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Lifting discrete series by Dirac cohomology

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Abstract. Dirac cohomology of a discrete series representation is an analogue of the highest weight vectors of a highest weight representation. We employ Dirac cohomology for the induction and character lifting of discrete series as well as determining the lifting of L-packets of discrete series.

Mathematics Subject Classification 2020: 22E47, 22E46

Key Words and Phrases: Dirac cohomology, discrete series, Dirac induction, L-packet, character lifting

1. Introduction

I first met Joe Wolf at University of California Berkeley in the spring of 1987. It was my 2nd year in graduate school at MIT. At the beginning of the spring semester I arrived at Berkeley as an exchange student for the special year on representation theory at Mathematical Science Research Institute. Besides attending all the talks at MSRI, I attended the weekly Lie group seminar at the mathematics department. Joe was the organizer and he arranged many talks that were friendly to graduate students. Each time after seminar we went to an Italian restaurant for pizza and beer. Joe was very patient to students for questions and soon I got to know him well. Many years later we discussed using partial Dirac operators for geometric construction of tempered representations. The discussions resulted in publication of a joint paper "Partial Dirac cohomology and tempered representations"- [CHW23] together with another long time friend Meng-Kiat Chuah.

Dirac operators were employed for geometric construction of discrete series by Parthasarathy [Par72], Atiyah and Schmid [AS77], and it was extended to tempered representations by Wolf [Wol74]. In the 1990's, Vogan made a conjecture on the algebraic property of the Dirac operators in semisimple Lie algebra setting. This conjecture was proved by Pandžić and myself in 2002 [HP02]. Kostant extended the concept of Dirac cohomology and Vogan's conjecture to the more general setting of the cubic Dirac operator [Kos03]. Further extensions to affine Lie algebras, Lie superalgebras, and affine Hecke algebras have been achieved by many authors (we refer the reader to [Hua15, Remark 2.4] and references therein).

In the further development in the setting of Harish-Chandra modules, there are two basic problems: the classification of irreducible unitary representations with nonzero Dirac cohomology (Dirac series), and calculation of the Dirac cohomology for a given representation. In addition, various applications have been found in the theory of automorphic forms [HP06], in studying branching laws [HPZ13], geometric

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quantization [CH16], orthogonality relations of Harish-Chandra modules [HMS20], lifting of characters [Hua21] and decomposition of tensor product [Hua24].

The lifting of representations between reductive algebraic groups is an important problem in representation theory. The endoscopic transfer in the Langlands functionality and the theta correspondence in Howe's reductive dual pairs are primary examples. The aim of this paper is determining the lifting of L-packets of discrete series representations by Dirac cohomology. We emphasize using Dirac operators with respect to the maximal tori. This setting has the advantage that Dirac cohomology really resembles the highest weight vectors and is suitable for induction. We include the basic definitions, results and references with proofs for convenience of the reader.

2. Preliminaries on Dirac cohomology and Dirac index

This section contains necessary definitions and theorems used in the rest of the paper. Let G be a reductive Lie group with a Cartan involution θ. Denote by $\mathfrak{g}_0$ its Lie algebra and assume that $K = G^\theta$ is a maximal compact subgroup of G. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition for the complexified Lie algebra of G. Let B be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$, which restricts to the Killing form on the semisimple part $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $C(\mathfrak{p})$ the Clifford algebra of $\mathfrak{p}$ with respect to B. Then one can consider the following version of the Dirac operator:

$$D = \sum_{i=1}^{n} Z_i \otimes Z_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p});$$

here $Z_1, \ldots, Z_n$ is an orthonormal basis of $\mathfrak{p}$ with respect to the invariant symmetric bilinear form B. It follows that D is independent of the choice of the orthonomal basis $Z_1, \ldots, Z_n$ and it is invariant under the diagonal adjoint action of K.

The Dirac operator D is a square root of Laplace operator associated to the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. To explain this, we start with a Lie algebra map

$$\alpha : \mathfrak{k} \longrightarrow C(\mathfrak{p})$$

which is defined by the adjoint map $\mathrm{ad} : \mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ composed with the embedding of $\mathfrak{so}(\mathfrak{p})$ into $C(\mathfrak{p})$ using the identification $\mathfrak{so}(\mathfrak{p}) \simeq \bigwedge^2 \mathfrak{p}$. The explicit formula for α is (see [HP06, Section 2.3.3])

$$\alpha(X) = -\frac{1}{4} \sum_j [X, Z_j] Z_j.$$

Using α we can embed the Lie algebra $\mathfrak{k}$ diagonally into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, by

$$X \longmapsto X_\Delta = X \otimes 1 + 1 \otimes \alpha(X).$$

This embedding extends to $U(\mathfrak{k})$. We denote the image of $\mathfrak{k}$ by $\mathfrak{k}_\Delta$, and then the image of $U(\mathfrak{k})$ is the enveloping algebra $U(\mathfrak{k}_\Delta)$ of $\mathfrak{k}_\Delta$.

Let $C_\mathfrak{g}$ be the Casimir operator for $\mathfrak{g}$, given by $C_\mathfrak{g} = \sum Z_i^2 - \sum W_j^2$, where W_j is an orthonormal basis for $\mathfrak{k}_0$ with respect to the inner product $-B$, where B is the Killing form. Let $C_\mathfrak{k} = -\sum W_j^2$ be the Casimir operator for $\mathfrak{k}$. The image of $C_\mathfrak{k}$ under Δ is denoted by $C_{\mathfrak{k}_\Delta}$.

In the following we denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ containing a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ so that $\mathfrak{t}^*$ is embedded into $\mathfrak{h}^*$, and by $W_\mathfrak{g}$ and $W_\mathfrak{k}$ the Weyl groups of $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{t})$ respectively.

Then

$$D^2 = -C_{\mathfrak{g}} \otimes 1 + C_{\mathfrak{k}_\Delta} + \left(\|\rho_c\|^2 - \|\rho\|^2\right) 1 \otimes 1,$$

where ρ and ρ_c are half sums of positive roots for $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{t})$ respectively.

For any admissible $(\mathfrak{g}, K)$-module X, we consider the action of the Dirac operator D on $X \otimes S$, with S the spinor module for the Clifford algebra $C(\mathfrak{p})$.

Definition 2.1. The Dirac cohomology is defined as follows:

$$H_D(X) := \operatorname{Ker} D / \operatorname{Im} D \cap \operatorname{Ker} D.$$

It follows from the above identity for D^2 that $H_D(X)$ is a finite-dimensional module for the spin double cover $\widetilde{K}$ of K. In case X is unitary,

$$H_D(X) = \ker D = \ker D^2,$$

since D is self-adjoint with respect to a natural Hermitian inner product on $X \otimes S$.

Theorem 2.2 ([HP02]). *Let X be an admissible $(\mathfrak{g}, K)$-module with an infinitesimal character corresponding to parameter $\Lambda \in \mathfrak{h}^*$. Suppose that $H_D(X)$ contains a representation of $\widetilde{K}$ with infinitesimal character λ. Then Λ and $\lambda \in \mathfrak{t}^* \subseteq \mathfrak{h}^*$ are conjugate under $W_{\mathfrak{g}}$.*

Now we assume that G is a connected semisimple Lie group with finite center. We assume that rank of G is equal to that of K. Then G has a compact Cartan subgroup T. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the complexified Lie algebra of G. Since $\mathfrak{p}$ is even-dimensional, the spin module S decomposes as $S^+ \oplus S^-$, with the $\mathfrak{k}$-submodules $S^\pm$ being the even respectively odd part of S.

Let $X = X_\pi$ be the Harish-Chandra module of an admissible representation π of G or a $(\mathfrak{g}, K)$-module of finite length.

Definition 2.3. The spin index $I(X)$ is defined to be the following virtual $\widetilde{K}$-module:

$$I(X) = X \otimes S^+ - X \otimes S^-.$$

The Dirac operator D induces the action of the following $\widetilde{K}$-equivariant operators

$$D^\pm : X \otimes S^\pm \longrightarrow X \otimes S^\mp.$$

Since D^2 acts by a scalar on each $\widetilde{K}$-type, most of $\widetilde{K}$-modules in $X \otimes S^+$ are the same as in $X \otimes S^-$. Thus, $I(X)$ is a virtual $\widetilde{K}$-module, an integer combination of finitely many $\widetilde{K}$-modules.

Lemma 2.4. *The spin index is equal to the Euler characteristic of Dirac cohomology, i.e.,*

$$I(X) = H_D^+(X) - H_D^-(X).$$

Proof. As we mentioned above, $X \otimes S$ is decomposed into a direct sum of eigenspaces for D^2:

$$X \otimes S = \sum_\lambda (X \otimes S)_\lambda = (X \otimes S^+)_\lambda \oplus (X \otimes S^-)_\lambda.$$

For $\lambda \neq 0$, D induces an isomorphism of $\widetilde{K}$-modules

$$(X \otimes S^+)_\lambda \longrightarrow (X \otimes S^-)_\lambda.$$

It follows that

$$X \otimes S^+ - X \otimes S^- = (X \otimes S^+)_0 - (X \otimes S^-)_0.$$

Since D is a differential on $\operatorname{Ker} D^2$ and the corresponding cohomology is exactly the Dirac cohomology $H_D(X)$, Lemma 2.4 follows from the Euler–Poincaré principle. □

Therefore, the spin index $I(X)$ is also called *Dirac index*, since it is equal to the index of D^+, in the sense of index for a Fredholm operator. It is equal to the Euler characteristic of Dirac cohomology $H_D(X)$ as shown in Lemma 2.4. We recall a theorem from [Hua15, Theorem 10.1].

Theorem 2.5. *Suppose that an irreducible Harish-Chandra module X has regular infinitesimal character. Then we have*

$$\operatorname{Hom}_{\widetilde{K}}\Big(H_D^+(X), H_D^-(X)\Big) = 0.$$

In particular, it follows that the Dirac index $I(X) = 0$ if and only if the Dirac cohomology $H_D(X) = 0$.

Note that the above definition of $I(X)$ makes sense for any $(\mathfrak{g}, K)$-module (or virtual $(\mathfrak{g}, K)$-module) X of finite length that admits an infinitesimal character. Moreover, if X satisfies the weaker condition of having generalised infinitesimal character, the definition still makes sense. In this case, Pandžić and Somberg [PS16] defined "higher order Dirac cohomology". Then the spin index $I(X)$ is equal to the Euler characteristic of the higher order Dirac cohomology (cf. [MPV17, (3.6), (3.7) p. 1474]).

We further note that the proof of Theorem 2.5 needs the setting of cubic Dirac operators and relation between the Dirac cohomology and Lie algebra cohomology (cf. [Hua15]). It is natural to use Dirac index when we deal with distribution characters of elliptic representations on the maximal tori. It is also crucial for study of the orthogonality relations [HMS20; HX12; HX22] and the translation principle [MPV17].

3. Cubic Dirac operators and associated cohomology

In this section we recall Kostant's cubic Dirac operator and the basic properties of the corresponding Dirac cohomology. The purpose is the calculation of Dirac cohomology of finite-dimensional representations with respect to an equal rank subgroup. It will be used for calculation of Dirac cohomology of L-packets of discrete series.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra with Killing form B. Let $\mathfrak{r} \subset \mathfrak{g}$ be a reductive Lie subalgebra such that $B|_{\mathfrak{r}\times\mathfrak{r}}$ is non-degenerate. Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be the orthogonal decomposition with respect to B. Then the restriction $B|_{\mathfrak{s}}$ is also non-degenerate. Denote by $C(\mathfrak{s})$ the Clifford algebra of $\mathfrak{s}$ with

$$uu' + u'u = -2B(u, u')$$

for all $u, u' \in \mathfrak{s}$. The above choice of sign is the same as in [HP06], but different from the definition in [Kos99], as well as in [Kos03]. The two different choices of signs have no essential difference since the two bilinear forms are equivalent over $\mathbb{C}$. Now fix an orthonormal basis $Z_1, \dots, Z_m$ of $\mathfrak{s}$. Kostant [Kos99] defines the cubic Dirac operator D by

$$D = \sum_{i=1}^{m} Z_i \otimes Z_i + 1 \otimes v \in U(\mathfrak{g}) \otimes C(\mathfrak{s}).$$

Here $v \in C(\mathfrak{s})$ is the image of the fundamental 3-form $w \in \bigwedge^3(\mathfrak{s}^*)$,

$$w(X, Y, Z) = \frac{1}{2} B(X, [Y, Z]),$$

under the Chevalley map $\bigwedge(\mathfrak{s}^*) \to C(\mathfrak{s})$ and the identification of $\mathfrak{s}^*$ with $\mathfrak{s}$ by the Killing form B. Explicitly,

$$v = \frac{1}{2} \sum_{1 \leq i < j < k \leq m} B([Z_i, Z_j], Z_k) Z_i Z_j Z_k.$$

The cubic Dirac operator has a good square in analogy with the Dirac operator associated with the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ in Section 2. We have a similar Lie algebra map

$$\alpha : \mathfrak{r} \longrightarrow C(\mathfrak{s})$$

which is defined by the adjoint map $\mathrm{ad} : \mathfrak{r} \to \mathfrak{so}(\mathfrak{s})$ composed with the embedding of $\mathfrak{so}(\mathfrak{s})$ into $C(\mathfrak{s})$ using the identification $\mathfrak{so}(\mathfrak{s}) \simeq \bigwedge^2 \mathfrak{s}$. The explicit formula for α is (see [HP06, Section 2.3.3])

$$\alpha(X) = -\frac{1}{4} \sum_j [X, Z_j] Z_j, \quad X \in \mathfrak{r}. \tag{1}$$

Using α we can embed the Lie algebra $\mathfrak{r}$ diagonally into $U(\mathfrak{g}) \otimes C(\mathfrak{s})$, by

$$X \longmapsto X_\Delta = X \otimes 1 + 1 \otimes \alpha(X).$$

This embedding extends to $U(\mathfrak{r})$. We denote the image of $\mathfrak{r}$ by $\mathfrak{r}_\Delta$, and then the image of $U(\mathfrak{r})$ is the enveloping algebra $U(\mathfrak{r}_\Delta)$ of $\mathfrak{r}_\Delta$. Let $\Omega_\mathfrak{g}$ (resp. $\Omega_\mathfrak{r}$) be the Casimir elements for $\mathfrak{g}$ (resp. $\mathfrak{r}$). The image of $\Omega_\mathfrak{r}$ under Δ is denoted by $\Omega_{\mathfrak{r}_\Delta}$.

Let $\mathfrak{h}_\mathfrak{r}$ be a Cartan subalgebra of $\mathfrak{r}$ which is contained in $\mathfrak{h}$. It follows from Kostant's calculation ([Kos99, Theorem 2.16]) that

$$D^2 = -\Omega_\mathfrak{g} \otimes 1 + \Omega_{\mathfrak{r}_\Delta} - \left(\|\rho\|^2 - \|\rho_\mathfrak{r}\|^2\right) 1 \otimes 1, \tag{2}$$

where $\rho_\mathfrak{r}$ denotes the half sum of positive roots for $(\mathfrak{r}, \mathfrak{h}_\mathfrak{r})$. We also note the sign difference with Kostant's formula due to our choice of bilinear form for the definition of the Clifford algebra $C(\mathfrak{s})$.

We denote by W the Weyl group associated to the root system $\Delta(\mathfrak{g}, \mathfrak{h})$ and $W_\mathfrak{r}$ the Weyl group associated to the root system $\Delta(\mathfrak{r}, \mathfrak{h}_\mathfrak{r})$. The following theorem due to Kostant is an extension of Vogan's conjecture on the symmetric pair case which is proved in [HP02]. (See [Kos03, Theorems 4.1 and 4.2] or [HP06, Theorem 4.1.4]).

Theorem 3.1. *There is an algebra homomorphism*

$$\zeta : Z(\mathfrak{g}) \longrightarrow Z(\mathfrak{r}) \cong Z(\mathfrak{r}_\Delta)$$

such that for any $z \in Z(\mathfrak{g})$ *one has*

$$z \otimes 1 - \zeta(z) = Da + aD \quad \textit{for some} \ \ a \in U(\mathfrak{g}) \otimes C(\mathfrak{s}).$$

Moreover, ζ is determined by the following commutative diagram:

$$\begin{tikzcd} Z(\mathfrak{g}) \arrow[r, "\zeta"] \arrow[d, "\eta"'] & Z(\mathfrak{r}) \arrow[d, "\eta_{\mathfrak{r}}"'] \\ P(\mathfrak{h}^*)^W \arrow[r, "\mathrm{Res}"] & P(\mathfrak{h}_{\mathfrak{r}}^*)^{W_{\mathfrak{r}}}. \end{tikzcd}$$

Here the vertical maps η and $\eta_{\mathfrak{r}}$ are Harish-Chandra isomorphisms.

Definition 3.2. Let S be a spin module of $C(\mathfrak{s})$. Consider the action of D on $V \otimes S$

$$D : V \otimes S \longrightarrow V \otimes S \tag{3}$$

with $\mathfrak{g}$ acting on V and $C(\mathfrak{s})$ on S. The *Dirac cohomology* of V is defined to be the $\mathfrak{r}$-module

$$H_D(V) := \operatorname{Ker} D/(\operatorname{Ker} D \cap \operatorname{Im} D).$$

The following theorem is a consequence of the above theorem.

Theorem 3.3 ([HP06; Kos03]). *Let V be a $\mathfrak{g}$-module with $Z(\mathfrak{g})$ infinitesimal character χ_Λ. Suppose that an $\mathfrak{r}$-module N is contained in the Dirac cohomology $H_D(V)$ and has $Z(\mathfrak{r})$ infinitesimal character χ_λ. Then $\lambda = w\Lambda$ for some $w \in W$.*

Suppose that V_λ is a finite-dimensional representation with highest weight $\lambda \in \mathfrak{h}^*$. Kostant [Kos03] calculated the Dirac cohomology of V_λ with respect to any equal rank quadratic subalgebra $\mathfrak{r}$ of $\mathfrak{g}$. Assume that $\mathfrak{h} \subset \mathfrak{r} \subset \mathfrak{g}$ is the Cartan subalgebra for both $\mathfrak{r}$ and $\mathfrak{g}$. Define $W(\mathfrak{g},\mathfrak{h})^1$ to be the subset of the Weyl group $W(\mathfrak{g},\mathfrak{h})$ by

$$W(\mathfrak{g},\mathfrak{h})^1 = \Big\{ w \in W(\mathfrak{g},\mathfrak{h}) \,\Big|\, w(\rho) \text{ is } \Delta^+(\mathfrak{r},\mathfrak{h}) - \text{dominant} \Big\}.$$

This is the same as the subset of elements $w \in W(\mathfrak{g},\mathfrak{h})$ that map the positive Weyl $\mathfrak{g}$-chamber into the positive $\mathfrak{r}$-chamber. There is a bijection $W(\mathfrak{r},\mathfrak{h}) \times W(\mathfrak{g},\mathfrak{h})^1 \to W(\mathfrak{g},\mathfrak{h})$ given by $(w,\tau) \mapsto w\tau$. Kostant [Kos03] proved the following result.

Proposition 3.4 (Kostant [Kos03]). *One has*

$$H_D(V_\lambda) = \bigoplus_{w \in W(\mathfrak{g},\mathfrak{h})^1} E_{w(\lambda+\rho)-\rho_{\mathfrak{r}}}.$$

This result is important for understanding the analogy between Dirac cohomology and highest weight vectors.

4. Dirac induction for compact Lie groups

In this section we recall some results from [Hua11, Section 5]. For convenience of the reader we also include our ideas of the proofs. We retain the notation of the previous section. Let G be a connected compact Lie group and R be a connected and closed subgroup of the same rank. We now calculate the Dirac cohomology $H_D(L^2(G))$ for a compact Lie group G. Recall that

$$D = D(\mathfrak{g},\mathfrak{r}) = \sum_k W_k \otimes W_k - \frac{1}{2} \sum_{i<j<k} B([W_i, W_j], W_k) \otimes W_i W_j W_k.$$

We denote by $\mathbb{D}$ the action of the cubic Dirac operator D on $L^2(G)\otimes S$,

$$\mathbb{D}\colon L^2(G)\otimes S \longrightarrow L^2(G)\otimes S,$$

with the differentiation from right on the C^∞-functions on G. Then $\mathbb{D}$ is a densely defined symmetric operator. It can be extended as a close operator. The kernel of $\mathbb{D}$ is a closed subspace of the Hilbert space $L^2(G)\otimes S$. Then the following theorem is a consequence of the Peter–Weyl theorem and Proposition 3.4.

Theorem 4.1 ([Hua11, Theorem 5.1])**.** *Let G be a connected compact Lie group with a closed connected subgroup R of equal rank. Then as a $G\times\widetilde{R}$-module, there is an orthogonal sum decomposition of*

$$H_D\Big(L^2(G)\Big) = \ker\mathbb{D} = \sum_{\lambda\in\widehat{G}} V_\lambda \otimes H_D(V_\lambda^*) = \sum_{\lambda\in\widehat{G}} V_\lambda\otimes\left(\sum_{w\in W^1} U_{w\cdot\lambda^*}\right).$$

Here λ denotes the highest weight of the irreducible G-module V_λ and λ^* is the highest weight for the contragredient representation V_λ^*.

We apply the theorem to the case when R is the maximal torus T.

Corollary 4.2 ([Hua11, Corollary 5.2])**.** *As a $G\times\widetilde{T}$-module,*

$$H_D\Big(L^2(G)\Big) = \ker\mathbb{D} = \bigoplus_{\lambda\in\widehat{G}} V_\lambda\otimes\left(\sum_{w\in W}\mathbb{C}_{w(\lambda^*+\rho)}\right).$$

Naturally, we have the bijection of two sets of parameters for $\widehat{G}$,

$$\widehat{T}/W \longleftrightarrow \Big\{\text{regular elements in } \widehat{T}+\rho\Big\}/W.$$

Here the left hand side is the set of parameters for highest weights and the right hand side is the set of parameters for infinitesimal characters (of the dual representations).

Remark 4.3 (Generalized Borel–Weil Theorem)**.** It follows from the above theorem that $\mathrm{Hom}_{\widetilde{R}}(U_{\mu*},\ker\mathbb{D})$ gives an irreducible G-module V_λ, provided that the irreducible $\widetilde{R}$-module U_{μ^*} in $H_D(V_{\lambda^*})$ (i.e., there is a $w\in W^1$ such that $\mu^*+\rho(\mathfrak{r}) = w(\lambda^*+\rho(\mathfrak{g}))$. If we identify the L^2-sections of the spinor bundle $L^2(G/R, S\otimes U_\mu)$ with $\mathrm{Hom}_{\widetilde{R}}(U_\mu^*, L^2(G)\otimes S)$, then D induces a closed operator on the L^2-sections

$$D_\mu\colon L^2(G)\otimes_R(S\otimes U_\mu)\longrightarrow L^2(G)\otimes_R(S\otimes U_\mu).$$

Then $\ker D_\mu$ gives an geometric construction of V_λ. This is Kostant's generalization of the Borel–Weil Theorem (cf. [HP06; Kos03]).

Let $\mathrm{R}(G)$ denote the character ring of G. We define the Dirac induction from $\mathrm{R}(\widetilde{R})$ to $\mathrm{R}(G)$ by

$$\text{D-Ind}(U) = \sum_{\alpha\in\widehat{G}} \dim\mathrm{Hom}_{\widetilde{R}}(H_D(V_\alpha), U)V_\alpha.$$

To simplify the notation, we note that the intertwining number of two G-modules,

$$\tau(A,B) = \dim\mathrm{Hom}_G(A,B),$$

extends to a symmetric bilinear form on $\mathrm{R}(G)$, which is denoted by $\langle\ ,\ \rangle_G$. Then we have

$$\text{D-Ind}(U) = \sum_{\alpha\in\widehat{G}} \langle H_D(V_\alpha), U\rangle_{\widetilde{R}} V_\alpha.$$

This induction is in the same spirit of Bott's definition of the formally induced representation from $\mathrm{R}(R)$ to the completion of $\mathrm{R}(G)$ as

$$A \longmapsto \sum_{\alpha\in\widehat{G}} \langle V_\alpha|_R, A\rangle_R V_\alpha.$$

We note that Bott's induction involves an infinite sum. The Dirac induction from $\mathrm{R}(\widetilde{R})$ to $\mathrm{R}(G)$ is defined by

$$\text{D-Ind}(U) = \sum_{\alpha\in\widehat{G}} \langle H_D(V_\alpha), U\rangle_{\widetilde{R}} V_\alpha,$$

which is always a finite sum. This is due to the fact that there are only finitely many irreducible G-modules V_α such that $\langle H_D(V_\alpha), U\rangle_{\widetilde{R}} \neq 0$. We note that both definitions of Dirac cohomology and Dirac induction extend to the character ring $\mathrm{R}(G)$ and $\mathrm{R}(\widetilde{R})$, respectively.

Proposition 4.4. *Suppose that R is a connected closed subgroup of a connected compact Lie group G of the same rank. Then the following two identities hold:*

(i) $\text{D-Ind}(H_D(V)) = |W^1|V$ *for* $V \in \mathrm{R}(G)$.
(ii) $H_D(\text{D-Ind}(U)) = \sum_{w\in W^1} w\cdot U$ *for* $U \in \mathrm{R}(\widetilde{R})$.

Proof. For an irreducible G-module V or an irreducible $\widetilde{R}$-module U, the two identities follow from Proposition 4.3 on Dirac cohomology of irreducible G-modules. It is clear that these two identities extend to $V \in \mathrm{R}(G)$ and $U \in \mathrm{R}(\widetilde{R})$. □

The same argument in the above proof gives another two identities in the following proposition.

Proposition 4.5. *Suppose that R is a connected closed subgroup of a connected compact Lie group G of the same rank. Then the following two identities hold:*

(i) $\langle U_1, U_2\rangle_{\widetilde{R}} = \langle \text{D-Ind}(U_1), \text{D-Ind}(U_2)\rangle_G$ *for* $U_1, U_2 \in \mathrm{R}(\widetilde{R})$.
(ii) $\langle V_1, V_2\rangle_G = |W^1|^2\langle H_D(V_1), H_D(V_2)\rangle_{\widetilde{R}}$ *for* $V_1, V_2 \in \mathrm{R}(G)$.

5. Normalized Dirac operators and associated cohomology

In this section we first recall from [HPR06] (see also [HP06]) the definition and some basic properties of the cubic Dirac operators and the Dirac cohomology of unitary representations. Then we calculate the Dirac cohomology of the discrete series with respect to the normalized Dirac operator $D = \widetilde{D}(\mathfrak{g}, \mathfrak{t})$. This is crucial for using Dirac cohomology for induction and lifting of discrete series in the following sections as we used for geometric quantization in [CH16].

Let $\mathfrak{g}$ be any complex reductive Lie algebra, with a fixed invariant nondegenerate symmetric bilinear form B, and let $\mathfrak{r}$ be a quadratic subalgebra of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$, where $\mathfrak{s}$ is the orthogonal of $\mathfrak{r}$ with respect to B. Let $\mathfrak{r}_1$ be another quadratic subalgebra of $\mathfrak{g}$ with orthogonal $\mathfrak{s}_1$. We assume that $\mathfrak{r}_1 \subseteq \mathfrak{r}$, and hence $\mathfrak{s} \subseteq \mathfrak{s}_1$. In particular,

$$\mathfrak{g} = \mathfrak{r}_1 \oplus \mathfrak{s}_1 = \mathfrak{r}_1 \oplus \mathfrak{s} \oplus (\mathfrak{r} \cap \mathfrak{s}_1).$$

We will specialize to the case when $\mathfrak{r}$ is $\mathfrak{k}$ and then $\mathfrak{r}_1$ will be $\mathfrak{t}$ contained in $\mathfrak{k}$.

As in [HP06, Section 9.3] we write down the Dirac operator $D(\mathfrak{g}, \mathfrak{r}_1)$, using an orthonormal basis for $\mathfrak{s}_1$ formed by orthonormal bases Z_i for $\mathfrak{s}$ and Z'_j for $\mathfrak{r} \cap \mathfrak{s}_1$. Let $U(\cdot)$ be the universal enveloping algebra and $C(\cdot)$ be the Clifford algebra. Identifying

$$U(\mathfrak{g}) \otimes C(\mathfrak{s}_1) = U(\mathfrak{g}) \otimes C(\mathfrak{s}) \mathbin{\overline{\otimes}} C(\mathfrak{r} \cap \mathfrak{s}_1), \tag{4}$$

where $\overline{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product, we can write

$$\begin{aligned} D(\mathfrak{g}, \mathfrak{r}_1) = \sum_i Z_i \otimes Z_i \otimes 1 + \sum_j Z'_j \otimes 1 \otimes Z'_j \\ + \frac{1}{2} \sum_{i<j<k} B([Z_i, Z_j], Z_k) \otimes Z_i Z_j Z_k \otimes 1 \\ + \frac{1}{2} \sum_{i<j} \sum_k B([Z_i, Z_j], Z'_k) \otimes Z_i Z_j \otimes Z'_k \\ + \frac{1}{2} \sum_{i<j<k} B\big([Z'_i, Z'_j], Z'_k\big) \otimes 1 \otimes Z'_i Z'_j Z'_k. \end{aligned} \tag{5}$$

We note that the Kostant's original definition (cf. [Kos03]) uses exterior multiplication to define the cubic term. In the present case, we can use Clifford multiplication instead, because there is no difference between exterior and Clifford multiplication for orthogonal vectors.

Note also that the terms with Z_i, Z'_j and Z'_k do not appear in (5), because $B([Z_i, Z'_j], Z'_k) = B(Z_i, [Z'_j, Z'_k]) = 0$, as $[Z'_j, Z'_k] \in \mathfrak{r}$ is orthogonal to $\mathfrak{s}$.

We consider $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ as the subalgebra $U(\mathfrak{g}) \otimes C(\mathfrak{s}) \otimes 1$ of $U(\mathfrak{g}) \otimes C(\mathfrak{s}) \mathbin{\overline{\otimes}} C(\mathfrak{r} \cap \mathfrak{s}_1)$. In view of this, we see that the first and third summands in (5) combine to give $D(\mathfrak{g}, \mathfrak{r})$, Kostant's cubic Dirac operator corresponding to $\mathfrak{r} \subset \mathfrak{g}$.

The remaining three summands in (5) come from the cubic Dirac operator corresponding to $\mathfrak{r}_1 \subset \mathfrak{r}$. However, this is an element of the algebra $U(\mathfrak{r}) \otimes C(\mathfrak{r} \cap \mathfrak{s}_1)$, and this algebra has to be embedded into $U(\mathfrak{g}) \otimes C(\mathfrak{s}) \mathbin{\overline{\otimes}} C(\mathfrak{r} \cap \mathfrak{s}_1)$ diagonally, by

$$\Delta : U(\mathfrak{r}) \otimes C(\mathfrak{r} \cap \mathfrak{s}_1) \cong U(\mathfrak{r}_\Delta) \mathbin{\overline{\otimes}} C(\mathfrak{r} \cap \mathfrak{s}_1) \subset U(\mathfrak{g}) \otimes C(\mathfrak{s}) \mathbin{\overline{\otimes}} C(\mathfrak{r} \cap \mathfrak{s}_1).$$

Here $U(\mathfrak{r}_\Delta)$ is embedded into $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ by a diagonal embedding while the factor $C(\mathfrak{r} \cap \mathfrak{s}_1)$ remains unchanged.

We will denote $\Delta(D(\mathfrak{r}, \mathfrak{r}_1))$ by $D_\Delta(\mathfrak{r}, \mathfrak{r}_1)$. In case when there are several subalgebras and confusion might arise, we will use the more precise notation $\Delta_{\mathfrak{g},\mathfrak{r}}$ instead of Δ and give up the notation $D_\Delta(\cdot)$.

Theorem 5.1 ([HP06, Theorem 9.4.1]). *With notation as above, $D(\mathfrak{g}, \mathfrak{r}_1)$ decomposes as $D(\mathfrak{g}, \mathfrak{r}) + D_\Delta(\mathfrak{r}, \mathfrak{r}_1)$. Moreover, the summands $D(\mathfrak{g}, \mathfrak{r})$ and $D_\Delta(\mathfrak{r}, \mathfrak{r}_1)$ anti-commute.*

The anti-commuting property given in this theorem can be applied to relate the Dirac cohomologies of the various Dirac operators involved. For convenience, we define the cohomology of any linear operator A on a vector space V to be the vector space $H(A) = \ker A/(\operatorname{Im} A \cap \ker A)$. We call the operator A semisimple, if V is the (algebraic) direct sum of eigenspaces of A.

Lemma 5.2 ([HP06, Lemma 9.4.2]). *Let A and B be anti-commuting linear operators on an arbitrary vector space V.*

- (i) *Assume that A^2 is semisimple, and denote by V_λ the eigenspace of A^2 with eigenvalue λ. Then the cohomology $H(A+B)$ of $A+B$ on V is the same as the cohomology of the restriction of $A+B$ to $V_0 = \operatorname{Ker} A^2$.*
- (ii) *Assume that A^2 is semisimple, and that $\operatorname{Ker} A^2 = \operatorname{Ker} A = H(A)$; so $\operatorname{Ker} A \cap \operatorname{Im} A = 0$. Then $H(A+B)$ is equal to the cohomology of B restricted to the cohomology (i.e., kernel) of A.*
- (iii) *Assume that A^2 and B are semisimple. Then $H(A+B)$ is the cohomology (i.e., the kernel) of B acting on $H(A)$.*

From now on we specialize to the case when $\mathfrak{r} = \mathfrak{k}$ contains $\mathfrak{r}_1 = \mathfrak{t}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexification of the Cartan decomposition. This is the same as the orthogonal decomposition with respect to the Killing form B. In this case, the expression of the Dirac operator in (5) appears as

$$\begin{aligned} D(\mathfrak{g},\mathfrak{t}) = {} & \sum_i Z_i \otimes Z_i \otimes 1 + \sum_j Z'_j \otimes 1 \otimes Z'_j \\ & + \frac{1}{2} \sum_{i<j} \sum_k B([Z_i, Z_j], Z'_k) \otimes Z_i Z_j \otimes Z'_k \\ & + \frac{1}{2} \sum_{i<j<k} B\Big([Z'_i, Z'_j], Z'_k\Big) \otimes 1 \otimes Z'_i Z'_j Z'_k. \end{aligned}$$

This is due to the fact $B([Z_i, Z_j], Z_k) = 0$ for $Z_i, Z_j, Z_k \in \mathfrak{p}$. We have

$$D(\mathfrak{g},\mathfrak{t}) = D(\mathfrak{g},\mathfrak{k}) + D_\Delta(\mathfrak{k},\mathfrak{t}).$$

Let $\mathfrak{s}$ be the orthocomplement of $\mathfrak{t}$ in $\mathfrak{g}$. Let S be the simple module of the Clifford algebra $C(\mathfrak{s})$. If V is a $(\mathfrak{g}, K)$-module, then $D(\mathfrak{g},\mathfrak{t})$ acts on $V \otimes S$. We denote by $H_D(\mathfrak{g},\mathfrak{t};V)$ the cohomology of $V \otimes S$ with respect to $D(\mathfrak{g},\mathfrak{t})$; analogous notation will be used for other Dirac operators. We note that $H_D(\mathfrak{g},\mathfrak{t};V)$ is in fact the cohomology of the operator $D(\mathfrak{g},\mathfrak{t})$ on $V \otimes S$.

Proposition 5.3 ([HP06, Proposition 9.4.3]). *Let V be an irreducible unitary $(\mathfrak{g}, K)$-module. Then*

$$H_D(\mathfrak{g},\mathfrak{t};V) = H_D(\mathfrak{k},\mathfrak{t};H_D(\mathfrak{g},\mathfrak{k};V)).$$

In other words, the Dirac cohomology $H_D(\mathfrak{g},\mathfrak{t};V)$ can be calculated "in stages", as the $D(\mathfrak{k},\mathfrak{t})$-cohomology of the $D(\mathfrak{g},\mathfrak{k})$-cohomology.

Let V be the Harish-Chandra module of a unitary representation of G. There is also a unitary structure on the spin module S [HP06, Section 2.3.9]. Then the G-invariant hermitian form on V induces a $\mathfrak{g}$-invariant hermitian form on $V \otimes S$. It follows from the unitarity of V and the property of the hermitian form on S [HP06, Proposition 2.3.10] that $D(\mathfrak{g},\mathfrak{k})$ is symmetric and $D_\Delta(\mathfrak{k},\mathfrak{t})$ is skew-symmetric with respect to this form. We define the normalized Dirac operator

$$D = \widetilde{D}(\mathfrak{g}, \mathfrak{t}) = D(\mathfrak{g}, \mathfrak{k}) + iD_\Delta(\mathfrak{k}, \mathfrak{t}).$$

Then D is symmetric. Note that both $D(\mathfrak{g}, \mathfrak{k})^2$ and $-D_\Delta(\mathfrak{k}, \mathfrak{t})^2$ are positive definite, so $\widetilde{D}(\mathfrak{g}, \mathfrak{t})^2 = D(\mathfrak{g}, \mathfrak{k})^2 - D_\Delta(\mathfrak{k}, \mathfrak{t})^2$ is also positive definite. Then D is an elliptic differential operator. This is the very purpose of introducing the normalization. The Dirac cohomology $H_D(V)$ defined by the normalized $D = \widetilde{D}(\mathfrak{g}, \mathfrak{t})$ is $\ker D = \ker D^2$. We also note that $iD_\Delta(\mathfrak{k}, \mathfrak{t})$ and $D_\Delta(\mathfrak{k}, \mathfrak{t})$ define the same Dirac cohomology. It follows from Lemma 5.2 that $H_D(V)$ can also be calculated in the same way as

$$H_D(V) = H_D(\mathfrak{k}, \mathfrak{t}; H_D(\mathfrak{g}, \mathfrak{k}; V))$$

in Proposition 5.3. In particular, $H_D(V) = \ker D = H_D(\mathfrak{k}, \mathfrak{t}; H_D(\mathfrak{g}, \mathfrak{k}; V))$. In other words, $\widetilde{D}(\mathfrak{g}, \mathfrak{t})$ and $D(\mathfrak{g}, \mathfrak{t})$ define the same Dirac cohomology.

For any admissible $(\mathfrak{g}, K)$-module V, we know that $H_D(\mathfrak{g}, \mathfrak{k}; V)$ is a finite dimensional $\widetilde{K}$-module. Since $\mathfrak{k}$ and $\mathfrak{t}$ have equal rank, $H_D(\mathfrak{k}, \mathfrak{t}; H_D(\mathfrak{g}, \mathfrak{k}; V))$ is given by the Kostant theorem [HP06, Theorem 5.2.1]. This gives $H_D(\mathfrak{g}, \mathfrak{t}; V)$ explicitly provided that we know $H_D(\mathfrak{g}, \mathfrak{k}; V)$ explicitly.

Let $V = A_{\mathfrak{b}}(\lambda)$ be a discrete series representation with Harish-Chandra parameter $\lambda + \rho$. Here $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ is a θ-stable Borel subalgebra of $\mathfrak{g}$ (where $\mathfrak{g}^\theta = \mathfrak{k}$) containing a Cartan subalgebra $\mathfrak{t}$, and ρ is half the sum of positive roots determined by $\mathfrak{n}$. Then the Dirac cohomology of V with respect to $D(\mathfrak{g}, \mathfrak{k})$ consists of a single $\widetilde{K}$-type $V(\mu)$, whose highest weight is $\mu = \lambda + \rho_n$, where $\rho_n = \rho(\mathfrak{n} \cap \mathfrak{p})$. This is obtained from the highest weight of the lowest K-type of V, $\lambda + 2\rho_n$, by shifting by $-\rho_n$ (the lowest weight of S).

This result can be proved as follows: it is shown in [HP02, Proposition 5.4] that this $\widetilde{K}$-type is contained in the Dirac cohomology. Since V has a unique lowest K-type, and since $-\rho_n$ is the lowest weight of the spin module, with multiplicity one, it follows that any other $\widetilde{K}$-type has strictly larger highest weight, and thus cannot contribute to the Dirac cohomology. We note that the analytic Weyl group $W_K = N(T)/T$ is isomorphic to the algebraic Weyl group $W_{\mathfrak{k}} = W(\Delta(\mathfrak{k}, \mathfrak{t}))$. We now apply the above mentioned Kostant's formula to calculate the Dirac cohomology with respect to $D(\mathfrak{k}, \mathfrak{t})$:

$$H_D(\mathfrak{k}, \mathfrak{t}; V(\mu)) = \ker D(\mathfrak{k}, \mathfrak{t}) = \bigoplus_{w \in W_{\mathfrak{k}}} \mathbb{C}_{w(\mu + \rho_{\mathfrak{k}})}.$$

It follows from $\mu + \rho_{\mathfrak{k}} = \lambda + \rho$ that

$$H_D(\mathfrak{k}, \mathfrak{t}; V(\mu)) = \bigoplus_{w \in W_{\mathfrak{k}}} \mathbb{C}_{w(\lambda + \rho)}.$$

As a consequence of Proposition 3.7, we obtain

Proposition 5.4. *The Dirac cohomology $H_D(\mathfrak{g}, \mathfrak{t}; A_{\mathfrak{b}}(\lambda))$ of the discrete series representation $A_{\mathfrak{b}}(\lambda)$ is*

$$\operatorname{Ker} D = \sum_{w \in W_K} \mathbb{C}_{w(\lambda + \rho)}.$$

The above proposition says the Dirac cohomology with respect to the normalized Dirac operator $\widetilde{D}(\mathfrak{g}, \mathfrak{t})$ of a discrete series is the direct sum of the characters corresponding to all possible Harish-Chandra parameters. It indicates that the Dirac cohomology is analogous to the highest weight vector of a highest weight module.

6. Dirac induction for discrete series

In this section we first review from [CH16, Section 3] the Dirac cohomology of $L^2(G)$ with respect to the normalized Dirac operator $D = \widetilde{D}(\mathfrak{g}, \mathfrak{t})$ defined in the previous section. It is closely related to the calculation of Dirac cohomology of $L^2(G)$ with respect to the Dirac operator $D(\mathfrak{g}, \mathfrak{k})$ in [Hua15]. Then we define Dirac induction for the construction of discrete series. This new version of induction is modified from the induction from [Hua15] by inducing from character of a compact Cartan subgroup.

Let G be a connected semisimple Lie group having a finite center. Let Θ be a Cartan involution of G. Then the group $K = G^\Theta$ of fixed points of Θ is a maximal compact subgroup of G. Let T be a maximal torus of K. In this section we assume that T is also a Cartan subgroup of G. Then G and K have the same rank. As proved by Harish-Chandra, this equal rank condition is a necessary and sufficient condition for G to have discrete series representations. The classification and parametrization of the discrete series was obtained by Harish-Chandra. We now describe the Harish-Chandra parameters for discrete series representations.

Let $\widehat{T}$ denote the dual group of T. Then $\widehat{T}$ is isomorphic to the weight lattice Λ in $i\mathfrak{t}_0^*$ with respect to a root system $\Delta(\mathfrak{k}, \mathfrak{t})$. For the moment we let ρ be the half sum of positive roots, relative to an arbitrary ordering of the roots in $\Delta(\mathfrak{g}, \mathfrak{t})$. Then

$$\Lambda_\rho = \Lambda + \rho$$

does not depend on the particular choice of positive root system. Any two possible choices for ρ differ by a sum of roots, and hence by an element in Λ. We note that W_K acts on Λ_ρ, since W_K preserves the lattice Λ and any translation of ρ differs from ρ by a sum of roots. However, the Weyl group $W_\mathfrak{g} = W(\Delta(\mathfrak{g}, \mathfrak{t}))$ may not act on Λ_ρ. The following is a fundamental result of Harish-Chandra [Har65; Har66].

Theorem 6.1. *The equivalence classes of discrete series representations $\mathcal{H}_\lambda$ of G are in bijection with the W_K-orbits of regular elements λ in Λ_ρ.*

We may and will assume that the Harish-Chandra parameter λ of a discrete series $\mathcal{H}_\lambda$ is $\Delta^+(\mathfrak{k}, \mathfrak{t})$-dominant. We define $\Delta^+(\mathfrak{g}, \mathfrak{t})$ so that λ is dominant, then the discrete series representation $\mathcal{H}_\lambda$ has K-finite vectors V_λ which is equal to $A_\mathfrak{b}(\lambda - \rho)$ as described in the previous section. The space V_λ of the K-finite vectors of $\mathcal{H}_\lambda$ is called the Harish-Chandra module of $\mathcal{H}_\lambda$. The K-finite vectors are among C^∞-vectors. Now we consider the normalized Dirac operator $D = \widetilde{D}(\mathfrak{g}, \mathfrak{t})$ acting on K-finite vectors of $\mathcal{H}_\lambda$. Then D is a densely defined symmetric operator

$$D\colon \mathcal{H}_\lambda \otimes S \longrightarrow \mathcal{H}_\lambda \otimes S.$$

We recall a standard fact in functional analysis that a densely defined symmetric operator is closable and its closure is also symmetric. If A^* denotes the adjoint of a densely defined symmetric operator A, then the closure $\overline{A}$ of A is equal to $(A^*)^*$ and $\overline{A}$ is also symmetric [MV97, Lemma 20.1]. Thus, the above D is closable and the closure $\overline{D}$ is also symmetric. It follows that $\ker \overline{D}$ is a closed subspace of the Hilbert space $\mathcal{H}_\lambda \otimes S$. We define the Dirac cohomology $H_D(\mathcal{H})$ of an irreducible unitary representation $\mathcal{H}$ to be $\ker \overline{D}$. The following proposition shows that the Dirac cohomology of an irreducible unitary representation is equal to the Dirac cohomology of its Harish-Chandra module.

Proposition 6.2 ([CH16, Proposition 3.2])**.** *The kernel of* $\overline{D}\colon \mathcal{H}_\lambda \otimes S \to \mathcal{H}_\lambda \otimes S$ *coincides with the kernel of* $D\colon V_\lambda \otimes S \to V_\lambda \otimes S$. *It follows that*

$$\operatorname{Ker} \overline{D} = H_D(V_\lambda) = \sum_{w \in W_K} \mathbb{C}_{w\lambda}.$$

Recall that the Dirac operator D is in $U(\mathfrak{g}) \otimes C(\mathfrak{s})$. We first consider

$$D\colon C^\infty(G) \otimes S \longrightarrow C^\infty(G) \otimes S$$

with Z_i and Z_i' acting on $C^\infty(G)$ by differentiation on the right, i.e., as left invariant vector fields on G. Then D induces a densely defined symmetric operator on the Hilbert space $L^2(G) \otimes S$, and the closure of D defines a closed symmetric operator

$$\mathbb{D}\colon L^2(G) \otimes S \longrightarrow L^2(G) \otimes S.$$

Then $\ker \mathbb{D}$ is a closed subspace in $L^2(G) \otimes S$. We define the Dirac cohomology $H_D(L^2(G))$ of $L^2(G)$ to be $\ker \mathbb{D}$. It follows from the fact that D is T-invariant that $\ker \mathbb{D}$ is a $G \times T$-module. Let $\widehat{G}_{d.s.}$ denotes the set of equivalence classes of discrete series representations of G.

Theorem 6.3 ([CH16, Theorem 3.3])**.** *We have the following orthogonal sum decomposition as* $G \times T$*-representations:*

$$\operatorname{Ker} \mathbb{D} = \sum_{\lambda \in \widehat{G}_{d.s.}} \mathcal{H}_\lambda \otimes \left(\sum_{w \in W_K} \mathbb{C}_{-w\lambda} \right).$$

Remark 6.4. It follows from the above theorem that $\operatorname{Hom}_{\widetilde{T}}(\mathbb{C}_\lambda^*, \ker \mathbb{D})$ is the discrete series $\mathcal{H}_\lambda$, provided $\mathbb{C}_\lambda^* = H_D(\mathcal{H}_\lambda^*)$. We can identify

$$\operatorname{Hom}_{\widetilde{T}}\Big(\mathbb{C}_\lambda^*, L^2(G) \otimes\Big)$$

with L^2-sections of the line bundle $L^2(G/T, S \otimes \mathbb{C}_\lambda)$. Then Dirac operator D induces a densely defined symmetric operator D_λ on the corresponding Hilbert space:

$$D_\lambda\colon L^2(G/T, S \otimes \mathbb{C}_\lambda) \longrightarrow L^2(G/T, S \otimes \mathbb{C}_\lambda).$$

Then $\ker D_\lambda$ is a discrete series representation with Harish-Chandra parameter λ, if $\lambda \in \Lambda_\rho$ is regular, or zero otherwise.

We define the Dirac induction

$$\text{D-Ind}\colon \Lambda_\rho \longrightarrow \widehat{G}_{d.s.} \cup \{0\}$$

by $\text{D-Ind}(\mathbb{C}_\lambda) = \mathcal{H}_\lambda$ the discrete series representation with Harish-Chandra parameter λ provided $\lambda \in \Lambda_\rho$ is regular with respect to $\Delta(\mathfrak{g}, \mathfrak{t})$. If λ is not regular, then we make the convention that $\text{D-Ind}(\mathbb{C}_\lambda) = 0$.

Proposition 6.5. *There are two identities relating Dirac cohomology and Dirac induction:*

(i) $\text{D-Ind}(H_D(\mathcal{H}_\lambda)) = |W_K| \mathcal{H}_\lambda$ *for* $\lambda \in \widehat{G}_{d.s.}$.

(ii) $H_D(\text{D-Ind}(\mathbb{C}_\lambda)) = \bigoplus_{w\in W_K} \mathbb{C}_{w\lambda}$ *provided* $\lambda \in \Lambda_\rho$ *is regular with respect to* $\Delta(\mathfrak{g}, \mathfrak{t})$.

Proof. For an irreducible discrete series representation V of G or U of M, the two identities follow from Proposition 5.6 on Dirac cohomology for discrete series. ☐

The same argument in the above proof gives another two identities in the following proposition.

Proposition 6.6. *Suppose that M is a connected closed subgroup of a connected semisimple Lie group G with a common compact Cartan subgroup T. Then the following two identities hold:*

(i) $\langle U_1, U_2\rangle_M = \langle \text{D-Ind}(U_1), \text{D-Ind}(U_2)\rangle_G$ *for* $U_1, U_2 \in \widehat{M}_{d.s.}$.

(ii) $\langle V_1, V_2\rangle_G = |W^1|^2\langle H_D(V_1), H_D(V_2)\rangle_M$ *for* $V_1, V_2 \in \widehat{G}_{d.s.}$.

7. Lifting of L-packets of discrete series

We retain the notation in the previous section. Let G be a connected noncompact semisimple Lie groups with finite center. Suppose that θ is a Cartan involution of G. Then $K = G^\theta$ is a maximal compact subgroup of G. Let M be a θ-stable reductive subgroup of G. Then $K_M = M^\theta$ is a maximal compact subgroup of M. We assume that there exists a compact Cartan subgroup T for both M and G. Furthermore, we also assume that $\rho(\mathfrak{g}) - \rho(\mathfrak{m}) \in \widehat{T} = \Lambda$. We define the

$$\text{Lift}_M^G : \widehat{M}_{d.s.} \longrightarrow \widehat{G}_{d.s.} \cup \{0\}$$

as the composition of taking one character from the Dirac cohomology with respect to $\widetilde{D}(\mathfrak{r}, \mathfrak{t})$ and Dirac induction from $\widehat{T}$ to discrete series of G. Namely, for any $\sigma \in \widehat{M}_{d.s.}$, if $\mathbb{C}_\lambda \in H_D(\sigma)$, then

$$\text{Lift}_M^G(\sigma) = \text{D-Ind}_T^G(\mathbb{C}_\lambda).$$

This is well-defined since the Weyl group W_{K_M} is contained in the Weyl group W_K and the set of elements in the W_{K_M}-orbit of λ is contained in the set of elements in the W_K-orbit of λ. It follows from Theorem 5.4 that

$$\text{Lift}_M^G(\sigma) = \frac{1}{|W_{K_M}|}\,\text{D-Ind}_T^G(H_D(\sigma)).$$

It is difficult to calculate Dirac cohomology with respect to $D(\mathfrak{g}, \mathfrak{m})$ except for finite dimensional representations. Here $\mathfrak{m}$ is the complexified Lie algebra of M. We define

$$\text{Tran}_M^G : \widehat{G}_{d.s.} \longrightarrow \widehat{M}_{d.s.}$$

by $\text{Tran}_M^G(\pi) = \frac{1}{|W_{K_M}|}\text{D-Ind}_T^M(H_D(\pi))$, a direct sum of irreducible discrete series of M. It plays the role of Dirac cohomology with respect to Dirac operator $D(\mathfrak{g}, \mathfrak{m})$. It is straightforward to check that this is the adjoint to the induction operator. We summarize this as a lemma.

Lemma 7.1. *Suppose that both M and G have a common maximal torus T. Then for any $U \in \widehat{M}_{d.s.}$ and $V \in \widehat{G}_{d.s.}$, we have*

$$\left\langle U, \text{Tran}_M^G(V)\right\rangle_M = \left\langle \text{Lift}_M^G(U), V\right\rangle_G.$$

Thus, Tran_M^G *is right adjoint to* Lift_M^G.

An example of L-packet is the set of discrete series representations with a given infinitesimal character and given central character. The discrete series representations of $SL(2,\mathbb{R})$ are grouped into L-packets with two elements. Let $L_M(\sigma)$ denote the L-packet of discrete series of M containing σ. Clearly, we have the following statement for the lifting of L-packets of discrete series.

Proposition 7.2. *Suppose that the reductive connected subgroup* M *of a connected semisimple Lie group* G *share a compact Cartan subgroup. Then for any* $\sigma \in \widehat{M}_{d.s.}$, *we have*

$$\mathrm{Lift}(L_M(\sigma)) \subseteq L_G(\mathrm{Lift}(\sigma)).$$

As the final remark, we note that the idea of lifting characters by Dirac cohomology is useful in various more general settings. Suppose that G and G' are two connected semisimple Lie groups with isomorphic maximal tori

$$T_G \cong T_{G'}.$$

Then it is possible to extend the definition of lifting of discrete series characters from G to that of G'. For example, the transfer of discrete series of two fold cover $Mp(2n,\mathbb{R})$ to the discrete series of $SO(p,q)$ with $p+q=2n+1$. In this case, the lifting coincides with the theta correspondence in Howe dual pair as verified by Adams–Barbasch [AB98].

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