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# Non-singular weakly symmetric nilmanifolds

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**Abstract.** A Riemannian manifold is called weakly symmetric if any two points in  $M$  can be interchanged by an isometry. We give a complete classification of simply connected non-singular weakly symmetric nilmanifolds. Besides previously known examples, there are new families with 3-dimensional center, and a one-parameter family of dimensions 14.

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*Key Words and Phrases:* non-singular nilmanifold, weakly symmetric manifold

*The authors dedicate this paper to the memory of Joe Wolf. Among his many remarkable contributions to geometry and Lie theory, are deep results in the theory of weakly symmetric and geodesic orbit spaces. This research is a small contribution to the topic.*

## 1. Introduction

A complete Riemannian manifold  $M$  is called *weakly symmetric* if any two points in  $M$  can be interchanged by an isometry. It is called *geodesic orbit*, GO for short, if every geodesic is the image of a one-parameter group of isometries. Both concepts have been studied extensively. In [BKV97] it was shown that weakly symmetric spaces are GO. But the converse does not hold. Clearly, both conditions imply that  $M$  is homogeneous, and so  $M = G/H$  where  $G$  is the full isometry group.

Weakly symmetric spaces fall into several very different categories. When the group  $G$  is reductive (e.g., when  $M$  is compact and simply connected), such spaces have been classified in [Vin01; Wol07; Yak04]. Another category is the one of weakly symmetric nilmanifolds. It is well known that for a nilpotent Lie group both the GO and the weakly symmetric condition imply that it is either abelian or 2-step nilpotent, see [Gor96, Theorem 2.2]. In this case, one finds a partial classification in [Wol07; Yak05]. This classification is based on the fact that weakly symmetric spaces are commutative [Sel56], and the partial classification of the former is obtained from the classification of the latter.

To state our results, we recall the following standard construction. Let  $(M, g)$  be a metric 2-step nilpotent Lie group and  $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$  the corresponding metric Lie algebra. Let  $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$ ,  $\mathfrak{a} = \mathfrak{n}^\perp$ . Here we can assume that  $\mathfrak{z}$  is the center of  $\mathfrak{n}$  since otherwise  $\mathfrak{n}$  is the product of  $\mathfrak{n}' \subset \mathfrak{n}$  with an abelian Lie algebra. The Lie brackets of  $\mathfrak{n}$  are encoded in the endomorphisms

$$J_Z \in \mathfrak{so}(\mathfrak{a}) \text{ with } \langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle, \text{ for all } X, Y \in \mathfrak{a} \text{ and } Z \in \mathfrak{z} \quad (1)$$

which defines a linear subspace:

$$V = \text{Span}\{J_Z \mid Z \in \mathfrak{z}\} \subset \mathfrak{so}(\mathfrak{a}).$$

Moreover,  $V$  inherits an inner product from the push-forward of the inner product on  $\mathfrak{z} \subset \mathfrak{n}$ . Conversely, we can start with a Euclidean vector space  $\mathfrak{a}$ , a linear subspace  $V \subset \mathfrak{so}(\mathfrak{a})$  and an inner product  $\langle \cdot, \cdot \rangle$  on  $V$ . This defines a metric nilpotent Lie algebra  $\mathfrak{n}$  by setting  $\mathfrak{n} = V \oplus \mathfrak{a}$  with  $V = [\mathfrak{n}, \mathfrak{n}]$  and Lie brackets  $[\mathfrak{a}, \mathfrak{a}] \subset V$  given by (1). Furthermore, it inherits an inner product by declaring the decomposition to be orthogonal and using the given inner products on  $V$  and  $\mathfrak{a}$ . There is a one-to-one correspondence between pairs  $(V, \langle \cdot, \cdot \rangle)$  with  $V \subset \mathfrak{so}(\mathfrak{a})$  and simply connected metric 2-step nilpotent Lie groups. We call  $(V, \langle \cdot, \cdot \rangle)$  a *WS-pair* (resp. *GO-pair*) if the corresponding simply connected nilpotent metric Lie group is weakly symmetric (resp. geodesic orbit). We call an inner product  $\langle \cdot, \cdot \rangle$  on a subspace  $V \subset \mathfrak{so}(\mathfrak{a})$  *admissible* if  $(V, \langle \cdot, \cdot \rangle)$  is a WS-pair, and *standard* if  $\langle \cdot, \cdot \rangle$  is the restriction of a bi-invariant inner product on  $\mathfrak{so}(\mathfrak{a})$ . If  $(V, \langle \cdot, \cdot \rangle)$  is a WS-pair, then a standard inner product is automatically admissible.

The case where  $V$  is a subalgebra of  $\mathfrak{so}(\mathfrak{a})$ , allowing any bi-invariant metric on  $V$ , corresponds to the case where the nilpotent Lie group is naturally reductive, see [Gor96, Theorem 2.5]. These are all GO, but WS ones constitute a small subset of them [Lau04, Theorem 5]. A second class of special examples are 2-step nilpotent Lie algebras constructed from Clifford modules, so called (generalized) H-type algebras. We call such examples to be of *Clifford type*, allowing a non-standard admissible inner product on  $V$ . In [Zil96, Example 6], [BRV98, Theorem 1] one finds a classification of WS-pairs of Clifford type with a standard inner product, and in [Lau99, Corollary 5.11] a classification with any admissible inner product. There two families, and one pair in dimension 31, which are GO but not WS.

A 2-step nilpotent Lie algebra is called *non-singular* if all nonzero elements of  $V$  are invertible. All 2-step nilpotent Lie algebras of Clifford type have this property. But the class of non-singular Lie algebras is surprisingly large and is not classified, unless  $\dim V = 1$  or  $2$ . For the current state of knowledge on non-singular 2-step nilpotent Lie algebras, we refer the reader to [Ebe94; LO14] and the bibliographies therein. Our goal is to classify 2-step nilpotent non-singular WS-pairs. In [NZ26] the authors classified all non-singular GO-pairs. Since weakly symmetric pairs are GO, we can use this classification in order to determine which ones are weakly symmetric.

In the following,  $\mathbb{H}$  denotes the algebra of quaternions with the standard inner product,  $L_q$  and  $R_q$  the left and the right multiplications by  $q \in \mathbb{H}$ , and  $i, j, k$  the standard basis of the space of imaginary quaternions  $\text{Im } \mathbb{H}$ . Furthermore,  $J_1, \dots, J_7$  are seven anticommuting complex structures on  $\mathbb{R}^8$ .

**Theorem 1.1.** *Let  $(V, \langle \cdot, \cdot \rangle)$  be a non-singular WS-pair. Then it is either of Clifford type, or belongs to one of the following:*

- (a)  $\dim V = 1$  and  $V = \mathbb{R}J$  with  $J$  a non-singular skew-symmetric matrix.
- (b)  $\dim V = 2$ . We identify  $\mathfrak{a}$  with  $\mathbb{H}^p$  and let  $V = \text{Span}(J_1, J_2)$ , with

$$J_1 = \text{diag}(L_{a_1}, L_{a_2}, \dots, L_{a_p}), \quad J_2 = \text{diag}(L_{b_1}, L_{b_2}, \dots, L_{b_p}),$$

where, for each  $s = 1, \dots, p$ , the quaternions  $a_s, b_s \in \text{Im } \mathbb{H}$  are linearly independent. The inner product  $\langle \cdot, \cdot \rangle$  on  $V$  is arbitrary.

(c)  $\dim V = 3$ . We identify  $\mathfrak{a}$  with  $\mathbb{H}^p$  and let  $V = \text{Span}(J_1, J_2, J_3)$  with

$$J_1 = \text{diag}(\lambda_1 L_1, \dots, \lambda_p L_p), \quad J_2 = \text{diag}(\mu_1 L_1, \dots, \mu_p L_p), \\ J_3 = \text{diag}(\mu_1 L_k, \dots, \mu_p L_k),$$

where  $\lambda_s, \mu_s \in \mathbb{R}$  are nonzero. The inner product on  $V$  is such that the elements  $J_1, J_2$  and  $J_3$  are mutually orthogonal,  $\|J_2\| = \|J_3\|$ , and  $\|J_1\|$  is arbitrary.

(d)  $\dim V = 6$ . Then  $\dim \mathfrak{a} = 8$  and  $V = \text{Span}(J_1, J_2, J_3, J_4, J_5, J')$ , where  $J' = J_7 \cos \theta + J_6 J_7 \sin \theta$ ,  $\theta \in (0, \pi/2)$ . The inner product on  $V$  is such that its restriction to  $\text{Span}(J_1, J_2, J_3, J_4, J_5)$  is standard,  $J' \perp J_i$ ,  $i = 1, \dots, 5$ , and  $\|J'\|$  is arbitrary.

Notice that in the non-Clifford case, a non-singular nilmanifold with  $\dim \mathfrak{z} \in \{1, 2, 6\}$  is WS if and only if it is GO, see Theorem 2.1 below. But if  $\dim V = 3$  or 7, there are non-isomorphic families which are GO, but neither WS nor of Clifford type. We remark that if we allow some of  $a_s, b_s$  to be linearly dependent in case (b), or some of  $\lambda_s, \mu_s$  to be zero in case (c), the resulting nilpotent Lie group is still WS, but is no longer non-singular. Notice also that in case (d), different values of  $\theta$  give non-isomorphic Lie algebras since the maxima of the maximal eigenvalues of the matrices  $J^2$ ,  $J \in V$ , with  $\text{Tr}(J^2) = -1$ , are distinct [NZ26, Section 3]. Note that we can view all of the nilpotent Lie algebras in Theorem 1.1 as deformations of the corresponding ones of Clifford type by changing the structure constants.

In Section 5 we discuss the relation of the above classification with previously known results.

## 2. Preliminaries

Given a subspace  $V \subset \mathfrak{so}(\mathfrak{a})$  with an inner product  $\langle \cdot, \cdot \rangle$ , denote by  $\mathbf{N}(V)$  its orthogonal normalizer:

$$\mathbf{N}(V) = \{N \in \mathbf{O}(\mathfrak{a}) \mid NVN^{-1} \subset V, \text{ and } (\text{Ad}_N)|_V \in \mathbf{O}(V)\}.$$

and by  $\mathbf{C}(V)$  the centralizer:

$$\mathbf{C}(V) = \{N \in \mathbf{O}(\mathfrak{a}) \mid JN = NJ \text{ for all } J \in V\}.$$

Clearly,  $\mathbf{C}(V)$  is a normal subgroup of  $\mathbf{N}(V)$ , and hence on the Lie algebra level we have a decomposition into ideals:

$$\mathfrak{N}(V) = \mathbf{C}(V) \oplus \mathfrak{P}(V).$$

In this decomposition,  $\mathfrak{P}(V)$  is the orthogonal complement of  $\mathbf{C}(V)$  relative to the restriction of the standard inner product on  $\mathfrak{so}(\mathfrak{a})$  to  $\mathfrak{N}(V)$ . As  $\mathfrak{N}(V)$  is a subalgebra of  $\mathfrak{so}(\mathfrak{a})$  and  $\mathbf{C}(V)$  is its ideal, the subspace  $\mathfrak{P}(V)$  must be an ideal of  $\mathfrak{N}(V)$  as well. We call  $\mathfrak{P}(V)$  the *pure normalizer* which clearly acts effectively on  $V$ .

The isometry group of a left invariant metric on  $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{a}$  is equal to

$$K = \{(A, B) \in \mathbf{O}(\mathfrak{z}) \times \mathbf{O}(\mathfrak{a}) \mid BJ_Z B^{-1} = J_{AZ} \text{ with } J_Z \in V\},$$

see [Wol63, Theorem 4.2]. Thus the condition for the nilmanifold to be weakly symmetric is equivalent to the following: for any  $J \in V$  and any  $X \in \mathfrak{a}$ , there exists an element  $N \in \mathbf{N}(V)$  such that

$$NX = -X \quad \text{and} \quad NJ = -JN. \tag{2}$$

For simplicity, we will also write the latter equation as  $N \cdot J = -J$ , where  $N \cdot J = N^t J N$ .

We start by recalling the classification of non-singular GO 2-step nilpotent Lie algebras.

**Theorem 2.1** ([NZ26, Theorem 1]). *Let  $(V, \langle \cdot, \cdot \rangle)$  be a 2-step nilpotent non-singular GO-pair. Then it is either of Clifford type, or belongs to one of the following:*

- (a)  $\dim V = 1$ . Then  $V$  is spanned by a non-singular skew-symmetric matrix.
- (b)  $\dim V = 2$ . We identify  $\mathfrak{a}$  with  $\mathbb{H}^p$  and then  $V = \text{Span}(J_1, J_2)$  with

$$J_1 = \text{diag}(L_{a_1}, L_{a_2}, \dots, L_{a_p}), \quad J_2 = \text{diag}(L_{b_1}, L_{b_2}, \dots, L_{b_p}),$$

where  $a_s, b_s \in \text{Im } \mathbb{H}$  are linearly independent.

- (c)  $\dim V = 3$ . Two cases are possible:

- (i) Identify  $\mathfrak{a}$  with  $\mathbb{H}^p$ . Then  $V = \text{Span}(J_1, J_2, J_3)$  with

$$J_1 = \text{diag}(L_{a_1}, L_{a_2}, \dots, L_{a_p}),$$

$$J_2 = \text{diag}(L_{b_1}, L_{b_2}, \dots, L_{b_p}),$$

$$J_3 = \text{diag}(L_{c_1}, L_{c_2}, \dots, L_{c_p}),$$

where, for every  $s = 1, \dots, p$ , the quaternions  $\{a_s, b_s, c_s\}$  are a basis of  $\text{Im } \mathbb{H}$ .

- (ii) Let  $\mathfrak{a} = \mathbb{H}^p \oplus W$ ,  $p \geq 0$ , and  $\rho: \mathfrak{so}(3) \rightarrow \mathfrak{so}(W)$  be a quaternionic representation on  $W = \mathbb{R}^{4q}$ ,  $q > 1$ , with no 4-dimensional sub-representations. Then  $V = \text{Span}(J_1, J_2, J_3)$  with

$$J_1 = \text{diag}(\lambda_1 L_i, \dots, \lambda_p L_i, \rho(\mathbf{i})),$$

$$J_2 = \text{diag}(\lambda_1 L_j, \dots, \lambda_p L_j, \rho(\mathbf{j})),$$

$$J_3 = \text{diag}(\lambda_1 L_k, \dots, \lambda_p L_k, \rho(\mathbf{k})),$$

where  $\lambda_s \neq 0$  and where we identify  $\mathfrak{so}(3)$  with  $\text{Im } \mathbb{H}$ .

- (d)  $\dim V = 6$ . Then  $\dim \mathfrak{a} = 8$  and  $V = \text{Span}(J_1, J_2, J_3, J_4, J_5, J')$ , where  $J' = J_7 \cos \theta + J_6 J_7 \sin \theta$ ,  $\theta \in (0, \pi/2)$ .
- (e)  $\dim V = 7$ . Then  $\dim \mathfrak{a} = 8$  and  $V = \text{Span}(J_1, J_2, J_3, J_4, J_5, J_6, J')$ , where  $J' = J_7 \cos \theta + J_6 J_7 \sin \theta$ ,  $\theta \in (0, \pi/2)$ .

Admissible inner products for the GO-pairs in Theorem 2.1 are as follows:

- If  $\dim V \in \{1, 2\}$  or  $\dim V = 3$  in case (c-i) (these are subspaces of so-called *centralizer type*), then any inner product on  $V$  is admissible.
- If  $\dim V = 3$  in case (c-ii), the inner product is standard.
- If  $\dim V = 6, 7$ , then the restriction of the inner product to  $\text{Span}(J_1, J_2, \dots, J_5)$  is standard, and  $J' \perp J_i$ , respectively  $J_6, J' \perp J_i$ , for  $i = 1, \dots, 5$ .

We also state the classification of GO-pairs of Clifford type. Let  $\langle \cdot, \cdot \rangle$  be an inner product on  $V$  and let  $S$  be the symmetric operator defined by  $\langle J, J \rangle = (S J, J)$  for  $J \in V$ , where  $(\cdot, \cdot)$  is a standard inner product. Let  $\lambda_1, \dots, \lambda_q$  be the eigenvalues of  $S$  with multiplicities  $m_1, \dots, m_q$ , respectively. We say that the inner product  $\langle \cdot, \cdot \rangle$  has *eigenvalue type*  $(m_1, \dots, m_q)$ ; so a standard inner product has eigenvalue type  $(\dim V)$ .

**Theorem 2.2** ([Zil96], [BRV98], [Lau99, Corollary 5.11]). *A WS-pair  $(V, \langle \cdot, \cdot \rangle)$  is of Clifford type precisely in the following cases.*

- (a)  $\dim V \in \{1, 2\}$  and  $\langle \cdot, \cdot \rangle$  is arbitrary.
- (b)  $\dim V = 3$  and the eigenvalue type of  $\langle \cdot, \cdot \rangle$  is either (3) or (2, 1).
- (c)  $\dim V = 5$ ,  $\dim \mathfrak{a} = 8$  and  $\langle \cdot, \cdot \rangle$  is standard.
- (d)  $\dim V = 6$ ,  $\dim \mathfrak{a} = 8$  and the eigenvalue type of  $\langle \cdot, \cdot \rangle$  is either (6) or (5, 1).
- (e)  $\dim V = 7$ ,  $\dim \mathfrak{a} = 8$  and the eigenvalue type of  $\langle \cdot, \cdot \rangle$  is either (7) or (5, 2).
- (f)  $\dim V = 7$ ,  $\dim \mathfrak{a} = 16$ , the inner product  $\langle \cdot, \cdot \rangle$  is standard, and  $V$  is spanned by pairwise anticommuting complex structures  $K_i$ ,  $i = 1, \dots, 7$  such that  $K_1 \dots K_7 = \pm \text{Id}_{\mathfrak{a}}$ .

Notice that by Theorem 2.1, the nilpotent Lie groups in (c) and (f) admit no deformations of the structure constants (which of course preserves the condition of being non-singular) which are also GO (and a fortiori, WS), whereas the remaining ones do.

### 3. Examples are weakly symmetric

We start by discussing the examples in our Theorem 1.1.

Suppose  $\dim V = 1$ . We can identify  $\mathfrak{a}$  with  $\mathbb{C}^p$  such that  $J$  acts by multiplication by an imaginary number in each coordinate. If  $P(X) = \bar{X}$ , then conjugation by  $P$  takes  $J$  to  $-J$ . We can now choose an element  $Q$  in maximal torus  $\text{diag}(e^{i\theta_1}, \dots, e^{i\theta_p})$  which lies in the centralizer, with  $Q(PX) = -X$  for all  $X \in \mathfrak{a}$ . Thus  $QP$  satisfy property (2). The inner product  $\langle \cdot, \cdot \rangle$  on  $V$  is clearly arbitrary.

Suppose  $\dim V = 2$  with  $J_1 = \text{diag}(L_{a_1}, \dots, L_{a_p})$ ,  $J_2 = \text{diag}(L_{b_1}, \dots, L_{b_p})$ , where  $a_s, b_s$  imaginary quaternions. For each  $s$ , there exists a unit quaternion  $q_s$  which is orthogonal to  $a_s$  and  $b_s$ , and which anti-commutes with both  $a_s$  and  $b_s$ . Hence  $P = \text{diag}(L_{q_1}, \dots, L_{q_p})$  induces the action of  $-\text{Id}$  on  $V$ . We can furthermore chose unit quaternions  $r_s$  such that  $R_{r_s}(L_{q_s}X_s) = -X_s$ , where  $X_s \in \mathbb{H}$  is the corresponding component of  $X \in \mathfrak{a}$ . Defining  $Q = \text{diag}(R_{r_1}, \dots, R_{r_p})$  it follows that  $Q(PX) = -X$ . Since  $Q$  also commutes with  $P$ , this implies that  $N = QP \in \mathcal{O}(\mathfrak{a})$  satisfies (2). The inner product  $\langle \cdot, \cdot \rangle$  on  $V$  is arbitrary.

Suppose  $\dim V = 3$ . We have  $V = \text{Span}(J_1, J_2, J_3)$  with

$$\begin{aligned} J_1 &= \text{diag}(\lambda_1 L_1, \dots, \lambda_p L_1), \\ J_2 &= \text{diag}(\mu_1 L_j, \dots, \mu_p L_j), \\ J_3 &= \text{diag}(\mu_1 L_k, \dots, \mu_p L_k), \end{aligned}$$

where  $\lambda_s, \mu_s \in \mathbb{R}$  are nonzero. Given an arbitrary  $J = aJ_1 + bJ_2 + cJ_3 \in V$  with  $a, b, c \in \mathbb{R}$  and  $X \in \mathfrak{a}$ , let  $r$  be a unit imaginary quaternion orthogonal to both  $i$  and  $bj + ck$ . Then the element  $P = \text{diag}(L_r, L_r, \dots, L_r) \in \mathcal{O}(\mathfrak{a})$  anti-commutes with  $J$ . Moreover,  $P$  lies in  $\mathcal{N}(V)$  since conjugation by  $P$  sends  $J_1$  to  $-J_1$  and acts as an orthogonal transformation with determinant  $-1$  on  $\text{Span}(J_2, J_3)$ . We can now chose an element in the centralizer of  $V$  of the form  $Q = \text{diag}(R_{r_1}, \dots, R_{r_p})$ , for some unit quaternions  $r_i$ , such that  $Q(PX) = -X$ . Thus  $N = QP$  satisfies the equations  $N \cdot J = -J$  and  $NX = -X$ , as required.

Finally, notice that  $\mathcal{N}(V)$  acts by isometries in an inner product on  $V$  if and only if  $J_1, J_2$  and  $J_3$  are mutually orthogonal and  $\|J_2\| = \|J_3\|$ .

Suppose  $\dim V = 6$ . Here we have  $V = \text{Span}(J_1, J_2, J_3, J_4, J_5, J')$  with  $J' = J_7 \cos \theta + J_6 J_7 \sin \theta$ ,  $\theta \in (0, \pi/2)$ , where  $J_1, \dots, J_7$  are pairwise anticommuting complex structures on  $\mathbb{R}^8$ . By [NZ26, Lemma 7] the subgroup  $\text{Sp}(2) \subset \text{SO}(8)$ , with its standard embedding, lies in  $\mathbf{N}(V)$  and acts on  $\text{Span}(J_1, J_2, J_3, J_4, J_5) \subset V$  as  $\text{SO}(5)$  under the two-fold cover  $\text{Sp}(2) \rightarrow \text{SO}(5)$ . Furthermore,  $J_6$  clearly also lies in  $\mathbf{N}(V)$  and acts, via conjugation, as  $-\text{Id}$  on  $V$ . For a generic  $J \in V$  the subgroup of  $\text{SO}(5)$  that commutes with  $J$  is equal to  $\text{SO}(4) \subset \text{SO}(5)$ . Its lift under the two-fold cover is  $\text{Sp}(1)\text{Sp}(1) \subset \text{Sp}(2)$  with its block embedding. Since the action of  $\text{Sp}(2)$  on  $\mathfrak{a} = \mathbb{H}^2$  is standard,  $\text{Sp}(1)\text{Sp}(1) \subset \text{Sp}(2)$  acts on each component of  $\mathbb{H}^2 = \mathbb{H} \oplus \mathbb{H}$  separately. Thus for each  $X \in \mathfrak{a}$ , there exists a  $Q \in \text{Sp}(1)\text{Sp}(1)$  with  $Q(J_6 X) = -X$  and hence  $N = QJ_6$  satisfies (2).

The subgroup of  $\mathbf{N}(V)$  generated by  $\text{Sp}(2)$  and  $J_6$  acts by isometries in an inner product on  $V$  if and only if its restriction to  $\text{Span}(J_1, J_2, J_3, J_4, J_5)$  is standard and  $J' \perp J_i$ , for  $i = 1, \dots, 5$ .

Notice that in all of these examples, the eigenvalue type of  $\langle \cdot, \cdot \rangle$  on  $V$  is the same as those of Clifford type, and we simply deform the structure constants of  $\mathfrak{n}$ .

One can show that the moduli space of non-isomorphic Lie algebras in case (b) of Theorem 1.1 depends on  $2p - 3$  real parameters when  $p > 1$ . To see this, one computes the Pfaffian  $\text{Pf}(x_1, x_2) = \prod_{i=1}^p \|x_1 a_i + x_2 b_i\|^2$  of the pencil  $x_1 J_1 + x_2 J_2$  and denotes  $\alpha_i$  the complex root with positive imaginary part of the quadratic polynomial  $t \mapsto \|ta_i + b_i\|^2$ . Then for generic  $a_i, b_i \in \mathbb{H}$ , two algebras of the form (b) of Theorem 1.1 are isomorphic if and only if the so constructed unordered  $p$ -tuples  $\{\alpha_1, \dots, \alpha_p\}$  can be mapped one onto another by a real Möbius transformation in the upper half-plane (that is, by a hyperbolic isometry). Mapping  $\alpha_1$  to  $i$  and  $\alpha_2$  to a positive multiple of  $i$ , all the remaining  $p - 2$  are fixed up to reflection in the imaginary axis (see [LO14, Sections 2.1, 3] for details). Similarly, computing the Pfaffian one can show that moduli space of isomorphism classes of Lie algebras in case (c) of Theorem 1.1 at a generic point is  $(p - 1)$ -dimensional, and corresponds to the equivalence classes of  $p$ -tuples  $\{\lambda_1 : \mu_1, \dots, \lambda_p : \mu_p\} \subset \mathbb{R}P^1$  under projective transformations of  $\mathbb{R}P^1$ .

#### 4. Proof of Theorem 1.1

We now need to show that the remaining GO-pairs in Theorem 2.1 are not WS-pairs.

First consider the GO-pairs in case (c-ii) of Theorem 2.1. It is sufficient to show that  $\mathbf{N}(V) \subset \text{O}(\mathbb{H}^p) \times \text{O}(W) \subset \text{O}(\mathfrak{a})$ . Indeed, this fact implies that if the condition (2) is satisfied for  $(V, \langle \cdot, \cdot \rangle)$ , then it is also satisfied with  $\mathfrak{a}$  replaced by  $W$ , and  $V$  replaced by its projection  $V'$  to  $\mathfrak{so}(W)$ . But such a  $V'$  is the image of a skew-symmetric representation of the algebra  $\mathfrak{su}(2)$  on  $W$  which is the direct sum of representations of dimension at least 8, and then no such space  $V'$  can be WS by [Lau04, Theorem 5 and Theorem 3]. The proof of the fact that  $\mathbf{N}(V) \subset \text{O}(\mathbb{H}^p) \times \text{O}(W)$  is a Schur-type argument. Let  $1 \leq r \leq p$ , and let  $W_s \subset W$  be an irreducible  $\mathfrak{so}(3)$ -submodule of dimension  $4m (\geq 8)$ , with  $\rho_s$  the corresponding representation. For  $N \in \mathbf{N}(V)$  denote  $N_{rs}$  the  $4 \times (4m)$  block of the matrix  $N$  positioned in the rows corresponding to the  $r$ -th factor  $\mathbb{H}$  of  $\mathbb{H}^p$  and the columns corresponding to  $W_s$ . For any  $a = 1, 2, 3$ , we have  $JN = NJ_a$ , for some  $J \in V$ , and so for any  $q \in \text{Im } \mathbb{H}$  there exists  $q' \in \text{Im } \mathbb{H}$  such that  $\lambda_r L_{q'} N_{rs} = N_{rs} \rho_s(q)$ . Let  $W' = \text{Ker } N_{rs} \subset W_s$ . Then the subspace  $W'$  is

nontrivial as  $4m > 4$  and  $\rho_s$ -invariant. As  $\rho_s$  is irreducible, we obtain  $W' = W$ , and so  $N_{rs} = 0$ . Thus the subspace  $W \subset \mathfrak{a}$  is  $\mathbf{N}$ -invariant, as required.

Next we consider the GO-pairs in case (c-i) of Theorem 2.1. For  $s = 1, \dots, p$ , denote  $\mathbb{H}_s$  the  $s$ -th factor of  $\mathbb{H}^p$  and define the (non-singular) maps  $A_s : \mathfrak{z} \rightarrow \text{Im } \mathbb{H}$  such that  $J_Z = \text{diag}(L_{A_1 Z}, \dots, L_{A_p Z})$ . Then any  $N \in \mathbf{N}(V)$  induces the map  $\phi_N : \mathfrak{z} \rightarrow \mathfrak{z}$  such that  $N \cdot J_Z = J_{\phi_N Z}$ . The map  $\phi_N$  is orthogonal relative to both the given inner product on  $\mathfrak{z}$  and the standard one (inherited from the restriction of the Killing form on  $\mathfrak{so}(\mathfrak{a})$  to  $V$ ). Let  $N_{rs}$  be the  $4 \times 4$  block of the matrix  $N$  positioned in the rows corresponding to  $\mathbb{H}_r$  and the columns corresponding to  $\mathbb{H}_s$ . For all  $Z \in \mathfrak{z}$  we have

$$L_{A_r Z} N_{rs} = N_{rs} L_{A_s(\phi_N Z)}. \tag{3}$$

We prove the following.

**Lemma 4.1.** *In the above notation, for the GO-pairs in case (c-i) of Theorem 2.1, the following holds.*

- (a) *Suppose  $N_{rs} \neq 0$  for some  $s, r = 1, \dots, p$ . Then there exist  $w_{rs}, w'_{rs} \in \mathbb{H}$  with  $\|w_{rs}\| = 1$ ,  $w'_{rs} \neq 0$ , such that  $N_{rs} = L_{w_{rs}} R_{w'_{rs}}$ . Moreover,  $w_{rs}^{-1}(A_r Z)w_{rs} = A_s(\phi_N Z)$ , for all  $Z \in \mathfrak{z}$ , and we have  $\det(A_s A_r^{-1}) = \det(\phi_N) (\in \{\pm 1\})$ .*
- (b) *The group  $\mathbf{N}(V)/\mathbf{C}(V)$  cannot be discreet.*

**Proof.** To prove assertion (a), for some fixed  $s, r = 1, \dots, p$ , denote  $Q = N_{rs} \neq 0$  and define the linear map  $\psi : \mathbb{H} \rightarrow \mathbb{H}$  by  $\psi|_{\text{Im } \mathbb{H}} = (A_s \circ \phi_N \circ A_r^{-1})|_{\text{Im } \mathbb{H}}$  on  $\text{Im } \mathbb{H}$  and by  $\psi(1) = 1$ ; note that  $\psi$  is non-singular. From (3) we obtain  $L_u Q = Q L_{\psi(u)}$ , for all  $u \in \mathbb{H}$ . If  $X \in \text{Ker } Q$ , then also  $L_{\psi(u)} X \in \text{Ker } Q$  which implies that the subspace  $\text{Ker } Q \subset \mathbb{H}$  is  $L_v$ -invariant, for all  $v \in \mathbb{H}$ . It follows that  $Q$  is non-singular, since  $Q \neq 0$ . Then  $Q L_{\psi(uv)} = L_{uv} Q = L_u L_v Q = Q L_{\psi(u)} L_{\psi(v)} = Q L_{\psi(u)\psi(v)}$ , for all  $u, v \in \mathbb{H}$ , and so  $\psi : \mathbb{H} \rightarrow \mathbb{H}$  is an isomorphism. Then  $\psi(u) = w^{-1} u w$  for some unit  $w \in \mathbb{H}$ , which gives  $L_u Q L_w^{-1} = Q L_w^{-1} L_u$ , for all  $u \in \mathbb{H}$ . Thus  $Q L_w^{-1}$  commutes with all the left multiplications; it follows that it is a right multiplication, so that  $Q = L_w R_{w'}$ , for some  $w' \in \mathbb{H}$ . Then  $w^{-1}(A_r Z)w = A_s(\phi_N Z)$ , for all  $Z \in \mathfrak{z}$ , as required. In particular,  $\det(A_s A_r^{-1}) = \det(\phi_N)$ , and  $\det(\phi_N) \in \{\pm 1\}$ , as  $\phi_N$  is orthogonal.

For assertion (b), suppose that the group  $\mathbf{N}(V)/\mathbf{C}(V)$  is discreet. Since both groups are compact, this means that  $\mathbf{N}(V)/\mathbf{C}(V)$  is finite. As the map  $N \mapsto \phi_N$  is well-defined on the cosets from  $\mathbf{N}(V)/\mathbf{C}(V)$ , its image is a finite subset of  $\text{O}(\mathfrak{z})$ . By (2), any  $Z \in \mathfrak{z}$  belongs to the  $(-1)$ -eigenspace of  $\phi_N$ , for some  $N \in \mathbf{N}(V)$ , and so there exists an open subset  $U \subset \mathfrak{z}$  such that the condition  $N \cdot J_Z = -J_Z$  for  $Z \in U$  is only satisfied with some  $N \in \mathbf{N}(V)$  such that  $\phi_N = -\text{Id}$ . From assertion (a) it follows that for  $s, r = 1, \dots, p$ , the block  $N_{rs}$  of such an  $N$  can only be nonzero if  $\det(A_s A_r^{-1}) = -1$ . Consider the orthogonal decomposition  $\mathfrak{a} = \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{b}_3$ , where

$$\mathfrak{b}_1 = \bigoplus_{r: \det(A_1 A_r^{-1})=1} \mathbb{H}_r, \quad \mathfrak{b}_2 = \bigoplus_{r: \det(A_1 A_r^{-1})=-1} \mathbb{H}_r, \quad \mathfrak{b}_3 = \bigoplus_{r: \det(A_1 A_r^{-1}) \notin \{\pm 1\}} \mathbb{H}_r.$$

Then  $N\mathfrak{b}_1 \subset \mathfrak{b}_2$ ,  $N\mathfrak{b}_2 \subset \mathfrak{b}_1$  and  $N\mathfrak{b}_3 \subset \mathfrak{b}_3$ . Clearly  $\mathfrak{b}_1 \neq 0$  and moreover, as  $N$  is orthogonal, we have  $\dim \mathfrak{b}_1 = \dim \mathfrak{b}_2$  and

$$N = \begin{pmatrix} 0 & Q_1 & 0 \\ Q_2 & 0 & 0 \\ 0 & 0 & Q_3 \end{pmatrix},$$

relative to the decomposition  $\mathfrak{a} = \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{b}_3$ , where the matrices  $Q_1, Q_2$  and  $Q_3$  are orthogonal. But then the second equation of (2) is violated if we take  $X$  to be a nonzero vector from  $\mathfrak{b}_1$ .  $\square$

Consider the pure normalizer  $\mathfrak{P}(V) \subset \mathfrak{so}(\mathfrak{a})$ . From [NZ26, Lemma 2(d)] it is isomorphic to a subalgebra of  $\mathfrak{so}(3)$  and lies in the subspace  $\bigoplus_{s=1}^p \mathfrak{so}(\mathbb{H}_s)$ . With our choice of the basis, any element  $\text{diag}(R_{u_1}, \dots, R_{u_p})$ , where  $u_s \in \text{Im } \mathbb{H}$ , lies in the centralizer  $\mathcal{C}(V)$ , and so  $\mathfrak{P}(V)$  must be a subalgebra of the algebra  $\bigoplus_{s=1}^p \mathfrak{so}(3)$ , spanned by the elements  $\text{diag}(L_{u_1}, \dots, L_{u_p})$ , where  $u_s \in \text{Im } \mathbb{H}$ . Moreover, the subalgebra  $\mathfrak{P}(V)$  cannot be trivial, by Lemma 4.1(b). Let  $P = \text{diag}(L_{u_1}, \dots, L_{u_p}) \in \mathfrak{P}(V)$  be nonzero. Then  $\|u_1\| = \dots = \|u_p\|$  as  $V$  must be  $\text{ad}_P$ -invariant. Choosing an appropriate basis in each coordinate, we can assume that  $P = \text{diag}(L_1, \dots, L_1)$ . As  $V$  is an  $\text{ad}_P$ -module, we obtain  $V = \text{Span}(J_1, J_2, J_3)$ , where

$$\begin{aligned} J_1 &= \text{diag}(\lambda_1 L_1, \dots, \lambda_p L_1), \\ J_2 &= \text{diag}(\mu_1 L_j, \dots, \mu_p L_j), \\ J_3 &= \text{diag}(\mu_1 L_k, \dots, \mu_p L_k), \end{aligned}$$

for some nonzero constants  $\lambda_s, \mu_s$ ,  $s = 1, \dots, p$  (by additionally rotating the subspaces  $\text{Span}(j, k) \subset \mathbb{H}_s$  by a choice of the basis, if necessary). Moreover, the action of  $\text{ad}_P$  on  $V$  must be skew-symmetric, and so an admissible inner product  $\langle \cdot, \cdot \rangle$  on  $V$  must satisfy  $J_1 \perp J_2, J_3$  and  $\|J_2\| = \|J_3\|$ , as in (c) of Theorem 1.1.

Finally, we consider the case where  $\dim V = 7$ . By Theorem 2.1(e), we have  $\dim \mathfrak{a} = 8$  and  $V = \text{Span}(J_1, J_2, J_3, J_4, J_5, J_6, J')$ , where  $J' = J_7 \cos \theta + J_6 J_7 \sin \theta$ ,  $\theta \in (0, \pi/2)$ . We prove that for  $J_6 \in V$ , there is no  $N \in \mathbf{N}(V)$  such that  $N \cdot J_6 = -J_6$ , and hence condition (2) cannot possibly be satisfied. It is sufficient to show this assuming the inner product on  $V$  to be standard; we take  $\|J\|^2 = -\frac{1}{8} \text{Tr } J^2$  for  $J \in V$ , so that the above basis for  $V$  is orthonormal (recall that  $\text{Tr}(J_i J_6 J_7) = 0$  for all  $i = 1, \dots, 5$ ).

For a vector  $u = (u_1, \dots, u_5) \in \mathbb{R}^5$ , denote  $J_u = \sum_{i=1}^5 u_i J_i$ . Take an element  $J = J_u + a J_6 + b J' \in V$  with  $a, b \in \mathbb{R}$ . We have  $J^2 = -(\|u\|^2 + a^2 + b^2)I_8 + 2b \sin \theta J_u J_6 J_7$ . The operator  $A = J_u J_6 J_7$  is symmetric, has zero trace, and  $A^2 = \|u\|^2 I_8$ , and so the eigenvalues of  $A$  are  $\pm \|u\|$ , both of multiplicity 4 (provided  $u \neq 0$ ). Therefore  $J^2$  has two eigenvalues,  $-(\|u\|^2 + a^2 + b^2) \pm 2b \|u\| \sin \theta = -\|J\|^2 \pm 2 \sin \theta \langle J, J' \rangle (\sum_{i=1}^5 \langle J, J_i \rangle^2)^{1/2}$ , both of multiplicity 4 (provided  $b \|u\| \neq 0$ ). For  $N \in \mathbf{N}(V)$ , the conjugation by  $N$  acts orthogonally on  $V$  and preserves the eigenvalues of  $J^2$ . It follows that it preserves the function  $\Phi(J) = \text{Tr}((J^2 + \|J\|^2 I_8)^2) = 4 \sin^2 \theta \langle J, J' \rangle^2 \sum_{i=1}^5 \langle J, J_i \rangle^2$  for  $J \in V$ . Hence for any  $J \in V$  and  $N \in \mathbf{N}(V)$  we have

$$\langle J, J' \rangle^2 \sum_{i=1}^5 \langle J, J_i \rangle^2 = \langle J, N \cdot J' \rangle^2 \sum_{i=1}^5 \langle J, N \cdot J_i \rangle^2. \quad (4)$$

Taking  $J \perp J'$  in (4) we obtain  $\langle J, N \cdot J' \rangle^2 \sum_{i=1}^5 \langle J, N \cdot J_i \rangle^2 = 0$ , and so  $J \perp N \cdot J'$ . It follows that the subspace  $\mathbb{R} J' \subset V$  is  $N$ -invariant, and so  $N \cdot J' = \pm J'$ . Then the subspace  $(J')^\perp = \text{Span}(J_1, J_2, J_3, J_4, J_5, J_6)$  is also  $N$ -invariant, and so  $\sum_{i=1}^6 \langle J, J_i \rangle^2 = \sum_{i=1}^6 \langle J, N \cdot J_i \rangle^2$  which, together with (4) and the fact that  $N \cdot J' = \pm J'$  gives  $\langle J, N \cdot J_6 \rangle = \pm \langle J, N \cdot J_6 \rangle$ . Thus the subspace  $\mathbb{R} J_6 \subset V$  is also  $N$ -invariant, and  $N \cdot J_6 = \pm J_6$ . Then the subspace  $\text{Span}(J_1, J_2, J_3, J_4, J_5) = (\text{Span}(J_6, J')^\perp) \cap V$  is also  $N$ -invariant. Since for any orthonormal basis  $\{v^1, \dots, v^5\} \subset \mathbb{R}^5$  we have  $J_{v^1} \dots J_{v^5} = \pm J_1 \dots J_5$  we obtain  $N \cdot (J_1 \dots J_5) = \pm J_1 \dots J_5$ . But  $J_1 \dots J_5 J_6 J_7 = \pm I_8$  which implies  $N \cdot (J_6 J_7) = \pm J_6 J_7$ . Since we already know that  $N \cdot J_6 = \pm J_6$ ,

this gives  $N \cdot J_7 = \pm J_7$ . Hence for any  $N \in \mathbf{N}(V)$  there exist  $\varepsilon_6, \varepsilon_7, \varepsilon' \in \{\pm 1\}$  such that  $N \cdot J_6 = \varepsilon_6 J_6$ ,  $N \cdot J_7 = \varepsilon_7 J_7$  and  $N \cdot J' = \varepsilon' J'$ . These equations give  $\varepsilon_7 J_7 \cos \theta + \varepsilon_6 \varepsilon_7 J_6 J_7 \sin \theta = \varepsilon' (J_7 \cos \theta + \varepsilon_6 \varepsilon_7 J_6 J_7 \sin \theta)$ , and so  $\varepsilon' = \varepsilon_7 = \varepsilon_6 \varepsilon_7$ . But then  $\varepsilon_6 = 1$ , and so for no  $N \in \mathbf{N}(V)$  we can have  $N \cdot J_6 = -J_6$ , in violation of (2).

This completes the proof of Theorem 1.1.

### 5. Appendix

We discuss the relation of our classification with the known partial classification of WS nilmanifolds given in [Yak05, Theorem 5.7], [Wol07, Theorem 15.4.10] and [Vin01, Table 3]. We will be specifically interested in placing the WS nilmanifold obtained from the WS-pair in part (d) of Theorem 1.1 in this context. The said classification is based on the fact that a WS space is commutative in the sense of [Sel56].

It is known that any commutative nilmanifold is at most 2-step nilpotent, and in the above references, the authors give the classification of all irreducible, maximal commutative nilmanifolds  $(N \rtimes H)/H$ , where  $N$  is a nilpotent group with a left-invariant metric and  $H$  is a group of isometric automorphisms of  $\mathfrak{n}$ . We can also assume that the center  $\mathfrak{z}$  of the Lie algebra  $\mathfrak{n}$  of  $N$  and coincides with the derived algebra of  $\mathfrak{n}$ . A nilmanifold  $(N \rtimes H)/H$  is called *irreducible*, if  $H$  acts irreducibly on  $\mathfrak{a}$ . Given a commutative nilmanifold  $(N \rtimes H)/H$ , let  $\mathfrak{z}_0 \subset \mathfrak{z}$  be an  $H$ -module, with  $Z_0 \subset N$  the corresponding subgroup. Then the space  $(N/Z_0 \rtimes H)/H$  is again commutative. This process of passing from  $N$  to  $N/Z_0$  is called a central reduction. A *maximal* commutative nilmanifold is a one that cannot be obtained by a central reduction from a bigger one. After giving the classification of irreducible, maximal commutative nilmanifolds, the authors prove that all of them, except for one, are weakly symmetric (notice however that almost none of them are non-singular).

This exceptional space is given by  $\mathfrak{a} = \mathbb{H}^p$ ,  $H = \mathrm{Sp}(p)$  and  $\mathfrak{z} = S_0^2 \mathbb{H}^p \oplus \mathrm{Im} \mathbb{H}$ , where  $S_0^2 \mathbb{H}^p$  is the  $\mathrm{Sp}(p)$ -module of  $\mathbb{H}$ -hermitian  $p \times p$  matrices with trace 0, and  $\mathrm{Im} \mathbb{H}$  is the trivial  $\mathrm{Sp}(p)$ -module. In our notation,  $\mathrm{Im} \mathbb{H} \subset \mathfrak{so}(4p) = \mathfrak{so}(\mathfrak{a})$  is the subalgebra  $\mathfrak{sp}(1) = \{R_i, R_j, R_k\}$  of right multiplications on  $\mathbb{H}^p$  and  $\mathrm{Sp}(p) \subset \mathbf{N}(V)$  acts by left multiplication.

The nilmanifold in [Lau98, Theorem 5] is constructed by taking the central reduction by the subspace  $S_0^2 \mathbb{H}^2 \subset \mathfrak{z}$  and then choosing the inner product on  $\mathrm{Im}(\mathbb{H})$  whose eigenvalue type is  $(1, 1, 1)$ . This prevents the group of all isometric automorphisms to be bigger than  $\mathrm{Sp}(2)$ . The resulting nilmanifold is commutative, but not WS. However, if one takes the eigenvalue type  $(2, 1)$  instead, one obtains a WS nilmanifold since  $\mathbf{N}(V)$  also contains the subgroup  $\mathrm{O}(2)$ . This example is a special case of nilmanifolds in part (d) of Theorem 1.1.

In case  $p = 2$ , the above maximal nilmanifold admits an alternative description in terms of pairwise commuting complex structures  $J_1, \dots, J_7$  on  $\mathfrak{a} = \mathbb{R}^8$ , as follows. The space  $V \subset \mathfrak{so}(8)$  is the sum of the 5-dimensional  $\mathrm{Sp}(2)$ -module  $V_1$  spanned by the by  $J_1, \dots, J_5$ , and the 3-dimensional trivial module  $\mathrm{Span}(J_6, J_7, J_6 J_7)$ , where  $\mathrm{Sp}(2)$  ( $= \mathrm{Spin}(5)$ ) can be viewed as the connected Lie group with the Lie algebra spanned by the elements  $J_i J_j$ ,  $1 \leq i < j \leq 5$ . The resulting nilmanifold is commutative and not-WS. However some of its central reductions *are* weakly symmetric.

Consider all central reductions such that the resulting subspace  $V \subset \mathfrak{so}(8)$  is the given by  $V = V_1 \oplus V'$ , where  $V'$  is a proper subspace of  $\mathrm{Span}(J_6, J_7, J_6 J_7) \simeq \mathfrak{sp}(1)$ . The inner product  $\langle \cdot, \cdot \rangle$  on  $V$  is such that its restriction to  $V_1$  is standard and

that  $V' \perp V_1$ . All the resulting nilmanifolds are commutative, but some are weakly-symmetric, and some are not. Let  $J' = J_7 \cos \theta + J_6 J_7 \sin \theta$ . Up to isometry, all possible cases are as follows:

- If  $V' = 0$ , then the pair  $(V, \langle \cdot, \cdot \rangle)$  is WS, and is of Clifford type [Zil96], [BRV98], [Lau99, Corollary 5.11].
- If  $V' = \mathbb{R} \cdot J'$ , the pair  $(V, \langle \cdot, \cdot \rangle)$  is WS, for any  $\theta \in [0, \pi/2]$ . If  $\theta \in (0, \pi/2)$ , this is the WS-pair in part (d) of Theorem 1.1. The same proof also works for  $\theta = 0, \pi/2$  (but in the latter case, the subspace  $V$  is not non-singular). If  $\theta = 0$ , the WS-pair of Clifford type [Lau99, Corollary 5.11].
- If  $V' = \text{Span}(J_6, J')$ ,  $\theta \in (0, \pi/2)$ , then the pair  $(V, \langle \cdot, \cdot \rangle)$  is not WS, for any inner product on  $V'$ , see Section 3.
- If  $V' = \text{Span}(J_6, J_7)$ , then the pair  $(V, \langle \cdot, \cdot \rangle)$  is WS if and only if the restriction of  $\langle \cdot, \cdot \rangle$  to  $V'$  is standard, see [Lau99, Corollary 5.11]. The resulting WS-pair is of Clifford type. Note that in this case,  $\mathbf{N}(V) \supset \text{Sp}(2) \times \text{U}(1)$ .
- If  $V' = \text{Span}(J_6, J_6 J_7)$ , then the pair  $(V, \langle \cdot, \cdot \rangle)$  is WS, for any inner product on  $V'$ , but  $V$  is not non-singular. In this case, the group  $\mathbf{N}(V)$  contains the element  $J_7$ , the conjugation by which acts as  $-\text{Id}$  on  $V$ . Then an argument similar to the one in the case  $\dim V = 6$  in Section 3 shows the pair is WS.

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