a normal, Θ-compatible subgroup of G, and G/G_{nc} is a compact Lie group.

Remark 1.7. If G is connected and K_1 is the subgroup of G with Lie algebra

$$\mathfrak{k} = \mathfrak{z}_{\mathfrak{k}} \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_l,$$

then K_1 is also a normal subgroup of G, and we have

$$G = K_1 G_{nc}.$$

Corollary 1.8. *Assume that $\mu_{\mathfrak{p}} \colon X \to \mathfrak{p}$ is constant and let $I := \bigcap_{x \in X} G_x$ denote the ineffectivity of the G-action on X. Then the following statements hold:*

1. *I is an normal subgroup of G which is compatible with the Cartan decomposition.*
2. *I contains the non-compact factor of G.*
3. *G/I is a compact Lie group. In particular every G-orbit is compact.*

Proof. By assumption, we have $X = \mathcal{M}_{\mathfrak{p}}(\beta)$ for a K-fixed point $\beta \in \mathfrak{p}$. Corollary 1.4 implies that all isotropy groups of G are Θ-compatible. Since intersections of Θ-compatible subgroups of G are Θ-compatible, this proves (1).

To prove (2), we need to show that the Lie algebra $\mathfrak{i}$ of I contains $\mathfrak{p}$. It suffices to consider the case where X is a G-orbit. Since $\mu_{\mathfrak{p}}$ is constant, we have: $T_x Y = \ker \mathrm{d}\,\mu_{\mathfrak{p}}(x) = \mathfrak{p} \cdot x^{\perp}$ for $Y := G \cdot x$. This implies $\mathfrak{p} \cdot x = 0$ for all $x \in X$, and thus $\mathfrak{p} \subset \mathfrak{i}$. Finally, part (3) follows from (2) or Corollary 1.8. $\square$

Remark 1.9. Under the assumption of Proposition 1.6, the G-action on X is given by the induced action of the compact group G/I. In particular, if G is a compatible subgroup of $U^{\mathbb{C}}$ without connected compact factors, then the group G acts on X as the finite group G/I. Moreover, if G is also connected, the G-action is trivial.

In the complex case, the ineffectivity $I$is a normal complex subgroup of G. Therefore, in the complex case, the assumption in Proposition 1.6 always implies that G/I is finite. If G is connected, the G-action on X is trivial.

2. The Slice Theorem

We will use the Slice Theorem of Luna type for Hamiltonian actions several times. It is valid for the action of a Θ-compatible subgroup G of $U^{\mathbb{C}}$ on Z or slightly more generally for the G-action on a G-invariant submanifold X of Z. The precise statement is as follows.

Recall that the points in a twisted product $G \times^H Y$ are the H-orbits in $G \times Y$ where H-acts by $(h, g, y) \mapsto (gh^{-1}, hy)$. In the following we set $[g, y] := H \cdot (g, y) \in G \times^H Y$ and identify Y with the subset $\{[e, y] : y \in Y\}$ of $G \times^H Y$.

Theorem 2.1 (Slice Theorem). *Let X be a G-invariant submanifold of Z and $x \in \mathcal{M}_{\mathfrak{p}}(\beta_0)$. Then the following statements hold:*

(1) *$G_x = K_x \exp(\mathfrak{p}_x)$ is a Θ-compatible subgroup of G (Corollary 1.4).*
(2) *For any G_x-equivariant splitting $T_xX = \mathfrak{g} \cdot x \oplus W$ of the G_x-representation T_xX, there exist:*
    (a) *an open, G_x-invariant neighborhood U of 0 in W,*
    (b) *an open, G-invariant neighborhood Ω of x in X,*

    and a G-equivariant diffeomorphism $\Psi \colon G \times^{G_x} U \to \Omega$ such that $\Psi([e, 0]) = x$.
(3) *In the complex case, the splitting can be chosen to be $\mathbb{C}$-linear, and the map Ψ is then a G-equivariant biholomorphism.*

Proof. This shown in [HS07a, Section 14]. □

Remarks 2.2.

(1) In the complex case, the Slice Theorem 2.1 has been proven in [HL94] and also in [Sja95].
(2) The group G_x acts on T_xX by the isotropy representation. Complete reducibility on T_xX for $x \in \mathcal{M}_{\mathfrak{p}}(\beta_0)$ follows from the fact that every $\alpha \in \mathfrak{p}_x$ is represented on $W = T_xX$ as a self-adjoint operator with respect to the K_x-invariant inner product $(\cdot\,,\cdot)_x$ on T_xX (see e.g. [HS07a, Lemma 14.7]).
(3) The proof of the Slice Theorem 2.1 can be reduced to the case where $\beta_0 = 0$. This is done using the shifting method which is also available for the G-action on X and is as follows. For any $\beta_0 = \mu_{\mathfrak{p}}(x)$ we replace Z be $Z \times O$, where $O = U \cdot \beta_0$. The manifold O is a coadjoint orbit and it is a generalized $U^{\mathbb{C}}$-homogeneous flag manifold. The action of U on O is Hamiltonian with momentum map $\mu_O(y) = -y$. Since O is a compact Kähler manifold, the U-action extends to a holomorphic $U^{\mathbb{C}}$-action. In particular the group G acts on O by holomorphic transformations and there is exactly one closed G-orbit which is a K-orbit (see [Wol69]) in O. It turns out that $K \cdot \beta_0$ is the unique closed G-orbit in O. On $Z \times O$ we have the product Kähler form with product momentum map $\mu + \mu_O$. The zero fiber contains (x, β_0). In the case where β_0 is a K-fixed point, we can replace Z be $Z \times O$ and identify X with the submanifold $X \times \{\beta_0\}$ of $Z \times O$. Then we can use the Slice Theorem 2.1 for the zero fiber of the new momentum mapping.

(4) If $x \in X$ is a G-fixed point, then $\beta_0 := \mu_{\mathfrak{p}}(x)$ is a K-fixed point. The Slice Theorem 2.1 is in this case a linearization theorem for the G-action in a G-invariant neighborhood of x. It implies that the set of G-fixed points X^G is a smooth submanifold of X.
(5) If in the complex case x is a K-fixed point, then it is $G = K^{\mathbb{C}}$-fixed. In this case the G-action can be holomorphically linearized.
(6) For a commutative Θ-compatible subgroup of G the Slice Theorem 2.1 holds at every $x \in X$. In particular it holds for every subgroup $\Gamma_\beta := \exp(\mathbb{R}\beta)$ where $\beta \in \mathfrak{p}$.
(7) If Y is a G-invariant subset of Z and $x \in \mathcal{M}_{\mathfrak{p}}(\beta_0) \cap Y$ then the Slice theorem for Z gives by pulling back a Slice Theorem for Y. The map $\Psi\colon G \times^{G_x} U_Y \to \Omega \cap Y$ is a G-equivariant homeomorphism, where U is identified with $\{[e,u] \in G \times^{G_x} U : u \in U\}$ and $U_Y := \Psi^{-1}(Y \cap \Psi(U))$.

If x is a $\Gamma_\beta := \exp(\mathbb{R}\beta)$ fixed point, then the group Γ_β acts on the tangent space T_xX via the isotropy representation $g \mapsto (\mathrm{d}\,g)(x)$. Here, $(\mathrm{d}\,g)(x)\colon T_xX \to T_xX$ denotes the derivative of $g\colon X \to X$, $y \mapsto g \cdot y$, at x for $g \in \Gamma$. The linear $\mathbb{R}$-action given by $(t,v) \mapsto \mathrm{d}(\exp(t\beta))(x) \cdot v$ on T_xX corresponds to the linearized vector field $\beta_{T_xX} = (\mathrm{d}\,\beta_X)(x)$. We view $\mathrm{d}\,\beta_X(x)$ as a linear operator on T_xX.

Applying the Slice Theorem 2.1 to the Γ_β-action at the fixed point $x \in X^\beta$ leads to the following linearization result.

Proposition 2.3. *Let X be a smooth G-invariant submanifold of Z and let $x \in X^\beta$. Assume there is given a Θ-compatible subgroup I^β such that $\Gamma_\beta \subset I^\beta \subset G_x$. Then the following holds:*

(1) *The tangent space decomposes I^β-invariantly as*

$$T_xX = W^{\beta-} \oplus W^\beta \oplus W^{\beta+},$$

where I^β-acts on T_xX via the isotropy representation, and:
  (i) *$W^{\beta-}$ is the sum of the eigenspaces corresponding to strictly negative eigenvalues,*
  (ii) *$W^{\beta+}$ is the direct sum of the eigenspaces corresponding to strictly positive eigenvalues, and*
  (iii) *$W^\beta = T_xX^\beta$ is the sum of the eigenspaces corresponding to the zero eigenvalue*

of the linear endomorphism $(\mathrm{d}\,\beta_X)(x)\colon T_xX \to T_xX$

Moreover, there exists an open I^β-stable neighborhood Ω of x in X, an open I^β invariant neighborhood U of $0 \in T_xX$, and an I^β-equivariant diffeomorphism $\Psi\colon U \to \Omega$.

(2) *In the complex setting, Ψ is biholomorphic and the decomposition of T_xX is complex linear.*

Remark 2.4. Since $G_x \subset G^\beta$ and G^β is the centralizer of β,the linear mappings $\mathrm{d}\,h(x)\colon T_xX \to T_x$ for $h \in I^\beta$ all commute with $\mathrm{d}\,\beta_X(x)$. Consequently, all eigenspaces of $\mathrm{d}\,\beta_X(x)$ remain invariant under the linearized action of I^β.

Remark 2.5. It is important to note that $W^{\beta-}$ consists exactly of those points $y \in T_xX$ for which the limit

$$\lim_{t \to +\infty} \exp(t\beta) \cdot y$$

exists (see Section 4). Moreover, the limit exists if and only if there exists some sequence $t_k \to +\infty$ such that the limit

$$\lim_{k \to +\infty} \exp(t_k\beta) \cdot y$$

exists.

These kinds of limits generally fail to hold without a Hamiltonian assumption. This failure extends even to holomorphic $G = \mathbb{C}^*$-actions at $\mathbb{C}^*$-fixed points. In particular, there exist compact complex $\mathbb{C}^*$-surfaces that admit an orbit whose closure is a singular complex curve with an isolated singularity, which is locally reducible at the fixed point (see, e.g., [Hau96]).

3. Hilbert quotients

Let H be a subgroup of G, and M a subset of an H-stable subset Y of Z. We define the saturation of M in Y with respect to H as

$$\mathcal{S}_H(M)(Y) := \{z \in Y : \overline{H \cdot z} \cap M \neq \emptyset\},$$

where $\overline{H \cdot z}$ denotes the closure of the orbit $H \cdot z$ in Y. In general, $\mathcal{S}_H(M)(Y)$ is a proper subset of $\mathcal{S}_H(M)(Z) \cap Y$. However, if Y is closed in Z, these sets coincide.

The set $\mathcal{S}_G(\mu_{\mathfrak{p}})(Y)$ is called the set of semistable points of Y with respect to $\mu_{\mathfrak{p}}$. It has been shown in [HS07b] that $\mathcal{S}_G(\mu_{\mathfrak{p}}(\beta))(Z)$ is open in Z for every $\beta \in \mathfrak{p}$. In the smooth complex case, this result is also contained in [Kir84]. For a closed G-stable subset Y of Z, this implies that $\mathcal{S}_G(\mu_{\mathfrak{p}}(\beta))(Y)$ is open in Y. Since we have fixed X, we will usually write $\mathcal{S}_G(\mu_{\mathfrak{p}}(\beta))$ instead of $\mathcal{S}_G(\mu_{\mathfrak{p}}(\beta))(X)$. The set of semistable points plays a central role in [HS07a]. One reason for this is the following quotient theorem:

Let $Y \subset Z$ be G-invariant, and let $x, y \in Y$. We define a relation $\sim$ on Y by declaring $x \sim y$ if and only if

$$Y \cap \overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset.$$

If this relation is an equivalence relation, we denote the corresponding quotient by $Y /\!\!/ G$ and call it the topological Hilbert quotient of Y by the action of G.

Theorem 3.1 (Quotient Theorem [HS07a]). *Let $\beta_0 \in \mathfrak{p}$ be a K-fixed point. Then the topological Hilbert quotient $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p}}(\beta_0)) /\!\!/ G$ exists and has the following properties.*

1. *Every fiber of the quotient map $\pi \colon \mathcal{S}_G(\mathcal{M}_{\mathfrak{p}}(\beta_0)) \to \mathcal{S}_G(\mathcal{M}_{\mathfrak{p}}(\beta_0)) /\!\!/ G$ contains a unique closed G-orbit. Any other orbit in the fiber has strictly larger dimension.*
2. *The closure of every G-orbit in a fiber of π contains the closed G-orbit.*
3. *Every fiber of π intersects $\mathcal{M}_{\mathfrak{p}}(\beta_0)$ in a unique K-orbit which lies in the unique closed G-orbit.*
4. *The inclusion $\mathcal{M}_{\mathfrak{p}} \hookrightarrow \mathcal{S}_G(\mathcal{M}_{\mathfrak{p}}(\beta_0))$ induces a homeomorphism $\mathcal{M}_{\mathfrak{p}}/K \cong \mathcal{S}_G(\mathcal{M}_{\mathfrak{p}}(\beta_0)) /\!\!/ G$.*
5. *In the complex case, the quotient $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p}}(\beta_0)) /\!\!/ G$ is a complex space and the quotient map $\pi \colon \mathcal{S}_G(\mathcal{M}_{\mathfrak{p}}(\beta_0)) \to \mathcal{S}_G(\mathcal{M}_{\mathfrak{p}}(\beta_0)) /\!\!/ G$ is holomorphic.*

Remarks 3.2. After shifting, we may assume that $\beta_0 = 0$ and $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p}}(\beta_0)) = \mathcal{S}_G(\mathcal{M}_{\mathfrak{p}})$. If this set is non-empty, then the following statements hold:

(1) $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p}})$ coincides with an open stratum in the sense of [HSS08]. In the complex case, this stratum is the minimal stratum associated with the norm square of the (shifted) momentum map in the sense of [Kir84].

(2) For $x, y \in \mathcal{S}_G(\mathcal{M}_{\mathfrak{p}})$, we have $\pi(x) = \pi(y)$ if and only if $f(x) = f(y)$ holds for every continuous G-invariant function $f \colon \mathcal{S}_G(\mathcal{M}_{\mathfrak{p}}) \to \mathbb{R}$. In other words, the quotient $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p}})/G$ is determined by the algebra of G-invariant continuous functions on $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p}})$. It is worth noting that, in the real algebraic setting, it is not sufficient to use G-invariant polynomials to separate the π-fibers. A typical example is given by the $\mathbb{R}$-action on $\mathbb{R} \setminus \{0\}$ defined by $t \cdot x = t^2 x$.

(3) In the complex case, the sheaf of holomorphic functions on the quotient is the sheaf of invariant holomorphic functions on $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p}})$, and the quotient map π is a Stein map. That is, preimages of Stein subspaces of the quotient are Stein subspaces of $\mathcal{S}_G(\mathcal{M}_{\mathfrak{p}})$ (see [HMP98] for more details). Such Hilbert quotients are called analytic Hilbert quotients.

4. Parabolic subgroups

Let G be a Θ-compatible subgroup of $U^{\mathbb{C}}$ and $G = K \exp \mathfrak{p}$ the Cartan decomposition of G.

Let $\beta \in \mathfrak{p}$. We have the vector field β_G whose one-parameter subgroup is given by $(t, y) \mapsto \exp(t\beta) y \exp(-t\beta)$. Then,

$$G^{\beta} = \{y \in G : \beta_G(y) = 0\} = \{y \in G : \exp(t\beta) y \exp(-t\beta) = y \ \text{ for all } t \in \mathbb{R}\}$$

is the centralizer of β in G. We have (see for example [BGH13, Section 2.3]) the parabolic subgroup

$$G^{\beta+} := \Big\{ y \in G : \lim_{t \to -\infty} \exp(t\beta) y \exp(-t\beta) \text{ exists} \Big\}$$

with unipotent radical

$$R^{\beta+} := \Big\{ y \in G : \lim_{t \to -\infty} \exp(t\beta) y \exp(-t\beta) = e \Big\}.$$

The group $G^{\beta+}$ is the semi-direct product of G^{β} with $R^{\beta+}$ and we have the projection $\pi^{\beta+} \colon G^{\beta+} \to G^{\beta}$, $\pi^{\beta+}(y) := \lim_{t \to -\infty} \exp(t\beta) y \exp(-t\beta)$. The opposite of $G^{\beta+}$ is the parabolic subgroup $G^{\beta-} = \Theta(G^{\beta+}) = G^{\beta} \cdot R^{\beta-}$, where

$$R^{\beta-} := \Big\{ y \in G : \lim_{t \to +\infty} \exp(t\beta) y \exp(-t\beta) = e \Big\}.$$

Remark 4.1. Let Q be a parabolic subgroup of G with Levi decomposition $Q = R \rtimes L$. The exponential map

$$\exp \colon \mathfrak{r} \longrightarrow R$$

is an L-equivariant isomorphism, where the action is given by conjugation. That is, for all $h \in L$ and $\xi \in \mathfrak{r}$, the following holds:

$$\exp(\mathrm{Ad}(h)\xi) = h \exp(\xi) h^{-1}.$$

Example 4.2. For the group $G = \mathrm{SL}_n(\mathbb{R})$, one possible choice for the subspace $\mathfrak{p}$ is the space of symmetric matrices of trace zero. By conjugating with an element of $K = \mathrm{SO}_n(\mathbb{R})$, we may assume that $\beta \in \mathfrak{p}$ is a diagonal matrix.

If all eigenvalues of β are distinct, then G^β consists exactly of the diagonal matrices in G. In this case, the subgroups $R^{\beta-}$ and $R^{\beta+}$ correspond to the lower triangular matrices and upper triangular matrices, respectively.

If the eigenvalues of β have multiplicities, then G^β consists of block diagonal matrices, and analogous adjustments apply to $R^{\beta-}$ and $R^{\beta+}$.

We introduce subsets of a G-invariant closed subset X of Z, analogous to G^β, $G^{\beta+}$, and $G^{\beta-}$. For $\beta \in \mathfrak{p}$, consider the vector field β_X on X along G-orbits, whose one-parameter subgroup is given by $(t, y) \mapsto \exp(t\beta)\cdot y$. The set of $\Gamma_\beta := \exp \mathbb{R}\beta$-fixed points is

$$X^\beta = \{y \in X : \beta_X(y) = 0\} = X \cap Z^\beta.$$

For a real number r, define

$$X(\beta, r) := \Big\{x \in X : \overline{\exp \mathbb{R}\beta \cdot x} \cap (\mu^\beta)^{-1}(r) \neq \emptyset\Big\}.$$

Since X is closed in Z, these subsets are obtained by intersecting the analogous subsets of Z with X.

If non-empty, the set $X(\beta, r)$ can be interpreted as a set of semistable points in X for the action of the subgroup $\Gamma_\beta = \exp \mathbb{R}\beta$, in the sense of Section 3 (see [HSS08, Proposition 4.3] and the correction in arxiv:math/0611491v2). Note that Γ_β is a Θ-compatible subgroup of G.

Since $\operatorname{grad} \mu^\beta(y) = \beta_Z(y) = \beta_X(y)$ holds for every $y \in X$ and $\beta \in \mathfrak{p}$, the set

$$X_r^\beta := X(\beta, r) \cap Z^\beta = X(\beta, r) \cap X^\beta$$

is both open and closed in X^β. We have $G^\beta = K^\beta \exp \mathfrak{p}^\beta$ and the map $\mu_{\mathfrak{p}}^\beta$ is K^β-invariant. This implies that G^β stabilizes X_r^β, and thus also the subset

$$X_r^{\beta-} := \Big\{y \in X(\beta, r) : \lim_{t\to+\infty} \exp t\beta \cdot y \text{ exists in } X(\beta, r)\Big\}.$$

Remark 4.3. For future reference, we note the following:

(1) The Slice Theorem 2.1 implies that $X_r^{\beta-}$ is well-defined (see Proposition 2.3 and Remark 2.4 or the above reference [HSS08]). If $g \in G^{\beta-}$ and $y \in X^{\beta-}$, then

$$\begin{aligned}\lim_{t\to+\infty} \exp(t\beta)\cdot(g\cdot y) &= \lim_{t\to+\infty}\Big(\exp(t\beta)\, g \lim_{t\to+\infty}\exp(-t\beta)\Big)\Big(\lim_{t\to+\infty}\exp(t\beta)\cdot y\Big)\\ &= h\cdot x,\end{aligned}$$

where $h \in G^\beta$ and $x \in X_r^\beta$. This shows that $X_r^{\beta-}$ is a $G^{\beta-}$-stable set.

Similarly, we define the $G^{\beta+}$-stable set

$$X_r^{\beta+} := \Big\{y \in X(\beta, r) : \lim_{t\to-\infty} \exp t\beta \cdot y \text{ exists in } X(\beta, r)\Big\}.$$

(2) The map $p^{\beta-} : X_r^{\beta-} \to X_r^\beta$, defined by $p^{\beta-}(y) = \lim_{t\to+\infty} \exp t\beta \cdot y$, is well-defined. The calculation in (1) shows that for $g = rh \in G^{\beta-} = R^{\beta-}$ and $y \in X^{\beta-}$, we have

$$p^{\beta-}(g\cdot y) = h \cdot p^{\beta-}(y).$$

In other words, $p^{\beta-}\colon X_r^{\beta-} \to X_r^{\beta}$ is a $G^{\beta-}$-equivariant fibration, where on X^β we consider the $G^{\beta-}$-action given by the homomorphism $G^{\beta-} \to G^\beta$, $rh \mapsto h$, i.e., the map $p^{\beta-}\colon X_r^{\beta-} \to X_r^{\beta}$ is $R^{\beta-}$-invariant and G^β-equivariant.

Remark 4.4. One can define the above subsets for any G-invariant subset X of Z. In this case we have for $X_r^\beta = Z_r^\beta \cap X$ and $X_r^{\beta-} \subset Z_r^{\beta-} \cap X$. If X is closed in Z, then $X_r^{\beta-} = Z_r^{\beta-} \cap X$ holds and similar for the $+$-case.

In the next section, we will also require the following result regarding the Hessian of μ^β at critical points.

Lemma 4.5. *Let $x \in X^\beta$ be a smooth point and let $v \in T_x(X)$ be an eigenvector of $\beta \in \mathfrak{p}_x$ with eigenvalue $\lambda(\beta)$. Suppose $\gamma(t)$ is a smooth curve in X such that $\gamma(0) = x$ and $\frac{\mathrm{d}}{\mathrm{d}\,t}\gamma(0) = v$. Then*

$$\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}\Big(\mu_{\mathfrak{p}}^\beta \circ \gamma\Big)(0) = \lambda(\beta)\|v\|^2.$$

Proof. This follows from a direct calculation (see [HSS08, Proposition 2.5]). ☐

Remark 4.6. A calculation analogous to that in the proof of [HSS08, Proposition 2.5] shows that the eigenspaces of $(\mathrm{d}\,\beta_X)(x)$ are orthogonal with respect to the Hessian of μ^β at the point x. Consequently, Lemma 4.5 implies that μ^β is a Morse–Bott function.

Moreover, if x is a maximum of $\mu^\beta(X)$, then $W^{\beta+} = \{0\}$.

5. Extreme points

In this section X denotes a compact G-invariant subset of Z. Unless otherwise stated, we do not assume X to be smooth. As before, the group G is assumed to be compatible with Θ.

Let σ be an extreme point of the convex envelope E of $\mu_{\mathfrak{p}}(X)$ and K^σ the centralizer of σ in K.

Remark 5.1. Let $\mathfrak{a}$ be a maximal subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. Then $A = \exp \mathfrak{a}$ is a Θ-compatible subgroup of G. The convex hull P of $\mu_{\mathfrak{a}}(X)$ is simpler to analyze than E and captures all relevant information about the extreme points of E. Specifically, P is the convex hull of the finite subset $\mu_{\mathfrak{a}}(X^A)$ ([BGH16, Proposition 3.1]) and is therefore a convex polytope. In particular, every face of P es exposed.

An extreme point of E is a face of E of zero dimension. Theorem 0.2 in [BGH16] applied to extreme points of E shows that every extreme point σ of E lies in the K-orbit of an extreme point of P. Up to the action of the Weyl group on $\mathfrak{a}$, this extreme point of P is uniquely determined by σ.

In the smooth complex connected case, it is well known that $\mu_{\mathfrak{a}}(X)$ is convex ([Ati82; GS82]). This remains the case if X is only assumed to be an irreducible analytic subset of Z. In the general compact case, however, it is only known that $\mu_{\mathfrak{a}}(X)$ is a finite union of convex polytopes ([BGH16, Proposition 3.1]; see also [HS10]) and the question of convexity remains open.

Every face of E is exposed ([BGH16, Theorem 0.2]). In particular, the set C_σ of $\beta \in \mathfrak{p}$ which expose σ is non-empty. By definition $\beta \in \mathfrak{p}$ exposes σ if

$$E \cap H^+_{\langle \beta, \sigma \rangle}(\beta) = \{\sigma\}$$

where for any $r \in \mathbb{R}$ we define

$$H^+_r(\beta) := \{\alpha \in \mathfrak{p} : \langle \beta, \alpha \rangle \geq r\}.$$

The set C_σ is a K^σ-invariant convex cone in $\mathfrak{p}$. Since K^σ is compact and acts linearly on $\mathfrak{p}$, the set $C_\sigma^{K^\sigma}$ of K^σ-fixed points in C_σ is non empty.

For an extreme point σ of E and $\beta \in C_\sigma$, we introduce the following notation. Let $r := \max \mu^\beta(X)$ and define:

$$X^\beta_{\max} := X^\beta_r, \quad X^{\beta-}_{\max} := X^{\beta-}_r$$

(see Section 4).

We also define the map

$$p^{\beta-} : X^{\beta-}_{\max} \longrightarrow X^\beta_{\max}, \quad y \longmapsto \lim_{t \to +\infty} \exp(t\beta) \cdot y.$$

Next, we summarize the fundamental properties of the fibers of extreme points in E:

Proposition 5.2. *For $\beta \in \mathfrak{p}$ and $x \in X^\beta_{\max}$ we have*

$$\beta_X(x) = 0.$$

Moreover, using the notation of Proposition 2.3, it follows that $W^{\beta+} = \{0\}$ if X is smooth at x.

Proof. Since x is a maximum of the function $\mu^\beta | G \cdot x$ it follows that $\beta_X(x) = 0$. The result $W^{\beta+} = \{0\}$ is then a consequence of Remark 4.6. □

Proposition 5.3. *Let σ be an extreme point of E.*

1. *For every $\beta \in C_\sigma$ we have $\mu_{\mathfrak{p}}^{-1}(\sigma) = X^\beta_{\max}$.*
2. *$X^\beta_{\max}$ is open and closed in X^β and G^β-stable.*
3. *For $\beta \in C_\sigma$ every G^β-orbit in $X^\beta_{\max}$ is compact and $X^\beta_{\max} \subset X^{\mathfrak{p}^\beta}$.*
4. *For $\beta \in C_\sigma$ and all $x \in X^\beta_{\max}$ we have $\mathfrak{r}^{\beta+} \cdot x = \{0\}$*
5. *For $\beta \in C_\sigma^{K^\sigma}$ we have $K^\sigma = K^\beta$.*
6. *For $\beta \in C_\sigma^{K^\sigma}$ and $x \in X^\beta_{\max}$ we have $\xi_X(x) \neq 0$ for all non zero $\xi \in \mathfrak{r}^{\beta-}$.*

Proof.

- Part (1) is a special case of [BGH16, Lemma 3.1].
- Since μ^β is locally constant on X^β, we obtain the first part of (2). The function μ^β is K^β-invariant, and X^β is G^β-stable. Because $G^\beta = K^\beta \exp(\mathfrak{p}^\beta)$ and $\exp(\mathfrak{p}^\beta)$ is connected, it follows that $X^\beta_{\max}$ is G^β-stable.
- Since $\mu_{\mathfrak{p}} | X^\beta_{\max}$ is constant by (1), the restricted momentum map for the G^β-action on $X^\beta_{\max}$ is constant. Proposition 1.6 now implies (3).
- For (4), we apply Proposition 2.3 in the case where $X = G \cdot x$. Using the notation from there, we have $W^{\beta+} = \{0\}$ for every $x \in X^\beta_{\max}$. This implies

$$\mathfrak{r}^{\beta+} \cdot x = \{0\}.$$

- If β is K^σ-fixed, then $K^\sigma \subset K^\beta$. Since $\mu_{\mathfrak{p}}$ is K-equivariant and K^σ is the K-isotropy group at σ, we also have $K^\beta \subset K^\sigma$. This shows (5).

- Let $\xi \in \mathfrak{r}^{\beta -}$ such that $\xi_X(x) = 0$. Since $\mathfrak{r}^{\beta +} \cdot x = \{0\}$ and $\theta(\xi) \in \mathfrak{r}^{\beta +}$, we obtain
$$(\xi + \theta(\xi))_X(x) = 0.$$
This implies that $\xi + \theta(\xi) \in \mathfrak{k}_x \subset \mathfrak{k}^\sigma = \mathfrak{k}^\beta \subset \mathfrak{g}^\beta \subset \mathfrak{g} = \mathfrak{r}^{\beta -} + \mathfrak{g}^\beta + \mathfrak{r}^{\beta +}$. Since the last sum is direct, we conclude that $\xi = 0$.

□

Remarks 5.4.

(1) Since $X^\beta_{\max} \subset X^{\mathfrak{p}^\beta}$ and $G^\beta = K^\beta \cdot \exp(\mathfrak{p}^\beta)$, it follows that
$$G^\beta \cdot x = K^\beta \cdot x \quad \text{for all} \quad x \in X^\beta_{\max}.$$

(2) For $\beta \in C_\sigma^{K^\sigma}$, Proposition 5.3 (5) shows that $X^\beta_{\max}$ is K^β-stable. Furthermore, Proposition 5.3 (3) implies that the ineffectivity
$$I^\beta := \bigcap_{z \in X^\beta_{\max}} (G^\beta)_z$$
of the G^β-action on $X^\beta_{\max}$ is a closed normal subgroup of G^β. The Lie algebra of I^β contains the ideal
$$\left[\mathfrak{p}^\beta, \mathfrak{p}^\beta\right] \oplus \mathfrak{p}^\beta$$
of $\mathfrak{g}^\beta$.

Corollary 5.5. *Let σ be an extreme point of E and $\beta \in C_\sigma$. Then the following statements hold:*

(1) *For every $x \in X^\beta_{\max}$, the orbit $G \cdot x$ is a K-orbit.*
(2) *$X^{\beta -}_{\max}$ is open in X.*
(3) *If X is connected and we are in the complex case, then $X^{\beta -}_{\max}$ is dense in X.*

Proof. Let $x \in X^\beta_{\max}$ be a fixed point.

By Proposition 5.3 (4), we have
$$R^{\beta +} \cdot x = \{x\}.$$
Since $G^{\beta +} = G^\beta R^{\beta +}$ is a parabolic subgroup of G, it follows that $G = KG^{\beta +}$. This implies
$$G \cdot x = KG^\beta \cdot x.$$
Applying Proposition 5.3 (3) again, we obtain
$$G \cdot x = K \cdot G^\beta \cdot x = K \cdot x,$$
which proves (1).

For (2), we apply Remark 4.6. It implies, in particular, that $X^{\beta +}$ is an open Morse stratum for μ^β.

Since in the complex case the open Morse stratum is unique, we also obtain (3). □

Corollary 5.6. *Assume that G is connected and that we are in the complex case. Then the following statements hold:*

(1) *For $\beta \in C_\sigma$, we have $X^\beta_{\max} \subset X^{G^\beta}$.*
(2) *For $\beta \in C_\sigma^{K^\sigma}$, we have $G^\beta = G^\sigma$.*

Proof. Since G^β is connected for a connected complex reductive group $G = K^{\mathbb{C}}$, $\mathfrak{p}^\beta = i\mathfrak{k}^\beta$, and $G^\beta = K^\beta \exp(\mathfrak{p}^\beta)$ it follows that

$$X^{\mathfrak{p}^\beta} = X^{\mathfrak{g}^\beta} = X^{G^\beta}. \qquad \square$$

Remark 5.7. As shown in [BGH16, Lemma 3.4], if $\beta \in C_\sigma^{K^\sigma}$ then bundle

$$p^{\beta-} : X^{\beta-} \longrightarrow X^\beta_{\max}$$

is independent of the choice of $\beta \in C_\sigma^{K^\sigma}$.

6. The structure theorem

In this section, we prove a local structure theorem at any point $x \in \mu_{\mathfrak{p}}^{-1}(\sigma)$, where σ is an extreme point of the convex envelope E of $\mu_{\mathfrak{p}}(X)$, and X is a G-invariant compact submanifold of Z.

To proceed, fix an element $\beta \in C_\sigma^{K^\sigma}$ (see Proposition 5.3(5)). Let I^β be the ineffectivity of the G^β-action on $X^\beta_{\max}$. By Corollary 1.8, I^β is a Θ-compatible normal subgroup of G^β. Furthermore, its Lie algebra contains the ideal $[\mathfrak{p}^\beta, \mathfrak{p}^\beta] + \mathfrak{p}^\beta$, and I^β is a Θ-compatible normal subgroup of G^β that contains the non-compact factor of G^β.

Remark 6.1. In many cases, we have $G^\beta = I^\beta$, and consequently, $G^{\beta-} = R^{\beta-} \rtimes G^\beta$ holds as well. For example this occurs in the following situations:

1. The group G^β is connected and does not contain a non-trivial normal compact subgroup.
2. The group G is connected, and we are in the complex case (Corollary 5.6).
3. The preimage $(p^{\beta-})^{-1}(\sigma)$ is a singleton, i.e., $(p^{\beta-})^{-1}(\sigma) = \{x\}$.

Proposition 6.2. *The map*

$$p^{\beta-} : X^{\beta-}_{\max} \longrightarrow X^\beta_{\max}$$

realizes $X^\beta_{\max}$ as the topological Hilbert quotient of $X^{\beta-}_{\max}$ with respect to the Γ_β and also the I^β-action.

Proof. Let $I^\beta = L^\beta \exp(\mathfrak{m}^\beta)$ be the Cartan decomposition of the Θ-compatible group I^β. The momentum map

$$\mu_{\mathfrak{m}} : X^\beta_{\max} \longrightarrow \mathfrak{m}$$

is constant with value $\beta_0 \in \mathfrak{m}$ and satisfies

$$\mu_{\mathfrak{m}}^{-1}(\beta_0) = X^\beta_{\max}.$$

Since L^β acts trivially on $X^\beta_{\max}$, the claim follows from Theorem 3.1. $\square$

Remark 6.3. Proposition 6.2 can also be verified directly using Proposition 6.4 below. Note that the only closed Γ^β-orbit in $W^{\beta-}$ is $\{0\}$. Since I^β contains Γ^β, $\{0\}$ is also the only closed orbit of I^β.

Moreover, the only Γ_β-invariant neighborhood of zero in $W^{\beta-}$ (where $\Gamma_\beta := \exp(\mathbb{R}\beta)$) is the entire space $W^{\beta-}$.

Since I^β is Θ-compatible and fixes $X^\beta_{\max}$ pointwise we may apply the Slice Theorem 2.1 at a point $x \in X^\beta_{\max}$. For the I^β-representation on T_xX there exist an I^β-equivariant splitting:

$$T_xX = W^{\beta-} \oplus W^\beta$$

(i) W^β is the submodule of I^β-fixed points, and
(ii) $W^{\beta-}$ is the I^β-submodule on which the linearized vector field $(d\beta_X)(x)$-acts with strictly negative eigenvalues.

Proposition 6.4. *By the Slice Theorem 2.1, we obtain the following. There exists:*

(i) *an open neighborhood U of 0 in W^β,*
(ii) *an open neighborhood V of x in $X^\beta_{\max}$,*
(iii) *a smooth isomorphism $\psi\colon U \to V$, and*
(iv) *an I^β-equivariant smooth isomorphism $\Psi\colon W^{\beta-} \times U \to (p^{\beta-})^{-1}(V)$,*

such that the following diagram commutes:

$$\begin{tikzcd}
W^{-}\beta_{-} \times U \arrow[r, "\Psi"] \arrow[d, "\pi_U"'] & (p^{\beta-})^{-1}(V) \arrow[d, "p^{\beta-}"] \\
U \arrow[r, "\psi"] & V
\end{tikzcd}$$

where π_U denotes the projection onto the second factor U.

Proof. By the Slice Theorem (Proposition 2.3), there exists an open I^β-stable neighborhood Ω_1 of $0 \in T_xX$ and an open I^β-stable neighborhood Ω_2 of x in X and a I^β-equivariant diffeomorphism $\Psi\colon \Omega_1 \to \Omega_2$ with $\Psi(0) = x$.

Since Γ_β is contained in I^β and the weights of $d\beta_X(x)$ are strictly negative we may assume, after possibly shrinking Ω_1, that $\Omega_1 = W^{\beta-} \times U$ for some open neighborhood U of 0 in W^β. Because I^β preserves the fibers of π_U, the set $W^{\beta-} \times U$ is I^β-invariant. Moreover, since Ψ maps the set of I^β-fixed points into the set of I^β-fixed points, its image Ω_2 is I^β-invariant.

Since $\Psi|U$ has constant rank, it follows that $V := \Psi(U)$ is an open subset of $X^\beta_{\max}$ and that the map $\psi\colon U \to V$, defined by $\psi(y) = \Psi(y)$ is a diffeomorphism. Furthermore, we have $\Omega_2 = (p^{\beta-})^{-1}(V)$.

To verify the commutativity of the diagram, note that for any $w \in p^{\beta-})^{-1}(V)$ there exists $y \in V$ such that $p^{\beta-}(w) = y \in V$. Since $V \subset \Omega_2$, there exists $t_1 \in \mathbb{R}$ such that $w_1 := \exp(t_1\beta) \cdot w \in \Omega_2$. By the definition of Ψ, we have $w_1 = \Psi(z_1)$ for some $z_1 \in W^{\beta-} \times U$. Thus, $\Psi(z_1) = w_1$ which shows that $\Omega_2 = (p^{\beta-})^{-1}(V)$. This completes the proof of Proposition 6.4. □

Remark 6.5. The fibers of $p^{\beta-}$ are invariant under the $R^{\beta-}$-action, which is free (Proposition 5.3 (6)). Moreover, over the connected component of $X^\beta_{\max}$ containing x, each fiber is I^β-equivariantly isomorphic to the I^β-representation $W^{\beta-}$, i.e., $p^{\beta-}$ is a locally trivial fibration with typical fiber $W^{\beta-}$.

Additionally, in a product chart defined by Ψ, we obtain a smooth action of $R^{\beta-}$ on $W^{\beta-} \times U$ such that Ψ is $R^{\beta-}$-equivariant. In other words, this yields a family of $R^{\beta-}$-actions on $W^{\beta-}$ parametrized by U.

Lemma 6.6. *The tangent space $\mathfrak{r}^{\beta-} \cdot x$ of the $R^{\beta-}$-orbit through x is an I^β-submodule of $W^{\beta-}$. The differential $\varphi_x\colon \mathfrak{r}^{\beta-} \to \mathfrak{r}^{\beta-} \cdot x$ of the orbit map $\Phi_x\colon R^{\beta-} \to X$, $g \mapsto g \cdot x$, is an I^β-equivariant isomorphism.*

Proof. For $h \in I^\beta$ and $g \in R^{\beta-}$ we have $hgh^{-1} \cdot x = h \cdot (g \cdot x)$. This shows that φ_x is an I^β-equivariant surjective map. Proposition 5.3 (6) implies injectivity of φ_x. Since $W^{\beta-}$ is the tangent space of the fiber of $(p^{\beta-})^{-1}(x)$ which contains $R^{\beta-} \cdot x$ we also have $\mathfrak{r}^{\beta-} \cdot x \subset W^{\beta-}$. □

Remark 6.7. If β is represented on a finite-dimensional Euclidean vector space W as a semisimple operator with strictly negative eigenvalues, then we have

$$\Gamma_\beta \cdot D = W$$

for any neighborhood D of 0 in W. For example, this is the case for the representation $\beta \mapsto \operatorname{ad}(\beta)$ on $\mathfrak{r}^{\beta-}$.

Since $\exp\colon \mathfrak{r}^{\beta-} \to R^{\beta-}$ is a Γ_β-equivariant diffeomorphism, we obtain

$$\Gamma_\beta \cdot N = R^{\beta-}$$

for any neighborhood N of $e \in R^{\beta-}$.

Theorem 6.8 (Local Structure Theorem C)**.** *Let X be a compact G-invariant submanifold of Z, σ an extreme point of the convex envelope of $\mu_\mathfrak{p}(X)$ in $\mathfrak{p}$, $\beta \in C_\sigma^{K^\sigma}$ and $x \in (p^{\beta-})^{-1}(\sigma)$.*

Let $W^{\beta-} = \mathfrak{r}^{\beta-} \cdot x \oplus F$ be an I^β-equivariant splitting and $\Psi\colon W^{\beta-} \times U \to \Omega$ the diffeomorphism obtained by the Slice Theorem 2.1. Then we can choose the open neighborhood U of $0 \in W^\beta$ such that the map

$$\Phi\colon R^{\beta-} \times F \times U \longrightarrow (p^{\beta-})^{-1}(V), \qquad (g, v, u) \longmapsto g \cdot \Psi(v, u)$$

is a diffeomorphism.

The map is $R^{\beta-} \rtimes I^\beta$-equivariant where the action on $R^{\beta-} \times F \times U$ is given by $(r, h), (g, v, u) \mapsto (rgh^{-1}, h \cdot v, u)$.

Proof. By definition, the map Φ is equivariant.

We note that β is represented as a semisimple operator on $\mathfrak{r}^{\beta-} \oplus F$ with strictly negative eigenvalues.

Let $\gamma\colon \mathbb{R} \to I^\beta$ be the one-parameter group defined by $\gamma(t) = \exp(t\beta)$. For every $(g, v, u) \in R^\beta \times F \times U$, we have

$$\lim_{t\to+\infty} \gamma(t) \cdot (g, v, u) = \lim_{t\to+\infty} \Big(\gamma(t) g \gamma(t)^{-1},\ \gamma(t) \cdot v, u\Big) = (e, 0, u).$$

Additionally,

$$\lim_{t\to+\infty} \gamma(t) \cdot \Phi(g, v, u) = \lim_{t\to+\infty} \Big(\gamma(t) g \gamma(t)^{-1}\Big) \cdot \Psi(\gamma(t) \cdot v, u) = \Psi(0, u).$$

The differential of Φ at the point $(e, 0, 0)$ is an isomorphism.

Every neighborhood of $(e, 0, 0)$ contains a product neighborhood $B := N \times D \times \widetilde{U}$ of $(e, 0, 0)$ such that $\Phi|_B$ is a diffeomorphism. We have $\Gamma_\beta \cdot B = R^{\beta-} \times F \times \widetilde{U}$. Since Φ is Γ_β-equivariant, it is a local diffeomorphism on $R^{\beta-} \times F \times \widetilde{U}$. After possibly shrinking U, we may assume that $\widetilde{U} = U$. Since $\Psi(\{e\} \times \{0\} \times U) = \psi(U) = V$, the map is surjective.

It remains to show that

$$\Phi\colon R^{\beta-} \times F \times U \longrightarrow (p^{\beta-})^{-1}(V)$$

is injective. For $a, b \in R^{\beta-} \times F \times U$, there exists a $t \in \mathbb{R}$ such that $\gamma(t) \cdot a, \gamma(t) \cdot b \in B$. If $\Phi(\gamma(t) \cdot a) = \Phi(\gamma(t) \cdot b)$, then $\gamma(t) \cdot a = \gamma(t) \cdot b$ due to the injectivity of Φ on B. This shows $a = b$. $\square$

Remark 6.9.

(1) In the complex case and for a connected G we have $I^\beta = G^\beta$. The map Φ is biholomorphic and the inclusion $R^{\beta-} \times F \subset G^{\beta-} \times F$ induces a $G^{\beta-}$-equivariant biholomorphic map

$$R^{\beta-} \times F \times U \longrightarrow \left(G^{\beta-} \times^{G^\beta} F\right) \times U.$$

(2) In the real case the inclusion $R^{\beta-} \times F \times U \subset (I^{\beta-} \times F) \times U$ induces an $I^{\beta-}$-equivariant diffeomorphism

$$R^{\beta-} \times F \times U \longrightarrow \left(\left(R^{\beta-} \rtimes I^\beta\right) \times^{I^\beta} F\right) \times U.$$

(3) If we replace $X^\beta_{\max}$ by some union C of its connected components, then in a similar way as above we can define I^β for this union C. Then Theorem 6.8 holds for this modified I^β and any $x \in C$.

(4) In the complex connected case $X^\beta_{\max}$ is connected (see, e.g., [Ati82]).

Corollary 6.10. *Let $\{x\}$ be a connected component of $X^\beta_{\max}$ (i.e., an isolated point). Then the following statements hold:*

(1) *I^β contains the identity component $(G^\beta)_0$ of G^β.*

(2) *The preimage $(p^{\beta-})^{-1}(x)$ is the connected component of $X^{\beta-}_{\max}$ containing x.*

(3) *$(p^{\beta-})^{-1}(x)$ is I^β-equivariantly diffeomorphic to the I^β-representation $W^{\beta-} = \mathfrak{r}^{\beta-} \times F$.*

(4) *$(p^{\beta-})^{-1}(x)$ is $R^{\beta-} \rtimes I^\beta$-equivariantly diffeomorphic with $(R^{\beta-} \rtimes I^\beta) \times^{I^\beta} F$.*

Furthermore, if $X^\beta_{\max} = \{x\} = \mu_{\mathfrak{p}}^{-1}(\sigma)$, then $I^\beta = G^\beta$ and $X^{\beta-}_{\max}$ is $G^{\beta-}$-equivariantly diffeomorphic to the G^β-representation $W^{\beta-} = \mathfrak{r}^{\beta-} \times F$.

7. Quotients

In this section, let X be a G-invariant, compact, and smooth submanifold of Z. Let σ be an extreme point of the convex envelope of $\mu_P(X)$, and let $\beta \in C_\sigma^{K^\sigma}$.

As a consequence of the Local Structure Theorem C, we obtain:

Corollary 7.1. *The action of $R^{\beta-}$ on $X^{\beta-}_{\max}$ is proper and free.*

Proof. We use the notation of Theorem 6.8. The $R^{\beta-}$-action on $R^{\beta-} \times F \times U$ by left-translation on the first component is proper and free. This implies that the $R^{\beta-}$-action on $X^{\beta-}_{\max}$ is free and locally over $X^\beta_{\max}$ it is also proper.

Let (g_l, y_l) be a sequence in $R^{\beta-} \times X^{\beta-}_{\max}$ such that $g_l \cdot y_l$ and y_l converge in $X^{\beta-}_{\max}$.

Since $p^{\beta-}$ is $R^{\beta-}$-invariant, we have $p^{\beta-}(g_l \cdot y_l) = p^{\beta-}(y_l)$, and both sequences converge to some $x \in X^\beta_{\max}$. By Theorem 6.8, we may assume that $x \in V$ and $g_l \cdot y_l$ and y_l lie in $(p^{\beta-})^{-1}(V)$.

Since the $R^{\beta-}$-action is proper on $(p^{\beta-})^{-1}(V)$, the sequence g_l has a convergent subsequence in $R^{\beta-}$. This shows that the action on $X^{\beta-}_{\max}$ is proper. ☐

The G^βmax-action on $X^{\beta-}_{\max}$ induces an $G^{\beta-}/R^{\beta-}$-action on the quotient $X^{\beta-}_{\max}/R^{\beta-}$. On $X^{\beta}_{\max}$ the group G^β and therefore also I^β act by the artificial $R^{\beta-} \rtimes G^\beta$-action given by the projection $R^{\beta-} \rtimes G^\beta \to G^\beta$. Since $p^{\beta-}: X^{\beta-}_{\max} \to X^{\beta}_{\max}$ is $G^{\beta-}$-equivariant we obtain, after the identification of $G^{\beta-}/R^{\beta-}$ with G^β, a G^β-equivariant map $p\colon X^{\beta-}_{\max}/R^{\beta-} \to X^{\beta}_{\max}$.

With this notations and conventions Theorem 6.8 now implies the following:

Theorem 7.2. *Let X be a compact G-invariant submanifold of Z. Then the diagram*

$$\begin{tikzcd}
& X^{\beta-}_{\max} \arrow[dl, "q"'] \arrow[dr, "p^{\beta-}"] & \\
X^{\beta-}_{\max}/R^{\beta-} \arrow[rr, "p"] & & X^{\beta}_{\max}
\end{tikzcd}$$

commutes, where $q\colon X^{\beta-}_{\max} \to X^{\beta-}_{\max}/R^{\beta-}$ denotes the quotient map. The following statements hold:

1. *$X^{\beta-}_{\max}/R^{\beta-}$ is a G^β-manifold.*
2. *The quotient map q is $G^{\beta-}$-equivariant.*
3. *The map q realizes $X^{\beta-}_{\max}$ as an $R^{\beta-}$-principal bundle.*
4. *The bundle map $p^{\beta-}: X^{\beta-}_{\max} \to X^{\beta}_{\max}$ is a topological Hilbert quotient with respect to the I^β-action, with typical fiber an I^β-representation $W^{\beta-}$ that only depends on the connected components of $X^{\beta}_{\max}$.*
5. *The map p induced by $p^{\beta-}$ is a topological Hilbert quotient with respect to the I^β-action, with typical fiber an I^β-representation F that only depends on the connected components of $X^{\beta}_{\max}$.*

In the complex connected case, we have $G^\beta = I^\beta$, the fibrations are holomorphic bundles and the maps p and $p^{\beta-}$ are analytic Hilbert quotients (see Remarks 3.2).

Proof. A formal calculation confirms that all maps in the diagram are $G^{\beta-}$-equivariant.

Parts (1)–(3) of Theorem 7.2 follow directly from Corollary 7.1.

For a point $x \in X^{\beta}_{\max}$, we use the slice chart introduced in the Local Structure Theorem C:

$$\Phi\colon R^{\beta-} \times F \times U \longrightarrow (p^{\beta-})^{-1}(V).$$

The map Ψ is $R^{\beta-} \rtimes I^\beta$-equivariant, where the action on $R^{\beta-} \times F \times U$ is given by:

$$(r,h)\cdot(g,v,u) \longmapsto \Big(rgh^{-1}, h\cdot v, u\Big).$$

In this notation: the map $p^{\beta-}$ is given by the projection:

$$\pi_U\colon R^{\beta-} \times F \times U \longrightarrow U.$$

The map q is given by the projection:

$$\pi_{F\times U}\colon R^{\beta-} \times F \times U \longrightarrow F \times U.$$

The map p is given by the projection:

$$p_U\colon F \times U \longrightarrow U.$$

Since I^β contains Γ_β and β acts on $\mathfrak{r}^{\beta-} \times F$ as a self-adjoint operator with strictly negative eigenvalues, we obtain (4)–(5). The complex case follows analogously from these observations. $\square$

Remark 7.3. In the complex connected case, the structure group of the bundle

$$p^{\beta-}: X^{\beta-}_{\max} \longrightarrow X^\beta_{\max}$$

is the group of G^β-equivariant biholomorphic maps $s\colon W^{\beta-} \to \mathfrak{W}^{\beta-}$. This group is a Lie group and contains, as a retract, the group $\mathrm{GL}_{G^\beta}(W^{\beta-})$ of G^β-equivariant complex linear isomorphisms. Since $p^{\beta-}$ is an analytic Hilbert quotient, it is a Stein map. In particular, the Oka principle can be applied. This implies that for an open Stein submanifold Q of $X^{\beta-}_{\max}$, the structure group of the bundle $p^{\beta-}: X^{\beta-}_{\max} \to X^\beta_{\max}$ can be reduced to the linear group $\mathrm{GL}_{G^\beta}(W^{\beta-})$. Consequently, the bundle $p^{\beta-}: X^{\beta-}_{\max} \to X^\beta_{\max}$ is, over Q, a holomorphic G^β-vector bundle with typical fiber $W^{\beta-} = \mathfrak{r}^{\beta-} \times F$.

The analogous statement also holds for the bundle

$$p\colon X^{\beta-}_{\max}/R^{\beta-} \longrightarrow X^\beta_{\max}.$$

Over a Stein submanifold of $X^\beta_{\max}$, this bundle is a holomorphic vector bundle with typical fiber F (see [Hei89] for details). We do not know of any example where the vector bundle structure is not given globally.

8. Compact orbits

Let X be a compact G-invariant subset of Z, and consider the function $\eta\colon Z \to \mathbb{R}$ defined by $\eta(y) = \|\mu_{\mathfrak{p}}(y)\|^2$. The gradient of η at $x \in X$ is given by

$$\operatorname{grad} \eta(x) = 2\beta_X(x),$$

where $\beta := \mu_{\mathfrak{p}}(x)$. Note that the flow of η is along G-orbits.

We now consider the maximal strata in the sense of the work of Kirwan in [Kir84] associated with the function η from the point of view developed so far.

Proposition 8.1. *Let $x \in X$ be such that $\eta(x)$ is maximal, and set $\beta := \mu_{\mathfrak{p}}(x)$. Then β is an extreme point of the convex envelope E of $\mu_{\mathfrak{p}}(X)$, and $x \in X^\beta_{\max}$.*

Proof. This follows from the general theory developed in [Kir84] (see also [HSS08] for the general case). For the sake of completeness we give a more direct proof here.

Since x is a maximum of $\eta|_{G\cdot x}$, it follows that $x \in X^\beta$. Moreover, we have

$$\mu^\beta_{\mathfrak{p}}(x) = \langle \mu_{\mathfrak{p}}(x), \beta\rangle = \eta(x) = \|\beta\|^2.$$

Since η attains its maximum at x, we obtain $\eta(y) \leq \eta(x)$ for every $y \in X$. This implies

$$\mu^\beta_{\mathfrak{p}}(y) = \langle \mu_{\mathfrak{p}}(y), \beta\rangle = \Big\langle \mu_{\mathfrak{p}}(y), \mu_{\mathfrak{p}}(x)\Big\rangle \leq \eta(x) = \mu^\beta_{\mathfrak{p}}(x),$$

and thus $x \in X^\beta_{\max}$.

The affine hyperplane $H_{\|\beta\|^2} := \{\alpha \in \mathfrak{p} : \langle \alpha, \beta\rangle \geqq \|\beta\|^2\}$ through β with normal vector β intersects the sphere of radius $\|\beta\|$ centered at the origin only at β. Since

$$\mu_{\mathfrak{p}}(X) \subset E \subset H^-_{\|\beta\|^2} := \Big\{\alpha \in \mathfrak{p} : \langle \alpha, \beta\rangle \leq \|\beta\|^2\Big\},$$

it follows that β is an extreme point of E. $\square$

Lemma 8.2. *The following statements hold:*

1. *Every G-orbit in X has a compact orbit in its closure.*
2. *Every compact G-orbit in Z is a K-orbit.*

Proof. For $z \in X$, let $Y := \overline{G \cdot z}$ be the closure of the G-orbit through z. Since Y is compact, there exists an extreme point σ of the convex envelope of $\mu_{\mathfrak{p}}(Y)$. Since $\sigma = \mu_{\mathfrak{p}}(y)$ for some $y \in Y$, Corollary 5.5 applied to $X = G \cdot y$ implies that $G \cdot y = K \cdot y$. ☐

Remark 8.3. Even in the smooth compact complex case, not every compact G-orbit in X can be realized as an orbit through a point in $X^{\beta}_{\max}$ for some β.

The simplest case occurs when the compact set X contains exactly one compact G-orbit. We collect some properties that hold in this case:

Lemma 8.4. *Assume that X contains exactly one compact G-orbit Y. Then:*

1. *$\mu_{\mathfrak{p}}(Y)$ is a K-orbit in $\mathfrak{p}$.*
2. *This K-orbit coincides with the set of extreme points of the convex envelope E of $\mu_{\mathfrak{p}}(X)$.*
3. *If σ is an extreme point, then σ exposes itself, and $R^{\beta-}$ acts freely on $X^{\sigma-}_{\max}$.*

Proof. By Proposition 8.1, the set of extreme points of E is a K-orbit. The freeness of the $R^{\beta-}$-action follows from $\sigma \in C_{\sigma}^{K^{\sigma}}$ and Proposition 5.3(6). ☐

Finally, we note the following:

Lemma 8.5. *Let X be a compact smooth G-invariant submanifold of Z with exactly one compact orbit. Then*

$$X = G \cdot X^{\beta-}_{\max}.$$

Proof. The set $G \cdot X^{\beta-}_{\max}$ is open in X, and $Y = X \setminus G \cdot X^{\beta-}_{\max}$ is compact. If we assume that Y is non-empty, Lemma 8.2 leads to a contradiction. ☐

9. The projective case

In this section we give a proof of the result of Brion–Luna–Vust ([BLV86]).

Assume that $G = K^{\mathbb{C}}$ is a connected complex reductive group with maximal compact subgroup K. Let V be a finite dimensional unitary representation of K, and set $X := \mathbb{P}(V)$. The action of K is Hamiltonian with respect to the standard Kähler metric on $\mathbb{P}(V)$ and we have a momentum map $\mu\colon X \to i\mathfrak{k}$. For a given linear subspace W of V we write in the following $\mathbb{P}(W)$ for the image of $W \setminus \{0\}$ in $\mathbb{P}(V)$.

Let $G \cdot x$ be a compact orbit in $\mathbb{P}(V)$. Then there is an irreducible subrepresentation W of V such that $G \cdot x \subset \mathbb{P}(W)$. Note that $\mathbb{P}(W)$ contains exactly one compact G-orbit.

Let β be an extreme point of the convex envelope of $\mu(\mathbb{P}(W))$. Note that $\mathbb{P}(W)^{\beta}_{\max}$ is an isolated fixed point of G^{β} in $\mathbb{P}(W)$.

For W we have the orthogonal decomposition $W = W_{\beta_0} \oplus W_{\beta_1} \cdots \oplus W_{\beta_m}$ into β-eigenspaces with eigenvalues β_j where $W_{\beta_0} = W^\beta$ is the line in W which corresponds to x.

The Slice Theorem 2.1 shows that $\mathbb{P}(W)^{\beta-}_{\max}$ can be identified with a G^β-representation. Let w_0 be a basis of W_{β_0}. For $w = w_0 \oplus w_1 \oplus \cdots \oplus w_m \in W \setminus \{0\} \subset W_{\beta_0} \oplus W_{\beta_1} \cdots \oplus W_{\beta_m}$ we have for its image $[w]$ in $\mathbb{P}(W)$ that

$$\exp t\beta \cdot [w] = \Big[w_0 \oplus e^{\beta_1 - \beta_0} w_1 \oplus \cdots \oplus e^{\beta_m - \beta_0} w_m\Big]$$

holds for all $t \in \mathbb{R}$. Note that $w_0 \neq 0$ is a basis of W_{β_0} and that $\beta_j - \beta_0 < 0$ for $j = 1, \ldots, m$. This shows that

$$\mathbb{P}(W)^{\beta-}_{\max} = [w_0 \oplus W_{\beta_1} \cdots \oplus W_{\beta_m}] = \mathbb{P}(W) \setminus \mathbb{P}\Big((W_{\beta_0})^{\perp_W}\Big).$$

Since eigenspaces corresponding to different eigenvalues are orthogonal we have the orthogonal decomposition

$$W = W_{\beta_0} \oplus \mathfrak{r}^{\beta-} \cdot w_0 \oplus (\mathfrak{g} \cdot w_0)^{\perp_W}$$

The representation F in Theorem 6.8 is now given as a subset of $\mathbb{P}(W)$ by

$$\mathbb{P}\Big(w_0 \oplus (\mathfrak{g} \cdot w_0)^\perp\Big) = \mathbb{P}\Big(W_{\beta_0} \oplus (\mathfrak{g} \cdot w_0)^\perp\Big) \setminus \mathbb{P}\Big((W_{\beta_0})^\perp\Big).$$

Theorem 6.8 shows the following.

Theorem 9.1. *The restriction*

$$\Phi \colon R^{\beta-} \times \Big(\mathbb{P}\Big(W_{\beta_0} \oplus (\mathfrak{g} \cdot w_0)^{\perp_W}\Big) \setminus \mathbb{P}\Big((\mathfrak{g} \cdot w_0)^{\perp_W}\Big)\Big) \longrightarrow \mathbb{P}(W) \setminus \mathbb{P}\Big((W_{\beta_0})^{\perp_W}\Big)$$

of the action map is an isomorphism.

In the general case we have a orthogonal decomposition $V = W \oplus U$, $W = W_{\beta_0} \oplus (W_{\beta_0})^{\perp_W}$, $W_{\beta_0}^{\perp_V} = W_{\beta_0}^{\perp_W} \oplus U$ and $(\mathfrak{g} \cdot w_0)^{\perp_V} = (\mathfrak{g} \cdot w_0)^{\perp_W} \oplus U$. Note that $\mathbb{P}(U)$ is a G-invariant subset of $\mathbb{P}(V)$. The following is shown in [BLV86].

Corollary 9.2. *The restriction*

$$\Phi \colon R^{\beta-} \times \Big(\mathbb{P}\Big(W_{\beta_0} \oplus (\mathfrak{g} \cdot w_0)^{\perp_V}\Big) \setminus \mathbb{P}\Big((W_{\beta_0})^{\perp_V}\Big)\Big) \longrightarrow \mathbb{P}(V) \setminus \mathbb{P}\Big((W_{\beta_0})^{\perp_V}\Big)$$

of the action map is an isomorphism.

Proof. Let $z \in \mathbb{P}(W_{\beta_0} \oplus (W_{\beta_0})^{\perp_W} \oplus U) \setminus \mathbb{P}((W_{\beta_0})^{\perp_V})$. As above let w_0 be a basis of W_{β_0}. Since $z \notin \mathbb{P}((W_{\beta_0})^{\perp_V})$ we have $[z] = [w_0 + w_1 + u]$ with $w_1 \in (\mathbb{C}w_0)^{\perp_W}$ and $u \in U$.

Using the result in the irreducible case there are $r \in R^{\beta-}$, $\alpha \in \mathbb{C}^*$ and $w \in (W_{\beta_0})^{\perp_W}$ such that

$$r \cdot [(\alpha w_0 + w) + u] = [r \cdot (\alpha w_0 + w) + r \cdot u] = [w_0 + w_1 + r \cdot u].$$

This shows that $r \cdot [(\alpha w_0 + w) + r^{-1} \cdot u] = [(w_0 + w_1) + u] = z$.

For injectivity, assume that $r \cdot [w_0 + a_1 + b_1] = [w_0 + a_2 + b_2]$, where a_1, $a_2 \in (W_{\beta_0})^{\perp_W}$ and $b_1, b_2 \in U$. It follows that $r \cdot (w_0 + a_1 + b_1) = \lambda(w_0 + a_2 + b_2)$ with $\lambda \in \mathbb{C}^*$. This shows $r \cdot (w_0 + a_1) = \lambda(w_0 + a_2)$ and $r \cdot b_1 = \lambda b_2$. Theorem 9.1 implies $r = e$ and we obtain $[w_0 + a_1 + b_1] = [\alpha w_0 + a_2 + b_2]$ □

References

[Ati82] Michael F. Atiyah, *Convexity and commuting Hamiltonians*, Bull. Lond. Math. Soc. **14** (1982), no. 1, pp. 1–15.

[BGH13] Leonardo Biliotti, Alessandro Ghigi, and Peter Heinzner, *Polar orbitopes*, Commun. Anal. Geom. **21** (2013), no. 3, pp. 579–606.

[BGH16] Leonardo Biliotti, Alessandro Ghigi, and Peter Heinzner, *Invariant convex sets in polar representations*, Isr. J. Math. **213** (2016), pp. 423–441.

[BLV86] Michel Brion, Dominique Luna, and Thierry Vust, *Espaces homogènes sphériques*, Invent. Math. **84** (1986), pp. 617–632.

[GM18] Daniel Greb and Christian Miebach, *Hamiltonian actions of unipotent groups on compact Kähler manifolds*, Épijournal de Géom. Algébr., EPIGA **2** (2018), article no. 10 (30 pages).

[GM23] Daniel Greb and Christian Miebach, *Momentum maps and the Kähler property for base spaces of reductive principal bundles*, Proc. Edinb. Math. Soc., II. Ser. **66** (2023), no. 1, pp. 218–230.

[GS82] Victor W. Guillemin and Shlomo Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), no. 3, pp. 491–513.

[Hau96] Jürgen Hausen, *Holomorphe $\mathbb{C}^*$-Operationen auf komplexen Flächen*, Konstanzer Schriften in Mathematik und Informatik, no. 11, Universität Konstanz, 1996. `https://kops.uni-konstanz.de/entities/publication/e9508715-be11-457d-aaf5-777dc39bc011`.

[Hei89] Peter Heinzner, *Fixpunktmengen kompakter Gruppen in Teilgebieten Steinscher Mannigfaltigkeiten*, J. Reine Angew. Math. **402** (1989), pp. 128–137.

[HL94] Peter Heinzner and Frank Loose, *Reduction of Complex Hamiltonian G-Spaces*, Geom. Funct. Anal. **4** (1994), no. 3, pp. 288–297.

[HMP98] Peter Heinzner, Luca Migliorini, and Marzia Polito, *Semistable quotients*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) **26** (1998), no. 2, pp. 233–248.

[HS10] Peter Heinzner and Patrick Schützdeller, *Convexity properties of gradient maps*, Adv. Math. **225** (2010), pp. 1119–1133.

[HS07a] Peter Heinzner and Gerald W. Schwarz, *Cartan decomposition of the moment map*, Math. Ann. **337** (2007), pp. 197–232.

[HSS08] Peter Heinzner, Gerald W. Schwarz, and Henrik Stötzel, *Stratifications with respect to actions of real reductive groups*, Compos. Math. **144** (2008), no. 1, pp. 163–185.

[HS07b] Peter Heinzner and Henrik Stötzel, *Semistable points with respect to real forms*, Math. Ann. **338** (2007), no. 1, pp. 1–9.

[Kir84] Frances C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, no. 31, Princeton University Press, 1984.

[KKS15] Friedrich Knop, Bernhard Krötz, and Henrik Schlichtkrull, *The local structure theorem for real spherical varieties*, Compos. Math. **151** (2015), no. 11, pp. 2145–2159.

[MS10] Christian Miebach and Henrik Stötzel, *Spherical gradiant manifolds*, Ann. Inst. Fourier **60** (2010), no. 6, pp. 2235–2260.

[Sja95] Reyer Sjamaar, *Holomorphic slices, symplectic reduction and multiplicities of representations*, Ann. Math. (2) **141** (1995), no. 1, pp. 87–129.

[Wol69] Joseph A. Wolf, *The action of a real semisimple group on a complex flag manifold. I: Orbit structure and holomorphic arc components*, Bull. Am. Math. Soc. **75** (1969), pp. 1121–1237.

Peter Heinzner, Fakultät für Mathematik, Ruhr Universität Bochum, Universitätsstrasse 150, 44780 Bochum, Deutschland; peter.heinzner@rub.de

Christian Zöller, Fakultät für Mathematik, Ruhr Universität Bochum, Universitätsstrasse 150, 44780 Bochum, Deutschland; christian.zoeller@rub.de

Received March 25, 2025
Accepted January 23, 2026

In final form June 17, 2026