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On the intertwining differential operators between vector bundles over the real projective space of dimension two

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Abstract. The main objective of this paper is twofold. One is to classify and construct $\mathrm{SL}(3, \mathbb{R})$ -intertwining differential operators between vector bundles over the real projective space $\mathbb{R}\mathbb{P}^2$. It turns out that two kinds of operators appear. We call them Cartan operators and PRV operators. The second objective is then to study the representations realized on the kernel of those operators both in the smooth and holomorphic setting. A key machinery is the BGG resolution. In particular, by exploiting some results of Davidson–Enright–Stanke and Enright–Joseph, the irreducible unitary highest weight modules of $\mathrm{SU}(1, 2)$ at the (first) reduction points are classified by the image of Cartan operators and kernel of PRV operators.

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Key Words and Phrases: intertwining differential operator, generalized Verma module, Cartan component, PRV component, BGG resolution, unitary highest weight module

1. Introduction

In this paper we study from a modern point of view the very classical topic of real projective geometry in dimension two; this was already studied in Greek mathematics, and in the Renaissance it became part of the theory of perspective in art. Here there is (from the point of view of Sophus Lie) the 8-dimensional group $G = \mathrm{SL}(3, \mathbb{R})$ of projective transformations acting on G/P with (in modern language) P a parabolic subgroup. As such it falls into the class of parabolic geometries, and also the kind of structure that Joseph A. Wolf studied in depth, namely flag manifolds. Our aim in this paper is to illustrate in detail several aspects of this geometry in finding some invariants associated with homogeneous vector bundles on G/P ; these are G -intertwining differential operators for the natural action on sections, and we shall see how they fit into the very general theory of parabolic geometry. This involves by a natural duality of Verma modules and the celebrated BGG resolution, and we will see natural constructions of representations of G in spaces of sections. It turns out that it is related to the classification of some irreducible unitary highest weight representations of another real form $G_u = \mathrm{SU}(1, 2)$ of $G_{\mathbb{C}} = \mathrm{SL}(3, \mathbb{C})$. The method uses some aspects of the theory of the Rankin–Cohen operators, known from the theory of automorphic forms; this allows quite explicit formulas for the differential operators in question; as such we feel that the explicit case of 2-dimensional projective geometry deserves attention. In the following, we shall describe our work in more detail.

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1.1. Classification and construction

First, we briefly introduce some notation; more detailed notation will be introduced in Section 3. Let G be a real reductive group and $P = MAN_+$ a Langlands decomposition of a parabolic subgroup P . We write $\text{Irr}(M)_{\text{fin}}$ for the set of equivalence classes of irreducible finite-dimensional representations of M . Likewise, let $\text{Irr}(A)$ denote the set of characters of A . Then, for the outer tensor product $\varpi \boxtimes \nu \boxtimes \text{triv}$ of $\varpi \in \text{Irr}(M)_{\text{fin}}$, $\nu \in \text{Irr}(A)$, and the trivial representation triv of N_+ , we put

$$I(\varpi, \nu) := \text{Ind}_P^G(\varpi \boxtimes \nu \boxtimes \text{triv})$$

for an unnormalized parabolically induced representation of G .

Let $\text{Diff}_G(I(\sigma, \lambda), I(\varpi, \nu))$ denote the space of G -intertwining differential operators $\mathcal{D}: I(\sigma, \lambda) \rightarrow I(\varpi, \nu)$. In what follows, we often refer to them as IDOs.

In this paper, we consider the following problem for the case

$$(G, P) = (\text{SL}(3, \mathbb{R}), P_{1,2}),$$

where $P_{1,2}$ is a maximal parabolic subgroup of G with partition $3 = 1 + 2$.

Problem 1.1. Do the following.

(A1) Classify $(\sigma, \lambda), (\varpi, \nu) \in \text{Irr}(M)_{\text{fin}} \times \text{Irr}(A)$ such that

$$\text{Diff}_G(I(\sigma, \lambda), I(\varpi, \nu)) \neq \{0\}.$$

(A2) For $(\sigma, \lambda), (\varpi, \nu) \in \text{Irr}(M)_{\text{fin}} \times \text{Irr}(A)$, determine the dimension

$$\dim \text{Diff}_G(I(\sigma, \lambda), I(\varpi, \nu)).$$

(A3) For $(\sigma, \lambda), (\varpi, \nu) \in \text{Irr}(M)_{\text{fin}} \times \text{Irr}(A)$, construct generators

$$\mathcal{D} \in \text{Diff}_G(I(\sigma, \lambda), I(\varpi, \nu)).$$

For recent developments on Problem 1.1, see, for instance, the introduction of [KØ25] and references therein. The most outstanding recent advance in Problem 1.1 is the invention of the so-called F-method [KP16a; KP16b]. This is a powerful machinery proposed by T. Kobayashi, which reformulates the problems (A1), (A2), and (A3) to solving a certain system of partial differential equations. As such, the F-method allows one to achieve the construction and classification of IDOs, simultaneously. The method plays a pivotal role especially in taking care of the differential symmetry breaking operators (DSBOs), which are G' -IDOs for $G' \subset G$ from a G -representation to a G' -representation. For instance, in [Kub], the first author tackles Problem 1.1 for DSBOs \mathcal{D} by using the F-method.

It is no harm to apply the machinery to the present situation. In fact, it is used in our earlier work [KØ25] to solve Problem 1.1 for the case $(G, P) = (\text{SL}(n, \mathbb{R}), P_{1,n-1})$ with $\dim \sigma = 1$ (line bundle case). However, in this paper, we take a more classical approach to the problem, which stems from the duality between the space of IDOs and that of homomorphisms between generalized Verma modules, as it appears simpler in the present case. Combining the duality with the idea of “standard map” of generalized Verma modules, we shall solve (A1) and (A2) in Section 4. It turned out that, for any $m, \ell \in \mathbb{Z}_{\geq 0}$, there exists a non-zero IDO \mathcal{D} with order $|\ell - m|$. See Theorems 4.1 and 5.1 for more details.

To solve (A3), one needs to compute P -invariants of generalized Verma modules. In doing so, there are, in principle, two steps to consider. The first is to compute MA -invariants of the tensor product of three irreducible finite-dimensional representations of MA . (On this matter, see [DES90, Theorems 4.3 and 5.5] and [EG11, Theorem 3] as relevant works). The second is the evaluation of the \mathfrak{n}_+ -invariance of the MA -invariants, where \mathfrak{n}_+ is the complexified Lie algebra of N_+ . This step can be thought of as computing a *conformal weight* in conformal geometry.

The first step for the present work can be reduced to computing the projection of the tensor product of two irreducible finite-dimensional representations of $SL(2, \mathbb{R})$ onto its irreducible components. Such projections are known as Gordan's transvectants, or equivalently, Rankin–Cohen brackets (see, for instance, [Cle25; El 06] for a relationship between them). For this purpose, one can make use of the explicit formula of Molchanov [Mol15]. We, nonetheless, compute the MA -invariance in a more elementary way (see Section 5.3).

The evaluation of the \mathfrak{n}_+ -invariance in the second step is in general not easy, especially for a vector bundle case. In the present work, we succeeded to avoid such a computation by showing that any IDO in the present case comes from the standard map between generalized Verma modules. This is done in Section 4.6.

It is noteworthy that only the so-called Cartan component and PRV component among all $SL(2, \mathbb{R})$ -irreducible components contribute to the construction of IDOs. This verifies some insight made by Kable in [Kab18, Introduction] on his *automatic conformal invariance* result for second order systems on vector bundles. In this paper, we refer to the IDOs \mathcal{D} coming from the Cartan component and PRV component as the *Cartan operator* and *PRV operator*, respectively.

1.2. BGG resolution

The kernel $\text{Ker } \mathcal{D}$ and image $\text{Im } \mathcal{D}$ of IDOs \mathcal{D} are naturally representations of G . These representations are closely related to the composition structure of the parabolically induced representations. Indeed, in [KØ25], we exploited the composition structure studied by Howe–Lee [HL99] and van Dijk–Molchanov [DM99] to investigate the representations on $\text{Ker } \mathcal{D}$ and $\text{Im } \mathcal{D}$ for the case $(G, P) = (SL(n, \mathbb{R}), P_{1, n-1})$ with $\dim \sigma = 1$. (See Remark 6.5 for more details.)

As opposed to the line bundle case, the composition structure for a vector bundle case seems not fully investigated. Then, in the present work, we study $\text{Ker } \mathcal{D}$ and $\text{Im } \mathcal{D}$ via the BGG resolution. A BGG resolution, or more generally, BGG sequence [CD01; ČSS01] is a powerful tool to construct IDOs \mathcal{D} , which are also referred to as *equivariant differential operators*, *invariant differential operators*, or *conformally covariant differential operators* (in conformal geometry), among others. Eastwood–Gover [EG11] carefully studied BGG resolutions for the case $(G, P) = (SL(n, \mathbb{R}), P_{1, n-1})$. With the aid of their results, we shall show that the Cartan operators and PRV operators are the first and second BGG operators, respectively. This in particular shows that every irreducible finite-dimensional representation of G is realized in the kernel $\text{Ker } \mathcal{D}$ with a suitable parity condition.

It is remarked that, as M for $P = MAN_+$ is $M \simeq SL^\pm(2, \mathbb{R})$, the representations $\sigma \in \text{Irr}(M)_{\text{fin}}$ involve some parity. BGG sequences are well studied in parabolic geometry; nonetheless, such a condition seems not to be paid attention to as it is automatically fixed by the irreducible finite-dimensional representation realized in $\text{Ker } \mathcal{D}$. The IDOs \mathcal{D} are in fact “rigid” with respect to the parity of $\sigma \in \text{Irr}(M)_{\text{fin}}$

(cf. [KØ19, Lemma 2.17]). So, if the BGG resolution is utilized to determine the representations in $\text{Ker } \mathcal{D}$ and $\text{Im } \mathcal{D}$, then one should be careful for the parity of representations of M . We shall return and discuss this point in more detail in Section 6.1 (especially after Corollary 6.4).

1.3. Irreducible unitary highest weight modules

As there exists a duality between the space of IDOs \mathcal{D} and that of homomorphisms between generalized Verma modules, the IDOs \mathcal{D} are determined by the pair of complex Lie algebras $(\mathfrak{g}, \mathfrak{p})$, where \mathfrak{g} and \mathfrak{p} are the complexified Lie algebras of $G = \text{SL}(3, \mathbb{R})$ and the parabolic subgroup P . So, the operators \mathcal{D} are intertwining operators also for another real form $G_u = \text{SU}(1, 2)$.

More precisely speaking, let K be a maximal compact subgroup of G_u . Then, via the Borel embedding $G_u/K \hookrightarrow G_{\mathbb{C}}/P_{\mathbb{C}}$, where $P_{\mathbb{C}}$ is the complexification of a parabolic subgroup P , IDOs \mathcal{D} are regarded as G_u -intertwining differential operators on the space of holomorphic sections over G_u/K . In this context, IDOs \mathcal{D} are more often referred to as *covariant differential operators* (see, for instance, [HJ82; HJ83; Jak85]).

In the above setting with $G_{\mathbb{C}}/\bar{P}_{\mathbb{C}}$, where $\bar{P}_{\mathbb{C}}$ is the complexification of the opposite parabolic subgroup \bar{P} to P , Davidson–Enright–Stanke [DES90; DES91] uses IDOs \mathcal{D} to classify unitary highest weight modules at the so-called reduction points. In their study, the aforementioned PRV components play a central role. (For a connection between unitary highest weight modules and PRV components, see also Enright–Joseph [EJ90], Enright–Wallach [EW97], Pandžić–Prlić–Souček–Tuček [Pan+23], and Pandžić–Prlić–Savin–Souček–Tuček [Pan+25], among others.)

There is only one reduction point (first reduction point) for $G_u = \text{SU}(1, 2)$. Then, at the end of the paper, we shall classify those unitary representations by the kernel of PRV operators and image of Cartan operators via the BGG resolution. This is done in Theorem 7.1.

1.4. Organization of the paper

Now we describe the rest of the paper. There are seven sections including this introduction. In Section 2, we recall the duality between the space of IDOs \mathcal{D} and that of homomorphisms between generalized Verma modules. We also review quickly the notion of standard map between generalized Verma modules. Then, in Section 3, we specialize the framework considered in Section 2 to $(G, P) = (\text{SL}(3, \mathbb{R}), P_{1,2})$. In particular, we fix some notation for $\text{Irr}(M)_{\text{fin}}$ and $\text{Irr}(A)$, carefully.

Sections 4 and 5 are devoted to the classification of parameters ((A1) and (A2)) and explicit construction of IDOs \mathcal{D} ((A3)), respectively. The main results, namely, Theorems 4.1 and 5.1 solve Problem 1.1 for the present case. In addition, we also study the algebraic counterpart of Problem 1.1 for generalized Verma modules. These are achieved in Theorems 4.2 and 5.3.

The representations in the kernel $\text{Ker } \mathcal{D}$ and image $\text{Im } \mathcal{D}$ are studied in Section 6. This is done via the BGG resolution in Corollary 6.4. In this section we carefully investigate the effect of the parity conditions to $\text{Ker } \mathcal{D}$ and image $\text{Im } \mathcal{D}$.

The last section, Section 7, is for the classification of the irreducible unitary highest weight modules of $\text{SU}(1, 2)$ at the (first) reduction points. We first quickly review a general theory of the Borel embedding $G_u/K \hookrightarrow G_{\mathbb{C}}/\bar{P}_{\mathbb{C}}$. Then we give a

classification of such representations in terms of the kernel of PRV operators and image of Cartan operators via the BGG resolution. This is achieved in Theorem 7.1.

2. Preliminaries

The aim of this section is to recall from the literature the so-called duality theorem (Theorem 2.1) and the notion of standard map between generalized Verma modules. Much of the material in this section is taken from [KØ25]. In later sections, we shall apply those ideas to the case $(G, P) = (\mathrm{SL}(3, \mathbb{R}), P_{1,2})$.

2.1. Duality theorem

Let G be a real reductive Lie group and $P = MAN_+$ a Langlands decomposition of a parabolic subgroup P of G . We denote by $\mathfrak{g}(\mathbb{R})$ and $\mathfrak{p}(\mathbb{R}) = \mathfrak{m}(\mathbb{R}) \oplus \mathfrak{a}(\mathbb{R}) \oplus \mathfrak{n}_+(\mathbb{R})$ the Lie algebras of G and $P = MAN_+$, respectively.

For a real Lie algebra $\mathfrak{h}(\mathbb{R})$, we write \mathfrak{h} and $\mathcal{U}(\mathfrak{h})$ for its complexification and the universal enveloping algebra of \mathfrak{h} , respectively. For instance, $\mathfrak{g}, \mathfrak{p}, \mathfrak{m}, \mathfrak{a}$, and \mathfrak{n}_+ are the complexifications of $\mathfrak{g}(\mathbb{R}), \mathfrak{p}(\mathbb{R}), \mathfrak{m}(\mathbb{R}), \mathfrak{a}(\mathbb{R})$, and $\mathfrak{n}_+(\mathbb{R})$, respectively.

For $\lambda \in \mathfrak{a}^* \simeq \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}(\mathbb{R}), \mathbb{C})$, we denote by \mathbb{C}_λ the one-dimensional representation of A defined by $a \mapsto a^\lambda := e^{\lambda(\log a)}$. For a finite-dimensional representation (σ, V) of M and $\lambda \in \mathfrak{a}^*$, we denote by σ_λ the outer tensor product representation $\sigma \boxtimes \mathbb{C}_\lambda$. As a representation on V , we define $\sigma_\lambda: ma \mapsto a^\lambda \sigma(m)$. By letting N_+ act trivially, we regard σ_λ as a representation of P . Let $\mathcal{V} := G \times_P V \rightarrow G/P$ be the G -equivariant vector bundle over the real flag variety G/P associated with the representation (σ_λ, V) of P . We identify the Fréchet space $C^\infty(G/P, \mathcal{V})$ of smooth sections with

$$C^\infty(G, V)^P := \left\{ f \in C^\infty(G, V) : f(gp) = \sigma_\lambda^{-1}(p)f(g) \text{ for } p \in P \right\},$$

the space of P -invariant smooth functions on G . Then, via the left regular representation L of G on $C^\infty(G)$, we realize the parabolically induced representation $\pi_{(\sigma, \lambda)} = \mathrm{Ind}_P^G(\sigma_\lambda)$ on $C^\infty(G/P, \mathcal{V})$. We denote by R the right regular representation of G on $C^\infty(G)$.

Similarly, for a finite-dimensional representation (η_ν, W) of MA , we define the induced representation $\pi_{(\eta, \nu)} = \mathrm{Ind}_P^G(\eta_\nu)$ on the space $C^\infty(G/P, \mathcal{W})$ of smooth sections for a G -equivariant vector bundle $\mathcal{W} := G \times_P W \rightarrow G/P$. We write $\mathrm{Diff}_G(\mathcal{V}, \mathcal{W})$ for the space of intertwining differential operators $\mathcal{D}: C^\infty(G/P, \mathcal{V}) \rightarrow C^\infty(G/P, \mathcal{W})$.

Let $\mathfrak{g}(\mathbb{R}) = \mathfrak{n}_-(\mathbb{R}) \oplus \mathfrak{m}(\mathbb{R}) \oplus \mathfrak{a}(\mathbb{R}) \oplus \mathfrak{n}_+(\mathbb{R})$ be the Gelfand–Naimark decomposition of $\mathfrak{g}(\mathbb{R})$, and write $N_- = \exp(\mathfrak{n}_-(\mathbb{R}))$. We identify N_- with the open Bruhat cell N_-P of G/P via the embedding $\iota: N_- \hookrightarrow G/P, \bar{n} \mapsto \bar{n}P$. Via the restriction of the vector bundle $\mathcal{V} \rightarrow G/P$ to the open Bruhat cell $N_- \xrightarrow{\iota} G/P$, we regard $C^\infty(G/P, \mathcal{V})$ as a subspace of $C^\infty(N_-) \otimes V$.

We view intertwining differential operators $\mathcal{D}: C^\infty(G/P, \mathcal{V}) \rightarrow C^\infty(G/P, \mathcal{W})$ as differential operators $\mathcal{D}': C^\infty(N_-) \otimes V \rightarrow C^\infty(N_-) \otimes W$ such that the restriction $\mathcal{D}'|_{C^\infty(G/P, \mathcal{V})}$ to $C^\infty(G/P, \mathcal{V})$ is a map $\mathcal{D}'|_{C^\infty(G/P, \mathcal{V})}: C^\infty(G/P, \mathcal{V}) \rightarrow C^\infty(G/P, \mathcal{W})$

(see the diagram below).

$$\begin{array}{ccc} C^\infty(N_-) \otimes V & \xrightarrow{\mathcal{D}'} & C^\infty(N_-) \otimes W \\ \iota^* \uparrow & & \uparrow \iota^* \\ C^\infty(G/P, \mathcal{V}) & \xrightarrow{\mathcal{D}=\mathcal{D}'|_{C^\infty(G/P, \mathcal{V})}} & C^\infty(G/P, \mathcal{W}) \end{array}$$

In particular, we regard $\text{Diff}_G(\mathcal{V}, \mathcal{W})$ as

$$\text{Diff}_G(\mathcal{V}, \mathcal{W}) \subset \text{Diff}_{\mathbb{C}}(C^\infty(N_-) \otimes V, C^\infty(N_-) \otimes W) \quad (1)$$

$$\simeq \text{Diff}_{\mathbb{C}}(C^\infty(N_-)) \otimes \text{Hom}_{\mathbb{C}}(V, W)$$

$$\simeq \text{Diff}_{\mathbb{C}}(C^\infty(N_-)) \otimes V^\vee \otimes W, \quad (2)$$

where $\text{Diff}_{\mathbb{C}}(C^\infty(N_-))$ denotes the space of differential operators on $C^\infty(N_-)$ and $V^\vee := \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$.

For a finite-dimensional representation (σ_λ, V) of MA , we write $((\sigma_\lambda)^\vee, V^\vee)$ for the contragredient representation of (σ_λ, V) . By letting \mathfrak{n}_+ act on V^\vee trivially, we regard the infinitesimal representation $d\sigma^\vee \boxtimes \mathbb{C}_{-\lambda}$ of $(\sigma_\lambda)^\vee$ as a \mathfrak{p} -module. The induced module

$$M_{\mathfrak{p}}(V^\vee) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V^\vee$$

is called a generalized Verma module, provided that V is a simple \mathfrak{p} -module. Via the diagonal action of P on $M_{\mathfrak{p}}(V^\vee)$, we regard $M_{\mathfrak{p}}(V^\vee)$ as a (\mathfrak{g}, P) -module. For the proof of the following theorem, see, for instance, [CS90; KP16a; KR00].

Theorem 2.1 (Duality theorem). *There is a natural linear isomorphism*

$$\mathcal{D}: \text{Hom}_P(W^\vee, M_{\mathfrak{p}}(V^\vee)) \xrightarrow{\sim} \text{Diff}_G(\mathcal{V}, \mathcal{W}), \quad (3)$$

where, for $\varphi \in \text{Hom}_P(W^\vee, M_{\mathfrak{p}}(V^\vee))$ and $F \in C^\infty(G/P, \mathcal{V}) \simeq C^\infty(G, V)^P$, the element $\mathcal{D}(\varphi)F \in C^\infty(G/P, \mathcal{W}) \simeq C^\infty(G, W)^P$ is given by

$$\langle \mathcal{D}(\varphi)F, w^\vee \rangle = \sum_j \langle dR(u_j)F, v_j^\vee \rangle \quad \text{for } w^\vee \in W^\vee, \quad (4)$$

where $\varphi(w^\vee) = \sum_j u_j \otimes v_j^\vee \in M_{\mathfrak{p}}(V^\vee)$.

Remark 2.2. By the Frobenius reciprocity, the isomorphism (3) is equivalent to

$$\mathcal{D}: \text{Hom}_{\mathfrak{g}, P}(M_{\mathfrak{p}}(W^\vee), M_{\mathfrak{p}}(V^\vee)) \xrightarrow{\sim} \text{Diff}_G(\mathcal{V}, \mathcal{W}). \quad (5)$$

2.2. Standard map

To introduce the notion of standard map between generalized Verma modules, we first reparametrize them in terms of infinitesimal characters.

Let \mathfrak{g} be a complex simple Lie algebra. Fix a Cartan subalgebra \mathfrak{h} and write $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{h})$ for the set of roots of \mathfrak{g} with respect to \mathfrak{h} . Choose a positive system Δ^+ and denote by Π the set of simple roots of Δ . Let \mathfrak{b} denote the Borel subalgebra of \mathfrak{g} associated with Δ^+ , namely, $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, where \mathfrak{g}_α is the root space for $\alpha \in \Delta^+$.

Let $\langle \cdot, \cdot \rangle$ denote the inner product on \mathfrak{h}^* induced from a non-degenerate symmetric bilinear form on \mathfrak{g} . For $\alpha \in \Delta$, we write $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$. Also, write s_α

for the root reflection with respect to $\alpha \in \Delta$. As usual, we let $\rho = (1/2) \sum_{\alpha \in \Delta^+} \alpha$ be half the sum of the positive roots.

Let $\mathfrak{p} \supset \mathfrak{b}$ be a standard parabolic subalgebra of \mathfrak{g} . Write $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}_+$ for the Levi decomposition of \mathfrak{p} . We let $\Pi(\mathfrak{l}) = \{\alpha \in \Pi : \mathfrak{g}_\alpha \subset \mathfrak{l}\}$.

Now we put

$$\mathbf{P}_\Gamma^+ := \{\mu \in \mathfrak{h}^* : \langle \mu, \alpha^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Pi(\mathfrak{l})\}. \quad (6)$$

For $\mu \in \mathbf{P}_\Gamma^+$, let $E(\mu - \rho)$ be the finite-dimensional simple $\mathcal{U}(\mathfrak{l})$ -module with highest weight $\mu - \rho$. By letting \mathfrak{n}_+ act trivially, we regard $E(\mu - \rho)$ as a $\mathcal{U}(\mathfrak{p})$ -module. Then the induced module

$$N_{\mathfrak{p}}(\mu) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} E(\mu - \rho) \quad (7)$$

is the generalized Verma module with highest weight $\mu - \rho$. If $\mathfrak{p} = \mathfrak{b}$, then $N(\mu) \equiv N_{\mathfrak{b}}(\mu)$ is the (ordinary) Verma module with highest weight $\mu - \rho$.

Let $\mu, \eta \in \mathbf{P}_\Gamma^+$. It follows from a theorem by BGG–Verma (Theorem 2.4 below) that if $\text{Hom}_{\mathfrak{g}}(N_{\mathfrak{p}}(\mu), N_{\mathfrak{p}}(\eta)) \neq \{0\}$, then $\text{Hom}_{\mathfrak{g}}(N(\mu), N(\eta)) \neq \{0\}$ (cf. [Dix96, Theorem 7.6.23] and [Hum08, Theorem 5.1]).

Conversely, suppose that there exists a non-zero \mathfrak{g} -homomorphism $\varphi : N(\mu) \rightarrow N(\eta)$. Let $\text{pr}_\mu : N(\mu) \rightarrow N_{\mathfrak{p}}(\mu)$ denote the canonical projection map. Then we have $\varphi(\text{Ker}(\text{pr}_\mu)) \subset \text{Ker}(\text{pr}_\eta)$ [Lep77, Proposition 3.1]. Thus, the map φ induces a \mathfrak{g} -homomorphism $\varphi_{\text{std}} : N_{\mathfrak{p}}(\mu) \rightarrow N_{\mathfrak{p}}(\eta)$ such that the following diagram commutes:

$$\begin{array}{ccc} N(\mu) & \xrightarrow{\varphi} & N(\eta) \\ \text{pr}_\mu \downarrow & \circlearrowleft & \downarrow \text{pr}_\eta \\ N_{\mathfrak{p}}(\mu) & \xrightarrow{\varphi_{\text{std}}} & N_{\mathfrak{p}}(\eta). \end{array}$$

The map φ_{std} is called the *standard map* from $N_{\mathfrak{p}}(\mu)$ to $N_{\mathfrak{p}}(\eta)$ associated with φ [Lep77, p. 501]. As $\dim \text{Hom}_{\mathfrak{g}}(N(\mu), N(\eta)) \leq 1$, the standard map φ_{std} is unique up to scalar. It is known that the standard map φ_{std} could be zero, and even if $\varphi_{\text{std}} = 0$, there could be another non-zero map from $N_{\mathfrak{p}}(\mu)$ to $N_{\mathfrak{p}}(\eta)$. Any homomorphism that is not standard is called a *non-standard map*.

It is known when the standard map φ_{std} is zero. To state the criterion, we introduce the notion of a link between two weights.

Definition 2.3 (Bernstein–Gelfand–Gelfand). Let $\mu, \eta \in \mathfrak{h}^*$ and $\beta_1, \dots, \beta_t \in \Delta^+$. Set $\eta_0 := \eta$ and $\eta_i := s_{\beta_i} \cdots s_{\beta_1} \eta$ for $1 \leq i \leq t$. We say that the sequence $(\beta_1, \dots, \beta_t)$ links η to μ if it satisfies the following two conditions:

- (1) $\eta_t = \mu$;
- (2) $\langle \eta_{i-1}, \beta_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $i \in \{1, \dots, t\}$.

Let $L(\mu)$ denote the unique irreducible quotient of the Verma module $N(\mu)$.

Theorem 2.4 (BGG–Verma). *The following conditions on $(\mu, \eta) \in (\mathfrak{h}^*)^2$ are equivalent:*

- (i) $\text{Hom}_{\mathfrak{g}}(N(\mu), N(\eta)) \neq \{0\}$;
- (ii) $L(\mu)$ is a composition factor of $N(\eta)$;
- (iii) there exists a sequence $(\beta_1, \dots, \beta_t)$ with $\beta_t \in \Delta^+$ that links η to μ .

The criterion on the vanishing of the standard map φ_{std} is first studied by Lepowsky [Lep77] and then Boe refined Lepowsky's criterion [Boe85]. The next theorem is a version of Boe's criterion [Boe85, Theorem 3.3].

Theorem 2.5. *Let $\mu, \eta \in \mathfrak{h}^*$ and suppose that there exists a non-zero \mathfrak{g} -homomorphism $\varphi \in \text{Hom}_{\mathfrak{g}}(N(\mu), N(\eta))$. Then the following two conditions on (μ, η) are equivalent:*

- (i) *the standard map $\varphi_{\text{std}}: N_{\mathfrak{p}}(\mu) \rightarrow N_{\mathfrak{p}}(\eta)$ associated with φ is non-zero;*
- (ii) *for all sequences $(\beta_1, \dots, \beta_t)$ linking η to μ , we have $\eta_1 \in \mathbf{P}_1^+$.*

3. Specialization to $(\text{SL}(3, \mathbb{R}), P_{1,2})$

The aim of this short section is to introduce the necessary notation for the rest of the paper. The expositions of this section are mainly a specialization of [KØ25] to $n = 3$.

3.1. Notation

Let $G = \text{SL}(3, \mathbb{R})$ with Lie algebra $\mathfrak{g}(\mathbb{R}) = \mathfrak{sl}(3, \mathbb{R})$. We put

$$N_j^+ := E_{1,j+1}, \quad N_j^- := E_{j+1,1}, \quad \text{for } j \in \{1, 2\}, \quad (8)$$

and

$$H_0 := \frac{1}{3}(2E_{1,1} - E_{2,2} - E_{3,3}) = \frac{1}{3} \text{diag}(2, -1, -1),$$

where $E_{i,j}$ denote the matrix units. We normalize H_0 as $\widetilde{H}_0 := \frac{3}{2}H_0$, namely,

$$\widetilde{H}_0 = \frac{1}{2}(2E_{1,1} - E_{2,2} - E_{3,3}) = \frac{1}{2} \text{diag}(2, -1, -1). \quad (9)$$

Let

$$\mathfrak{n}_+(\mathbb{R}) = \text{Ker}(\text{ad}(H_0) - \text{id}) = \text{Ker}(\text{ad}(\widetilde{H}_0) - \frac{3}{2} \text{id}), \quad (10)$$

$$\mathfrak{n}_-(\mathbb{R}) = \text{Ker}(\text{ad}(H_0) + \text{id}) = \text{Ker}(\text{ad}(\widetilde{H}_0) + \frac{3}{2} \text{id}). \quad (11)$$

Then we have

$$\mathfrak{n}_{\pm}(\mathbb{R}) = \text{span}_{\mathbb{R}}\{N_1^{\pm}, N_2^{\pm}\}.$$

For $X, Y \in \mathfrak{g}(\mathbb{R})$, let $\text{Tr}(X, Y) = \text{Trace}(XY)$ denote the trace form of $\mathfrak{g}(\mathbb{R})$. Then N_i^+ and N_j^- satisfy $\text{Tr}(N_i^+, N_j^-) = \delta_{i,j}$. In what follows, we identify the dual $\mathfrak{n}_-(\mathbb{R})^{\vee}$ with $\mathfrak{n}_+(\mathbb{R})^{\vee} \simeq \mathfrak{n}_+(\mathbb{R})$ via the trace form $\text{Tr}(\cdot, \cdot)$.

We put $\mathfrak{a}(\mathbb{R}) := \mathbb{R}\widetilde{H}_0$ and

$$\mathfrak{m}(\mathbb{R}) := \left\{ \begin{pmatrix} 0 & \\ & X \end{pmatrix} : X \in \mathfrak{sl}(2, \mathbb{R}) \right\} \simeq \mathfrak{sl}(2, \mathbb{R}). \quad (12)$$

We have $\mathfrak{m}(\mathbb{R}) \oplus \mathfrak{a}(\mathbb{R}) = \text{Ker}(\text{ad}(\widetilde{H}_0))$ and the decomposition $\mathfrak{g}(\mathbb{R}) = \mathfrak{n}_-(\mathbb{R}) \oplus \mathfrak{m}(\mathbb{R}) \oplus \mathfrak{a}(\mathbb{R}) \oplus \mathfrak{n}_+(\mathbb{R})$ is a Gelfand–Naimark decomposition of $\mathfrak{g}(\mathbb{R})$. The subalgebra $\mathfrak{p}(\mathbb{R}) := \mathfrak{m}(\mathbb{R}) \oplus \mathfrak{a}(\mathbb{R}) \oplus \mathfrak{n}_+(\mathbb{R})$ is a maximal parabolic subalgebra of $\mathfrak{g}(\mathbb{R})$ and the nilpotent radicals $\mathfrak{n}_{\pm}(\mathbb{R})$ are abelian.

Let $P = N_G(\mathfrak{p}(\mathbb{R}))$, the normalizer of $\mathfrak{p}(\mathbb{R})$ in G . We write $P = MAN_+$ for the Langlands decomposition of P corresponding to $\mathfrak{p}(\mathbb{R}) = \mathfrak{m}(\mathbb{R}) \oplus \mathfrak{a}(\mathbb{R}) \oplus \mathfrak{n}_+(\mathbb{R})$. Then $A = \exp(\mathfrak{a}(\mathbb{R})) = \exp(\mathbb{R}\widetilde{H}_0)$ and $N_+ = \exp(\mathfrak{n}_+(\mathbb{R}))$. The group M is given by

$$M = \left\{ \begin{pmatrix} (\det(g))^{-1} & \\ & g \end{pmatrix} : g \in \text{SL}^{\pm}(2, \mathbb{R}) \right\} \simeq \text{SL}^{\pm}(2, \mathbb{R}).$$

For a closed subgroup J of G , we denote by $\text{Irr}(J)$ and $\text{Irr}(J)_{\text{fin}}$ the sets of equivalence classes of irreducible representations of J and irreducible finite-dimensional representations of J , respectively.

For $\lambda \in \mathbb{C}$, we define a one-dimensional representation $\mathbb{C}_\lambda = (\chi^\lambda, \mathbb{C})$ of $A = \exp(\mathbb{R}\widetilde{H}_0)$ by

$$\chi^\lambda: \exp(t\widetilde{H}_0) \mapsto \exp(\lambda t). \quad (13)$$

Then $\text{Irr}(A)$ is given by

$$\text{Irr}(A) = \{\mathbb{C}_\lambda : \lambda \in \mathbb{C}\} \simeq \mathbb{C}.$$

For $\alpha \in \{\pm\} \simeq \mathbb{Z}/2\mathbb{Z}$, a one-dimensional representation \mathbb{C}_α of M is defined by

$$\begin{pmatrix} \det(g)^{-1} \\ g \end{pmatrix} \mapsto \text{sgn}^\alpha(\det(g)),$$

where

$$\text{sgn}^\alpha(\det(g)) = \begin{cases} 1 & \text{if } \alpha = +, \\ \text{sgn}(\det(g)) & \text{if } \alpha = -. \end{cases}$$

To describe $\text{Irr}(M)_{\text{fin}}$, let $S^m(\mathbb{C}^2)$ denote the space of symmetric tensors on \mathbb{C}^2 of homogeneous degree m . We then write $\text{Pol}^m(\mathbb{C}^2) := S^m((\mathbb{C}^2)^\vee)$, the space of polynomial functions on \mathbb{C}^2 of homogeneous degree m . Since

$$\text{Pol}^m(\mathbb{C}^2) = S^m((\mathbb{C}^2)^\vee) \simeq S^m(\mathbb{C}^2)^\vee, \quad (14)$$

the two spaces $S^m(\mathbb{C}^2)$ and $\text{Pol}^m(\mathbb{C}^2)$ are dual to each other.

Let sym_2^m and poly_2^m denote the irreducible representations of $\text{SL}(2, \mathbb{R})$ on $S^m(\mathbb{C}^2)$ and $\text{Pol}^m(\mathbb{C}^2)$, respectively. By the representation theory of $\mathfrak{sl}(2, \mathbb{C})$, we have

$$\begin{aligned} \text{Irr}(\text{SL}(2, \mathbb{R}))_{\text{fin}} &= \left\{ (\text{sym}_2^m, S^m(\mathbb{C}^2)) : m \in \mathbb{Z}_{\geq 0} \right\} \\ &= \left\{ (\text{poly}_2^m, \text{Pol}^m(\mathbb{C}^2)) : m \in \mathbb{Z}_{\geq 0} \right\}. \end{aligned}$$

Then $\text{Irr}(M)_{\text{fin}} = \text{Irr}(\text{SL}^\pm(2, \mathbb{R}))_{\text{fin}}$ is given by

$$\text{Irr}(M)_{\text{fin}} = \{\mathbb{C}_\alpha \otimes \varpi : (\alpha, \varpi) \in \{\pm\} \times \text{Irr}(\text{SL}(2, \mathbb{R}))_{\text{fin}}\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_{\geq 0}.$$

Since $\text{Irr}(P)_{\text{fin}} \simeq \text{Irr}(M)_{\text{fin}} \times \text{Irr}(A)$, the set $\text{Irr}(P)_{\text{fin}}$ can be parametrized by

$$\text{Irr}(P)_{\text{fin}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{C}.$$

For $(\alpha, \text{poly}_2^m, \lambda) \in \text{Irr}(P)_{\text{fin}}$, we write

$$I(\text{poly}_2^m, \lambda)^\alpha := \text{Ind}_P^G((\mathbb{C}_\alpha \otimes \text{poly}_2^m) \boxtimes \mathbb{C}_\lambda) \quad (15)$$

for the unnormalized parabolically induced representation $\text{Ind}_P^G((\mathbb{C}_\alpha \otimes \text{poly}_2^m) \boxtimes \mathbb{C}_\lambda)$ of G . Likewise, for $(\alpha, \text{sym}_2^m, \lambda) \in \text{Irr}(P)_{\text{fin}}$, we write

$$M_{\mathfrak{p}}(\text{sym}_2^m, \lambda)^\alpha := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (S^m(\mathbb{C}^2)_\alpha \boxtimes \mathbb{C}_\lambda), \quad (16)$$

where $S^m(\mathbb{C}^2)_\alpha$ stands for the M -representation

$$S^m(\mathbb{C}^2)_\alpha := \mathbb{C}_\alpha \otimes S^m(\mathbb{C}^2).$$

By the duality theorem (Theorem 2.1) and (14), we have

$$\begin{aligned} \text{Diff}_G(I(\text{poly}_2^m, \lambda)^\alpha, I(\text{poly}_2^\ell, \nu)^\beta) \\ \simeq \text{Hom}_{\mathfrak{g}, P}(M_{\mathfrak{p}}(\text{sym}_2^\ell, -\nu)^\beta, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^\alpha). \end{aligned} \quad (17)$$

Via the equivalence (17), we shall classify and construct both intertwining differential operators

$$\mathcal{D} \in \text{Diff}_G\left(I(\text{poly}_2^m, \lambda)^\alpha, I(\text{poly}_2^\ell, \nu)^\beta\right)$$

and (\mathfrak{g}, P) -homomorphisms

$$\varphi \in \text{Hom}_{\mathfrak{g}, P}\left(M_{\mathfrak{p}}(\text{sym}_2^\ell, \nu)^\beta, M_{\mathfrak{p}}(\text{sym}_2^m, \lambda)^\alpha\right)$$

in Sections 4 and 5.

4. Classification of parameters

The aim of this section is to classify the parameters $(\alpha, \beta; m, \ell; \lambda, \nu) \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}_{\geq 0})^2 \times \mathbb{C}^2$ such that

$$\text{Diff}_G\left(I(\text{poly}_2^m, \lambda)^\alpha, I(\text{poly}_2^\ell, \nu)^\beta\right) \neq \{0\}.$$

By the duality theorem (Theorem 2.1), it is equivalent to classify the parameters such that

$$\text{Hom}_{\mathfrak{g}, P}\left(M_{\mathfrak{p}}(\text{sym}_2^\ell, -\nu)^\beta, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^\alpha\right) \neq \{0\}.$$

The classification is given in Theorems 4.1 and 4.2 for the former and latter, respectively.

4.1. Classification of parameters

We start by stating the main results. We put

$$\Lambda_1 := \left\{(\alpha, \beta; m, \ell; \lambda, \nu) \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}_{\geq 0})^2 \times \mathbb{C}^2 : (18) \text{ holds}\right\},$$

$$\Lambda_2 := \left\{(\alpha, \beta; m, \ell; \lambda, \nu) \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}_{\geq 0})^2 \times \mathbb{C}^2 : (19) \text{ holds}\right\},$$

and

$$k := \ell - m \in \mathbb{Z}_{\geq 0}, \quad \beta - \alpha = k, \quad (\lambda, \nu) = \left(\frac{1}{2}(2 - m - 2k), \frac{1}{2}(2 - m + k)\right), \quad (18)$$

$$k := m - \ell \in \mathbb{Z}_{\geq 0}, \quad \beta - \alpha = k, \quad (\lambda, \nu) = \left(\frac{1}{2}(4 + m - 2k), \frac{1}{2}(4 + m + k)\right). \quad (19)$$

Here, we understand $\beta - \alpha = k$ for $k \in \mathbb{Z}$ as

$$\beta - \alpha = (-1)^k, \quad (20)$$

where $\beta - \alpha \in \{\pm 1\} \equiv \{\pm 1\}$ is defined by

$$\beta - \alpha = \begin{cases} + & \text{if } \alpha = \beta, \\ - & \text{otherwise.} \end{cases}$$

We set

$$\Lambda := \Lambda_1 \cup \Lambda_2.$$

The parameters are classified as follows.

Theorem 4.1. *The following conditions on $(\alpha, \beta; m, \ell; \lambda, \nu) \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}_{\geq 0})^2 \times \mathbb{C}^2$ are equivalent:*

- (i) $\text{Diff}_G(I(\text{poly}_2^m, \lambda)^\alpha, I(\text{poly}_2^\ell, \nu)^\beta) \neq \{0\}$;
- (ii) $\dim \text{Diff}_G(I(\text{poly}_2^m, \lambda)^\alpha, I(\text{poly}_2^\ell, \nu)^\beta) = 1$;
- (iii) *one of the following two cases holds:*
 - (a) $(\alpha, m, \lambda) = (\beta, \ell, \nu)$;
 - (b) $(\alpha, \beta; m, \ell; \lambda, \nu) \in \Lambda$.

The next theorem is the algebraic counterpart of Theorem 4.1 via the duality theorem.

Theorem 4.2. *The following conditions on $(\alpha, \beta; m, \ell; \lambda, \nu) \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}_{\geq 0})^2 \times \mathbb{C}^2$ are equivalent:*

- (i) $\text{Hom}_{\mathfrak{g}, P}(M_{\mathfrak{p}}(\text{sym}_2^{\ell}, -\nu)^{\beta}, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^{\alpha}) \neq \{0\}$;
- (ii) $\dim \text{Hom}_{\mathfrak{g}, P}(M_{\mathfrak{p}}(\text{sym}_2^{\ell}, -\nu)^{\beta}, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^{\alpha}) = 1$;
- (iii) *one of the following two cases holds:*
 - (a) $(\alpha, m, \lambda) = (\beta, \ell, \nu)$;
 - (b) $(\alpha, \beta; m, \ell; \lambda, \nu) \in \Lambda$.

4.2. Strategy of the proof of Theorems 4.1 and 4.2

Our basic strategy to prove Theorems 4.1 and 4.2 is to reduce the space

$$\text{Diff}_G(I(\text{poly}_2^m, \lambda)^{\alpha}, I(\text{poly}_2^{\ell}, \nu)^{\beta})$$

as much as possible. More precisely, observe that as $M_0 \simeq \text{SL}(2, \mathbb{R})$, we have

$$(M_0, \text{Ad}_{\mathfrak{n}_-}, \mathfrak{n}_-) \simeq (\text{SL}(2, \mathbb{R}), \text{std}, \mathbb{C}^2), \quad (21)$$

whose equivalence is simply given by

$$z_1 N_1^- + z_2 N_2^- \longmapsto z_1 e_1 + z_2 e_2,$$

where e_j for $j = 1, 2$ are the standard unit vectors. Equivalence (21) leads to

$$(M_0, \text{Ad}_{\mathfrak{n}_-}^m, S^m(\mathfrak{n}_-)) \simeq (\text{SL}(2, \mathbb{R}), \text{sym}_2^m, S^m(\mathbb{C}^2)).$$

Observation 4.3. By the duality theorem and (14), we have

$$\begin{aligned} & \text{Diff}_G(I(\text{poly}_2^m, \lambda)^{\alpha}, I(\text{poly}_2^{\ell}, \nu)^{\beta}) \\ & \simeq \text{Hom}_{\mathfrak{g}, P}(M_{\mathfrak{p}}(\text{sym}_2^{\ell}, -\nu)^{\beta}, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^{\alpha}) \\ & \simeq \text{Hom}_P(S^{\ell}(\mathbb{C}^2)_{\beta} \boxtimes \mathbb{C}_{-\nu}, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^{\alpha}) \\ & = \text{Hom}_{MA}(S^{\ell}(\mathbb{C}^2)_{\beta} \boxtimes \mathbb{C}_{-\nu}, (M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^{\alpha})^{\mathfrak{n}_+}) \\ & \subset \text{Hom}_{MA}(S^{\ell}(\mathbb{C}^2)_{\beta} \boxtimes \mathbb{C}_{-\nu}, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^{\alpha}) \\ & = \text{Hom}_{MA}(S^{\ell}(\mathbb{C}^2)_{\beta} \boxtimes \mathbb{C}_{-\nu}, S(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2)_{\alpha} \otimes \mathbb{C}_{-\lambda}). \quad (22) \end{aligned}$$

Since the MA -representation $S(\mathfrak{n}_-)$ decomposes irreducibly into

$$S(\mathfrak{n}_-) \Big|_{MA} = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} S^k(\mathfrak{n}_-),$$

the space of MA -homomorphisms in (22) can be further simplified to

$$\begin{aligned}
(22) &= \text{Hom}_{MA} \left(S^\ell(\mathbb{C}^2)_\beta \boxtimes \mathbb{C}_{-\nu}, S(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2)_\alpha \otimes \mathbb{C}_{-\lambda} \right) \\
&= \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \text{Hom}_{MA} \left(S^\ell(\mathbb{C}^2)_\beta \boxtimes \mathbb{C}_{-\nu}, S^k(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2)_\alpha \otimes \mathbb{C}_{-\lambda} \right) \\
&\subset \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \text{Hom}_M \left(S^\ell(\mathbb{C}^2)_\beta, S^k(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2)_\alpha \right) \\
&\subset \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \text{Hom}_{M_0} \left(S^\ell(\mathbb{C}^2), S^k(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2) \right) \\
&\simeq \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \text{Hom}_{\text{SL}(2, \mathbb{R})} \left(S^\ell(\mathbb{C}^2), S^k(\mathbb{C}^2) \otimes S^m(\mathbb{C}^2) \right).
\end{aligned}$$

Based on Observation 4.3, we first show the equivalence (i) \Leftrightarrow (iii) of Theorems 4.1 and 4.2 in stages by proceeding with the following four steps.

Step 1: Classify $(k, m, \ell) \in (\mathbb{Z}_{\geq 0})^3$ such that

$$\text{Hom}_{\text{SL}(2, \mathbb{R})} \left(S^\ell(\mathbb{C}^2), S^k(\mathbb{C}^2) \otimes S^m(\mathbb{C}^2) \right) \neq \{0\}.$$

Step 2: Classify $(\alpha, \beta; \ell, m, k) \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}_{\geq 0})^3$ such that

$$\text{Hom}_M \left(S^\ell(\mathbb{C}^2)_\beta, S^k(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2)_\alpha \right) \neq \{0\}.$$

Step 3: Classify $(\alpha, \beta; m, k, \ell; \lambda, \nu) \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}_{\geq 0})^3 \times \mathbb{C}^2$ such that

$$\text{Hom}_{MA} \left(S^\ell(\mathbb{C}^2)_\beta \otimes \mathbb{C}_{-\nu}, S^k(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2)_\alpha \otimes \mathbb{C}_{-\lambda} \right) \neq \{0\}.$$

Step 4: Classify $(\alpha, \beta; m, \ell; \lambda, \nu) \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}_{\geq 0})^2 \times \mathbb{C}^2$ such that

$$\text{Hom}_{\mathfrak{g}, P} \left(M_{\mathfrak{p}}(\text{sym}_2^\ell, -\nu)^\beta, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^\alpha \right) \neq \{0\}.$$

After achieving the equivalence (i) \Leftrightarrow (iii), we prove the whole equivalence in Section 4.7.

4.3. Step 1

We start with the classification of (k, m, ℓ) such that

$$\text{Hom}_{\text{SL}(2, \mathbb{R})} \left(S^\ell(\mathbb{C}^2), S^k(\mathbb{C}^2) \otimes S^m(\mathbb{C}^2) \right) \neq \{0\}.$$

Proposition 4.4. *The following conditions on $(k, m, \ell) \in (\mathbb{Z}_{\geq 0})^3$ are equivalent:*

- (i) $\text{Hom}_{\text{SL}(2, \mathbb{R})} \left(S^\ell(\mathbb{C}^2), S^k(\mathbb{C}^2) \otimes S^m(\mathbb{C}^2) \right) \neq \{0\}$;
- (ii) $m + k - \ell = 2d$ for some $d \in \{0, \dots, \min(k, m)\}$.

Proof. Recall the classical Clebsch–Gordan formula:

$$S^k(\mathbb{C}^2) \otimes S^m(\mathbb{C}^2) \Big|_{\text{SL}(2, \mathbb{R})} \simeq \bigoplus_{0 \leq d \leq \min(k, m)} S^{k+m-2d}(\mathbb{C}^2). \quad (23)$$

The assertion immediately follows from (23). \square

4.4. Step 2

The next goal is to classify $(\alpha, \beta; \ell, m, k)$ such that

$$\mathrm{Hom}_M(S^\ell(\mathbb{C}^2)_\beta, S^k(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2)_\alpha) \neq \{0\}.$$

A direct computation shows that

$$(M, \mathrm{Ad}_{\mathfrak{n}_-}^k, S^k(\mathfrak{n}_-)) \simeq (\mathrm{SL}^\pm(2, \mathbb{R}), \mathrm{sgn}^k \otimes \mathrm{sym}_2^k, S^k(\mathbb{C}^2)).$$

Thus, as an M -representation, we have

$$S^k(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2)_\alpha \simeq S^k(\mathbb{C}^2) \otimes S^m(\mathbb{C}^2)_{\alpha+k},$$

where $\alpha + k \in \{\pm\}$ is defined as

$$\alpha + k = \begin{cases} + & \text{if } \alpha = (-1)^k, \\ - & \text{if } \alpha = (-1)^{k+1}. \end{cases} \quad (24)$$

Proposition 4.5. *The following conditions on $(\alpha, \beta; \ell, m, k) \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}_{\geq 0})^3$ are equivalent:*

- (i) $\mathrm{Hom}_M(S^\ell(\mathbb{C}^2)_\beta, S^k(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2)_\alpha) \neq \{0\}$;
- (ii) *both of the following conditions hold:*
 - (a) $m + k - \ell = 2d$ for some $d \in \{0, \dots, \min(k, m)\}$;
 - (b) $\beta - \alpha = k$.

Proof. The proposition readily follows from Proposition 4.4 and the above observation. \square

4.5. Step 3

As a third step, we wish to classify $(\alpha, \beta; m, k, \ell; \lambda, \nu)$ such that

$$\mathrm{Hom}_{MA}(S^\ell(\mathbb{C}^2)_\beta \otimes \mathbb{C}_{-\nu}, S^k(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2)_\alpha \otimes \mathbb{C}_{-\lambda}) \neq \{0\}.$$

It follows from (11) that \widetilde{H}_0 acts on $S^k(\mathfrak{n}_-)$ by weight $-\frac{3}{2}k$, which shows that, as an MA -module, we have

$$S^k(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2)_\alpha \otimes \mathbb{C}_{-\lambda} \simeq (S^k(\mathbb{C}^2) \otimes S^m(\mathbb{C}^2)_{\alpha+k}) \boxtimes \mathbb{C}_{-(\lambda + \frac{3}{2}k)}. \quad (25)$$

Proposition 4.6. *The following conditions on $(\alpha, \beta; m, k, \ell; \lambda, \nu) \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}_{\geq 0})^3 \times \mathbb{C}^2$ are equivalent:*

- (i) $\mathrm{Hom}_{MA}(S^\ell(\mathbb{C}^2)_\beta \otimes \mathbb{C}_{-\nu}, S^k(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2)_\alpha \otimes \mathbb{C}_{-\lambda}) \neq \{0\}$;
- (ii) *All of the following conditions hold:*
 - (a) $m + k - \ell = 2d$ for some $d \in \{0, \dots, \min(k, m)\}$;
 - (b) $\beta - \alpha = k$;
 - (c) $\nu - \lambda = \frac{3}{2}k$.

Proof. Proposition 4.5 together with (25) concludes the proposition. \square

4.6. Step 4

In the final step, we wish to classify $(\alpha, \beta; m, \ell; \lambda, \nu)$ such that

$$\mathrm{Hom}_{\mathfrak{g}, P}\left(M_{\mathfrak{p}}(\mathrm{sym}_2^{\ell}, -\nu)^{\beta}, M_{\mathfrak{p}}(\mathrm{sym}_2^m, -\lambda)^{\alpha}\right) \neq \{0\}. \quad (26)$$

It follows from Observation 4.3 that if (26) holds, then the parameters $(\alpha, \beta; m, \ell; \lambda, \nu)$ satisfy the condition (ii) of Proposition 4.6 for some $k \in \mathbb{Z}_{\geq 0}$. Further, by [KØ19, Corollary 2.16], the parameter α can be chosen freely as far as $\beta - \alpha \in \mathbb{Z}_{\geq 0}$. Thus, it suffices to classify $(m, k, \ell; \lambda) \in (\mathbb{Z}_{\geq 0})^3 \times \mathbb{C}$ such that

$$\mathrm{Hom}_{\mathfrak{g}, P}\left(M_{\mathfrak{p}}\left(\mathrm{sym}_2^{\ell}, -(\lambda + \frac{3}{2}k)\right)^{\alpha+k}, M_{\mathfrak{p}}(\mathrm{sym}_2^m, -\lambda)^{\alpha}\right) \neq \{0\}$$

with condition

$$\frac{m+k-\ell}{2} \in \mathbb{Z} \cap [0, \min(k, m)].$$

Theorem 4.7. *The following conditions on $(m, k, \ell; \lambda) \in (\mathbb{Z}_{\geq 0})^3 \times \mathbb{C}$ are equivalent:*

- (i) $\mathrm{Hom}_{\mathfrak{g}, P}(M_{\mathfrak{p}}(\mathrm{sym}_2^{\ell}, -(\lambda + \frac{3}{2}k))^{\alpha+k}, M_{\mathfrak{p}}(\mathrm{sym}_2^m, -\lambda)^{\alpha}) \neq \{0\}$;
- (ii) *one of the following cases holds:*
 - (a) $(k, \ell) = (0, m)$ and $\lambda \in \mathbb{C}$;
 - (b) $\ell = m + k$ and $\lambda = \frac{1}{2}(2 - m - 2k)$;
 - (c) $\ell = m - k$ and $\lambda = \frac{1}{2}(4 + m - 2k)$.

Remark 4.8. Theorem 4.7 shows that only the largest component $S^{m+k}(\mathbb{C}^2)$ and the smallest component $S^{m-k}(\mathbb{C}^2)$ of the tensor product decomposition of $S^k(\mathbb{C}^2) \otimes S^m(\mathbb{C}^2)$ in (23) contribute to the construction of intertwining differential operators. These components $S^{m+k}(\mathbb{C}^2)$ and $S^{m-k}(\mathbb{C}^2)$ are called the *Cartan component* and *PRV component* of $S^k(\mathbb{C}^2) \otimes S^m(\mathbb{C}^2)$, respectively (cf. [Kum11]), where PRV comes from the names K. R. Parthasarathy, R. Ranga Rao, and V. S. Varadarajan.

If $m = 0$, then we regard $S^k(\mathbb{C}^2) = S^k(\mathbb{C}^2) \otimes S^0(\mathbb{C}^2)$ as both the Cartan component and PRV component.

We prove Theorem 4.7 by utilizing the idea of standard map between generalized Verma modules. Then we first rewrite the generalized Verma modules as in Section 2.2.

Take a Cartan subalgebra \mathfrak{h} of \mathfrak{g} such that $\mathfrak{h} := \{\mathrm{diag}(a_1, a_2, a_3) : \sum_{j=1}^3 a_j = 0\}$. Let $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{h})$ denote the set of roots of \mathfrak{g} with respect to \mathfrak{h} . As usual, we take sets of positive roots Δ^+ and simple roots Π as

$$\Delta^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3\} \quad \text{and} \quad \Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}.$$

We realize \mathfrak{h}^* as a subspace of \mathbb{C}^3 by identifying ε_j with the canonical basis e_j , and write elements in \mathfrak{h}^* in coordinates. For instance, half the sum of the positive roots $\rho = (1/2) \sum_{\alpha \in \Delta^+} \alpha$ is given by

$$\rho = \frac{1}{2}(2, 0, -2) = (1, 0, -1).$$

Likewise, the differential $d\chi \in \mathfrak{h}^*$ of the character χ of A with $\lambda = 1$ defined in (13) is expressed as

$$d\chi = \frac{1}{3}(2, -1, -1), \quad (27)$$

so that $d\chi^\lambda$ is given by

$$d\chi^\lambda = \lambda \cdot d\chi = \frac{\lambda}{3}(2, -1, -1).$$

A root reflection with respect to $\alpha \in \Delta$ is denoted by s_α . The Weyl group W of \mathfrak{g} is then given by

$$W = \{e, s_{\varepsilon_1 - \varepsilon_2}, s_{\varepsilon_2 - \varepsilon_3}, s_{\varepsilon_2 - \varepsilon_3} s_{\varepsilon_1 - \varepsilon_2}, s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_2 - \varepsilon_3}, s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_2 - \varepsilon_3} s_{\varepsilon_1 - \varepsilon_2}\} \quad (28)$$

with

$$s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_2 - \varepsilon_3} s_{\varepsilon_1 - \varepsilon_2} = s_{s_{\varepsilon_1 - \varepsilon_2}(\varepsilon_2 - \varepsilon_3)} = s_{\varepsilon_1 - \varepsilon_3} = s_{s_{\varepsilon_2 - \varepsilon_3}(\varepsilon_1 - \varepsilon_2)} = s_{\varepsilon_2 - \varepsilon_3} s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_2 - \varepsilon_3}. \quad (29)$$

Recall from Section 2.2 that, for $\mu \in \mathbf{P}_\mathfrak{l}^+$ for $\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{a}$, we write

$$N_{\mathfrak{p}}(\mu) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} E(\mu - \rho),$$

where $E(\mu - \rho)$ is the finite-dimensional simple $\mathcal{U}(\mathfrak{l})$ -module with highest weight $\mu - \rho$.

Write

$$\omega_1 := \frac{1}{2}(0, 1, -1).$$

Then the generalized Verma modules in consideration can be expressed as

$$\begin{aligned} M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^\alpha &= N_{\mathfrak{p}}(m\omega_1 - \lambda d\chi + \rho)^\alpha, \\ M_{\mathfrak{p}}\left(\text{sym}_2^\ell, -\left(\lambda + \frac{3}{2}k\right)\right)^\beta &= N_{\mathfrak{p}}\left(\ell\omega_1 - \left(\lambda + \frac{3}{2}k\right) d\chi + \rho\right)^\beta. \end{aligned}$$

Now we wish to classify $(m, k, \ell; \lambda)$ such that

$$\text{Hom}_{\mathfrak{g}, P}\left(N_{\mathfrak{p}}\left(\ell\omega_1 - \left(\lambda + \frac{3}{2}k\right) d\chi + \rho\right)^{\alpha+k}, N_{\mathfrak{p}}(m\omega_1 - \lambda d\chi + \rho)^\alpha\right) \neq \{0\} \quad (30)$$

with condition

$$\frac{m + k - \ell}{2} \in \mathbb{Z} \cap [0, \min(k, m)]. \quad (31)$$

We start with a necessary condition of (30).

Proposition 4.9. *If (30) holds, then one of the following conditions holds:*

- (a) $(k, \ell) = (0, m)$ and $\lambda \in \mathbb{C}$;
- (b) $\ell = m + k$ and $\lambda = \frac{1}{2}(2 - m - 2k)$;
- (c) $\ell = m - k$ and $\lambda = \frac{1}{2}(4 + m - 2k)$.

Proof. By assumption, the infinitesimal character of $N_{\mathfrak{p}}(\ell\omega_1 - (\lambda + \frac{3}{2}k) d\chi + \rho)^{\alpha+k}$ agrees with that of $N_{\mathfrak{p}}(m\omega_1 - \lambda d\chi + \rho)^\alpha$. Thus, there exists $w \in W$ such that

$$\ell\omega_1 - \left(\lambda + \frac{3}{2}k\right) d\chi + \rho = w(m\omega_1 - \lambda d\chi + \rho). \quad (32)$$

For a simple expression of both sides of (32), we normalize λ by

$$\tilde{\lambda} := \frac{2}{3}\lambda.$$

Then the weights $m\omega_1 - \lambda d\chi + \rho$ and $\ell\omega_1 - (\lambda + \frac{3}{2}k) d\chi + \rho$ are given in coordinates as

$$\begin{aligned} m\omega_1 - \lambda d\chi + \rho &= m\omega_1 - \frac{3}{2}\tilde{\lambda} d\chi + \rho \\ &= \frac{m}{2}(0, 1, -1) - \frac{\tilde{\lambda}}{2}(2, -1, -1) + (1, 0, -1) \\ &= \left(1 - \tilde{\lambda}, \frac{1}{2}(\tilde{\lambda} + m), \frac{1}{2}(\tilde{\lambda} - m - 2)\right) \end{aligned}$$

and

$$\begin{aligned} \ell\omega_1 - (\lambda + \frac{3}{2}k) d\chi + \rho &= \ell\omega_1 - \frac{3}{2}(\tilde{\lambda} + k) d\chi + \rho \\ &= \frac{\ell}{2}(0, 1, -1) - \frac{\tilde{\lambda}+k}{2}(2, -1, -1) + (1, 0, -1) \\ &= \left(1 - \tilde{\lambda} - k, \frac{1}{2}(\tilde{\lambda} + k + \ell), \frac{1}{2}(\tilde{\lambda} + k - \ell - 2)\right). \end{aligned}$$

Thus (32) amounts to

$$\left(1 - \tilde{\lambda} - k, \frac{1}{2}(\tilde{\lambda} + k + \ell), \frac{1}{2}(\tilde{\lambda} + k - \ell - 2)\right) = w\left(1 - \tilde{\lambda}, \frac{1}{2}(\tilde{\lambda} + m), \frac{1}{2}(\tilde{\lambda} - m - 2)\right).$$

Since the Weyl group element $s_{\varepsilon_i - \varepsilon_j}$ acts on $\mathfrak{h}^* \subset \mathbb{C}^3$ as the exchange of the i th entry with the j th entry, it follows from (28) that one of the following cases holds:

- (1) $1 - \tilde{\lambda} = 1 - \tilde{\lambda} - k$;
- (2) $1 - \tilde{\lambda} = \frac{1}{2}(\tilde{\lambda} + k + \ell)$;
- (3) $1 - \tilde{\lambda} = \frac{1}{2}(\tilde{\lambda} + k - \ell - 2)$.

We consider these cases separately.

Case 1. Since $1 - \tilde{\lambda} = 1 - \tilde{\lambda} - k$, we have $k = 0$. It then follows from (31) that $m = \ell$. Thus,

$$N_{\mathfrak{p}}\left(\ell\omega_1 - (\lambda + \frac{3}{2}k) d\chi + \rho\right)^{\alpha+k} = N_{\mathfrak{p}}(m\omega_1 - \lambda d\chi + \rho)^{\alpha}.$$

In this case, for any $\lambda \in \mathbb{C}$, we have

$$0 \neq \text{id} \in \text{Hom}_{\mathfrak{g}, P}(N_{\mathfrak{p}}(m\omega_1 - \lambda d\chi + \rho)^{\alpha}, N_{\mathfrak{p}}(m\omega_1 - \lambda d\chi + \rho)^{\alpha}), \quad (33)$$

where id denotes the identity map.

Case 2. In this case, we have either

$$\begin{aligned} 1 - \tilde{\lambda} &= \frac{1}{2}(\tilde{\lambda} + k + \ell), \\ \frac{1}{2}(\tilde{\lambda} + m) &= 1 - \tilde{\lambda} - k, \\ \frac{1}{2}(\tilde{\lambda} - m - 2) &= \frac{1}{2}(\tilde{\lambda} + k - \ell - 2), \end{aligned} \quad (34)$$

or

$$\begin{aligned} 1 - \tilde{\lambda} &= \frac{1}{2}(\tilde{\lambda} + k + \ell), \\ \frac{1}{2}(\tilde{\lambda} + m) &= \frac{1}{2}(\tilde{\lambda} + k - \ell - 2), \\ \frac{1}{2}(\tilde{\lambda} - m - 2) &= 1 - \tilde{\lambda} - k. \end{aligned} \quad (35)$$

If (34) holds, then a direct computation shows that

$$\begin{aligned} \tilde{\lambda} &= \frac{1}{3}(2 - m - 2k), \\ \ell &= m + k. \end{aligned} \quad (36)$$

If (35) holds, then $\ell = k - m - 2$. Since $\ell \geq 0$, this implies that $\min(k, m) = m$. It then follows from (31) that there exists $d \in \{0, \dots, m\}$ such that $\ell = k + m - 2d$. Thus,

$$k - m - 2 = \ell = k + m - 2d \geq k - m,$$

which is a contradiction. Therefore, (35) does not occur.

Case 3. As in Case 2, in this case, we have either

$$\begin{aligned} 1 - \tilde{\lambda} &= \frac{1}{2}(\tilde{\lambda} + k - \ell - 2), \\ \frac{1}{2}(\tilde{\lambda} + m) &= \frac{1}{2}(\tilde{\lambda} + k + \ell), \\ \frac{1}{2}(\tilde{\lambda} - m - 2) &= 1 - \tilde{\lambda} - k, \end{aligned} \tag{37}$$

or

$$\begin{aligned} 1 - \tilde{\lambda} &= \frac{1}{2}(\tilde{\lambda} + k - \ell - 2), \\ \frac{1}{2}(\tilde{\lambda} + m) &= 1 - \tilde{\lambda} - k, \\ \frac{1}{2}(\tilde{\lambda} - m - 2) &= \frac{1}{2}(\tilde{\lambda} + k + \ell). \end{aligned} \tag{38}$$

If (37) holds, then we have

$$\begin{aligned} \tilde{\lambda} &= \frac{1}{3}(4 + m - 2k), \\ \ell &= m - k. \end{aligned} \tag{39}$$

If (38) holds, then $\ell = -(m + k + 2) < 0$, which contradicts the condition that $\ell \geq 0$. Thus, this case does not happen.

The conditions (33), (36), and (39) conclude the proposition. \square

Now we are ready to prove Theorem 4.7.

Proof of Theorem 4.7. By Proposition 4.9, it suffices to show that condition (ii) of Theorem 4.7 implies (26), equivalently, (30). Since we already checked that the identity map id exists in condition (ii-a) in the proof of Proposition 4.9, we only consider conditions (ii-b) and (ii-c). We are going to show that the standard map is non-zero for these cases.

Condition (ii-b). As $\ell = m + k$ and $\lambda = \frac{1}{2}(2 - m - 2k)$, we have

$$\begin{aligned} \ell\omega_1 - (\lambda + \frac{3}{2}k) d\chi + \rho &= \frac{1}{3}(m - k + 1, m + 2k + 1, -(2m + k + 2)), \\ m\omega_1 - \lambda d\chi + \rho &= \frac{1}{3}(m + 2k + 1, m - k + 1, -(2m + k + 2)). \end{aligned}$$

Thus,

$$\ell\omega_1 - (\lambda + \frac{3}{2}k) d\chi + \rho = s_{\varepsilon_1 - \varepsilon_2}(m\omega_1 - \lambda d\chi + \rho).$$

Since

$$\langle m\omega_1 - \lambda d\chi + \rho, \varepsilon_1 - \varepsilon_2 \rangle = k \in \mathbb{Z}_{\geq 0},$$

the positive root $\varepsilon_1 - \varepsilon_2$ links $m\omega_1 - \lambda d\chi + \rho$ to $\ell\omega_1 - (\lambda + \frac{3}{2}k) d\chi + \rho$. Thus, it follows from Theorem 2.4 that there exists a non-zero \mathfrak{g} -homomorphism

$$\varphi: N\left(\ell\omega_1 - (\lambda + \frac{3}{2}k) d\chi + \rho\right) \longrightarrow N(m\omega_1 - \lambda d\chi + \rho)$$

between (ordinary) Verma modules $N(\ell\omega_1 - (\lambda + \frac{3}{2}k) d\chi + \rho)$ and $N(m\omega_1 - \lambda d\chi + \rho)$. Further, clearly, $\varepsilon_1 - \varepsilon_2$ is the only linking positive root. Thus, by Theorem 2.5, the standard map

$$\varphi_{\text{std}}: N_{\mathfrak{p}}\left(\ell\omega_1 - (\lambda + \frac{3}{2}k) d\chi + \rho\right)^{\alpha+k} \longrightarrow N_{\mathfrak{p}}(m\omega_1 - \lambda d\chi + \rho)^{\alpha}$$

associated with φ is non-zero.

Condition (ii-c). As $\ell = m - k$ and $\lambda = \frac{1}{2}(4 + m - 2k)$, we have

$$\begin{aligned}\ell\omega_1 - (\lambda + \tfrac{3}{2}k) d\chi + \rho &= \tfrac{1}{3}(- (m + k + 1), 2m - k + 2, -m + 2k - 1), \\ m\omega_1 - \lambda d\chi + \rho &= \tfrac{1}{3}(-m + 2k - 1, 2m - k + 2, -(m + k + 1)).\end{aligned}$$

Thus,

$$\ell\omega_1 - (\lambda + \tfrac{3}{2}k) d\chi + \rho = s_{\varepsilon_1 - \varepsilon_3}(m\omega_1 - \lambda d\chi + \rho).$$

Since

$$\langle m\omega_1 - \lambda d\chi + \rho, \varepsilon_1 - \varepsilon_3 \rangle = k \in \mathbb{Z}_{\geq 0},$$

the positive root $\varepsilon_1 - \varepsilon_3$ links $m\omega_1 - \lambda d\chi + \rho$ to $\ell\omega_1 - (\lambda + \frac{3}{2}k) d\chi + \rho$. As in (29), we have $s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_2 - \varepsilon_3} s_{\varepsilon_1 - \varepsilon_2} = s_{\varepsilon_2 - \varepsilon_3} s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_2 - \varepsilon_3} = s_{\varepsilon_1 - \varepsilon_3}$. However, one can check that under the condition $m - k \geq 0$, neither $(\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_2)$ nor $(\varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3)$ links $m\omega_1 - \lambda d\chi + \rho$ to $\ell\omega_1 - (\lambda + \frac{3}{2}k) d\chi + \rho$. Thus, $\varepsilon_1 - \varepsilon_3$ is the only linking positive root. Then, by Theorems 2.4 and 2.5, there exists a non-zero \mathfrak{g} -homomorphism

$$\varphi: N\left(\ell\omega_1 - (\lambda + \tfrac{3}{2}k) d\chi + \rho\right) \longrightarrow N(m\omega_1 - \lambda d\chi + \rho)$$

between the Verma modules and the associated standard map

$$\varphi_{\text{std}}: N_{\mathfrak{p}}\left(\ell\omega_1 - (\lambda + \tfrac{3}{2}k) d\chi + \rho\right)^{\alpha+k} \longrightarrow N_{\mathfrak{p}}(m\omega_1 - \lambda d\chi + \rho)^{\alpha}$$

is non-zero. Now the proposed assertion follows. \square

4.7. Proof of Theorems 4.1 and 4.2

To finish this section, we are going to prove Theorems 4.1 and 4.2. By the duality theorem, it suffices to show Theorem 4.2.

Proof of Theorem 4.2. The equivalence of (i) and (iii) follows from Theorem 4.7 and its preceding arguments. To show the equivalence of (ii) and (iii), for $\beta - \alpha = \ell - m$ and $\nu - \lambda = \frac{3}{2}|\ell - m|$, we have

$$\begin{aligned}\dim \text{Hom}_{\mathfrak{g}, P}\left(M_{\mathfrak{p}}(\text{sym}_2^{\ell}, -\nu)^{\beta}, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^{\alpha}\right) \\ \leq \dim \text{Hom}_{\text{SL}(2, \mathbb{R})}\left(S^{\ell}(\mathbb{C}^2), S^{|\ell - m|}(\mathbb{C}^2) \otimes S^m(\mathbb{C}^2)\right) \\ \leq 1.\end{aligned}$$

Now the desired equivalence follows. \square

Remark 4.10. If (30) holds, then the generalized Verma module $N_{\mathfrak{p}}(m\omega_1 - \lambda d\chi + \rho)$ is reducible. Indeed, it follows from a criterion of Bai–Xiao in [BX21, Theorem 5.1] that generalized Verma modules $N_{\mathfrak{p}}(m\omega_1 - \lambda d\chi + \rho)$ for $(m, \ell; \lambda, \nu)$ satisfying (18) or (19) are reducible.

5. Explicit formulas of \mathcal{D} and φ

Our next aim is to give explicit formulas for intertwining differential operators \mathcal{D} and (\mathfrak{g}, P) -homomorphisms φ . Those formulas are given in Theorems 5.1 and 5.3, respectively.

In the following, we define multiplication on the tensor product $A \otimes B \otimes C$ of three rings A , B , and C , componentwise, that is, for $a_j \in A$, $b_j \in B$, and $c_j \in C$ for

$j = 1, 2$, we define $(a_1 \otimes b_1 \otimes c_1)(a_2 \otimes b_2 \otimes c_2)$ by

$$(a_1 \otimes b_1 \otimes c_1)(a_2 \otimes b_2 \otimes c_2) := a_1 a_2 \otimes b_1 b_2 \otimes c_1 c_2.$$

If the rings in concern have unity 1, we skip writing 1 in the component. For instance, we abbreviate $a \otimes b \otimes 1$ to $a \otimes b$.

As usual, for $z, w \in \mathbb{C}$, the binomial coefficient $\binom{z}{w}$ is defined as

$$\binom{z}{w} := \begin{cases} \frac{z(z-1)\cdots(z-w+1)}{w!} & \text{if } w \in 1 + \mathbb{Z}_{\geq 0}, \\ 1 & \text{if } w = 0, \\ 0 & \text{otherwise.} \end{cases}$$

5.1. Explicit formulas of \mathcal{D}

We start with the explicit formulas of intertwining differential operators \mathcal{D} . As in (1), we understand $\mathcal{D} \in \text{Diff}_G(I(\text{poly}_2^m, \lambda)^\alpha, I(\text{poly}_2^\ell, \nu)^\beta)$ as a linear map

$$\mathcal{D}: C^\infty(\mathbb{R}^2) \otimes \text{Pol}^m(\mathbb{C}^2) \longrightarrow C^\infty(\mathbb{R}^2) \otimes \text{Pol}^\ell(\mathbb{C}^2) \quad (40)$$

via the diffeomorphism

$$\mathbb{R}^2 \xrightarrow{\sim} N_-, \quad (x_1, x_2) \longmapsto \exp(x_1 N_1^- + x_2 N_2^-). \quad (41)$$

In order to distinguish between $\text{Pol}^m(\mathbb{C}^2)$ on the source and $\text{Pol}^\ell(\mathbb{C}^2)$ on the target in (40), we write

$$\text{Pol}^m(\mathbb{C}^2) := \mathbb{C}^m[u_1, u_2] \quad \text{and} \quad \text{Pol}^\ell(\mathbb{C}^2) := \mathbb{C}^\ell[v_1, v_2].$$

Then we consider

$$\mathcal{D}: C^\infty(\mathbb{R}^2) \otimes \mathbb{C}^m[u_1, u_2] \longrightarrow C^\infty(\mathbb{R}^2) \otimes \mathbb{C}^\ell[v_1, v_2]. \quad (42)$$

With respect to the coordinate (41), the infinitesimal right translation $dR(N_j^-)$ for $j = 1, 2$ is given by $dR(N_j^-) = \frac{\partial}{\partial x_j}$. It then follows from the duality theorem (Theorem 2.1) that intertwining differential operators \mathcal{D} in (42) have constant coefficients only. Thus (2) reads

$$\text{Diff}_G\left(I(\text{poly}_2^m, \lambda)^\alpha, I(\text{poly}_2^\ell, \nu)^\beta\right) \subset \mathbb{C}\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right] \otimes \mathbb{C}^m[u_1, u_2]^\vee \otimes \mathbb{C}^\ell[v_1, v_2]. \quad (43)$$

Via the identification $\mathbb{C}^m[u_1, u_2]^\vee \simeq \mathbb{C}^m\left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right]$, we further regard (43) as

$$\text{Diff}_G\left(I(\text{poly}_2^m, \lambda)^\alpha, I(\text{poly}_2^\ell, \nu)^\beta\right) \subset \mathbb{C}\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right] \otimes \mathbb{C}^m\left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right] \otimes \mathbb{C}^\ell[v_1, v_2]. \quad (44)$$

For $\frac{\partial^m}{\partial u_1^{m-j} \partial u_2^j} \in \mathbb{C}^m\left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right]$, we write

$$\widetilde{\frac{\partial^m}{\partial u_1^{m-j} \partial u_2^j}} := \frac{1}{(m-j)! j!} \cdot \frac{\partial^m}{\partial u_1^{m-j} \partial u_2^j}.$$

For $(\alpha, \beta; m, \ell; \lambda, \nu) \in \Lambda_1$ with $k := \ell - m$, we define

$$\mathcal{D}_{m,k}^{m+k} \equiv \mathcal{D}_{m,\ell-m}^\ell: C^\infty(\mathbb{R}^2) \otimes \mathbb{C}^m[u_1, u_2] \longrightarrow C^\infty(\mathbb{R}^2) \otimes \mathbb{C}^{m+k}[v_1, v_2]$$

by

$$\mathcal{D}_{m,k}^{m+k} := \frac{1}{m!} \left(\frac{\partial}{\partial u_1} \otimes v_1 + \frac{\partial}{\partial u_2} \otimes v_2 \right)^m \left(\frac{\partial}{\partial x_1} \otimes v_1 + \frac{\partial}{\partial x_2} \otimes v_2 \right)^k, \quad (45)$$

which amounts to

$$\mathcal{D}_{m,k}^{m+k} = \sum_{j=0}^m \sum_{r=0}^k \binom{k}{r} \frac{\partial^k}{\partial x_1^{k-r} \partial x_2^r} \otimes \widetilde{\frac{\partial^m}{\partial u_1^{m-j} \partial u_2^j}} \otimes v_1^{m+k-j-r} v_2^{j+r}.$$

For

$$f(x_1, x_2) = \sum_{j=0}^m f_j(x_1, x_2) \otimes u_1^{m-j} u_2^j \in C^\infty(\mathbb{R}^2) \otimes \mathbb{C}^m[u_1, u_2],$$

we have

$$\mathcal{D}_{m,k}^{m+k} f(x_1, x_2) = \sum_{j=0}^m \sum_{r=0}^k \binom{k}{r} \frac{\partial^k f_j}{\partial x_1^{k-r} \partial x_2^r}(x_1, x_2) \otimes v_1^{m+k-j-r} v_2^{j+r}. \quad (46)$$

Similarly, for $(\alpha, \beta; m, \ell; \lambda, \nu) \in \Lambda_2$ with $k := m - \ell$, we define

$$\mathcal{D}_{m,k}^{m-k} \equiv \mathcal{D}_{m,m-\ell}^\ell: C^\infty(\mathbb{R}^2) \otimes \mathbb{C}^m[u_1, u_2] \longrightarrow C^\infty(\mathbb{R}^2) \otimes \mathbb{C}^{m-k}[v_1, v_2]$$

by

$$\mathcal{D}_{m,k}^{m-k} := \frac{1}{m!} \left(\frac{\partial}{\partial u_1} \otimes v_1 + \frac{\partial}{\partial u_2} \otimes v_2 \right)^{m-k} \left(\frac{\partial}{\partial x_1} \otimes \frac{\partial}{\partial u_2} - \frac{\partial}{\partial x_2} \otimes \frac{\partial}{\partial u_1} \right)^k, \quad (47)$$

which amounts to

$$\begin{aligned} \mathcal{D}_{m,k}^{m-k} &= \sum_{j=0}^{m-k} \sum_{r=0}^k (-1)^r \frac{\binom{m-k}{j} \binom{k}{k-r}}{\binom{m}{k+j-r}} \frac{\partial^k}{\partial x_1^{k-r} \partial x_2^r} \otimes \widetilde{\frac{\partial^m}{\partial u_1^{m-k-j+r} \partial u_2^{k+j-r}}} \otimes v_1^{m-k-j} v_2^j \\ &= \sum_{p=0}^m \sum_{k+j-r=p} (-1)^r \frac{\binom{m-k}{j} \binom{k}{k-r}}{\binom{m}{p}} \frac{\partial^k}{\partial x_1^{k-r} \partial x_2^r} \otimes \widetilde{\frac{\partial^m}{\partial u_1^{m-p} \partial u_2^p}} \otimes v_1^{m-k-j} v_2^j. \end{aligned}$$

For

$$f(x_1, x_2) := \sum_{p=0}^m f_p(x_1, x_2) \otimes u_1^{m-p} u_2^p \in C^\infty(\mathbb{R}^2) \otimes \mathbb{C}^m[u_1, u_2],$$

we have

$$\mathcal{D}_{m,k}^{m-k} f(x_1, x_2) = \sum_{p=0}^m \sum_{k+j-r=p} (-1)^r \frac{\binom{m-k}{j} \binom{k}{k-r}}{\binom{m}{p}} \frac{\partial^k f_p}{\partial x_1^{k-r} \partial x_2^r}(x_1, x_2) \otimes v_1^{m-k-j} v_2^j.$$

Theorem 5.1. *The following holds:*

$$\text{Diff}_G(I(\text{poly}_2^m, \lambda)^\alpha, I(\text{poly}_2^\ell, \nu)^\beta) = \begin{cases} \mathbb{C} \text{ id} & \text{if } (\alpha, m, \lambda) = (\beta, \ell, \nu), \\ \mathbb{C} \mathcal{D}_{m,\ell-m}^\ell & \text{if } (\alpha, \beta; m, \ell; \lambda, \nu) \in \Lambda_1, \\ \mathbb{C} \mathcal{D}_{m,m-\ell}^\ell & \text{if } (\alpha, \beta; m, \ell; \lambda, \nu) \in \Lambda_2, \\ \{0\} & \text{otherwise.} \end{cases}$$

Remark 5.2. The intertwining differential operators $\mathcal{D}_{0,\ell}^\ell$ are constructed in [KØ25]. In the cited paper, $\mathcal{D}_{0,\ell}^\ell$ are denoted by \mathcal{D}_ℓ . (See also [Kub, Remark 7.2].)

5.2. Explicit formulas of φ

Next we consider the explicit formulas of (\mathfrak{g}, P) -homomorphisms

$$\varphi \in \text{Hom}_{\mathfrak{g}, P} \left(M_{\mathfrak{p}}(\text{sym}_2^\ell, -\nu)^\beta, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^\alpha \right)$$

via the identifications

$$\begin{aligned} \text{Hom}_{\mathfrak{g}, P} \left(M_{\mathfrak{p}}(\text{sym}_2^\ell, -\nu)^\beta, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^\alpha \right) \\ \simeq \text{Hom}_P \left(S^\ell(\mathbb{C}^2)_\beta \boxtimes \mathbb{C}_{-\nu}, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^\alpha \right) \\ \subset \text{Hom}_{\mathbb{C}} \left(S^\ell(\mathbb{C}^2), S(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2) \right) \\ \simeq S(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2) \otimes \text{Pol}^\ell(\mathbb{C}^2). \end{aligned} \quad (48)$$

As in Section 5.1, we regard

$$\begin{aligned} \text{Pol}^\ell(\mathbb{C}^2) &\simeq \mathbb{C}^\ell[v_1, v_2], & S^\ell(\mathbb{C}^2) &\simeq \mathbb{C}^\ell \left[\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right], \\ S(\mathfrak{n}_-) &= \mathbb{C}[N_1^-, N_2^-], & S^m(\mathbb{C}^2) &\simeq \mathbb{C}^m \left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right]. \end{aligned}$$

We then identify the map

$$\varphi: S^\ell(\mathbb{C}^2) \longrightarrow S(\mathfrak{n}_-) \otimes S^m(\mathbb{C}^2)$$

with

$$\varphi: \mathbb{C}^\ell \left[\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right] \longrightarrow \mathbb{C}[N_1^-, N_2^-] \otimes \mathbb{C}^m \left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right],$$

which is, via (48), considered as

$$\varphi \in \mathbb{C}[N_1^-, N_2^-] \otimes \mathbb{C}^m \left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right] \otimes \mathbb{C}^\ell[v_1, v_2].$$

For $v_1^{\ell-p} v_2^p \in \mathbb{C}^\ell[v_1, v_2]$, we write

$$\widetilde{(v_1^{\ell-p} v_2^p)} := \frac{1}{(\ell-p)! p!} \cdot v_1^{\ell-p} v_2^p.$$

For $(\alpha, \beta; m, \ell; \lambda, \nu) \in \Lambda_1$ with $k := \ell - m$, we define

$$\varphi_{m,k}^{m+k} \equiv \varphi_{m,\ell-m}^\ell: \mathbb{C}^{m+k} \left[\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right] \longrightarrow \mathbb{C}^k[N_1^-, N_2^-] \otimes \mathbb{C}^m \left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right]$$

by

$$\varphi_{m,k}^{m+k} := \frac{1}{(m+k)!} \left(\frac{\partial}{\partial u_1} \otimes v_1 + \frac{\partial}{\partial u_2} \otimes v_2 \right)^m (N_1^- \otimes v_1 + N_2^- \otimes v_2)^k, \quad (49)$$

which amounts to

$$\begin{aligned} \varphi_{m,k}^{m+k} &= \sum_{j=0}^m \sum_{r=0}^k \frac{\binom{m}{j} \binom{k}{r}}{\binom{m+k}{j+r}} (N_1^-)^{k-r} (N_2^-)^r \otimes \frac{\partial^m}{\partial u_1^{m-j} \partial u_2^j} \otimes (v_1^{\widetilde{m+k-j-r}} v_2^{j+r}) \\ &= \sum_{p=0}^{m+k} \sum_{j+r=p} \frac{\binom{m}{j} \binom{k}{r}}{\binom{m+k}{p}} (N_1^-)^{k-r} (N_2^-)^r \otimes \frac{\partial^m}{\partial u_1^{m-j} \partial u_2^j} \otimes (v_1^{\widetilde{m+k-p}} v_2^p). \end{aligned}$$

For

$$v := \sum_{p=0}^{m+k} a_p \frac{\partial^{m+k}}{\partial v_1^{m+k-p} \partial v_2^p} \in \mathbb{C}^{m+k} \left[\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right],$$

we have

$$\varphi_{m,k}^{m+k}(v) = \sum_{p=0}^{m+k} \sum_{j+r=p} a_p \frac{\binom{m}{j} \binom{k}{r}}{\binom{m+k}{p}} (N_1^-)^{k-r} (N_2^-)^r \otimes \frac{\partial^m}{\partial u_1^{m-j} \partial u_2^j}.$$

For $(\alpha, \beta; m, \ell; \lambda, \nu) \in \Lambda_2$ with $k := m - \ell$, we define

$$\varphi_{m,k}^{m-k} \equiv \varphi_{m,m-\ell}^\ell: \mathbb{C}^{m-k} \left[\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right] \longrightarrow \mathbb{C}^k [N_1^-, N_2^-] \otimes \mathbb{C}^m \left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right]$$

by

$$\varphi_{m,k}^{m-k} := \frac{1}{(m-k)!} \left(\frac{\partial}{\partial u_1} \otimes v_1 + \frac{\partial}{\partial u_2} \otimes v_2 \right)^{m-k} \left(N_1^- \otimes \frac{\partial}{\partial u_2} - N_2^- \otimes \frac{\partial}{\partial u_1} \right)^k, \quad (50)$$

which amounts to

$$\varphi_{m,k}^{m-k} = \sum_{j=0}^{m-k} \sum_{r=0}^k (-1)^r \binom{k}{r} (N_1^-)^{k-r} (N_2^-)^r \otimes \frac{\partial^m}{\partial u_1^{m-k-j+r} \partial u_2^{k+j-r}} \otimes (v_1^{\widetilde{m-k-j}} v_2^j).$$

For

$$v := \sum_{j=0}^{m-k} a_j \frac{\partial^{m-k}}{\partial v_1^{m-k-j} \partial v_2^j} \in \mathbb{C}^{m-k} \left[\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right],$$

we have

$$\varphi_{m,k}^{m-k}(v) = \sum_{j=0}^{m-k} \sum_{r=0}^k a_j (-1)^r \binom{k}{r} (N_1^-)^{k-r} (N_2^-)^r \otimes \frac{\partial^m}{\partial u_1^{m-k-j+r} \partial u_2^{k+j-r}}.$$

Theorem 5.3. *The following holds:*

$$\mathrm{Hom}_{\mathfrak{g},P} \left(M_{\mathfrak{p}}(\mathrm{sym}_2^\ell, -\nu)^\beta, M_{\mathfrak{p}}(\mathrm{sym}_2^m, -\lambda)^\alpha \right) = \begin{cases} \mathbb{C} \mathrm{id} & \text{if } (\alpha, m, \lambda) = (\beta, \ell, \nu), \\ \mathbb{C} \varphi_{m,\ell-m}^\ell & \text{if } (\alpha, \beta; m, \ell; \lambda, \nu) \in \Lambda_1, \\ \mathbb{C} \varphi_{m,m-\ell}^\ell & \text{if } (\alpha, \beta; m, \ell; \lambda, \nu) \in \Lambda_2, \\ \{0\} & \text{otherwise.} \end{cases}$$

5.3. Proofs of the explicit formulas

We show the explicit formulas of (45), (47), (49), and (50), uniformly. To this end, we first construct

$$\begin{aligned} 0 \neq \psi &\in \mathrm{Hom}_{\mathrm{SL}(2,\mathbb{R})} \left(S^{m+k-2d}(\mathbb{C}^2), S^k(\mathbb{C}^2) \otimes S^m(\mathbb{C}^2) \right) \\ &\simeq \left(S^k(\mathbb{C}^2) \otimes S^m(\mathbb{C}^2) \otimes \mathrm{Pol}^{m+k-2d}(\mathbb{C}^2) \right)^{\mathrm{SL}(2,\mathbb{R})} \end{aligned}$$

for $d \in \{0, \dots, \min(k, m)\}$.

Let $\varepsilon_j \in (\mathbb{C}^2)^\vee$ denote the dual basis of $e_j \in \mathbb{C}^2$ for $j = 1, 2$, namely, $\varepsilon_j(e_i) = \delta_{i,j}$. We regard ε_j as the dual basis of N_j^- as well. Write

$$S^k(\mathbb{C}^2) = \mathbb{C}^k [N_1^-, N_2^-], \quad S^m(\mathbb{C}^2) = \mathbb{C}^m [e_1, e_2], \quad \mathrm{Pol}^{m+k-2d}(\mathbb{C}^2) = \mathbb{C}^{m+k-2d} [\varepsilon_1, \varepsilon_2].$$

Then we wish to construct

$$0 \neq \psi \in \left(\mathbb{C}^k [N_1^-, N_2^-] \otimes \mathbb{C}^m [e_1, e_2] \otimes \mathbb{C}^{m+k-2d} [\varepsilon_1, \varepsilon_2] \right)^{\mathrm{SL}(2,\mathbb{R})}.$$

As $\dim(\mathbb{C}^k [N_1^-, N_2^-] \otimes \mathbb{C}^m [e_1, e_2] \otimes \mathbb{C}^{m+k-2d} [\varepsilon_1, \varepsilon_2])^{\mathrm{SL}(2,\mathbb{R})} = 1$ (see (23)), we have

$$\left(\mathbb{C}^k [N_1^-, N_2^-] \otimes \mathbb{C}^m [e_1, e_2] \otimes \mathbb{C}^{m+k-2d} [\varepsilon_1, \varepsilon_2] \right)^{\mathrm{SL}(2,\mathbb{R})} = \mathbb{C} \psi \quad \text{for some } \psi \neq 0.$$

We write

$$N^- := (N_1^-, N_2^-), \quad e := (e_1, e_2), \quad \varepsilon := (\varepsilon_1, \varepsilon_2).$$

Then define

$$\psi_{m,k}^{m+k-2d}(N^-, e, \varepsilon) \in \mathbb{C}^k[N_1^-, N_2^-] \otimes \mathbb{C}^m[e_1, e_2] \otimes \mathbb{C}^{m+k-2d}[\varepsilon_1, \varepsilon_2]$$

by

$$\begin{aligned} & \psi_{m,k}^{m+k-2d}(N^-, e, \varepsilon) \\ & := (N_1^- \otimes \varepsilon_1 + N_2^- \otimes \varepsilon_2)^{k-d} (e_1 \otimes \varepsilon_1 + e_2 \otimes \varepsilon_2)^{m-d} (N_1^- \otimes e_2 - N_2^- \otimes e_1)^d. \end{aligned} \quad (51)$$

Proposition 5.4. *For $d \in \{0, \dots, \min(k, m)\}$, we have*

$$\left(\mathbb{C}^k[N_1^-, N_2^-] \otimes \mathbb{C}^m[e_1, e_2] \otimes \mathbb{C}^{m+k-2d}[\varepsilon_1, \varepsilon_2] \right)^{\mathrm{SL}(2, \mathbb{R})} = \mathbb{C} \psi_{m,k}^{m+k-2d}(N^-, e, \varepsilon).$$

Proof. It is clear from (51) that

$$\psi_{m,k}^{m+k-2d}(N^-, e, \varepsilon) \in \mathbb{C}^k[N_1^-, N_2^-] \otimes \mathbb{C}^m[e_1, e_2] \otimes \mathbb{C}^{m+k-2d}[\varepsilon_1, \varepsilon_2].$$

Thus, it suffices to show that $\psi_{m,k}^{m+k-2d}(N^-, e, \varepsilon)$ is $\mathrm{SL}(2, \mathbb{R})$ -invariant.

Observe that as

$$\left(\mathbb{C}^2 \otimes (\mathbb{C}^2)^\vee \right)^{\mathrm{SL}(2, \mathbb{R})} \simeq \mathrm{Hom}_{\mathrm{SL}(2, \mathbb{R})}(\mathbb{C}^2, \mathbb{C}^2) = \mathbb{C} \mathrm{id},$$

we have

$$e_1 \otimes \varepsilon_1 + e_2 \otimes \varepsilon_2 \in \left(\mathbb{C}^2 \otimes (\mathbb{C}^2)^\vee \right)^{\mathrm{SL}(2, \mathbb{R})}.$$

Therefore,

$$(e_1 \otimes \varepsilon_1 + e_2 \otimes \varepsilon_2)^n \in \left(\mathbb{C}^n[e_1, e_2] \otimes \mathbb{C}^n[\varepsilon_1, \varepsilon_2] \right)^{\mathrm{SL}(2, \mathbb{R})}. \quad (52)$$

On the other hand, it follows from the $\mathrm{SL}(2, \mathbb{R})$ -invariance of the determinant $\det \begin{pmatrix} x & w \\ y & z \end{pmatrix} = xz - wy$ that we have

$$N_1^- \otimes e_2 - N_2^- \otimes e_1 \in (\mathfrak{n}_- \otimes \mathbb{C}^2)^{\mathrm{SL}(2, \mathbb{R})},$$

which yields

$$(N_1^- \otimes e_2 - N_2^- \otimes e_1)^n \in \left(\mathbb{C}^n[N_1^-, N_2^-] \otimes \mathbb{C}^n[e_1, e_2] \right)^{\mathrm{SL}(2, \mathbb{R})}. \quad (53)$$

Now the $\mathrm{SL}(2, \mathbb{R})$ -invariance of $\psi_{m,k}^{m+k-2d}(N^-, e, \varepsilon)$ follows from (52) and (53). \square

We write

$$\partial_u := \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right) \quad \text{and} \quad v := (v_1, v_2).$$

Then, under the identifications $M_0 \simeq \mathrm{SL}(2, \mathbb{R})$ and

$$\mathbb{C}^m[e_1, e_2] \simeq S^m(\mathbb{C}^2) \simeq \mathbb{C}^m \left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right], \quad (54)$$

$$\mathbb{C}^{m+k-2d}[\varepsilon_1, \varepsilon_2] \simeq \mathrm{Pol}^{m+k-2d}(\mathbb{C}^2) \simeq \mathbb{C}^{m+k-2d}[v_1, v_2], \quad (55)$$

Proposition 5.4 shows that

$$\left(\mathbb{C}^k[N_1^-, N_2^-] \otimes \mathbb{C}^m \left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right] \otimes \mathbb{C}^{m+k-2d}[v_1, v_2] \right)^{M_0} = \mathbb{C} \psi_{m,k}^{m+k-2d}(N^-, \partial_u, v).$$

In particular, for $d = 0$, we have

$$\mathbb{C} \psi_{m,k}^{m+k}(N^-, \partial_u, v) = \left(\mathbb{C}^k[N_1^-, N_2^-] \otimes \mathbb{C}^m \left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right] \otimes \mathbb{C}^{m+k}[v_1, v_2] \right)^{M_0}$$

with

$$\psi_{m,k}^{m+k}(N^-, \partial_u, v) = \left(\frac{\partial}{\partial u_1} \otimes v_1 + \frac{\partial}{\partial u_2} \otimes v_2 \right)^m (N_1^- \otimes v_1 + N_2^- \otimes v_2)^k.$$

For $d = k$ with $m \geq k$, we have

$$\mathbb{C}\psi_{m,k}^{m-k}(N^-, \partial_u, v) = \left(\mathbb{C}^k[N_1^-, N_2^-] \otimes \mathbb{C}^m \left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right] \otimes \mathbb{C}^{m-k}[v_1, v_2] \right)^{M_0}$$

with

$$\psi_{m,k}^{m-k}(N^-, \partial_u, v) = \left(\frac{\partial}{\partial u_1} \otimes v_1 + \frac{\partial}{\partial u_2} \otimes v_2 \right)^{m-k} \left(N_1^- \otimes \frac{\partial}{\partial u_2} - N_2^- \otimes \frac{\partial}{\partial u_1} \right)^k.$$

Now we are ready to prove Theorems 5.1 and 5.3.

Proof of Theorem 5.3. By (48), we have

$$\text{Hom}_P \left(S^\ell(\mathbb{C}^2)_\beta \boxtimes \mathbb{C}_{-\nu}, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^\alpha \right) \subset \mathbb{C}[N_1^-, N_2^-] \otimes \mathbb{C}^m \left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right] \otimes \mathbb{C}^\ell[v_1, v_2].$$

Further, Theorem 4.2 and Observation 4.3 show that, for $(\alpha, \beta; m, \ell; \lambda, \nu) \in \Lambda$ with $k := |m - \ell|$, we have

$$\begin{aligned} \text{Hom}_P \left(S^\ell(\mathbb{C}^2)_\beta \boxtimes \mathbb{C}_{-\nu}, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^\alpha \right) \\ = \left(\mathbb{C}^k[N_1^-, N_2^-] \otimes \mathbb{C}^m \left[\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right] \otimes \mathbb{C}^\ell[v_1, v_2] \right)^{M_0}. \end{aligned} \quad (56)$$

Since

$$\varphi_{m,k}^{m+k} = \frac{1}{(m+k)!} \psi_{m,k}^{m+k}(N^-, \partial_u, v) \quad \text{and} \quad \varphi_{m,k}^{m-k} = \frac{1}{(m-k)!} \psi_{m,k}^{m-k}(N^-, \partial_u, v),$$

the arguments preceding this proof conclude Theorem 5.3. \square

Proof of Theorem 5.1. Theorem 5.1 just follows from Theorems 2.1 and 5.3. Indeed, by these theorems, we have

$$\begin{aligned} \mathcal{D}: \text{Hom}_P \left(S^\ell(\mathbb{C}^2)_\beta \boxtimes \mathbb{C}_{-\nu}, M_{\mathfrak{p}}(\text{sym}_2^m, -\lambda)^\alpha \right) &\xrightarrow{\sim} \text{Diff}_G \left(I(\text{poly}_2^m, \lambda)^\alpha, I(\text{poly}_2^\ell, \nu)^\beta \right), \\ \psi_{m,|m-\ell|}^\ell(N^-, \partial_u, v) &\longmapsto \psi_{m,|m-\ell|}^\ell(\text{d}R(N^-), \partial_u, v). \end{aligned}$$

As $\text{d}R(N_j^-) = \frac{\partial}{\partial x_j}$ for $j = 1, 2$, the differential operators $\psi_{m,|m-\ell|}^\ell(\text{d}R(N^-), \partial_u, v)$ are given by

$$\psi_{m,|m-\ell|}^\ell(\text{d}R(N^-), \partial_u, v) = \psi_{m,k}^{m+k}(\partial_x, \partial_u, v),$$

where $\partial_x := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)$. Since

$$\mathcal{D}_{m,k}^{m+k} = \frac{1}{m!} \psi_{m,k}^{m+k}(\partial_x, \partial_u, v) \quad \text{and} \quad \mathcal{D}_{m,k}^{m-k} = \frac{1}{m!} \psi_{m,k}^{m-k}(\partial_x, \partial_u, v),$$

the desired assertion holds. \square

5.4. Cartan operators and PRV operators

In Theorem 5.1, we constructed two families $\mathcal{D}_{m,k}^{m+k}$ and $\mathcal{D}_{m,k}^{m-k}$ of differential operators. To end this section, we study a relationship between $\mathcal{D}_{m,k}^{m+k}$ and $\mathcal{D}_{m,k}^{m-k}$. In order to distinguish between them, we write

$$\mathcal{C}_{m,k}^{m+k} := \mathcal{D}_{m,k}^{m+k} \quad \text{and} \quad \mathcal{P}_{m,k}^{m-k} := \mathcal{D}_{m,k}^{m-k}. \quad (57)$$

As remarked in Remark 4.8, the operators $\mathcal{C}_{m,k}^{m+k}$ and $\mathcal{P}_{m,k}^{m-k}$ are obtained from the Cartan component and PRV component of the tensor product of two irreducible finite-dimensional representations of $\mathrm{SL}(2, \mathbb{R})$, respectively. We then call $\mathcal{C}_{m,k}^{m+k}$ and $\mathcal{P}_{m,k}^{m-k}$ *Cartan operators* and *PRV operators*, respectively. We remark that $\mathcal{C}_{0,k}^k$ are also regarded as PRV operators as there is no tensor product decomposition in this case (see Remark 4.8).

To simplify notation, we write

$$I(m, \lambda)^\alpha := I(\mathrm{poly}_2^m, \lambda)^\alpha. \quad (58)$$

Then, by Theorem 5.1, we have

$$\mathcal{C}_{m,k+1}^{m+k+1} : I\left(m, -\frac{1}{2}(m+2k)\right)^\alpha \longrightarrow I\left(m+k+1, \frac{1}{2}(3-m+k)\right)^{\alpha+k+1} \quad (59)$$

and

$$\mathcal{P}_{m,k+1}^{m-k-1} : I\left(m, \frac{1}{2}(2+m-2k)\right)^\alpha \longrightarrow I\left(m-k-1, \frac{1}{2}(5+m+k)\right)^{\alpha+k+1}, \quad (60)$$

where, for $\alpha \in \{\pm\} \equiv \{\pm 1\}$ and $n \in \mathbb{Z}_{\geq 0}$, we define $\alpha + n \in \{\pm\}$ as in (24). We denote the kernel and image of $\mathcal{D} \in \{\mathcal{C}_{m,k+1}^{m+k+1}, \mathcal{P}_{m,k+1}^{m-k-1}\}$ in (59) and (60) by

$$\mathrm{Ker}(\mathcal{D})^\alpha \quad \text{and} \quad \mathrm{Im}(\mathcal{D})^{\alpha+k+1}.$$

We put

$$\mathrm{Cartan} := \left\{ \mathcal{C}_{m,k+1}^{m+k+1} : m, k \in \mathbb{Z}_{\geq 0} \right\}$$

and

$$\mathrm{PRV} := \left\{ \mathcal{P}_{m,k+1}^{m-k-1} : m, k, m-k-1 \in \mathbb{Z}_{\geq 0} \right\}.$$

Theorem 5.5. *There exists a bijection*

$$\Theta : \mathrm{Cartan} \longrightarrow \mathrm{PRV}$$

such that

$$\mathrm{Im}(\mathcal{C}_{m,k+1}^{m+k+1})^{\alpha+k+1} \subset \mathrm{Ker}\left(\Theta(\mathcal{C}_{m,k+1}^{m+k+1})\right)^{\alpha+k+1}.$$

Proof. Put

$$A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}) \rtimes \mathbb{R}^2.$$

Then, for $m, k \in \mathbb{Z}_{\geq 0}$, we set

$$(\tilde{m}, \tilde{k}) := (m+k+1, m) = A(m, k, 1)^t, \quad (61)$$

where the space \mathbb{R}^2 is realized as an affine subspace of \mathbb{R}^3 under the identification $(a, b)^t \simeq (a, b, 1)^t$. Then define

$$\Theta(\mathcal{C}_{m,k+1}^{m+k+1}) := \mathcal{P}_{\tilde{m}, \tilde{k}+1}^{\tilde{m}-\tilde{k}-1}.$$

We show that Θ satisfies the desired properties.

(1) *Well-definedness.* As $\tilde{m} = m + k + 1 \geq m + 1 = \tilde{k} + 1$, the PRV operator

$$\mathcal{P}_{\tilde{m}, \tilde{k}+1}^{\tilde{m}-\tilde{k}-1}: I\left(\tilde{m}, \frac{1}{2}(2 + \tilde{m} - 2\tilde{k})\right)^\alpha \longrightarrow I\left(\tilde{m} - \tilde{k} - 1, \frac{1}{2}(5 + \tilde{m} + \tilde{k})\right)^{\alpha+\tilde{k}+1} \quad (62)$$

is well-defined. Thus, $\Theta(\mathcal{C}_{m, k+1}^{m+k+1}) \in \text{PRV}$.

(2) *Inverse.* Observe that the inverse A^{-1} of the matrix A is

$$A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, for $m, k \in \mathbb{Z}_{\geq 0}$ with $m - k - 1 \in \mathbb{Z}_{\geq 0}$, we set

$$(\widehat{m}, \widehat{k}) := (k, m - k - 1) = A^{-1}(m, k, 1)^t. \quad (63)$$

By the choice of m, k , clearly we have $\widehat{m}, \widehat{k} \in \mathbb{Z}_{\geq 0}$. Thus the Cartan operator

$$\mathcal{C}_{\widehat{m}, \widehat{k}+1}^{\widehat{m}+\widehat{k}+1}: I\left(\widehat{m}, -\frac{1}{2}(\widehat{m} + 2\widehat{k})\right)^\alpha \longrightarrow I\left(\widehat{m} + \widehat{k} + 1, \frac{1}{2}(3 - \widehat{m} + \widehat{k})\right)^{\alpha+\widehat{k}+1}$$

is well-defined. We then define $\Theta^{-1}(\mathcal{P}_{m, k+1}^{m-k-1}) := \mathcal{C}_{\widehat{m}, \widehat{k}+1}^{\widehat{m}+\widehat{k}+1}$. By (61) and (63), the maps Θ and Θ^{-1} are clearly inverses to each other.

(3) *Im \subset Ker.* By (61), we have

$$\begin{aligned} \mathcal{P}_{\tilde{m}, \tilde{k}+1}^{\tilde{m}-\tilde{k}-1} &= \mathcal{P}_{m+k+1, m+1}^k, \\ I\left(\tilde{m}, \frac{1}{2}(2 + \tilde{m} - 2\tilde{k})\right)^{\alpha+k+1} &= I\left(m + k + 1, \frac{1}{2}(3 - m + k)\right)^{\alpha+k+1}, \\ I\left(\tilde{m} - \tilde{k} - 1, \frac{1}{2}(5 + \tilde{m} + \tilde{k})\right)^{\alpha+\tilde{k}+1} &= I\left(k, \frac{1}{2}(6 + 2m + k)\right)^{\alpha+m+k}. \end{aligned}$$

Then (62) reads

$$\mathcal{P}_{m+k+1, m+1}^k: I\left(m + k + 1, \frac{1}{2}(3 - m + k)\right)^{\alpha+k+1} \longrightarrow I\left(k, \frac{1}{2}(6 + 2m + k)\right)^{\alpha+m+k}. \quad (64)$$

By (59) and (60), we have

$$\begin{aligned} I\left(m, -\frac{1}{2}(m + 2k)\right)^\alpha &\xrightarrow{\mathcal{C}_{m, k+1}^{m+k+1}} I\left(m + k + 1, \frac{1}{2}(3 - m + k)\right)^{\alpha+k+1} \\ &\xrightarrow{\mathcal{P}_{m+k+1, m+1}^k} I\left(k, \frac{1}{2}(6 + 2m + k)\right)^{\alpha+m+k}. \end{aligned} \quad (65)$$

Clearly, the composition $\mathcal{P}_{m+k+1, m+1}^k \circ \mathcal{C}_{m, k+1}^{m+k+1}$ is an intertwining differential operator. However, Theorem 5.1 shows that

$$\text{Diff}_G\left(I\left(m, -\frac{1}{2}(m + 2k)\right)^\alpha, I\left(k, \frac{1}{2}(6 + 2m + k)\right)^{\alpha+m+k}\right) = \{0\}.$$

Therefore, we have

$$\text{Im}(\mathcal{C}_{m, k+1}^{m+k+1})^{\alpha+k+1} \subset \text{Ker}(\mathcal{P}_{m+k+1, m+1}^k)^{\alpha+k+1}.$$

This proves the proposition. \square

Remark 5.6. One can easily check that $(\mathcal{C}_{m,k+1}^{m+k+1}, \Theta(\mathcal{C}_{m,k+1}^{m+k+1}))$ is the only pair $(\mathcal{D}_1, \mathcal{D}_2)$ of non-identity and non-zero operators $\mathcal{D}_1, \mathcal{D}_2$ such that the composition $\mathcal{D}_2 \circ \mathcal{D}_1$ is defined as an intertwining differential operator.

In the next section, we shall show that

$$\mathrm{Im}(\mathcal{C}_{m,k+1}^{m+k+1})^{\alpha'} = \mathrm{Ker}(\mathcal{P}_{m+k+1,m+1}^k)^{\alpha'} \quad (66)$$

for appropriate $\alpha' \in \mathbb{Z}/2\mathbb{Z}$ for which the sequence (65) constitutes a BGG resolution.

6. BGG resolution

The aim of this section is to show that the sequence (65) is an exact sequence for appropriate $\alpha' \in \mathbb{Z}/2\mathbb{Z}$. As byproducts, we determine the G -representations on $\mathrm{Ker}(\mathcal{C}_{m,k+1}^{m+k+1})^{\alpha'}$ and $\mathrm{Im}(\mathcal{P}_{m+k+1,m+1}^k)^{\alpha'}$, and also show that the identity (66) holds for such α' . These are achieved in Theorem 6.1 and Corollary 6.4. We continue to denote our intertwining differential operators \mathcal{D} by $\mathcal{C}_{m,k+1}^{m+k+1}$ and $\mathcal{P}_{m,k+1}^{m-k-1}$ as in (57).

We remark that one may read off part of our results in this section in [ČGH12, Proposition 2.1] and [EG11, Section 4].

6.1. Main results

We start with the main results of this section; the proofs of them will be discussed in Section 6.2.

Let ϖ_1 and ϖ_2 be the first and second fundamental weights of $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, respectively, that is,

$$\varpi_1 := \frac{1}{3}(2, -1, -1) \quad \text{and} \quad \varpi_2 := \frac{1}{3}(1, 1, -2).$$

We denote by $V(m\varpi_1 + k\varpi_2)$ the irreducible finite-dimensional representation of \mathfrak{g} with highest weight $m\varpi_1 + k\varpi_2$.

Theorem 6.1 (BGG resolution). *There exists an injective G -intertwining operator*

$$\tilde{\mathcal{T}}: V(m\varpi_1 + k\varpi_2) \hookrightarrow I\left(m, -\frac{1}{2}(m + 2k)\right)^{m+k}$$

such that the following sequence is exact:

$$\begin{aligned} 0 \longrightarrow V(m\varpi_1 + k\varpi_2) &\xrightarrow{\tilde{\mathcal{T}}} I\left(m, -\frac{1}{2}(m + 2k)\right)^{m+k} \\ &\xrightarrow{\mathcal{C}_{m,k+1}^{m+k+1}} I\left(m + k + 1, \frac{1}{2}(3 - m + k)\right)^{m+1} \\ &\xrightarrow{\mathcal{P}_{m+k+1,m+1}^k} I\left(k, \frac{1}{2}(6 + 2m + k)\right)^+ \longrightarrow 0 \end{aligned} \quad (67)$$

with $I(k, \frac{1}{2}(6 + 2m + k))^+ = I(k, \frac{1}{2}(6 + 2m + k))^{2m+2}$ (see (24)).

It is known that the resolution (67) is a so-called BGG resolution for the pair $(G, P) = (\mathrm{SL}(3, \mathbb{R}), P_{1,2})$ (cf. [CD01; CSS01; EG11]). Theorem 6.1 shows that the Cartan operators $\mathcal{C}_{m,k+1}^{m+k+1}$ and PRV operators $\mathcal{P}_{m+k+1,m+1}^k$ are the first and second BGG operators, respectively.

Remark 6.2. As $V(0\varpi_1 + 0\varpi_2) = \mathbb{C}_{\text{triv}}$ is the trivial representation of G , if $(m, k) = (0, 0)$, then (67) can be thought of as the de Rham complex on \mathbb{R}^2 :

$$0 \longrightarrow \mathbb{C} \longrightarrow C^\infty(\mathbb{R}^2) \otimes \Lambda^0(\mathbb{C}^2) \xrightarrow{d} C^\infty(\mathbb{R}^2) \otimes \Lambda^1(\mathbb{C}^2) \xrightarrow{d} C^\infty(\mathbb{R}^2) \otimes \Lambda^2(\mathbb{C}^2) \longrightarrow 0$$

with $d = \mathcal{C}_{0,1}^1$ for the first differential and $d = \mathcal{P}_{1,1}^0$ for the second differential (see (45) and (47)).

Remark 6.3. As in (58), write

$$M_{\mathfrak{p}}(m, \lambda) := M_{\mathfrak{p}}(\text{sym}_2^m, \lambda).$$

Here, as opposed to Sections 4 and 5, we regard generalized Verma modules merely as \mathfrak{g} -modules; thus, we omit the parity condition $\alpha \in \mathbb{Z}/2\mathbb{Z}$. Then the BGG resolution of generalized Verma modules corresponding to (67) is given as

$$\begin{aligned} 0 \longrightarrow M_{\mathfrak{p}}\left(k, -\frac{1}{2}(6 + 2m + k)\right) &\xrightarrow{\varphi_{m+k+1, m+1}^k} M_{\mathfrak{p}}\left(m + k + 1, -\frac{1}{2}(3 - m + k)\right) \\ &\xrightarrow{\varphi_{m, k+1}^{m+k+1}} M_{\mathfrak{p}}\left(m, \frac{1}{2}(m + 2k)\right) \\ &\xrightarrow{\text{pr}} V(k\varpi_1 + m\varpi_2) \longrightarrow 0, \end{aligned} \quad (68)$$

where pr denotes the natural projection onto its irreducible quotient and $V(k\varpi_1 + m\varpi_2) = V(m\varpi_1 + k\varpi_2)^\vee$, the dual of $V(m\varpi_1 + k\varpi_2)$.

The next corollary is an immediate consequence of Theorem 6.1.

Corollary 6.4. *The following hold:*

- (1) $\text{Ker}(\mathcal{C}_{m, k+1}^{m+k+1})^{m+k} \simeq V(m\varpi_1 + k\varpi_2)$;
- (2) $\text{Im}(\mathcal{C}_{m, k+1}^{m+k+1})^{m+1} = \text{Ker}(\mathcal{P}_{m+k+1, m+1}^k)^{m+1}$;
- (3) $\text{Im}(\mathcal{P}_{m+k+1, m+1}^k)^+ = I(k, \frac{1}{2}(6 + 2m + k))^+$.

We remark that Corollary 6.4 does not fully determine the representations on the kernels and images of intertwining differential operators in consideration. Indeed, if we change the parity condition $m+k$ for $I(m, -\frac{1}{2}(m+2k))^{m+k}$ to $m+k+1$, then (67) does not conclude anything for the representation on the kernel $\text{Ker}(\mathcal{C}_{m, k+1}^{m+k+1})^{m+k+1}$ of $\mathcal{C}_{m, k+1}^{m+k+1}$ for

$$\mathcal{C}_{m, k+1}^{m+k+1} : I\left(m, -\frac{1}{2}(m+2k)\right)^{m+k+1} \longrightarrow I\left(m+k+1, \frac{1}{2}(3-m+k)\right)^m.$$

We shall show in Proposition 6.11 that $\text{Ker}(\mathcal{C}_{m, k+1}^{m+k+1})^{m+k+1} = \{0\}$ by making use of the ellipticity of Cartan operators $\mathcal{C}_{m, k+1}^{m+k+1}$.

Remark 6.5. In [KØ25], the composition structures of parabolically induced representations are utilized to determine the K -type structure of the G -representation on the kernel $\text{Ker}(\mathcal{C}_{0, k+1}^{k+1})^\alpha$ for any α for $\text{SL}(n, \mathbb{R})$. Further, the kernels $\text{Ker}(\mathcal{C}_{0, k+1}^{k+1})^\alpha$ are computed explicitly in the non-compact picture in [Kub]. (See Remark 5.2 for the difference of notation of differential operators between [Kub; KØ25] and this paper.)

6.2. Construction of $\tilde{\mathcal{T}}$

Remark that the construction of the desired $\tilde{\mathcal{T}}$ in Theorem 6.1 is well known in the context of parabolic geometry in great generality (see, for instance, [CD01; ČSS01]). In particular, the BGG resolution in concern is studied in, for instance, Čap–Gover–Hammerl [ČGH12] and Eastwood–Gover [EG11]. However, in contrast to the cited papers, we need to determine the character $\alpha \in \mathbb{Z}/2\mathbb{Z}$ of M explicitly. In doing so, we give a detailed account of $\tilde{\mathcal{T}}$ in this section.

6.2.1. Grading on $V(m\varpi_1 + k\varpi_2)$. We start with a grading on $V(m\varpi_1 + k\varpi_2)$ induced by the parabolic subgroup $P = MAN_+$. Recall from (9) that

$$\tilde{H}_0 = \frac{1}{2} \operatorname{diag}(2, -1, -1). \quad (69)$$

For $s \in \mathbb{C}$, we write $W[s]$ for the eigenspace of \tilde{H}_0 associated with eigenvalue s , namely,

$$W[s] := \left\{ v \in V(m\varpi_1 + k\varpi_2) : \tilde{H}_0 \cdot v = sv \right\},$$

where the dot (\cdot) denotes the action of \tilde{H}_0 on $V(m\varpi_1 + k\varpi_2)$. Since M commutes with $A = \exp(\mathbb{R}\tilde{H}_0)$, the space $W[s]$ is an MA -representation.

As $\mathfrak{m} \simeq \mathfrak{sl}(2, \mathbb{C})$, the restriction $V(m\varpi_1 + k\varpi_2)|_{\mathfrak{m}}$ decomposes into a direct sum of \mathfrak{m} -irreducible components. Let W_ℓ denote the irreducible component containing a unique (up to scalar) lowest weight vector v_ℓ of $V(m\varpi_1 + k\varpi_2)$. We then have $W_\ell = \mathcal{U}(\mathfrak{m})v_\ell$. Since A commutes with M , the group A acts on W_ℓ by scalar. Thus, $W_\ell \subset W[s_0]$ for some $s_0 \in \mathbb{C}$. Further, if $W_\ell \subsetneq W[s_0]$, then there would be another lowest weight vector $v'_\ell \notin \mathbb{C}v_\ell$, which contradicts the irreducibility of $V(m\varpi_1 + k\varpi_2)$. Therefore, we have $W_\ell = W[s_0]$.

It follows from (10) that we have

$$\mathfrak{n}_+ \cdot W[s] \subset W\left[s + \frac{3}{2}\right]. \quad (70)$$

Since $V(m\varpi_1 + k\varpi_2) = \mathcal{U}(\mathfrak{n}_+)v_\ell$, this yields a (non-irreducible) MA -decomposition

$$V(m\varpi_1 + k\varpi_2)|_{MA} = \bigoplus_{j=0}^{r_0} W\left[s_0 + \frac{3}{2}j\right]$$

with $W[s_0 + \frac{3}{2}r_0] \neq \{0\}$ for some $r_0 \in \mathbb{Z}_{\geq 0}$.

Lemma 6.6. *We have*

$$s_0 = -\frac{1}{2}(m + 2k) \quad \text{and} \quad r_0 = m + k.$$

Proof. Observe that the first and second fundamental weights ϖ_1 and ϖ_2 are

$$\varpi_1 = \frac{1}{3}(2, -1, -1)(= d\chi) \quad \text{and} \quad \varpi_2 = \frac{1}{3}(1, 1, -2)$$

in coordinates from Section 4.6. Then the highest weight $m\varpi_1 + k\varpi_2$ reads

$$m\varpi_1 + k\varpi_2 = \frac{1}{3}(2m + k, -m + k, -m - 2k).$$

As the longest Weyl group element is $s_{\varepsilon_1 - \varepsilon_3}$, the lowest weight of $V(m\varpi_1 + k\varpi_2)$ is

$$s_{\varepsilon_1 - \varepsilon_3}(m\varpi_1 + k\varpi_2) = -(k\varpi_1 + m\varpi_2)$$

(see, for instance, [GW09, Theorem 3.2.13]). In coordinates, we have

$$-(k\varpi_1 + m\varpi_2) = -\frac{1}{3}(m + 2k, m - k, -2m - k). \quad (71)$$

It follows from (69) that $\varpi_1(\widetilde{H}_0) = 1$ and $\varpi_2(\widetilde{H}_0) = \frac{1}{2}$, yielding

$$\begin{aligned} (m\varpi_1 + k\varpi_2)(\widetilde{H}_0) &= m + \frac{1}{2}k = \frac{1}{2}(2m + k), \\ -(k\varpi_1 + m\varpi_2)(\widetilde{H}_0) &= -(k + \frac{1}{2}m) = -\frac{1}{2}(m + 2k). \end{aligned}$$

Now, to determine s_0 , observe that

$$s_0 v_\ell = \widetilde{H}_0 \cdot v_\ell = -(k\varpi_1 + m\varpi_2)(\widetilde{H}_0) v_\ell = -\frac{1}{2}(m + 2k) v_\ell, \quad (72)$$

which shows that $s_0 = -\frac{1}{2}(m + 2k)$.

For r_0 , observe that as

$$\mathfrak{n}_+ \cdot W\left[-\frac{1}{2}(m + 2k) + \frac{3}{2}r_0\right] \subset W\left[-\frac{1}{2}(m + 2k) + \frac{3}{2}(r_0 + 1)\right] = \{0\},$$

the component $W\left[-\frac{1}{2}(m + 2k) + \frac{3}{2}r_0\right]$ contains a unique (up to scalar) highest weight vector v_h . As in (72), this shows that

$$W\left[-\frac{1}{2}(m + 2k) + \frac{3}{2}r_0\right] = W\left[\frac{1}{2}(2m + k)\right],$$

which forces $r_0 = m + k$. □

By Lemma 6.6, we obtain

$$V(m\varpi_1 + k\varpi_2)|_{MA} = \bigoplus_{j=0}^{m+k} W\left[-\frac{1}{2}(m + 2k) + \frac{3}{2}j\right]. \quad (73)$$

Now we claim that, as MA -representations, we have

$$W\left[-\frac{1}{2}(m + 2k)\right] \simeq (\mathbb{C}_{m+k} \otimes \text{poly}_2^m) \boxtimes \mathbb{C}_{-\frac{1}{2}(m+2k)}.$$

Toward this end, first we show its $\mathfrak{m} \oplus \mathfrak{a}$ -equivalence.

Proposition 6.7. *As $\mathfrak{m} \oplus \mathfrak{a}$ -modules, we have*

$$W\left[-\frac{1}{2}(m + 2k)\right] \simeq \text{poly}_2^m \boxtimes \mathbb{C}_{-\frac{1}{2}(m+2k)}.$$

Proof. As both $W\left[-\frac{1}{2}(m + 2k)\right]$ and $\text{poly}_2^m \boxtimes \mathbb{C}_{-\frac{1}{2}(m+2k)}$ are irreducible $\mathfrak{m} \oplus \mathfrak{a}$ -modules, it suffices to show that the lowest weights of these two modules are the same. The highest weight of poly_2^m is $m\omega_1$ with $\omega_1 = \frac{1}{2}(0, 1, -1)$; in particular, its lowest weight is $-m\omega_1$. Thus, the lowest weight of $\text{poly}_2^m \boxtimes \mathbb{C}_{-\frac{1}{2}(m+2k)}$ is

$$\begin{aligned} -m\omega_1 - \frac{1}{2}(m + 2k) d\chi &= \frac{1}{2}(0, -m, m) - \frac{1}{6}(m + 2k)(2, -1, -1) \\ &= -\frac{1}{3}(m + 2k, m - k, -2m - k). \end{aligned}$$

On the other hand, it follows from (71) that the lowest weight of $W\left[-\frac{1}{2}(m + 2k)\right]$, which is the lowest weight of $V(m\varpi_1 + k\varpi_2)$, is

$$-(k\varpi_1 + m\varpi_2) = -\frac{1}{3}(m + 2k, m - k, -2m - k).$$

Now the claim follows. □

Proposition 6.8. *As MA -modules, we have*

$$W\left[-\frac{1}{2}(m + 2k)\right] \simeq (\mathbb{C}_{m+k} \otimes \text{poly}_2^m) \boxtimes \mathbb{C}_{-\frac{1}{2}(m+2k)}. \quad (74)$$

Proof. We wish to determine the character of $M \simeq \mathrm{SL}(2, \mathbb{R})^\pm$ on $W[-\frac{1}{2}(m+2k)]$. Let v_ℓ be a lowest weight vector of $W[-\frac{1}{2}(m+2k)]$. Then $W[-\frac{1}{2}(m+2k)] = \mathcal{U}(\mathfrak{m})v_\ell$. It follows from a direct computation that

$$(M, \mathrm{Ad}, \mathfrak{m}) \simeq (\mathrm{SL}^\pm(2, \mathbb{R}), \mathrm{triv} \otimes \mathrm{Ad}, \mathfrak{sl}(2, \mathbb{C})).$$

Thus, to determine the character of M on $W[-\frac{1}{2}(m+2k)]$, it suffices to observe the character of M on v_ℓ .

Let $B = TN_{\mathbb{C}}^+$ be the Borel subgroup of $G_{\mathbb{C}} := \mathrm{SL}(3, \mathbb{C})$ consisting of upper triangular matrices with T the group of diagonal matrices and $N_{\mathbb{C}}^+$ the group of upper triangular matrices with 1 on the diagonal. As in [Sep07, Exercise 7.34], we write $\xi_{(a,b)}: T \rightarrow \mathbb{C}^\times$ for the character of T associated with $(a, b) := a\varpi_1 + b\varpi_2$ for $a, b \in \mathbb{Z}$. Then, by the Borel–Weil theorem and Weyl’s unitary trick, the irreducible representation $V(m\varpi_1 + k\varpi_2)$ of G can be realized as the space $\mathcal{O}(G_{\mathbb{C}}, \mathbb{C}_{-(k,m)})^B$ of B -invariant holomorphic functions on $G_{\mathbb{C}}$, namely,

$$\mathcal{O}(G_{\mathbb{C}}, \mathbb{C}_{-(k,m)})^B := \left\{ f \in \mathcal{O}(G_{\mathbb{C}}) : f(gtn) = \xi_{-(k,m)}(t^{-1})f(g) \text{ for } tn \in TN_{\mathbb{C}}^+ \right\},$$

where G acts on $\mathcal{O}(G_{\mathbb{C}}, \mathbb{C}_{-(k,m)})^B$ by left translation, i.e., $L(g_0)f(g) := f(g_0^{-1}g)$. Then put

$$f_\ell(g) := \det_1(g)^k \det_2(g)^m,$$

where $\det_r(g_{i,j}) := \det_{i,j \leq r}(g_{i,j})$. One can easily check that $f_\ell(g)$ is a lowest weight vector of $\mathcal{O}(G_{\mathbb{C}}, \mathbb{C}_{-(k,m)})^B$. A direct computation shows that, for $\tilde{h} := \begin{pmatrix} \det(h)^{-1} & & \\ & 1 & \\ & & h \end{pmatrix} \in M$ with $h \in \mathrm{SL}^\pm(2, \mathbb{R})$, we have

$$L(\tilde{h})f_\ell(g) = f_\ell(\tilde{h}^{-1}g) = \det(h)^{m+k} f_\ell\left(\begin{pmatrix} 1 & & \\ & h^{-1} & \\ & & \end{pmatrix} g\right),$$

which shows that \tilde{h} acts on $\mathcal{O}(G_{\mathbb{C}}, \mathbb{C}_{-(k,m)})^B$ as $\det(h)^{m+k} L\left(\begin{pmatrix} 1 & & \\ & h & \\ & & \end{pmatrix}\right)$. Now the proposed assertion follows. \square

6.2.2. Construction of $\tilde{\mathcal{T}}$. Put

$$W_+ := \bigoplus_{j=1}^{m+k} W\left[-\frac{1}{2}(m+2k) + \frac{3}{2}j\right],$$

the direct sum of the MA -components in (73) with $j > 0$, so that

$$V(m\varpi_1 + k\varpi_2) = W\left[-\frac{1}{2}(m+2k)\right] \oplus W_+.$$

This allows to identify the MA -representation $W[-\frac{1}{2}(m+2k)]$ with

$$W\left[-\frac{1}{2}(m+2k)\right] \simeq V(m\varpi_1 + k\varpi_2)/W_+. \tag{75}$$

It follows from (70) that W_+ is an \mathfrak{n}_+ -module; thus, W_+ is indeed a P -representation. Via the identification (75), we regard $W[-\frac{1}{2}(m+2k)]$ as an irreducible P -representation.

Regarding a G -representation $V(m\varpi_1 + k\varpi_2)$ as a P -representation, consider the induced representation from $V(m\varpi_1 + k\varpi_2)$:

$$T_P(m\varpi_1 + k\varpi_2) := \mathrm{Ind}_P^G(V(m\varpi_1 + k\varpi_2)).$$

Then, via the MA -projection

$$\mathrm{proj}: V(m\varpi_1 + k\varpi_2) \twoheadrightarrow V(m\varpi_1 + k\varpi_2)/W_+ \simeq W\left[-\frac{1}{2}(m+2k)\right] \tag{76}$$

and the MA -module equivalence (74), one can define a G -intertwining map

$$\text{Proj}: T_P(m\varpi_1 + k\varpi_2) \longrightarrow I\left(m, -\frac{1}{2}(m+2k)\right)^{m+k}, \quad F \longmapsto \text{proj} \circ F. \quad (77)$$

Now, let $\sigma: G \rightarrow \text{GL}(V(m\varpi_1 + k\varpi_2))$ denote the representation of G on $V(m\varpi_1 + k\varpi_2)$. For $v \in V(m\varpi_1 + k\varpi_2)$, define

$$\begin{aligned} \mathcal{T}: V(m\varpi_1 + k\varpi_2) &\longrightarrow C^\infty(G) \otimes V(m\varpi_1 + k\varpi_2) \\ v &\longmapsto f_v(g) := \sigma(g^{-1})v. \end{aligned}$$

By definition, the map \mathcal{T} is an injective intertwining operator:

$$\mathcal{T}: V(m\varpi_1 + k\varpi_2) \hookrightarrow C^\infty(G) \otimes V(m\varpi_1 + k\varpi_2). \quad (78)$$

Further, as $f_v(gp) = \sigma(p^{-1})f_v(g)$ for $p \in P$, we have $f_v(g) \in T_P(m\varpi_1 + k\varpi_2)$. Thus,

$$\mathcal{T}: V(m\varpi_1 + k\varpi_2) \hookrightarrow T_P(m\varpi_1 + k\varpi_2). \quad (79)$$

By composing \mathcal{T} in (78) with Proj in (77), we obtain an injective G -intertwining operator

$$\tilde{\mathcal{T}}: V(m\varpi_1 + k\varpi_2) \hookrightarrow I\left(m, -\frac{1}{2}(m+2k)\right)^{m+k}, \quad v \longmapsto (\text{Proj} \circ \mathcal{T})(v). \quad (80)$$

Proposition 6.9. *We have*

$$\text{Im } \tilde{\mathcal{T}} \subset \text{Ker}(\mathcal{C}_{m,k+1}^{m+k+1})^{m+k}.$$

Proof. The proposition can be shown in the same line with the proof for [KØ25, Theorem 6.5]. Indeed, let $d\sigma$ denote the differential of the representation σ of G on $V(m\varpi_1 + k\varpi_2)$. Then the representation $d\pi_\sigma$ of $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{l} \oplus \mathfrak{n}_+$ with $\mathfrak{l} := \mathfrak{m} \oplus \mathfrak{a}$ on the non-compact picture of the induced representation $T_P(m\varpi_1 + k\varpi_2) \subset C^\infty(N_-) \otimes V(m\varpi_1 + k\varpi_2)$ is given by

$$d\pi_\sigma(X)F(\bar{n}) = d\sigma\left((\text{Ad}(\bar{n}^{-1})X)_\mathfrak{l}\right)F(\bar{n}) - \left(dR\left((\text{Ad}(\cdot^{-1})X)_{\mathfrak{n}_-}\right)F\right)(\bar{n}) \quad (81)$$

for $X \in \mathfrak{g}$ and $F(\bar{n}) \in C^\infty(N_-) \otimes V(m\varpi_1 + k\varpi_2)$. Here $(\cdot)_\mathfrak{l}$ and $(\cdot)_{\mathfrak{n}_-}$ denote the projections $\mathfrak{g} \rightarrow \mathfrak{l}$ and $\mathfrak{g} \rightarrow \mathfrak{n}_-$, respectively, and dR is the infinitesimal right translation (see, for instance, [KØ25, (2.4)]). In particular, as \mathfrak{n}_- is abelian, for $U \in \mathfrak{n}_-$, we have

$$d\pi_\sigma(U)F(\bar{n}) = -dR(U)F(\bar{n}).$$

Further, a direct computation shows that

$$d\pi_\sigma(N_j^-)F(x_1, x_2) = -\frac{\partial}{\partial x_j}F(x_1, x_2) \quad (82)$$

in the local coordinates (41) for $F(x_1, x_2) \in C^\infty(\mathbb{R}^2) \otimes V(m\varpi_1 + k\varpi_2)$, where N_j^- ($j = 1, 2$) are the vectors of \mathfrak{n}_- defined in (8).

Now take a lowest weight vector $v_\ell \in V(m\varpi_1 + k\varpi_2)$. As N_- acts on v_ℓ trivially, we have $d\pi_\sigma(N_j^-)\mathcal{T}(v_\ell) = 0$ in the non-compact picture of $T_P(m\varpi_1 + k\varpi_2)$. It then follows from (82) that $\frac{\partial}{\partial x_j}v_\ell = 0$ for $j = 1, 2$ in the local coordinates (41). Thus, $\mathcal{T}(v_\ell)$ is a constant function in the non-compact picture of $T_P(m\varpi_1 + k\varpi_2)$; consequently, so is $\tilde{\mathcal{T}}(v_\ell) \in C^\infty(N_-) \otimes W[-\frac{1}{2}(m+2k)]$ in the non-compact picture of $I(m, -\frac{1}{2}(m+2k))^{m+k}$. It then follows from (46) that $\mathcal{C}_{m,k+1}^{m+k+1}\tilde{\mathcal{T}}(v_\ell) = 0$. Therefore,

$\tilde{\mathcal{T}}(v_\ell) \in \text{Ker}(\mathcal{C}_{m,k+1}^{m+k+1})$. Now the proposed assertion follows from the fact that $\tilde{\mathcal{T}}(V(m\varpi_1 + k\varpi_2)) = \mathcal{U}(\mathfrak{n}_+)\tilde{\mathcal{T}}(v_\ell)$ and the G -invariance of $\text{Ker}(\mathcal{C}_{m,k+1}^{m+k+1})$. \square

Remark 6.10. The image $\tilde{\mathcal{T}}(V(k\varpi_2))$ for $m = 0$ is explicitly given in [Kub, Proposition 8.5].

Now we are ready to show Theorem 6.1.

Proof of Theorem 6.1. It follows from Theorem 5.5 and Proposition 6.9 that the sequence (67) is a complex. Now one can compare (67) with the one in [EG11, Section 4] to conclude that it is in fact a resolution. \square

A similar idea to the proof of Proposition 6.9 shows the following.

Proposition 6.11. *We have:*

- (1) $\text{Ker}(\mathcal{C}_{m,k+1}^{m+k+1})^{m+k+1} = \{0\}$;
- (2) $\text{Im}(\mathcal{C}_{m,k+1}^{m+k+1})^m \simeq I(m, -\frac{1}{2}(m+2k))^{m+k+1}$.

Proof. The second statement follows from the first as the sequence

$$0 \longrightarrow I(m, -\frac{1}{2}(m+2k))^{m+k+1} \xrightarrow{\mathcal{C}_{m,k+1}^{m+k+1}} I(m+k+1, \frac{1}{2}(3-m+k))^m$$

is exact. Thus, it suffices to show the first assertion.

Assume the contrary, namely, $\text{Ker}(\mathcal{C}_{m,k+1}^{m+k+1})^{m+k+1} \neq \{0\}$. Observe that as the Cartan operator $\mathcal{C}_{m,k+1}^{m+k+1}$ is obtained from a Cartan component, it is an elliptic operator (cf. [Kal+96]). Thus, the kernel $\text{Ker}(\mathcal{C}_{m,k+1}^{m+k+1})^{m+k+1} \subset I(m, -\frac{1}{2}(m+2k))^{m+k+1}$ is finite-dimensional. In particular, the G -representation $\text{Ker}(\mathcal{C}_{m,k+1}^{m+k+1})^{m+k+1}$ has a non-zero lowest weight vector v_0 .

Let τ denote the MA -representation on the fiber $W := (\mathbb{C}_{m+k+1} \otimes \text{poly}_2^m) \boxtimes \mathbb{C}_{-\frac{1}{2}(m+2k)}$ of $I(m, -\frac{1}{2}(m+2k))^{m+k+1}$. Then, as in (81), the representation $d\pi_\tau$ of \mathfrak{g} on the non-compact picture of $I(m, -\frac{1}{2}(m+2k))^{m+k+1} \subset C^\infty(N_-) \otimes W$ is given by

$$d\pi_\tau(X)F(\bar{n}) = d\tau((\text{Ad}(\bar{n}^{-1})X)_\mathfrak{l})F(\bar{n}) - (dR((\text{Ad}(\cdot^{-1})X)_{\mathfrak{n}_-})F)(\bar{n})$$

for $X \in \mathfrak{g}$ and $F(\bar{n}) \in C^\infty(N_-) \otimes W$. In particular, for $Z \in \mathfrak{m} \oplus \mathfrak{a}$, we have

$$d\pi_\tau(Z)F(\bar{n}) = d\tau(Z)F(\bar{n}) - (dR(\text{Ad}(\cdot^{-1})Z - Z)F)(\bar{n}).$$

The same arguments as in the proof of Proposition 6.9 show that the lowest weight vector v_0 is a constant function in the non-compact picture. Thus, $d\pi_\tau(Z)v_0(\bar{n}) = d\tau(Z)v_0(\bar{n})$. As $v_0(\bar{n})$ is a lowest weight vector in $I(m, -\frac{1}{2}(m+2k))^{m+k+1}$, we have $d\tau(Z^-)v_0(\bar{n}) = d\pi_\tau(Z^-)v_0(\bar{n}) = 0$ for all negative root vectors $Z^- \in \mathfrak{m}$. Thus, $v_0(\bar{n})$ can be thought of as a lowest weight vector of $\text{poly}_2^m \boxtimes \mathbb{C}_{-\frac{1}{2}(m+2k)}$. Since the weight of $v_0(\bar{n})$ is the same as the lowest weight of $V(m\varpi_1 + k\varpi_2)$, this shows that $d\pi_\tau(\mathcal{U}(\mathfrak{g}))v_0(\bar{n}) \simeq V(m\varpi_1 + k\varpi_2)$ as G -modules; in particular, $d\pi_\tau(\mathcal{U}(\mathfrak{m} \oplus \mathfrak{a}))v_0(\bar{n}) \simeq W[-\frac{1}{2}(m+2k)]$ as MA -modules.

Now, let δ_τ denote the action of MA on $C^\infty(N_-) \otimes W$ induced by the left translation on $I(m, -\frac{1}{2}(m+2k))^{m+k+1}$, that is, for $F(\bar{n}) \in C^\infty(N_-) \otimes W$, we have

$$\delta_\tau(\ell)F(\bar{n}) = \tau(\ell)F(\ell^{-1}\bar{n}\ell).$$

As $v_0(\bar{n})$ is a constant function, we have $v_0(\ell^{-1}\bar{n}\ell) = v_0(\bar{n})$ for $\ell \in MA$. Thus,

$$\begin{aligned} \delta_\tau(\ell) d\pi_\tau(Z)v_0(\bar{n}) &= \tau(\ell) d\pi_\tau(Z)v_0(\ell^{-1}\bar{n}\ell) \\ &= \tau(\ell) d\tau(Z)v_0(\ell^{-1}\bar{n}\ell) \\ &= \tau(\ell) d\tau(Z)v_0(\bar{n}) \\ &= \tau(\ell) d\pi_\tau(Z)v_0(\bar{n}) \end{aligned}$$

for $Z \in \mathfrak{m} \oplus \mathfrak{a}$ and $\ell \in MA$, which yields that $d\pi_\tau(\mathcal{U}(\mathfrak{m} \oplus \mathfrak{a}))v_0(\bar{n}) \simeq W$ as MA -modules. Therefore, we have the isomorphisms of M -representations

$$\begin{aligned} \mathbb{C}_{m+k+1} \otimes \text{poly}_2^m &\simeq W|_M \\ &\simeq d\pi_\tau(\mathcal{U}(\mathfrak{m} \oplus \mathfrak{a}))v_0(\bar{n}) \\ &\simeq W[-\tfrac{1}{2}(m+2k)]|_M \\ &\simeq \mathbb{C}_{m+k} \otimes \text{poly}_2^m, \end{aligned}$$

which is a contradiction. Now the proposed assertion follows. \square

7. Unitary highest weight modules of $SU(1, 2)$

The aim of the last section is to exploit the results from the previous section to classify the irreducible unitary highest weight modules of $SU(1, 2)$ at the (first) reduction points. This is achieved in Theorem 7.1.

7.1. Preliminaries

Let $G_u := SU(1, 2)$. The maximal compact subgroup K is $K = S(U(1) \times U(2)) \simeq U(2)$. The complexified Lie algebra of G_u is $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and that of K is $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{a} \simeq \mathfrak{gl}(2, \mathbb{C})$. The Cartan decomposition of the complex Lie algebra \mathfrak{g} is then given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ with $\mathfrak{s} = \mathfrak{n}_+ \oplus \mathfrak{n}_-$. Let $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$ denote the complexification of G_u and K , respectively. Then $G_{\mathbb{C}} = SL(3, \mathbb{C})$ and $K_{\mathbb{C}} = GL(2, \mathbb{C})$. Write $N_{\mathbb{C}}^\pm$ for the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra \mathfrak{n}_\pm . Then $\bar{P}_{\mathbb{C}} := K_{\mathbb{C}}N_{\mathbb{C}}^-$ is a parabolic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\bar{\mathfrak{p}} = \mathfrak{k} \oplus \mathfrak{n}_- = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_-$. Via the Borel embedding, we realize G_u/K as an open subset of $N_{\mathbb{C}}^+\bar{P}_{\mathbb{C}}$, which is open dense in $G_{\mathbb{C}}/\bar{P}_{\mathbb{C}}$ (cf. [Wol72]):

$$G_u/K \stackrel{\text{open}}{\subset} N_{\mathbb{C}}^+\bar{P}_{\mathbb{C}} \stackrel{\text{open}}{\underset{\text{dense}}{\subset}} G_{\mathbb{C}}/\bar{P}_{\mathbb{C}}.$$

A holomorphic representation V of $K_{\mathbb{C}}$ is regarded as a $P_{\mathbb{C}} := K_{\mathbb{C}}N_{\mathbb{C}}^+$ and $\bar{P}_{\mathbb{C}}$ -representation by letting $N_{\mathbb{C}}^\pm$ act trivially. Then we write $\mathcal{V}_{\mathbb{C}} := G_{\mathbb{C}} \times_{\bar{P}_{\mathbb{C}}} V \rightarrow G_{\mathbb{C}}/\bar{P}_{\mathbb{C}}$ for the $G_{\mathbb{C}}$ -equivariant holomorphic vector bundle over $G_{\mathbb{C}}/\bar{P}_{\mathbb{C}}$. The restriction to the open set G_u/K defines a holomorphic vector bundle $\mathcal{V}_K := G_u \times_K V$ over G_u/K . We then obtain a $G_{\mathbb{C}}$ -representation $\mathcal{O}(G_{\mathbb{C}}/\bar{P}_{\mathbb{C}}, \mathcal{V}_{\mathbb{C}})$ and G_u -representation $\mathcal{O}(G_u/K, \mathcal{V}_K)$ on the spaces of holomorphic sections on $G_{\mathbb{C}}/\bar{P}_{\mathbb{C}}$ and G_u/K , respectively.

In the theory of Davidson–Enright–Stanke [DES90; DES91], it plays a role that the \mathfrak{k} -representation V of the holomorphic vector bundle $\mathcal{V}_{\mathbb{C}} = G_{\mathbb{C}} \times_{\bar{P}_{\mathbb{C}}} V$ is also

the inducing \mathfrak{k} -representation of a generalized Verma module. Indeed, for the space $\mathcal{O}(G_{\mathbb{C}}/\overline{P}_{\mathbb{C}}, \mathcal{V}_{\mathbb{C}})_K$ of K -finite vectors of $\mathcal{O}(G_{\mathbb{C}}/\overline{P}_{\mathbb{C}}, \mathcal{V}_{\mathbb{C}})$, we have

$$\mathcal{O}(G_{\mathbb{C}}/\overline{P}_{\mathbb{C}}, \mathcal{V}_{\mathbb{C}})_K = \text{Pol}(\mathfrak{n}_+) \otimes V \simeq S(\mathfrak{n}_-) \otimes V \simeq M_{\mathfrak{p}}(V).$$

In order to apply their theory, we consider $V = \text{sym}_2^m \boxtimes \mathbb{C}_{-\lambda}$ instead of $\text{poly}_2^m \boxtimes \mathbb{C}_{\lambda}$ for $\mathcal{V}_{\mathbb{C}}$ so that V is the same as the inducing \mathfrak{k} -representation of the generalized Verma module in the previous sections, where $\text{sym}_2^m \simeq (\text{poly}_2^m)^{\vee} \simeq \text{poly}_2^m$. Then, as in (58), for $V = \text{sym}_2^m \boxtimes \mathbb{C}_{-\lambda}$ we put

$$\mathcal{O}_{\overline{P}_{\mathbb{C}}}(m, \lambda) := \mathcal{O}(G_{\mathbb{C}}/\overline{P}_{\mathbb{C}}, \mathcal{V}_{\mathbb{C}}) \quad \text{and} \quad \mathcal{O}_K(m, \lambda) := \mathcal{O}(G_u/K, \mathcal{V}_K).$$

For the rest of this section, we resume the notation defined in Sections 4.6 and 6.1. For instance, the Cartan subalgebra \mathfrak{h} of \mathfrak{g} consists of trace-free diagonal matrices and ϖ_1 denotes the first fundamental weight. As in (6), we write $\mathbf{P}_{\mathfrak{k}}^+$ for the set of \mathfrak{k} -dominant integral weights. The sets of positive compact roots Δ_c^+ and positive noncompact roots Δ_n^+ are

$$\Delta_c^+ = \{\varepsilon_2 - \varepsilon_3\} \quad \text{and} \quad \Delta_n^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3\}.$$

7.2. Classification of irreducible unitary highest weight modules at reduction points

For $\mu \in \mathbf{P}_{\mathfrak{k}}^+$, put $\mathbb{L}(\mu) := \{\mu + z\varpi_1 : z \in \mathbb{R}\}$. According to the classification theory of Enright–Howe–Wallach [EHW83], there exists $\mu_0 \in \mathbb{L}(\mu)$ such that the irreducible highest weight module $L(\mu_0)$ corresponds to a limit of discrete series representation. We then put

$$\mathbb{L}(\mu_0)_u := \{\mu_0 + z\varpi_1 \in \mathbb{L}(\mu_0) : L(\mu_0 + z\varpi_1) \text{ is unitarizable}\}.$$

It follows from [EHW83] that $\mathbb{L}(\mu_0)_u$ takes the following form:



The right end point A is called the first reduction point. It is remarked that since $G_u = \text{SU}(1, 2)$ has split rank one, there exists only one reduction point for each line. Then we put

$$\widehat{(G_u)}_{h,r} := \left\{ \begin{array}{l} \text{irreducible unitary highest weight modules of } G_u \\ \text{at the (first) reduction points} \end{array} \right\} / \sim.$$

Now observe that, by the same arguments as in Section 6.2.2, one can define the injective G -intertwining operator

$$\tilde{\mathcal{T}} : V(m\varpi_1 + k\varpi_2) \hookrightarrow \mathcal{O}_{\overline{P}_{\mathbb{C}}}\left(m, \frac{1}{2}(m + 2k)\right). \quad (83)$$

Here, as we consider $(-\lambda d\chi, \mathbb{C})$ instead of $(\lambda d\chi, \mathbb{C})$ for the fiber of the induced representation, we have $\frac{1}{2}(m + 2k)$ in place of $-\frac{1}{2}(m + 2k)$ in (83). Then, as in (67),

we have an exact sequence

$$\begin{aligned} 0 \longrightarrow V(m\varpi_1 + k\varpi_2) &\xrightarrow{\tilde{\mathcal{T}}} \mathcal{O}_{\bar{P}_\mathbb{C}}\left(m, \frac{1}{2}(m+2k)\right) \\ &\xrightarrow{\mathcal{C}_{m,k+1}^{m+k+1}} \mathcal{O}_{\bar{P}_\mathbb{C}}\left(m+k+1, -\frac{1}{2}(3-m+k)\right) \\ &\xrightarrow{\mathcal{P}_{m+k+1,m+1}^k} \mathcal{O}_{\bar{P}_\mathbb{C}}\left(k, -\frac{1}{2}(6+2m+k)\right) \longrightarrow 0, \end{aligned}$$

which gives rise to an exact sequence for $\mathcal{O}_K(m, -\lambda)$:

$$\begin{aligned} 0 \longrightarrow V(m\varpi_1 + k\varpi_2) &\xrightarrow{\tilde{\mathcal{T}}} \mathcal{O}_K\left(m, \frac{1}{2}(m+2k)\right) \\ &\xrightarrow{\mathcal{C}_{m,k+1}^{m+k+1}} \mathcal{O}_K\left(m+k+1, -\frac{1}{2}(3-m+k)\right) \\ &\xrightarrow{\mathcal{P}_{m+k+1,m+1}^k} \mathcal{O}_K\left(k, -\frac{1}{2}(6+2m+k)\right) \longrightarrow 0, \end{aligned} \quad (84)$$

where the differential operators $\mathcal{C}_{m,k+1}^{m+k+1}$ and $\mathcal{P}_{m+k+1,m+1}^k$ are defined as in Section 5 via the local coordinates

$$\mathbb{C}^2 \xrightarrow{\sim} N_\mathbb{C}^+, \quad (z_1, z_2) \longmapsto \exp(z_1 N_1^+ + z_2 N_2^+). \quad (85)$$

We remark that the orders of $\mathcal{C}_{m,k+1}^{m+k+1}$ and $\mathcal{P}_{m+k+1,m+1}^k$ are $k+1$ and $m+1$, respectively.

Let $\text{Ker}_\mathcal{O}(\mathcal{C}_{m,k+1}^{m+k+1})$ denote the kernel of $\mathcal{C}_{m,k+1}^{m+k+1}$ as an operator in (84). The spaces $\text{Im}_\mathcal{O}(\mathcal{C}_{m,k+1}^{m+k+1})$ and $\text{Ker}_\mathcal{O}(\mathcal{P}_{m+k+1,m+1}^k)$ are defined, similarly. Then, combining the resolution (84) with the theory of Davidson–Enright–Stanke [DES90; DES91] on covariant differential operators, the set $\widehat{(G_u)}_{h,r}$ can be classified as follows.

Theorem 7.1. *We have*

$$\widehat{(G_u)}_{h,r} = \left\{ \text{Ker}_\mathcal{O}(\mathcal{C}_{0,1}^1) \right\} \cup \left\{ \text{Ker}_\mathcal{O}(\mathcal{P}_{k+1,1}^k) : k \in \mathbb{Z}_{\geq 0} \right\} \quad (86)$$

$$= \left\{ \text{Ker}_\mathcal{O}(\mathcal{C}_{0,1}^1) \right\} \cup \left\{ \text{Im}_\mathcal{O}(\mathcal{C}_{0,k+1}^{k+1}) : k \in \mathbb{Z}_{\geq 0} \right\}. \quad (87)$$

Proof. It follows from [DES90, Theorem 3.2] and [EJ90] that the irreducible unitary highest weight modules $\pi \in \widehat{(G_u)}_{h,r}$ are classified by the kernel of PRV operators whose orders are equal to the level of reduction of π . In the present case, the level of reduction is always 1 as there is only one reduction point. Since the order of the PRV operator $\mathcal{P}_{m+k+1,m+1}^k$ is $m+1$, we have $\text{Ker}_\mathcal{O}(\mathcal{P}_{k+1,1}^k) \in \widehat{(G_u)}_{h,r}$ for all $k \in \mathbb{Z}_{\geq 0}$. As remarked in the beginning of Section 5.4, the operators $\mathcal{C}_{0,k+1}^{k+1}$ are also PRV operators, which implies that $\text{Ker}_\mathcal{O}(\mathcal{C}_{0,1}^1) \in \widehat{(G_u)}_{h,r}$. Since the right-hand side of (86) exhausts all PRV operators with order 1, the identity (86) follows.

The second line (87) is the direct consequence of (86) and the following resolution:

$$\begin{aligned}
 0 \longrightarrow V(k\varpi_2) &\xrightarrow{\tilde{\mathcal{T}}} \mathcal{O}_K(0, k) \\
 &\xrightarrow{\mathcal{C}_{0, k+1}^{k+1}} \mathcal{O}_K\left(k+1, -\frac{1}{2}(3+k)\right) \\
 &\xrightarrow{\mathcal{P}_{k+1, 1}^k} \mathcal{O}_K\left(k, -\frac{1}{2}(6+k)\right) \longrightarrow 0.
 \end{aligned} \tag{88}$$

This concludes the theorem. \square

Remark 7.2. For $\pi \in \widehat{(G_u)_{h,r}}$, denote by π_K the space of K -finite vectors of π . Then the highest weights of $\text{Ker}_{\mathcal{O}}(\mathcal{C}_{0,1}^1)_K$ and $\text{Ker}_{\mathcal{O}}(\mathcal{P}_{k+1,1}^k)_K$ are given as follows.

If $m = 0$, then the BGG resolution (68) of generalized Verma modules reads

$$\begin{aligned}
 0 \longrightarrow M_{\mathfrak{p}}\left(k, -\frac{1}{2}(6+k)\right) &\xrightarrow{\varphi_{k+1,1}^k} M_{\mathfrak{p}}\left(k+1, -\frac{1}{2}(3+k)\right) \\
 &\xrightarrow{\varphi_{0, k+1}^{k+1}} M_{\mathfrak{p}}(0, k) \\
 &\xrightarrow{\text{pr}} V(k\varpi_1) \longrightarrow 0.
 \end{aligned}$$

By [DES90, Theorems 2.4 and 3.2], we have

$$\begin{aligned}
 \text{Ker}_{\mathcal{O}}(\mathcal{C}_{0,1}^1)_K &\simeq M_{\mathfrak{p}}(0, 0)/\text{Im}(\varphi_{0,1}^1) \simeq \mathbb{C}_{\text{triv}}, \\
 \text{Ker}_{\mathcal{O}}(\mathcal{P}_{k+1,1}^k)_K &\simeq M_{\mathfrak{p}}\left(k+1, -\frac{1}{2}(3+k)\right)/\text{Im}(\varphi_{k+1,1}^k).
 \end{aligned}$$

Then, in the coordinates used in Section 4.6, the following hold:

- (1) the highest weight of $\text{Ker}_{\mathcal{O}}(\mathcal{C}_{0,1}^1)_K$ is $(0, 0, 0)$;
- (2) the highest weight of $\text{Ker}_{\mathcal{O}}(\mathcal{P}_{k+1,1}^k)_K$ is

$$(k+1)\omega_1 - \frac{1}{2}(3+k) d\chi = (-1, 1, 0) + \frac{k}{3}(-1, 2, -1),$$

$$\text{where } \omega_1 = \frac{1}{2}(0, 1, -1) \text{ and } d\chi = \frac{1}{3}(2, -1, -1).$$

Remark 7.3. It follows from [EHW83] that a root system Q is associated to the first reduction point. (Also, see [DES90, Table 6.22]). In the present case, there are only two possibilities on Q , namely, $Q = \text{SU}(1, 1)$ or $\text{SU}(1, 2)$. The sets $\{\text{Ker}_{\mathcal{O}}(\mathcal{C}_{0,1}^1)\}$ and $\{\text{Ker}_{\mathcal{O}}(\mathcal{P}_{k+1,1}^k) : k \in \mathbb{Z}_{\geq 0}\}$ correspond to the cases $Q = \text{SU}(1, 2)$, $\text{SU}(1, 1)$, respectively. Here are some details.

In [DES90], the highest weights of π_K for $\pi \in \widehat{(G_u)_{h,r}}$ are given by

$$\lambda_{q'} + C_{q'},$$

which is associated to the root system $Q = \text{SU}(1, q')$ for $q' = 1, 2$ [DES90, Definition 6.5 and Proposition 6.6]. Here, $\lambda_{q'}$ is

$$\lambda_{q'} := -(3 - q')\varpi_1 + \varpi_{3-q'}$$

and $C_{q'}$ is an integral cone of \mathfrak{k} -dominant integral weights defined by

$$C_{q'} := \{k(-\varpi_1 + \varpi_{3-q'}) : k \in \mathbb{Z}_{\geq 0}\}$$

(see [DES90, Tables 6.21 and 6.22] for the full description). Now we consider the cases $Q = \mathrm{SU}(1, 2)$, $\mathrm{SU}(1, 1)$, separately.

- (1) $Q = \mathrm{SU}(1, 2)$: In this case, we have $\lambda_2 = (0, 0, 0)$ and $C_2 = \{(0, 0, 0)\}$. Thus, the corresponding highest weight is $(0, 0, 0)$. By Remark 7.2(1), this corresponds to $\mathrm{Ker}_{\mathcal{O}}(\mathcal{C}_{0,1}^1)$.
- (2) $Q = \mathrm{SU}(1, 1)$: In this case, we have

$$\lambda_1 = -2\varpi_1 + \varpi_2 = (-1, 1, 0)$$

and

$$C_1 = \{k(-\varpi_1 + \varpi_2) : k \in \mathbb{Z}_{\geq 0}\} = \left\{\frac{k}{3}(-1, 2, -1) : k \in \mathbb{Z}_{\geq 0}\right\}.$$

So, the highest weights have the form

$$(-1, 1, 0) + \frac{k}{3}(-1, 2, -1),$$

which is the highest weight of $\mathrm{Ker}_{\mathcal{O}}(\mathcal{P}_{k+1,1}^k)_K$ by Remark 7.2(2).

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