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Abstract. We propose a novel framework that unifies two fundamental concepts in mathematics: matrix-valued spherical functions and scalar modular forms. By extending the classical theory of modular forms to the matrix-valued setting, we introduce and study *modular spherical functions.* These are smooth functions defined on a connected unimodular Lie group G, a compact subgroup K and a discrete subgroup Γ, with values in endomorphism spaces of finite-dimensional vector spaces. Modular spherical functions are characterized as eigenfunctions of the algebra $D(G)^K$, consisting of all G-left invariant and K-right invariant differential operators on G, while satisfying specific transformation properties under the actions of G, K and Γ.

Focusing on the paradigmatic case $G = \mathrm{SL}(2, \mathbb{R})$, $K = \mathrm{SO}(2)$ and $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, we partially extend the characterization of modular spherical functions. We describe these functions in terms of polynomial eigenfunctions in two real variables for the n-Laplacian in the upper half-plane. The results include explicit bases for associated function spaces, recurrence relations for orthogonal polynomials, and analytic continuation. This work advances the study of modular spherical functions, opening new avenues in the representation theory of reductive Lie groups, orthogonal polynomials, and modular forms.

Mathematics Subject Classification 2020: 11F11, 11F30, 22E46, 30C10, 30C45, 42A24

Key Words and Phrases: matrix-valued spherical functions, upper half plane, modular functions, two variable polynomial eigenfunctions of the n-Laplacian, generalized Laguerre polynomials

1. Introduction

The theory of spherical functions has its roots in the classical works of Élie Cartan, Hermann Weyl, and Harish-Chandra, where scalar-valued spherical functions naturally arose in the study of functions on symmetric spaces G/K, with G a Lie group and K a compact subgroup. Over the past decades, significant progress has been made in extending these ideas to matrix-valued spherical functions, leading to profound connections with matrix orthogonal polynomials, time-band limiting problems, differential operators and related topics. A sense of these developments can be obtained from: [AT92; GV88; GPT02; GPT03; GT07; KT76; Tir03; VPT24].

The article [Tir74] is based on thesis work carried out under the supervision of Joe Wolf and contains, for the first time, an example of an operator valued spherical function. Building on the foundational work of R. Godement [God52] on spherical trace functions, we introduced in [Tir77] an intrinsic definition of operator valued spherical functions and established their main general properties.

Simultaneously, scalar modular forms have played a central role in number theory and analysis. Defined as holomorphic functions on the upper half-plane with specific transformation properties under $\mathrm{SL}(2,\mathbb{Z})$, modular forms are indispensable in understanding automorphic forms, elliptic curves, and L-functions.

This paper aims to bridge these two rich theories by introducing and studying *operator valued modular spherical functions*, which generalize modular forms to take values in operator spaces. These functions are defined on G, K and Γ and satisfy both the functional and differential equations characteristic of spherical functions, alongside modular transformation properties. The case $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2)$ and $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ provides an ideal setting for developing this theory, leveraging the structure of the upper half-plane and the spectral properties of modular forms.

We present explicit results for modular spherical functions, including their characterization as eigenfunctions of the Casimir operator, connections to polynomial eigenfunctions, and bases for function spaces such as $S_{n,k,m}$. Furthermore, we establish a correspondence between these functions and holomorphic sequences, providing a spectral interpretation that aligns with classical modular forms.

This work sets the stage for future studies in modular spherical functions, offering a unified perspective that merges the strengths of representation theory, modular forms, and matrix-valued analysis.

Outline of the paper

This paper is organized as follows. Section 2 introduces the foundational concepts and definitions, including the formal notion of modular spherical functions.

In Section 3, the necessary background on the groups $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2)$ and $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ is given. This section focuses on the restriction of the Casimir operator to the Borel subgroup and its connection to harmonic and polynomial eigenfunctions of the n-Laplacian in the upper half plane, providing explicit expressions and key results. In Section 3 we consider the linear spaces

$$S_{n,k,m} = \{f \in \mathbb{C}[x,y] : \Delta_n f = -(k+1)k\, f,\ \deg(f) \le m\}$$

for $n,k,m \in \mathbb{Z}$; $k,m \geq 0$. After a considerable amount of work we could find bases of these spaces given in Propositions 3.4, 3.6, 3.9, 3.12.

Section 4 explores the correspondence between some modular spherical functions and holomorphic sequences, relating these functions to holomorphic functions on the upper half-plane. This analysis includes bases for associated function spaces and spectral characterizations of eigenfunctions. Our main result Theorem 4.8, characterizes the analytic harmonic functions in the upper half plane in terms of pairs of sequences $\big((a_m)_{m\ge 0}, (b_m)_{m\ge 1}\big)$ of complex numbers satisfying the convergence conditions

$$\overline{\lim_{m\to\infty}}\ \sqrt[m]{|a_m|} \le 1, \quad \overline{\lim_{m\to\infty}}\ \sqrt[m]{|b_m|} \le 1.$$

Finally, Section 5 characterizes specific invariant holomorphic function spaces on the upper half-plane, emphasizing their transformation properties under some subgroups of the modular group Γ. This section connects modular spherical functions with automorphic forms and modular transformations, highlighting the broader implications of the results.

The paper develops a coherent framework for understanding modular spherical functions, combining tools from representation theory, harmonic analysis, and modular forms to establish new connections and results.

2. Definition

Let G be a connected unimodular Lie group, K a compact subgroup and Γ a discrete subgroup of G of finite covolume. Given a finite dimensional complex left Γ-module and right K-module M let us consider the following vector space

$$C^\infty(G,M)^{\Gamma\times K} = \{F \in C^\infty(G,M) : F(\gamma g k) = \gamma \cdot F(g)\cdot k,\ g \in G, \gamma \in \Gamma, k \in K\}.$$

Proposition 2.1. *Let $D(G)^K$ be the algebra of all differential operators on G left invariant by G and right invariant by K. If $F \in C^\infty(G,M)^{\Gamma\times K}$, then*

$$[DF](\gamma g k) = \gamma \cdot [DF](g)\cdot k$$

for all $g \in G, \gamma \in \Gamma, k \in K$.

Proof. Since D is left invariant by G and $F(\gamma g) = \gamma \cdot F(g)$ we have $[DF](\gamma g) = [D(\gamma \cdot F)](g) = \gamma \cdot [DF](g)$. Similarly, since D is right invariant by K and $F(gk) = F(g)\cdot k$ we have $[DF](gk) = [D(F\cdot k)](g) = [DF](g)\cdot k$. Finally

$$[DF](\gamma g k) = \gamma \cdot [DF](gk) = \gamma \cdot [DF](g)\cdot k. \qquad \square$$

Let (X,χ) be a finite dimensional complex representation of Γ and (V,σ) a finite dimensional complex representation of K such that σ is a multiple of $\pi \in \widehat{K}$. Consider $M = \mathrm{End}(X)\otimes \mathrm{End}(V)$ as a left Γ-module and a right K-module by defining $\gamma\cdot(S\otimes T) = \chi(\gamma)S\otimes T$ and $(S\otimes T)\cdot k = S\otimes T\sigma(k)$. From now on I_X will denote the identity transformation of X and I_V the identity transformation of V. Also we will make use of the following identifications: $\mathrm{End}(X) = \mathrm{End}(X)\otimes I_V$, $\mathrm{End}(V) = I_X \otimes \mathrm{End}(V)$.

Definition 2.2. A modular spherical (operator valued) function is a function $\Phi \in C^\infty(G,M)^{\Gamma\times K}$ such that:

(1) $\Phi(e) = I = I_X\otimes I_V$, the identity transformation of $X\otimes V$,
(2) $[D\Phi](g) = \Phi(g)[D\Phi](e)$, for all $D \in D(G)^K$ and $g\in G$.

Proposition 2.3. *Let $\Phi \in C^\infty(G,M)^{\Gamma\times K}$ be a modular spherical function. Then $D\Phi \in C^\infty(G,M)^{\Gamma\times K}$ for all $D \in D(G)^K$ and*

(1) $\Phi(k) = \sigma(k)$ *for all* $k\in K$ *and* $\Phi(\gamma) = \chi(\gamma)$ *for all* $\gamma \in \Gamma$,
(2) $[D\Phi](e) \in \mathrm{End}(X)\otimes \mathrm{End}_K(V)$ *for all* $D \in D(G)^K$.

Proof. In the first place we point out that $D\Phi \in C^\infty(G,M)^{\Gamma\times K}$ for all $D\in D(G)^K$ was proved in Proposition 2.1. By definition $\Phi(k) = \Phi(ek) = \Phi(e)\sigma(k) = \sigma(k)$ for all $k \in K$ and $\Phi(\gamma) = \Phi(\gamma e) = \chi(\gamma)\cdot\Phi(e) = I$ for all $\gamma\in\Gamma$. Also,

$$[D\Phi](e)\sigma(k) = [D\Phi](k) = \Phi(k)[D\Phi](e) = \sigma(k)[D\Phi](e),$$

the second equality follows from (2) in the definition of modular spherical function. $\square$

Corollary 2.4. *If $\Phi \in C^\infty(G,M)^{\Gamma\times K}$ is a modular spherical function and σ is irreducible, then the eigenvalue $[D\Phi](e) \in \mathrm{End}(X)\otimes I_V$.*

3. The case $(G,\ \Gamma,\ K) = (\mathrm{SL}(2,\mathbb{R}),\ \mathrm{SL}(2,\mathbb{Z}),\ \mathrm{SO}(2))$

Restriction of the Casimir operator of G to the Borel subgroup B

The isotropy subgroup of G at the imaginary unit i is K so the Borel subgroup $B = NA$ can be identified with $\mathcal{H}$, namely the mapping

$$\xi : p = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \longmapsto d^2 i + b$$

is a diffeomorphism of B onto $\mathcal{H}$. Furthermore we introduce in B the coordinate system $\{b, d\}$ by defining $b(p) = b$ and $d(p) = d$ for all $p \in B$.

It is convenient to consider the following one-parameter subgroups of $\mathrm{SL}(2;\mathbb{R})$:

$$\exp(tH) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},\ \exp(tV) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},\ \exp(tW) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

with

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have $[H, V] = 2W,\ [H, W] = 2V,\ [W, V] = 2H$. Then the Casimir of G has the expression $\Omega = H^2 + V^2 - W^2$.

Proposition 3.1. *Assume $F \in C^\infty(G)$ and $F(gk) = \delta_n(k)F(g)$ for all $g \in G, k \in K$, and let $f(b(p), d(p)) = F(p)$ for all $p \in B$. Then*

$$[\Omega F](p) = \left(4d^4 \frac{\partial^2 f}{\partial b^2} + d^2 \frac{\partial^2 f}{\partial d^2} - d \frac{\partial f}{\partial d} - 4nd^2 i \frac{\partial f}{\partial b} \right) (b(p), d(p)). \qquad (1)$$

for all $p \in B$.

Proof. We start computing $[W^2 F](p)$ for $p \in B$:

$$[W^2 F](p) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=0} F(p \exp(sW)) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=0} f(p)\delta_n(\exp(sW)) = -n^2 f(b(p), d(p)). \qquad (2)$$

Also

$$[H^2 F](p) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=0} F(p \exp(sH)) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=0} f(b(s), d(s))$$

where $b(s) = b(p \exp sH),\ d(s) = d(p \exp sH)$ for all $s \in \mathbb{R}$. From

$$p \exp sH = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} de^s & 0 \\ 0 & d^{-1}e^{-s} \end{pmatrix}$$

it follows that $b(s) = b,\ d(s) = de^s$. Hence

$$[H^2 F](p) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=0} f(b, de^s) = d^2 \frac{\partial^2 f}{\partial d^2}(b, d) + d \frac{\partial f}{\partial d}(b, d). \qquad (3)$$

Finally we compute:

$$[V^2 F](p) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=0} F(p \exp(sV)).$$

For all $s \in \mathbb{R}$ we have

$$p \exp sV = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}$$
$$= \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d\cosh s & d\sinh s \\ d^{-1}\sinh s & d^{-1}\cosh s \end{pmatrix}$$
$$= \begin{pmatrix} d\cosh s + bd^{-1}\sinh s & d\sinh s + bd^{-1}\cosh s \\ d^{-1}\sinh s & d^{-1}\cosh s \end{pmatrix}. \tag{4}$$

Let

$$\begin{pmatrix} d\cosh s + bd^{-1}\sinh s & d\sinh s + bd^{-1}\cosh s \\ d^{-1}\sinh s & d^{-1}\cosh s \end{pmatrix} = n(s)a(s)k(s) \tag{5}$$

where $n(s) \in N$, $a(s) \in A$ and $k(s) \in K$. Now consider the functions $b(s) = b(n(s))$, $d = d(a(s))$ defined for all $s \in \mathbb{R}$, and introduce $\theta(s) = \arcsin(k_{12}(s))$ for all $-\pi/2 < s < \pi/2$. Then

$$F(p \exp sV) = F(n(s)a(s)k(s)) = f(b(s), d(s))\delta_n(k(s)) = f(b(s), d(s))e^{in\theta(s)}.$$

Hence

$$\begin{aligned}[V^2F](p) = {}& \frac{\partial^2 f}{\partial b^2}(b(0), d(0))b'(0)^2 + 2\frac{\partial f}{\partial b\,\partial d}(b(0), d(0))b'(0)d'(0) \\ &+ \frac{\partial^2 f}{\partial d^2}(b(0), d(0))d'(0)^2 + \frac{\partial f}{\partial b}(b(0), d(0))b''(0) + \frac{\partial f}{\partial d}(b(0), d(0))d''(0) \\ &+ 2ni\frac{\partial f}{\partial b}(b(0), d(0))b'(0)\theta'(0) + 2ni\frac{\partial f}{\partial d}(b(0), d(0))d'(0)\theta'(0) \\ &+ nif(b(0), d(0))\theta''(0) - n^2 f(b(0), d(0))\theta'(0)^2.\end{aligned} \tag{6}$$

From (4) and (5) we get

$$\begin{pmatrix} d\cosh s + bd^{-1}\sinh s & d\sinh s + bd^{-1}\cosh s \\ d^{-1}\sinh s & d^{-1}\cosh s \end{pmatrix}$$
$$= \begin{pmatrix} d(s)\cos\theta(s) - b(s)d(s)^{-1}\sin\theta(s) & d(s)\sin\theta(s) + b(s)d(s)^{-1}\cos\theta(s) \\ -d(s)^{-1}\sin\theta(s) & d(s)^{-1}\cos\theta(s) \end{pmatrix}. \tag{7}$$

Hence

$$d^{-1}\sinh s = -d(s)^{-1}\sin\theta(s), \quad d^{-1}\cosh s = d(s)^{-1}\cos\theta(s). \tag{8}$$

From the first equation by evaluating at $s = 0$ we get $\theta(0) = 0$ and from the second one we get $d(0) = d$. Now by comparing the entries $(1, 2)$ of both matrices in (7) for $s = 0$ we obtain $b(0) = b$.

If we differentiate the first equation in (8) we get

$$d^{-1}\cosh s = d(s)^{-2}d'(s)\sin\theta(s) - d(s)^{-1}\theta'(s)\cos\theta(s), \tag{9}$$

and by evaluating at $s = 0$ we obtain $d^{-1} = -d^{-1}\theta'(0)$. Hence $\theta'(0) = -1$.

If we differentiate the second equation in (8) we get

$$d^{-1}\sinh s = -d(s)^{-2}d'(s)\cos\theta(s) - d(s)^{-1}\theta'(s)\sin\theta(s), \tag{10}$$

and by evaluating at $s = 0$ we obtain $0 = -d^{-2}d'(0)$, hence $d'(0) = 0$.

By looking at the entries $(1, 1)$ of (7) we get

$$d\cosh s + bd^{-1}\sinh s = d(s)\cos\theta(s) - b(s)d(s)^{-1}\sin\theta(s).$$

If we differentiate this identity we have

$$d\sinh s + bd^{-1}\cosh s = d'(s)\cos\theta(s) - d(s)\theta'(s)\sin\theta(s) - b'(s)d(s)^{-1}\sin\theta(s) \\ + b(s)d(s)^{-2}d'(s)\sin\theta(s) - b(s)d(s)^{-1}\theta'(s)\cos\theta(s). \quad (11)$$

By looking at the entries $(1,2)$ of (7) we get

$$d\sinh s + bd^{-1}\cosh s = d(s)\sin\theta(s) + b(s)d(s)^{-1}\cos\theta(s).$$

If we differentiate this identity we have

$$d\cosh s + bd^{-1}\sinh s = d'(s)\sin\theta(s) + d(s)\theta'(s)\cos\theta(s) + b'(s)d(s)^{-1}\cos\theta(s) \\ - b(s)d(s)^{-2}d'(s)\cos\theta(s) - b(s)d(s)^{-1}\theta'(s)\sin\theta(s). \quad (12)$$

If we evaluate (12) at $s=0$ we obtain $d = -d + b'(0)d^{-1}$, therefore $b'(0) = 2d^2$, because $d'(0) = 0$ and $\theta'(0) = -1$.

If we differentiate (9) we get

$$d^{-1}\sinh s = -2d(s)^{-3}d'(s)^2\sin\theta(s) + d(s)^{-2}d''(s)\sin\theta(s) \\ + 2d(s)^{-2}d'(s)\theta'(s)\cos\theta(s) + d(s)^{-1}\theta''(s)\cos\theta(s) \\ + d(s)^{-1}\theta'(s)^2\sin\theta(s),$$

and evaluating at $s=0$ we obtain $0 = d^{-1}\theta''(0)$, hence $\theta''(0) = 0$.

If we differentiate (10) we obtain

$$d^{-1}\cosh(s) = 2d(s)^{-3}d'(s)^2\cos\theta(s) - d(s)^{-2}d''(s)\cos\theta(s) \\ + 2d(s)^{-2}d'(s)\theta'(s)\sin\theta(s) - d(s)^{-1}\theta''(s)\sin\theta(s) \\ - d(s)^{-1}\theta'(s)^2\cos\theta(s).$$

At $s=0$ we get $d^{-1} = -d^{-2}d''(0) - d^{-1}\theta'(0)^2 = -d^{-2}d''(0) - d^{-1}$, hence $d''(0) = -2d$.

By differentiating (12) we get

$$d\sinh s + bd^{-1}\cosh s \\ = d''(s)\sin\theta(s) + 2d'(s)\theta'(s)\cos\theta(s) + d(s)\theta''(s)\cos\theta(s) \\ - d(s)\theta'(s)^2\sin\theta(s) + b''(s)d(s)^{-1}\cos\theta(s) - 2b'(s)d(s)^{-2}d'(s)\cos\theta(s) \\ - 2b'(s)d(s)^{-1}\theta'(s)\sin\theta(s) + 2b(s)d(s)^{-3}d'(s)^2\cos\theta(s) \\ - b(s)d(s)^{-2}d''(s)\cos\theta(s) + 2b(s)d(s)^{-2}d'(s)\theta'(s)\sin\theta(s) \\ - b(s)d(s)^{-1}\theta''(s)\sin\theta(s) - b(s)d(s)^{-1}\theta'(s)^2\cos\theta(s),$$

and at $s=0$ we obtain

$$bd^{-1} = b''(0)d^{-1} - bd^{-2}d''(0) - bd^{-1} = b''(0)d^{-1} + 2bd^{-2}d - bd^{-1} = b''(0)d^{-1} + bd^{-1}.$$

Therefore $b''(0) = 0$. Summarizing we obtained

$$\begin{array}{lll} \theta(0) = 0, & d(0) = d, & b(0) = b, \\ \theta'(0) = -1, & d'(0) = 0, & b'(0) = 2d^2, \\ \theta''(0) = 0, & d''(0) = -2d, & b''(0) = 0. \end{array}$$

Hence from (6) we get

$$[V^2F](p) = 4d^4\frac{\partial^2 f}{\partial b^2}(b,d) - 2d\frac{\partial f}{\partial d}(b,d) - 4nd^2 i\frac{\partial f}{\partial b}(b,d) - n^2 f(b,d). \quad (13)$$

Recall that $\Omega = H^2 + V^2 - W^2$. Then from (2), (3), and (13) we obtain

$$[\Omega F](p) = 4d^4 \frac{\partial^2 f}{\partial b^2}(b,d) + d^2 \frac{\partial^2 f}{\partial d^2}(b,d) - d\frac{\partial f}{\partial d}(b,d) - 4nd^2 i \frac{\partial f}{\partial b}(b,d). \tag{14}$$

The proposition is proved. ∎

In order to identify the operator in the right hand side of (14) we make the change of variables $x = b$, $y = d^2$, which corresponds to the coordinate system in the Borel subgroup B defined by $p \cdot i = y(p)i + x(p) \in \mathcal{H}$. Then we obtain

Theorem 3.2. *If $F \in C^\infty(G)$ and $F(gk) = \delta_n(k)F(g)$ for all $g \in G, k \in K$, let $f^*(x(p), y(p)) = f(b(p), d(p)) = F(p)$ for all $p \in B$. Then*

$$-\frac{1}{4}[\Omega F](p) = -y^2 \left(\frac{\partial^2 f^*}{\partial x^2} + \frac{\partial^2 f^*}{\partial y^2} \right)(x(p), y(p)) + iny \frac{\partial f^*}{\partial x}(x(p), y(p)).$$

Proof. By definition we have

$$\frac{\partial f}{\partial b} = \frac{\partial f^*}{\partial x}, \qquad \frac{\partial f}{\partial d} = 2d \frac{\partial f^*}{\partial y},$$

and

$$\frac{\partial^2 f}{\partial b^2} = \frac{\partial^2 f^*}{\partial x^2}, \qquad \frac{\partial^2 f}{\partial d \partial b} = 2d \frac{\partial^2 f^*}{\partial y \partial x}, \qquad \frac{\partial^2 f}{\partial d^2} = 2\frac{\partial f^*}{\partial y} + 4d^2 \frac{\partial^2 f^*}{\partial y^2}.$$

From (14) by putting $(x, y) = (x(p), y(p))$ we finally obtain

$$-\frac{1}{4}[\Omega F](p) = -y^2 \left(\frac{\partial^2 f^*}{\partial x^2}(x,y) + \frac{\partial^2 f^*}{\partial y^2}(x,y) \right) + iny \frac{\partial f^*}{\partial x}(x,y).$$ ∎

In the literature (cf. [GH11, Chapter 3, p. 83]) the operator $\Delta_n = -y^2 \Delta + iny \frac{\partial}{\partial x}$ is called the weight n-Laplace operator.

Polynomial eigenfunctions of Δ_n

We start by considering the case $n = 0$ and eigenvalue $\lambda = 0$.

Lemma 3.3. *Let $f = \sum_{p=0}^{n} a_p (x-a)^p (y-b)^{n-p}$. If $n \le 1$, then $\Delta f = 0$. For $n \ge 2$, $\Delta f = 0$ if and only if*

$$a_{p+2}(p+2)(p+1) + a_p(n-p)(n-p-1) = 0, \quad 0 \le p \le n,$$

where $a_{n+1} = a_{n+2} = 0$.

Proposition 3.4. *Let $f_0 = 1$ and*

$$f_n = \sum_{j \ge 0} (-1)^j \binom{n}{2j} (x-a)^{2j} (y-b)^{n-2j},$$

$$h_n = \sum_{j \ge 0} \frac{(-1)^j}{n} \binom{n}{2j+1} (x-a)^{2j+1} (y-b)^{n-2j-1}$$

for all $n \ge 1$. Then $\Delta f_n = 0$ for all $n \ge 0$ and $\Delta h_n = 0$ for all $n \ge 1$. Moreover, $\{f_j : 0 \le j \le n\} \cup \{h_j : 1 \le j \le n\}$ is a basis of all harmonic polynomials of degree less or equal n.

Proof. It is easy to check that f_n and h_n satisfy the conditions of the above lemma. On the other hand, let $\mathbb{C}_n[x,y]$ and $H_n[x,y]$ be, respectively, the linear subspaces of all polynomials of degree less or equal to n and of all harmonic polynomials of degree less or equal to n. It is well known that $\mathbb{C}_n[x,y] = (x^2+y^2)\mathbb{C}_{n-2}[x,y] \oplus H_n[x,y]$. Since $\dim \mathbb{C}_n[x,y] = (n+2)(n+1)/2$ it follows that $\dim H_n[x,y] = 2n+1$. This completes the proof. □

The case $n \neq 0$ and $\lambda = 0$ is more complex.

Lemma 3.5. *Let $f_m = \sum_{p=0}^m a_p x^p y^{m-p}$ be a homogeneous polynomial of degree m. If $m = 0$, then $\Delta_n f_0 = 0$. For $m \geq 1$, $\Delta_n f_m = 0$ if and only if*

$$-(p+2)(p+1)a_{p+2} - (m-p)(m-p-1)a_p + in(p+1)a_{p+1} = 0, \quad 0 \leq p \leq m,$$

where $a_{m+2} = a_{m+1} = 0$.

Finding the homogeneous polynomial $f_m = \sum_{p=0}^m a_p x^p y^{m-p}$ of degree $m \geq 1$ such that $\Delta_n f_m = 0$ is equivalent to solving the following system of linear equations

$$\begin{pmatrix} u_0 & v_0 & w_0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & u_1 & v_1 & w_1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & u_2 & v_2 & w_2 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & u_{m-2} & v_{m-2} & w_{m-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & u_{m-1} & v_{m-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & u_m \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ a_{m-2} \\ a_{m-1} \\ a_m \end{pmatrix} = 0,$$

with $u_p = -(m-p)(m-p-1), v_p = in(p+1), w_p = -(p+2)(p+1)$ for $0 \leq p \leq m$. We first observe that if $\Delta_n f_m = 0$ then $a_m = 0$ because $u_m = u_{m-1} = 0$ and $v_{m-1} \neq 0$. Moreover if $f_m \neq 0$ then $a_{m-1} \neq 0$ so we will choose $a_{m-1} = 1$. Hence the eigenvalues of the above upper tridiagonal matrix are

$$\lambda = -(m-p)(m-p-1), \quad 0 \leq p \leq m-1 \tag{15}$$

all with multiplicity 1.

By choosing $f_0 = 1$ for $1 \leq m \leq 7$ we obtain

$$\begin{aligned} f_0 &= 1, \\ f_1 &= y, \\ f_2 &= \frac{1}{2}iny^2 + xy, \\ f_3 &= -\frac{1}{6}(n^2+2)y^3 + inxy^2 + x^2y, \\ f_4 &= -\frac{1}{24}in(n^2+8)y^4 - \frac{1}{2}(n^2+2)xy^3 + \frac{3}{2}inx^2y^2 + x^3y, \\ f_5 &= \frac{1}{120}(n^4+20n^2+24)y^5 - \frac{1}{6}in(n^2+8)xy^4 - (n^2+2)x^2y^3 + 2inx^3y^2 + x^4y, \\ f_6 &= \frac{1}{720}in(n^4+40n^2+184)y^6 + \frac{1}{24}(n^4+20n^2+24)xy^5 \\ &\quad - \frac{5}{12}in(n^2+8)x^2y^4 - \frac{5}{3}(n^2+2)x^3y^3 + \frac{5}{2}inx^4y^2 + x^5y, \end{aligned}$$

$$f_7 = -\frac{1}{5.040}(n^6+70n^4+784n^2+720)y^7 + \frac{1}{120}in(n^4+40n^2+184)xy^6 + \frac{1}{8}(n^4+20n^2+24)x^2y^5 - \frac{5}{6}in(n^2+8)x^3y^4 - \frac{5}{2}(n^2+2)x^4y^3 + 3inx^5y^2 + x^6y.$$

We observe that for $1 \le m \le 7$

$$f_m = \sum_{j=1}^{m} i^{j-1} c_{m,j} p_{m,j}(n) x^{m-j} y^j,$$

where $c_{m,j} \in \mathbb{R}$ and $p_{m,j}(n)$ is a monic polynomial of degree $j-1$ in the variable n. Also $c_{m,1} = p_{m,1} = 1$.

Proposition 3.6. *Let $n \neq 0$. Let us consider the following homogeneous polynomials f_m of degree $m \ge 1$*

$$f_m = \sum_{j=1}^{m} i^{j-1} \frac{1}{j!}\binom{m-1}{m-j} q_{j-1}(n) x^{m-j} y^j,$$

where $q_j(n)$ is a monic polynomial of degree j in the undetermined n. The sequence of polynomials $(q_j)_{j\ge 0}$ satisfies the following three term recurrence relation:

$$nq_j = q_{j+1} - j(j+1)q_{j-1}, \quad j \ge 0, \quad q_0 = 1.$$

Then $\Delta_n f_m = 0$. Moreover the set $\{f_0\} \cup \{f_m : 1 \le m \le r\}$ is a basis of the linear space $\{f \in \mathbb{C}[x,y] : 0 \le \deg(f) \le r : \Delta_n f = 0\}$.

Proof. For $m \ge 1$ let

$$f_m = \sum_{j=1}^{m} i^{j-1} c_{m,j} p_{m,j}(n) x^{m-j} y^j, \quad c_{m,1} = p_{m,1} = 1.$$

Then $\frac{\partial f_m}{\partial x} = (m-1) f_{m-1}$. In fact, $\frac{\partial f_1}{\partial x} = 0$ and for $m \ge 2$ we have

$$\frac{\partial f_m}{\partial x} = (m-1) \sum_{j=1}^{m-1} i^{j-1}(m-j)(m-1)^{-1} c_{m,j} p_{m,j}(n) x^{m-j-1} y^j.$$

On the other hand

$$\Delta_n\left(\frac{\partial}{\partial x} f_m\right) = \frac{\partial}{\partial x}\big(\Delta_n f_m\big) = 0.$$

Hence $\sum_{j=1}^{m-1} i^{j-1}(m-j)(m-1)^{-1} c_{m,j} p_{m,j}(n) x^{m-j-1} y^j = f_{m-1}$ because it is also homogeneous of degree $m-1$ and the coefficient of $x^{m-2}y$ is 1. Hence

$$\sum_{j=1}^{m-1} i^{j-1} c_{m,j} p_{m,j}(n)(m-j) x^{m-j-1} y^j = (m-1) \sum_{j=1}^{m-1} i^{j-1} c_{m-1,j} p_{m-1,j}(n) x^{m-j-1} y^j,$$

equivalently

$$(m-j) c_{m,j} p_{m,j}(n) = (m-1) c_{m-1,j} p_{m-1,j}(n).$$

for all $1 \le j \le m-1$. Therefore

$$(m-j) c_{m,j} = (m-1) c_{m-1,j}, \qquad p_{m,j}(n) = p_{m-1,j}(n)$$

Hence

$$c_{m,j} = \frac{(m-1)}{(m-j)} c_{m-1,j} = \frac{(m-1)(m-2)}{(m-j)(m-j-1)} c_{m-2,j} = \cdots$$
$$= \frac{(m-1)(m-2)\cdots(m-(m-j))}{(m-j)(m-j-1)\cdots 1} c_{j,j} = c_{j,j}\binom{m-1}{m-j},$$

$$p_{m,j}(n) = p_{m-1.j}(n) = \cdots = p_{j,j}(n).$$

Let $p_j(n) = p_{j,j}(n)$. The recurrence relation satisfied by the sequence of polynomials $(p_j)_{j\geq 0}$ is consequence of Lemma 3.5: If $f_m = \sum_{p=0}^m a_p x^p y^{m-p}$, then

$$-(p+2)(p+1)a_{p+2} - (m-p)(m-p-1)a_p + in(p+1)a_{p+1} = 0, \quad 0 \leq p \leq m,$$

where $a_{m+2} = a_{m+1} = a_m = 0$. By changing the summation index p by $j = m-p$ we get $f_m = \sum_{j=1}^m a_{m-j} x^{m-j} y^j = \sum_{j=1}^m i^{j-1} c_{m,j} p_{m,j}(n) x^{m-j} y^j$. Hence $a_{m-j} = i^{j-1} c_{m,j} p_{m,j}(n)$ and

$$-(m-j+2)(m-j+1)a_{m-j+2} - j(j-1)a_{m-j} + in(m-j+1)a_{m-j+1} = 0, \quad 0 \leq p \leq m.$$

Therefore

$$(m-j+2)(m-j+1)c_{m,j-2}p_{j-2}(n) - j(j-1)c_{m,j}p_j(n) + n(m-j+1)c_{m,j-1}p_{j-1}(n) = 0. \quad (16)$$

By induction we will first prove that $c_{k,k} = 1/k!$ for $1 \leq k \leq m-1$. For $m = 2$ we have $f_2 = xy + ic_{2,2}p_{2,2}(n)y^2$. From

$$[\Delta_n f_2](x,y) = -2ic_{2,2}p_{2,2}(n)y^2 + iny^2 = 0.$$

Hence $-2c_{2,2} + 1 = 0$. Now assume $c_{k,k} = 1/k!$ for $1 \leq k \leq m-2$. From the inductive hypothesis we obtain

$$[\Delta_n f_{k+1,k+1}](x,y) = -\sum_{j=1}^{k-1} i^{j-1} c_{k+1,j} p_{k+1,j}(n)(k+1-j)(k-j)x^{k-j-1}y^{j+2}$$
$$- \sum_{j=2}^{k+1} i^{j-1} c_{k+1,j} p_{k+1,j}(n) j(j-1) x^{k+1-j} y^j$$
$$+ in \sum_{j=1}^{k} i^{j-1} c_{k+1,j} p_{k+1,j}(n)(k+1-j)x^{k-j}y^{j+1} = 0.$$

By looking at the coefficient of y^{k+1} we get

$$-2i^{k-2} c_{k+1,k-1} p_{k+1,k-1} - i^k (k+1)k c_{k+1,k+1} p_{k+1,k+1} + i^k c_{k+1,k} n p_{k+1,k} = 0.$$

Hence

$$2c_{k+1,k-1}p_{k-1} - (k+1)kc_{k+1,k+1}p_{k+1} + c_{k+1,k}np_k = 0.$$

Since

$$c_{k+1,k-1} = \frac{k(k-1)}{2} c_{k-1,k-1} = \frac{k}{2(k-2)!} \quad \text{and} \quad c_{k+1,k} = \frac{k}{1} c_{k,k} = \frac{1}{(k-1)!}$$

we obtain

$$\frac{k}{(k-2)!} p_{k-1} - (k+1)kc_{k+1,k+1}p_{k+1} + \frac{1}{(k-1)!} np_k = 0.$$

By using (17)

$$\frac{k}{(k-2)!}p_{k-1} - (k+1)kc_{k+1,k+1}\Big(k(k-1)p_{k-1} + np_k\Big) + \frac{1}{(k-1)!}np_k = 0$$

The coefficient of n^k is $-(k+1)kc_{k+1,k+1} + 1/(k-1)!$, thus $c_{k+1,k+1} = 1/(k+1)!$ completing the inductive step.

Thus, going back to (16) we obtain

$$\frac{(m-j+2)(m-j+1)}{(j-2)!}\binom{m-1}{m-j+2}p_{j-2}(n) - \frac{j(j-1)}{j!}\binom{m-1}{m-j}p_j(n) + \frac{n(m-j+1)}{(j-1)!}\binom{m-1}{m-j+1}p_{j-1}(n) = 0.$$

Hence

$$\frac{(m-j+2)(m-j+1)}{(j-2)!(m-j+2)!(j-3)!}p_{j-2}(n) - \frac{j(j-1)}{j!(m-j)!(j-1)!}p_j(n) + \frac{n(m-j+1)}{(j-1)!(m-j+1)!(j-2)!}p_{j-1}(n) = 0.$$

By simplifying we obtain

$$(j-1)(j-2)p_{j-2}(n) - p_j(n) + np_{j-1}(n) = 0. \tag{17}$$

The sequence of polynomials $q_j(n) = p_{j+1}(n)$ $(j \geq 0)$ satisfies the following three term recurrence relation:

$$nq_j = q_{j+1} - j(j+1)q_{j-1}, \quad j \geq 0, \quad q_0 = 1.$$

Finally:

$$f_m = \sum_{j=1}^{m} i^{j-1}\frac{1}{j!}\binom{m-1}{m-j}q_{j-1}(n)x^{m-j}y^j.$$

The last assertion in the statement of the proposition follows from (15). The proof is complete. ☐

A. Grünbaum [Grü25] found a similar expression of f_m in terms of another sequence $\tilde{q}_j$ of orthogonal polynomials. In fact he obtained

$$\tilde{f}_m(x,y) = \sum_{j=1}^{m}\frac{1}{(j-1)!}\binom{k}{j}\tilde{q}_{j-1}(in)x^{m-j}y^j, \qquad \text{for all } m \geq 1,$$

where $\tilde{q}_0 = 1, \tilde{q}_1(n) = n$ and $\tilde{q}_j(n) = n\tilde{q}_{j-1}(n) - (j-1)j\tilde{q}_{j-2}(n)$ for all $j \geq 2$. By Favard's theorem it follows that $(\tilde{q}_j)_{j\geq 0}$ is an orthogonal sequence with respect to a weight measure $d\mu(x)$ on the real line. Furthermore it is worth to observe that $\tilde{f}_m(x,y) = mf_m(x,y)$ and $\tilde{q}_j(in) = i^j q_j(n)$.

In Proposition 3.4 we found a basis of all homogeneous harmonic polynomials. Now we start to consider the polynomial eigenfunctions of the differential operator $-y^2\Delta$ with an eigenvalue $\lambda \neq 0$.

Lemma 3.7. *Let $f = \sum_{p=0}^{n} a_p x^p y^{n-p}$. Then $-y^2\Delta f = \lambda f$ if and only if*

$$a_{p+2}(p+2)(p+1) + a_p\Big((n-p)(n-p-1) + \lambda\Big) = 0, \quad 0 \leq p \leq n, \tag{18}$$

where $a_{n+1} = a_{n+2} = 0$.

In the following definition we introduce some homogeneous polynomials of degree n that will be eigenvectors of the differential operator $-y^2\Delta$.

Definition 3.8. For $2 \le j \le n$ let:

(1) If $n-j=2k$ we define

$$f_{j,n}(x,y)=\sum_{p=0}^{k} a_{2p,n}x^{2p}y^{n-2p},$$

where the sequence $(a_{0,n}, a_{2,n}, \dots, a_{n-j,n})$ is determined by the recurrence relation

$$a_{p+2,n}(p+2)(p+1)+a_{p,n}\Big((n-p)(n-p-1)-j(j-1)\Big)=0, \quad a_{0,n}=1.$$

(2) If $n-j=2k+1$ we define

$$h_{j,n}(x,y)=\sum_{p=0}^{k} a_{2p+1,n}x^{2p+1}y^{n-1-2p},$$

where the sequence $(a_{1,n}, a_{3,n}, \dots, a_{n-j,n})$ is determined by the recurrence relation

$$a_{p+2,n}(p+2)(p+1)+a_{p,n}\Big((n-p)(n-p-1)-j(j-1)\Big)=0, \quad a_{1,n}=1.$$

Proposition 3.9. *The possible nonzero eigenvalues of $-y^2\Delta$ in the space of homogeneous polynomials of degree $n \ge 2$ are $\lambda_j=-j(j-1)$, $2 \le j \le n$, all with multiplicity one and corresponding eigenvectors $f_{j,n}$ or $h_{j,n}$ according to $n-j=2k$ or $n-j=2k+1$, respectively. Explicitly, we have*

$$a_{2p,n}=\frac{(-1)^p}{(2p)!}\prod_{s=1}^{p}\Big((n-2s+2)(n-2s+1)-j(j-1)\Big), \quad p\ge 1, \quad a_0=1; \quad (19)$$

$$a_{2p+1,n}=\frac{(-1)^p}{(2p+1)!}\prod_{s=1}^{p}\Big((n-2s+1)(n-2s)-j(j-1)\Big), \quad p\ge 1, \quad a_1=1. \quad (20)$$

Proof. From (18) for $p=n$ we get $a_n(\lambda)=0$. Hence $a_n=0$. Now for $p=n-1$ we obtain $a_{n-1}(\lambda)=0$. Therefore $a_{n-1}=0$. If $p=n-2$ from Lemma 3.7 we get $a_{n-2}(2+\lambda)=0$, hence $\lambda=-2$ or $a_{n-2}=0$.

Assume $\lambda=-2$. We first observe that the following coefficient of the recurrence relation (18) satisfies

$$(n-p)(n-p-1)-2=(n-p-2)(n-p+1)\ge 4, \qquad 0\le p\le n-3.$$

Hence, from $a_{n-1}=0$, when $n=2k$ it follows $a_{n-5}=\dots=a_1=0$ and the sequence $(a_0, a_2, \dots, a_{n-2})$ is determined by the recurrence relation

$$a_{p+2}(p+2)(p+1)+a_p\Big((n-p)(n-p-1)-2\Big)=0$$

and the value of a_0. If we define

$$f_{2,n}(x,y)=\sum_{p=0}^{k-1} a_{2p}x^{2p}y^{n-2p}, \quad a_0=1,$$

then $f = a_0 f_{2,n}$. When $n = 2k+1$, from $a_{n-1} = 0$, it follows $a_{n-3} = a_{n-5} = \cdots = a_0 = 0$ and the sequence $(a_1, a_3, \ldots, a_{n-2})$ is determined by the recurrence relation

$$a_{p+2}(p+2)(p+1) + a_p\Big((n-p)(n-p-1) - 2\Big) = 0$$

and the value of a_1 If we define

$$h_{2,n}(x,y) = \sum_{p=0}^{k-1} a_{2p+1} x^{2p+1} y^{n-1-2p}, \quad a_1 = 1.$$

then $f = a_1 h_{2,n}$.

Now we generalize the above argument by considering $\lambda_j = -j(j-1)$, $2 \leq j \leq n$, and $f = \sum_{p=0}^{n-j} a_p x^p y^{n-p}$. Then $-y^2 \Delta f = \lambda_j f$ if and only if

$$a_{p+2}(p+2)(p+1) + a_p\Big((n-p)(n-p-1) + \lambda_j\Big) = 0, \quad 0 \leq p \leq n-j. \tag{21}$$

We first observe that the coefficient

$$(n-p)(n-p-1) + \lambda_j = (n-p-j)(n-p+j-1) \geq 2j, \quad 0 \leq p \leq n-j-3.$$

Hence, from $a_{n-j+1} = 0$, when $n - j = 2k$ it follows that $a_{n-j-1} = a_{n-j-3} = \cdots = a_1 = 0$ and the sequence $(a_0, a_2, \ldots, a_{n-j})$ is determined by the recurrence relation

$$a_{p+2}(p+2)(p+1) + a_p\Big((n-p)(n-p-1) + \lambda_j\Big) = 0$$

and the value of a_0. If we define

$$f_{j,n}(x,y) = \sum_{p=0}^{k} a_{2p,n} x^{2p} y^{n-2p}, \quad a_{0,n} = 1,$$

with

$$a_{2p,n} = \frac{(-1)^p}{(2p)!} \prod_{s=1}^{p} \Big((n-2s+2)(n-2s+1) - j(j-1)\Big), \quad p \geq 1, \tag{22}$$

then $f = a_0 f_{j,n}$.

When $n - j = 2k+1$, from $a_{n-j+1} = 0$, it follows that $a_{n-j-1} = a_{n-j-3} = \cdots = a_0 = 0$ and the sequence $(a_1, a_3, \ldots, a_{n-j})$ is determined by the recurrence relation

$$a_{p+2}(p+2)(p+1) + a_p\Big((n-p)(n-p-1) - j(j-1)\Big) = 0$$

and the value of a_1. If we define

$$h_{j,n}(x,y) = \sum_{p=0}^{k} a_{2p+1} x^{2p+1} y^{n-1-2p}, \quad a_1 = 1,$$

with

$$a_{2p+1,n} = \frac{(-1)^p}{(2p+1)!} \prod_{s=1}^{p} \Big((n-2s+1)(n-2s) - j(j-1)\Big), \quad p \geq 1, \tag{23}$$

then $f = a_1 h_{j,n}$.

Since we found $n-1$ different nonzero eigenvalues and the same number of corresponding nonzero eigenvectors plus two linearly independent eigenvectors of 0 eigenvalue, see Proposition 3.4, we proved that the linear operator defined by $-y^2\Delta$ in the space of homogeneous polynomials of degree $n \geq 2$ is diagonalizable with eigenvalues: 0 with multiplicity 2 and $j(j-1)$ for $2 \leq j \leq n$ with multiplicity 1. The proposition is proved. $\square$

The remaining case to be considered is to find the eigenvectors of Δ_n $n \neq 0$ of nonzero eigenvalues.

Lemma 3.10. *Let $f = \sum_{p=0}^{m} a_p x^p y^{m-p}$. Then $\Delta_n f = \lambda f$ $(\lambda \neq 0)$ if and only if*

$$a_{p+2}(p+2)(p+1) + a_p\Big((m-p)(m-p-1) + \lambda\Big) - ina_{p+1}(p+1) = 0, \qquad 0 \leq p \leq m \quad (24)$$

where $a_{m+1} = a_{m+2} = 0$. Moreover, the eigenvalues are

$$\lambda_k = -(k+1)k, \quad 0 \leq k \leq m-1 \qquad (25)$$

all with multiplicity 1.

Proof. Finding the homogeneous polynomial $f_m = \sum_{p=0}^{m} a_p x^p y^{m-p}$ of degree m such that $\Delta_n f_m = \lambda f_m$ is equivalent to solve the following system of linear equations

$$\begin{pmatrix} u_0 & v_0 & w_0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & u_1 & v_1 & w_1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & u_2 & v_2 & w_2 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & u_{m-2} & v_{m-2} & w_{m-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & u_{m-1} & v_{m-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & u_m \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ a_{m-2} \\ a_{m-1} \\ a_m \end{pmatrix} = \lambda \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ a_{m-2} \\ a_{m-1} \\ a_m \end{pmatrix},$$

with $u_p = -(m-p)(m-p-1)$, $v_p = in(p+1)$, $w_p = -(p+2)(p+1)$ for $0 \leq p \leq m$. We first observe that if $\Delta_n f_m = \lambda f_m$, then $m \geq 2$ because $\lambda \neq 0$. Besides $a_m = a_{m-1} = 0$ because $u_m = u_{m-1} = 0$. Hence the eigenvalues of the above upper tridiagonal matrix are

$$\lambda_k = u_{m-k-1} = -(k+1)k, \quad 0 \leq k \leq m-1$$

all with multiplicity 1. The proof is complete. □

Proposition 3.11. *The possible nonzero eigenvalues of Δ_n in the space of homogeneous polynomials of degree $m \geq 2$ are $\lambda_k = -(k+1)k$, $1 \leq k \leq m-1$, all with multiplicity one and corresponding eigenvectors*

$$f_{k,m} = \sum_{p=0}^{m-k-1} a_p x^p y^{m-p}$$

where the sequence $(a_0, a_1, \cdots, a_{m-k-1})$ is defined by the recurrence relation

$$a_{p+2}(p+2)(p+1) + a_p\Big((m-p)(m-p-1) - (k+1)k\Big) - ina_{p+1}(p+1) = 0, \quad a_0 = 1$$

for $0 \leq p \leq m-k-1$ and $a_{m,m-k} = a_{m,m-k+1} = 0$.

Proof. From Lemma 3.10 for $p = m$ we get $a_m\lambda = 0$. Hence $a_m = 0$. Now for $p = m-1$ we obtain $a_{m-1}\lambda = 0$. Therefore $a_{m-1} = 0$. If $p = m-2$ we get $a_{m-2}(2+\lambda) = 0$, hence $\lambda = -2$ or $a_{m-2} = 0$.

Assume $\lambda = -2$. We first observe that the following coefficient of the recurrence relation (24) satisfies

$$(m-p)(m-p-1) - 2 = (m-p-2)(m-p+1) \geq 4, \quad 0 \leq p \leq m-3.$$

Hence, if $f = \sum_{p=0}^{m-2} a_p x^p y^{m-p} \neq 0$ is a solution of $\Delta_n f = \lambda f$ the sequence $(a_0, a_1, \ldots, a_{m-2})$ is determined by the recurrence relation

$$a_{p+2}(p+2)(p+1) + a_p\Big((m-p)(m-p-1)+\lambda\Big) - ina_{p+1}(p+1) = 0 \qquad (26)$$

and the value of a_0.

We generalize this result by assuming that $a_m = a_{m-1} = \cdots = a_{m-k} = 0$ and $a_{m-k-1} \neq 0$ for $2 \leq k \leq m-1$. From (26) and $p = m-k-1$ we obtain $\lambda = \lambda_k = -(k+1)k$. By taking this value of λ we observe that the following coefficient of the recurrence relation (26) satisfies

$$(m-p)(m-p-1) - (k+1)k \geq 2(k+1), \quad 0 \leq p \leq m-k-2.$$

Hence, if $f = \sum_{p=0}^{m-2} a_p x^p y^{m-p} \neq 0$ is a solution of $\Delta_n f = \lambda_k f$ the sequence $(a_0, a_2, \ldots, a_{m-2})$ is determined by the recurrence relation

$$a_{p+2}(p+2)(p+1) + a_p\Big((m-p)(m-p-1)+\lambda_k\Big) - ina_{p+1}(p+1) = 0 \qquad (27)$$

and the value of a_0. We choose $a_{m-k-1} = 1$ and define

$$f_{k,m} = \sum_{p=0}^{m-k-1} a_p x^p y^{m-p}$$

where $(a_0, a_1, \cdots, a_{m-k-1})$ is defined by (27) for $0 \leq p \leq m-k-1$, $1 \leq k \leq m-1$ with $a_{m-k} = a_{m-k+1} = 0$. □

For $2 \leq m \leq 5$ and $\lambda_1 = -2$ we obtain

$$f_{1,2} = y^2,$$
$$f_{1,3} = \frac{1}{4} iny^3 + xy^2,$$
$$f_{1,4} = -\frac{1}{20}(n^2+4)y^4 + \frac{1}{2} inxy^3 + x^2y^2,$$
$$f_{1,5} = -\frac{i}{120}(n^3+14n)y^5 - \frac{3}{20}(n^2+4)xy^4 + \frac{3}{4} inx^2y^3 + x^3y^2.$$

We observe that for $2 \leq m \leq 5$

$$f_{1,m} = \sum_{p=0}^{m-2} i^p c_{m,p} p_{m,p}(n) x^{m-p-2} y^{p+2},$$

where $c_{m,p} \in \mathbb{R}$ and $p_{m,p}(n)$ is a monic polynomial of degree p in the variable n. Besides $c_{m,0} = p_{m,0} = 1$.

For $3 \leq m \leq 6$ and $\lambda_2 = -6$ we obtain

$$f_{2,3} = y^3,$$
$$f_{2,4} = \frac{1}{6} iny^4 + xy^3,$$
$$f_{2,5} = -\frac{1}{42}(n^2+6)y^5 + \frac{1}{3} inxy^4 + x^2y^3,$$
$$f_{2,6} = -\frac{1}{336} i(n^3+20n)y^6 - \frac{1}{14}(n^2+6)xy^5 + \frac{1}{2} inx^2y^4 + x^3y^3.$$

We observe that for $3 \le m \le 6$

$$f_{2,m} = \sum_{p=0}^{m-3} (-i)^p c_{m,p} p_{m,p}(n) x^{m-p-3} y^{p+3},$$

where $c_{m,p} \in \mathbb{R}$ and $p_{m,p}(n)$ is a monic polynomial of degree p in the variable n. Besides $c_{m,0} = p_{m,0} = 1$.

For $4 \le m \le 7$ and $\lambda_3 = -12$ we obtain

$$\begin{aligned}
f_{3,4} &= y^4,\\
f_{3,5} &= \frac{1}{8} iny^5 + xy^4,\\
f_{3,6} &= -\frac{1}{72}(n^2+8)y^6 + \frac{1}{4} inxy^5 + x^2y^4,\\
f_{3,7} &= -\frac{1}{720} i(n^3+26n)y^7 - \frac{1}{24}(n^2+8)xy^6 + \frac{3}{8} inx^2y^5 + x^3y^4.
\end{aligned}$$

We observe that for $4 \le m \le 7$

$$f_{3,m} = \sum_{p=0}^{m-4} i^p c_{m,p} p_{m,p}(n) x^{m-p-4} y^{p+4},$$

where $c_{m,p} \in \mathbb{R}$ and $p_{m,p}(n)$ is a monic polynomial of degree p in the variable n. Besides $c_{m,0} = p_{m,0} = 1$.

Proposition 3.12. *Let n, k be nonzero integers, $n \neq 0, k \ge 0$ and $\lambda_k = -(k+1)k$. Let us consider the following homogeneous polynomials $f_{k,m} \in \mathbb{C}[x,y]$ of degree $m \ge 2$*

$$f_{k,m} = \sum_{j=0}^{m-k-1} \frac{i^j}{j!} \binom{m-k-1}{j} \binom{2k+j+1}{j}^{-1} p_j(n) x^{m-k-j-1} y^{k+j+1}$$

where $p_j(n)$ is a monic polynomial of degree j in the undetermined n. The sequence of polynomials $(p_j)_{j\ge 0}$ satisfies the following three term recurrence relation:

$$np_j = p_{j+1} - j(2k+j+1)p_{j-1}, \quad j \ge 0, \quad p_0 = 1.$$

Then $\Delta_n f_{k,m} = -k(k+1) f_{k,m}$, $k \ge 1$. Moreover the set

$$\{f_0 = 1, f_1 = y\} \cup \{f_{k,m} : 1 \le k \le m-1\}$$

is a basis of the linear space $\{f \in \mathbb{C}[x,y] : 0 \le \deg(f) \le m : \Delta_n f = \lambda f, \lambda \in \mathbb{C}\}$.

Proof. For $1 \le k \le m-1$, let $\lambda_k = -k(k+1)$ and

$$f_{k,m} = \sum_{p=0}^{m-k-1} i^p c_{m,p} p_{m,p}(n) x^{m-k-p-1} y^{k+p+1}, \quad c_{m,0} = p_{m,0} = 1,$$

where $c_{m,p} \in \mathbb{R}$ and $p_{m,p}(n)$ is a monic polynomial of degree p in the variable n.

If $\Delta_n f_{k,m} = \lambda_k f_{k,m}$, then $\frac{\partial f_{k,m}}{\partial x} = (m-k-1) f_{k,m-1}$. In fact, $\frac{\partial f_{m-1,m}}{\partial x} = 0$ and for $k < m-1$ we have

$$\frac{\partial f_{k,m}}{\partial x} = (m-k-1) \sum_{p=0}^{m-k-2} i^p c_{m,p} p_{m,p}(n)(m-k-p-1)(m-k-1)^{-1} \times x^{m-k-p-2} y^{k+p+1}.$$

On the other hand

$$\Delta_n\left(\frac{\partial}{\partial x}f_{k,m}\right)=\frac{\partial}{\partial x}\big(\Delta_n f_{k,m}\big)=\lambda_k\frac{\partial}{\partial x}f_{k,m}.$$

Hence

$$\sum_{p=0}^{m-k-2} i^p c_{m,p}p_{m,p}(n)(m-k-p-1)(m-k-1)^{-1}x^{m-k-p-2}y^{k+p+1}=f_{k,m-1}$$

because it is also homogeneous of degree $m-1$ and the coefficient of $x^{m-k-2}y^{k+1}$ is 1. Therefore

$$\sum_{p=0}^{m-k-2} i^p c_{m,p}p_{m,p}(n)(m-k-p-1)(m-k-1)^{-1}x^{m-k-p-2}y^{k+p+1}$$
$$=f_{k,m-1}=\sum_{p=0}^{m-k-2} i^p c_{m-1,p}p_{m-1,p}(n)x^{m-k-p-2}y^{k+p+1}.$$

Equivalently

$$(m-k-p-1)c_{m,p}p_{m,p}(n)=(m-k-1)c_{m-1,p}p_{m-1,p}(n).$$

for all $0\le p\le m-k-2$. Therefore

$$(m-k-p-1)c_{m,p}=(m-k-1)c_{m-1,p},\qquad p_{m,p}(n)=p_{m-1,p}(n).$$

Hence

$$\begin{aligned}c_{m,p}&=\frac{(m-k-1)}{(m-k-p-1)}c_{m-1,p}=\frac{(m-k-1)(m-k-2)}{(m-k-p-1)(m-k-p-2)}c_{m-2,p}=\cdots\\&=\frac{(m-k-1)(m-k-2)\cdots(m-k-(m-k-p-1))}{(m-k-p-1)(m-k-p-2)\cdots 1}c_{k+p+1,p}\\&=c_{k+p+1,p}\binom{m-k-1}{p},\end{aligned}$$
$$p_{m,p}(n)=p_{m-1.p}(n)=\cdots=p_{p,p}(n).$$

Let $p_p(n)=p_{p,p}(n)$. The recurrence relation satisfied by the sequence of polynomials $(p_p)_{p\ge0}$ is a consequence of (24): If $f_{k,m}=\sum_{p=0}^{m-k-1}a_px^py^{m-p}$ satisfies $\Delta_nf_{k,m}=\lambda_kf_{k,m}$, $\lambda_k=-(k+1)k$, then for $0\le p\le m-k-1$ we have

$$-(p+2)(p+1)a_{p+2}-\Big((m-p)(m-p-1)+\lambda_k\Big)a_p+in(p+1)a_{p+1}=0$$

where $a_{m-k+1}=a_{m-k}=0$. From

$$\sum_{p=0}^{m-k-1}a_px^py^{m-p}=\sum_{p=0}^{m-k-1}i^{m-k-p-1}c_{m,m-k-p-1}p_{m-k-p-1}(n)x^py^{m-p}$$

we obtain $a_p=i^{m-k-p-1}c_{m,m-k-p-1}p_{m-k-p-1}(n)$. Therefore

$$\begin{aligned}&-(p+2)(p+1)i^{m-k-p-3}c_{m,m-k-p-3}p_{m-k-p-3}(n)\\&\qquad-\Big((m-p)(m-p-1)+\lambda_k\Big)i^{m-k-p-1}c_{m,m-k-p-1}p_{m-k-p-1}(n)\\&\qquad\qquad+in(p+1)i^{m-k-p-2}c_{m,m-k-p-2}p_{m-k-p-2}(n)=0.\end{aligned}$$

By putting $j = m-k-p-1 \geq 1$ we get $(m-p)(m-p-1)-(k+1)k = j(2k+j+1)$, $p+1 = m-k-j$ and

$$(m-k-j+1)(m-k-j)c_{m,j-2}p_{j-2}(n) - j(2k+j+1)c_{m,j}p_j(n) \\ + (m-k-j)c_{m,j-1}np_{j-1}(n) = 0. \quad (28)$$

From the coefficient of n^j we get: $j(2k+j+1)c_{m,j} = (m-k-j)c_{m,j-1}$. Thus

$$\begin{aligned} c_{m,j} &= \frac{(m-k-j)}{j(2k+j+1)}c_{m,j-1} = \frac{(m-k-j)(m-k-j+1)}{j(2k+j+1)(j-1)(2k+j)}c_{m,j-2} \\ &= \cdots = \frac{(m-k-j)(m-k-j+1)\cdots(m-k-1)}{j(2k+j+1)(j-1)(2k+j)\cdots 1(2k+2)}c_{m,0} \\ &= \frac{(m-k-1)!(2k+1)!}{j!(m-k-j-1)!(2k+j+1)!} = \frac{1}{j!}\binom{m-k-1}{j}\binom{2k+j+1}{j}^{-1}. \end{aligned}$$

Going back to (28) we obtain

$$\frac{(m-k-j+1)(m-k-j)}{(j-2)!}\binom{m-k-1}{j-2}\binom{2k+j-1}{j-2}^{-1}p_{j-2}(n) \\ - \frac{(2k+j+1)}{(j-1)!}\binom{m-k-1}{j}\binom{2k+j+1}{j}^{-1}p_j(n) \\ + \frac{(m-k-j)}{(j-1)!}\binom{m-k-1}{j-1}\binom{2k+j}{j-1}^{-1}np_{j-1}(n) = 0.$$

By simplifying we obtain

$$(j-1)(2k+j)p_{j-2}(n) - p_j(n) + np_{j-1}(n) = 0$$

for all $j \geq 1$.

The sequence of monic polynomials $(p_j)_{j\geq 0}$ in the variable n satisfies the following three term recurrence relation:

$$np_j = p_{j+1} - j(2k+j+1)p_{j-1}, \quad j \geq 0, \quad p_0 = 1.$$

Finally:

$$f_{k,m} = \sum_{j=0}^{m-k-1} \frac{\left((-1)^{k-1}i\right)^j}{j!}\binom{m-k-1}{j}\binom{2k+j+1}{j}^{-1}p_j(n)x^{m-k-j-1}y^{k+j+1}.$$

The last assertion in the statement of the proposition follows from (25). The proof is complete. □

4. Analytic eigenfunctions of Δ on the upper half plane

The goal of this section is to find a good parametrization of all real analytic eigenfunctions of Δ on $\mathcal{H} = \{(x,y) \in \mathbb{R}^2 : y > 0\}$. It is first convenient to recall some basic facts. For any power series $\sum_{m\geq 0} a_m(z-z_o)^m$ there exists a number R, $0 \leq R \leq \infty$, called the radius of convergence, with the following properties:

(1) The series converges absolutely for every $z \in \mathbb{C}$ with $|z-z_0| < R$. The convergence is uniform in every closed disc $|z-z_0| \leq \rho < R$.
(2) If $|z-z_0| > R$ the terms of the series are unbounded, and the series is consequently divergent.

(3) In $|z-z_0|<R$, the sum of the series is an analytic function. The derivatives can be obtained by termwise differentiation, and the derived series has the same radius of convergence.
(4) The radius of convergence is given by Cauchy–Hadamard formula

$$\frac{1}{R}=\overline{\lim_{m\to\infty}}\ \sqrt[m]{|a_m|}.$$

(5) If $f(z)$ is analytic in an open set $D\subset\mathbb{C}$ containing z_o, then the Taylor representation $f(z)=\sum_{m\geq 0}\frac{f^{(m)}(z_0)}{m!}(z-z_0)$ is valid in the largest open disc of center z_0 contained in D.

The situation is quite different if we consider a power series $\sum_{m\geq 0}a_m(x-x_o)^m$, $x,x_0\in\mathbb{R}$. For example the function

$$f(x)=\frac{1}{1+x^2}=1-x^2+x^4-x^6-\cdots$$

is analytic in the whole real line but the radius of convergence of the series is one.

We start by considering the case $n=k=0$. According to Proposition 3.4 we know that the homogeneous harmonic polynomials of degree m are linear combinations of f_m and h_m, where $f_0=1$ and

$$f_m=\sum_{j\geq 0}(-1)^j\binom{m}{2j}x^{2j}y^{m-2j},\quad h_m=\sum_{j\geq 0}\frac{(-1)^j}{m}\binom{m}{2j+1}x^{2j+1}y^{m-2j-1}$$

for all $m\geq 1$.

Lemma 4.1. *If $f\in H(\mathcal{H})$ is harmonic, then*

$$f(x,y)=a_0+\sum_{m\geq 1}\Big(a_mf_m(x,y-1)+b_mh_m(x,y-1)\Big) \tag{29}$$

where the power series is absolutely convergent in an open polydisc P centered at $(0,1)$. The sequences of complex numbers $(a_m)_{m\geq 0}$, $(b_m)_{m\geq 1}$ are unique and satisfy

$$\frac{1}{R}=\overline{\lim_{m\to\infty}}\ \sqrt[m]{|a_m|}<\infty,\qquad \frac{1}{R'}=\overline{\lim_{m\to\infty}}\ \sqrt[m]{|b_m|}<\infty.$$

Proof. The expression (29) follows from Proposition 3.4. By hypothesis $f(x,y)=\sum_{p,q\geq 0}a_{p,q}x^p(y-1)^q$ where the power series is absolutely convergent in an open polydisc P centered at $(0,1)$. Then

$$f(x,y)=a_0+\sum_{m\geq 1}\sum_{p+q=m}a_{p,q}x^p(y-1)^q=a_0+\sum_{m\geq 1}\Big(a_mf_m(x,y-1)+b_mh_m(x,y-1)\Big),$$

since

$$\Delta f=\Delta\sum_{p+q=m}\sum_{p+q=m}x^p(y-1)^q=\sum_{p+q=m}\Delta\sum_{p+q=m}x^p(y-1)^q=0.$$

Moreover both series $\sum_{m\geq 1}a_mf_m(x,y-1)$ and $\sum_{m\geq 1}b_mh_m(x,y-1)$ are absolutely convergent in P. By contradiction suppose that $\overline{\lim}_{m\to\infty}\sqrt[m]{|a_m|}=\infty$. Then either $\overline{\lim}_{m\to\infty}\sqrt[2m]{|a_{2m}|}=\infty$ or $\overline{\lim}_{m\to\infty}\sqrt[2m+1]{|a_{2m+1}|}=\infty$. If $\overline{\lim}_{m\to\infty}\sqrt[2m]{|a_{2m}|}=\infty$, for $y=1$ and $x\neq 0$ we have

$$\sum_{m\geq 1}|a_{2m}f_{2m}(x,0)|=\sum_{m\geq 1}|a_{2m}x^{2m}|=\infty,$$

because the radius of convergence of this power series is 0. But $\sum_{n\geq 1} a_{2m} f_{2m}(x,y)$ is absolutely convergent in the polydisc P, which leads to a contradiction.

Now we observe that

$$\sum_{m\geq 1} a_{2m+1} f_{2m+1}(x,y) = y \sum_{m\geq 1} a_{2m+1} \sum_{j\geq 0} (-1)^j \binom{2m+1}{2j} x^{2j} y^{2m-2j},$$

hence $\sum_{m\geq 1} a_{2m+1} \sum_{j\geq 0}(-1)^j \binom{2m+1}{2j} x^{2j} y^{2m-2j}$ is also absolutely convergent in P.

From $\overline{\lim}_{m\to\infty} \sqrt[2m1]{|a_{2m+1}|} = \infty$ and for $y = 1$ and $x \neq 0$ we get

$$\sum_{m\geq 1} |a_{2m+1} f_{2m+1}(x,0)| = \sum_{m\geq 1} (2m+1)|a_{2m+1} x^{2m}| = \infty,$$

because the radius of convergence of this power series is 0 since

$$\overline{\lim_{m\to\infty}} \sqrt[2m+1]{|(2m+1)a_{2m+1}|} = \lim_{m\to\infty} \sqrt[2m+1]{2m+1}\ \overline{\lim_{m\to\infty}} \sqrt[2m+1]{|a_{2m+1}|} = \infty,$$

contradicting the hypothesis that $\sum_{n\geq 1} a_{2m+1} f_{2m+1}(x,y)$ is absolutely convergent in the polydisc P.

Similarly one proves that $\overline{\lim}_{m\to\infty} \sqrt[m]{|b_m|} < \infty$. The proposition is proved. □

The proof of the following theorem will be completed in Theorem 4.6.

Theorem 4.2. *The linear space $H_0 = \{f \in H(\mathcal{H}) : \Delta f = 0\}$ of all harmonic analytic functions on the upper half plane is in a one to one correspondence with the complex vector space S of all ordered pairs (a,b) of sequences of complex numbers $a = (a_m)_{m\geq 0}$ and $b = (b_m)_{m\geq 1}$ such that*

$$\frac{1}{R} = \overline{\lim_{m\to\infty}} \sqrt[m]{|a_m|} < \infty, \qquad \frac{1}{R'} = \overline{\lim_{m\to\infty}} \sqrt[m]{|b_m|} < \infty.$$

The alluded bijection is given by

$$f(x,y) = \sum_{m\geq 0} \big(a_m f_m(x,y-1) + b_m h_m(x,y-1)\big) \quad \text{for all} \quad (x,y) \in P, \tag{30}$$

where P is an open polydisc centered at $(0,1)$ in which the power series is absolutely convergent.

Proof. The linear map $\mathfrak{S} : H_0 \to S$ given in $\mathfrak{S}(f) = \big((a_m)_{m\geq 0}, (b_m)_{m\geq 1}\big)$ defined by Lemma 4.1 is one to one by the principle of analytic continuation (see [Die60, (9.4.2) p. 202]). To prove surjectivity we will show that if $\big((a_m)_{m\geq 0}, (b_m)_{m\geq 1}\big) \in S$, then the analytic function f defined in (30) can be analytically continued to $\mathcal{H}$. This is equivalent to showing that the four series $\sum_{m\geq 0} a_m f_m(x,y-1)$, $\sum_{m\geq 0} b_m h_m(x,y-1))$, m even or odd can be analytically continued to $\mathcal{H}$.

Let us first consider the case of $\sum_{m\geq 0} a_m f_m(x,y-1)$ for m even. We have

$$\begin{aligned}\sum_{m\geq 0} a_{2m} f_{2m}(x,y-1) &= \sum_{m\geq 0} a_{2m} \sum_{j\geq 0} (-1)^j \binom{2m}{2j} x^{2j} (y-1)^{2m-2j} \\ &= \frac{1}{2} \sum_{m\geq 0} a_{2m} \Big((ix+y-1)^{2m} + (ix-y+1)^{2m} \Big).\end{aligned}$$

Let us study the analytic continuation of $F(x,y) = \sum_{m\geq 0} a_{2m}(ix+y-1)^{2m}$. By hypothesis, this power series is absolutely convergent in an open polydisc

$P = \{(x,y) \in \mathbb{R}^2 : |x| < \rho_1, |y-1| < \rho_2\}$. Now let $0 < |x_1| < \rho_1$, $|y_1 - 1| < \rho_2$ and consider

$$F_1(x,y) = \sum_{m\geq 0} a_{2m}\Big(i(x - x_1 + x_1) + y - y_1 + y_1 - 1\Big)^{2m}$$

which is absolutely convergent in any polydisc centered at (x_1, y_1) contained in P. We wonder if F_1 is absolutely convergent in a larger polydisc P_1 centered at (x_1, y_1). If this were the case, the function $F \smile F_1$ given by

$$(F \smile F_1)(u,v) = \begin{cases} F(u,v) & \text{if } (u,v) \in P \\ F_1(u,v) & \text{if } (u,v) \in P_1 \end{cases}$$

would be well defined and analytic on $P \cup P_1$.

To estimate the size of P_1 we look at F_1 as a power series of $(x-x_1)^p(y-y_1)^q$:

$$\begin{aligned} F_1(x,y) &= \sum_{m\geq 0} a_{2m} \sum_{j=0}^{2m} \binom{2m}{j} \Big(i(x - x_1 + x_1)\Big)^j (y - y_1 + y_1 - 1)^{2m-j} \\ &= \sum_{m\geq 0} a_{2m} \sum_{j=0}^{2m} \binom{2m}{j} \sum_{p=0}^{j} \sum_{q=0}^{2m-j} \binom{j}{p}\binom{2m-j}{q} x_1^{j-p}(y_1-1)^{2m-j-q}\Big(i(x-x_1)\Big)^p (y-y_1)^q. \end{aligned}$$

If $|x - x_1| < r$ and $|y - y_1| < r$, then

$$\begin{aligned} &\sum_{m\geq 0} |a_{2m}| \sum_{j=0}^{2m} \binom{2m}{j} \sum_{p=0}^{j} \sum_{q=0}^{2m-j} \binom{j}{p}\binom{2m-j}{q} |x_1|^{j-p}|y_1-1|^{2m-j-q} r^{p+q} \\ &\qquad = \sum_{m\geq 0} |a_{2m}| \sum_{j=0}^{2m} \binom{2m}{j} \sum_{p=0}^{j} \binom{j}{p} |x_1|^{j-p}\Big((|y_1 - 1| + r\Big)^{2m-j} r^p \\ &\qquad = \sum_{m\geq 0} |a_{2m}| \sum_{j=0}^{2m} \binom{2m}{j} \Big(|x_1| + r\Big)^j \Big(|y_1 - 1| + r\Big)^{2m-j} \\ &\qquad = \sum_{m\geq 0} |a_{2m}| \Big(|x_1| + |y_1 - 1| + 2r\Big)^{2m} < \infty \end{aligned}$$

if $|x_1| + |y_1 - 1| + 2r < R$.

Therefore the function $F_1(x,y)$ is analytic for all $|x - x_1| < r$, $|y - y_1| < r$. Hence $F_1(x,y)$ is analytic in

$$P_1 = \Big\{(x,y) \in \mathbb{R} : |x - x_1| <, \ |y - y_1| < \frac{1}{2}\Big(R - |x_1| - |y_1 - 1|\Big)\Big\}.$$

This leads us to think that analytic continuation of the function F, originally defined on P, along arcs in $\mathcal{H}$ could be proved. This would imply that F can be extended to the whole $\mathcal{H}$ because the upper half plane is simply connected. □

Example. If $h(z) = 1/z$, then h is a holomorphic function on $D = \mathbb{C} \setminus \{0\}$. Hence

$$f(x,y) = \frac{x}{x^2 + y^2}$$

is a harmonic function on D because it is the real part of the holomorphic function h. We have

$$
\begin{aligned}
f(x,y) &= \frac{x}{x^2+(y-1)^2+2(y-1)+1} = x - x\Big(x^2+(y-1)^2+2(y-1)\Big) \\
&\quad + x\Big(x^2+(y-1)^2+2(y-1)\Big)^2 - x\Big(x^2+(y-1)^2+2(y-1)\Big)^3 + \cdots \\
&= x - 2x(y-1) - x\Big(x^2+(y-1)^2\Big) \\
&\quad + x\Big(x^4+(y-1)^4+4(y-1)^2+2x^2(y-1)^2+4x^2(y-1)+4(y-1)^3\Big) \\
&\quad - 8x(y-1)^3\cdots \\
&= x - 2x(y-1) - x\Big(x^2+(y-1)^2-4(y-1)^2\Big) \\
&\quad + x\Big(x^4+(y-1)^4+2x^2(y-1)^2+4x^2(y-1)+4(y-1)^3\Big) \\
&\quad - 8x(y-1)^3\cdots \\
&= x - 2x(y-1) - x\Big(x^2+(y-1)^2-4(y-1)^2\Big) \\
&\quad + x\Big(4x^2(y-1)-4(y-1)^3\Big) + \cdots \\
&= x - 2x(y-1) - x\Big(x^2-3(y-1)^2\Big) + 4x\Big(x^2(y-1)-(y-1)^3\Big) + \cdots \\
&= h_1(x,y-1) - 2h_2(x,y-1) + 3h_3(x,y-1) - 4h_4(x,y-1) + \cdots
\end{aligned}
$$

Therefore

$$
f(x,y) = x\sum_{m\geq 0}(-1)^m\Big(x^2+(y-1))^2+2(y-1)\Big)^m = \sum_{m\geq 1}(-1)^{m+1}mh_m(x,y-1).
$$

It is not difficult to check that the power series in $(x, y-1)$ defining $f(x,y)$ is absolutely summable in the open polydisc $\{(x,y)\in\mathbb{R}^2 : |x|, |y-1| < 1/2\}$. In fact, if $|x|, |y-1| \leq r < 1/2$, then

$$
\sum_{m\geq 1}\sum_{j\geq 0}\binom{m}{2j+1}|x|^{2j+1}|y-1|^{m-2j-1} \leq \sum_{m\geq 1}\sum_{j\geq 0}\binom{m}{2j+1}r^m = \sum_{m\geq 1}2^{m-1}r^m < \infty,
$$

because the radius of convergence of the last power series is

$$
\Big(\overline{\lim_{n\to\infty}}\ \sqrt[n]{2^{n-1}}\Big)^{-1} = \frac{1}{2}.
$$

Lemma 4.3. *The following identities hold for all $n \geq 1$:*

$$
\sum_{j\geq 0}\binom{n}{2j} = \sum_{j\geq 0}\binom{n}{2j+1} = 2^{n-1}.
$$

More generally

$$
(x+y)^n + (x-y)^n = \begin{cases} 2\sum_{j\geq 0}\binom{n}{2j}x^{2j}y^{n-2j} & \text{if } n \text{ is even,} \\ 2\sum_{j\geq 0}\binom{n}{2j+1}x^{2j+1}y^{n-2j-1} & \text{if } n \text{ is odd.} \end{cases}
$$

$$
(x+y)^n - (x-y)^n = \begin{cases} 2\sum_{j\geq 0}\binom{n}{2j+1}x^{2j+1}y^{n-2j-1} & \text{if } n \text{ is even,} \\ 2\sum_{j\geq 0}\binom{n}{2j}x^{2j}y^{n-2j} & \text{if } n \text{ is odd.} \end{cases}
$$

We have

$$\begin{aligned} f(x,y) &= \sum_{m\geq 1}(-1)^{m+1}\sum_{j\geq 0}(-1)^j\binom{m}{2j+1}x^{2j+1}(y-1)^{m-2j-1} \\ &= i\sum_{m\geq 1}(-1)^m\sum_{j\geq 0}\binom{m}{2j+1}(ix)^{2j+1}(y-1)^{m-2j-1} \\ &= i\sum_{m\geq 1}\sum_{j\geq 0}\binom{2m}{2j+1}(ix)^{2j+1}(y-1)^{2m-2j-1} \\ &\quad - i\sum_{m\geq 1}\sum_{j\geq 0}\binom{2m-1}{2j+1}(ix)^{2j+1}(y-1)^{2m-2j-2} \\ &= \frac{i}{2}\sum_{m\geq 1}\Big((ix+y-1)^{2m}-(ix-y+1)^{2m}\Big) \\ &\quad - \frac{i}{2}\sum_{m\geq 1}\Big((ix+y-1)^{2m-1}+(ix-y+1)^{2m-1}\Big). \end{aligned}$$

Now we will consider the Taylor series expansion of the function f at $(0,b)$ with $0<b\leq 1$ by analyzing separately each of the four summands above. We start with

$$\begin{aligned} \sum_{m\geq 1}(ix+y-1)^{2m} &= \sum_{m\geq 1}\sum_{j=0}^{2m}\binom{2m}{j}(ix)^{2m-j}(y-b+b-1)^j \\ &= \sum_{m\geq 1}\sum_{j=0}^{2m}\sum_{k=0}^{j}\binom{2m}{j}\binom{j}{k}(b-1)^{j-k}(ix)^{2m-j}(y-b)^k. \end{aligned}$$

To estimate a polydisc of convergence of this power series in x and $y-b$ we take $|x|,|y-b|<r$ and look at

$$\begin{aligned} &\sum_{m\geq 1}\sum_{j=0}^{2m}\sum_{k=0}^{j}\binom{2m}{j}\binom{j}{k}(1-b)^{j-k}|x|^{2m-j}|y-b|^k \\ &\qquad < \sum_{m\geq 1}\sum_{j=0}^{2m}\sum_{k=0}^{j}\binom{2m}{j}\binom{j}{k}(1-b)^{j-k}r^{2m-j+k} \\ &\qquad = \sum_{m\geq 1}\sum_{j=0}^{2m}\binom{2m}{j}r^{2m-j}(r+1-b)^j = \sum_{m\geq 1}(2r+1-b)^{2m}<\infty \end{aligned}$$

if $2r+1-b<1$ or $r<b/2$. Therefore the Taylor series at $(0,b)$ of $\sum_{m\geq 1}(ix+y-1)^{2m}$ converges absolutely in the open polydisc $P=\{(x,y)\in\mathbb{R}^2 : |x|,|y-b|<b/2\}$. Now it is clear that the same happens with the other three summands, and hence with the function f.

Hence, if we consider the arc $\gamma(t)=(0,1-t)$, $0\leq t\leq\epsilon$ in $\mathcal{H}$, it is not difficult to see that the function f can be analytically continued along γ.

The unit disc

We will change our point of view, instead of working with the upper half plane we will do so with the open unit disc $\mathcal{D}=\{z\in\mathbb{C} : |z|<1\}$. Of course, both spaces are holomorphically equivalent and a holomorphic diffeomorphism of $\mathcal{D}$ onto $\mathcal{H}$ is provided by the bilinear map

$$w = i\frac{1+z}{1-z}.$$

In fact the image of $z(\theta) = e^{i\theta}$ is

$$\begin{aligned} w(z(\theta)) &= i\frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta} \\ &= i\frac{(1+\cos\theta+i\sin\theta)(1-\cos\theta+i\sin\theta)}{(1-\cos\theta)^2+\sin^2\theta} \\ &= \frac{-2\sin\theta}{(1-\cos\theta)^2+\sin^2\theta}. \end{aligned}$$

Hence $w(\pi) = 0$ and when the point $z(\theta)$ travels from $z(\pi)$ to $z(0)$ on the upper semi-circle its image $w(\theta)$ goes from $w(\pi) = 0$ to $w(0) = -\infty$. While when the point $z(\theta)$ travels from $z(\pi)$ to $z(0)$ on the lower semi-circle its image $w(\theta)$ goes from $w(\pi) = 0$ to $w(0) = \infty$. On the other hand $w(0) = i$ hence $\mathcal{D}$ is mapped onto $\mathcal{H}$.

A very important and well-known fact is the following: if $f(w)$ is a harmonic function and $w(z)$ is holomorphic then the composite function $f(w(z))$ is harmonic. In fact $\Delta f(w(z)) = |w'(z)|^2 \Delta f(w)$, see [Ahl53, p. 175]. If $f(z)$ is holomorphic in a domain $D \subset \mathbb{C}$, then $\mathrm{Re}(f(z))$ and $\mathrm{Im}(f(z))$ are harmonic in D. If $\phi(z)$ and $\psi(z)$ are harmonic functions such that $f(z) = \phi(z) + i\psi(z)$ is holomorphic, then ϕ and ψ are called conjugate harmonic functions, see [Ahl53, p. 40].

Lemma 4.4. *The polynomial functions* $f_m(x,y) = \sum_{j\geq 0}(-1)^j\binom{m}{2j}x^{2j}y^{m-2j}$ *and*

$$mh_m(x,y) = -\sum_{j\geq 0}(-1)^j\binom{m}{2j+1}x^{2j+1}y^{m-2j-1}$$

are conjugate harmonic for $m \geq 1$. *Moreover* $f(x,y) = f_m(x,y) - imh_m(x,y) = (x+iy)^m$.

Theorem 4.5. *The linear space* $H_0(\mathcal{D}) = \{f \in H(\mathcal{D}) : \Delta f = 0\}$ *of all harmonic analytic functions on the open unit disc is in a one to one correspondence with the complex vector space* S *of all pairs of sequences of complex numbers* $\big((a_m)_{m\geq 0}, (b_m)_{m\geq 1}\big)$ *such that*

$$\overline{\lim_{m\to\infty}}\ \sqrt[m]{|a_m|} \leq 1, \qquad \overline{\lim_{m\to\infty}}\ \sqrt[m]{|b_m|} \leq 1. \tag{31}$$

The alluded bijection is given by

$$f = a_0 + \sum_{m\geq 1}\Big(a_m f_m(x,y) + b_m mh_m(x,y)\Big) \quad \text{for all} \quad (x,y) \in \mathcal{D},$$

where the power series is absolutely convergent.

In particular $\{1\} \cup \{f_m(x,y), mh_m(x,y) : m \geq 1\}$ *is a basis of homogeneous polynomials of all harmonic polynomials in* $\mathbb{C}[x,y]$.

Proof. If $(a_m)_{m\geq 1}, (b_m)_{m\geq 1}$ satisfy (31) it is obvious that $(ca_m)_{m\geq 1}$ satisfy (31) for all $c \in \mathbb{C}$. By hypothesis for any $\epsilon > 0$ and any m sufficiently large we have $\sqrt[m]{|a_m|} \leq 1+\epsilon$ and $\sqrt[m]{|b_m|} \leq 1+\epsilon$. Hence $|a_m| + |b_m| \leq 2(1+\epsilon)^m$ and

$$\overline{\lim_{m\to\infty}}\ \sqrt[m]{|a_m|+|b_m|} \leq \overline{\lim_{m\to\infty}}\ \sqrt[m]{2}(1+\epsilon) = 1+\epsilon.$$

Thus S is a complex vector space.

Let $H(\mathcal{D})$ denote the space of all holomorphic functions in $\mathcal{D}$ and let $S_1 = \{(a_m)_{m\geq 0} : \overline{\lim}_{m\to\infty} \sqrt[m]{|a_m|} \leq 1\}$. The linear map $\rho : \sum_{m\geq 0} a_m z^m \mapsto (a_m)_{m\geq 0}$ from $H(\mathcal{D})$ into S_1 is one to one by the principle of analytic continuation. To prove surjectivity take $(a_m)_{m\geq 0} \in S_1$ and let $h(z) = \sum_{m\geq 0} a_m z^m$. Then the radius of convergence of this power series is greater than or equal to one. Thus h(z) is holomorphic in $\mathcal{D}$ and $(x+iy)^m = f_m(x,y) - imh_m(x,y)$, by Lemma 4.4. If a_m is real for all m, then $\sum_{m\geq 0} a_m f_m(x,y)$ is analytic and harmonic in $\mathcal{D}$, because it is the real part of $h(z)$.

In general when the given sequence in S is $(a_m + ib_m)_{m\geq 0}$ and $a_m, b_m \in \mathbb{R}$ for all $m \geq 0$, it follows that $a = (a_m)_{m\geq 0}$ and $b = (b_m)_{m\geq 0} \in S$. Therefore $\sum_{m\geq 1} a_m f_m$ and $\sum_{m\geq 1} m b_m h_m$ are analytic and harmonic in $\mathcal{D}$ as well as

$$f = a_0 + ib_0 + \sum_{m\geq 1} \Big(a_m f_m + m b_m h_m \Big).$$

Since $\mathfrak{S}(f) = (a,b)$ the theorem is proved. ∎

Theorem 4.6. *The linear space $H_0(\mathcal{H}) = \{f \in H(\mathcal{H}) : \Delta f = 0\}$ of all harmonic analytic functions on the upper half plane is in a one to one correspondence with the complex vector space S of all pairs of sequences of complex numbers $\big((a_m)_{m\geq 0}, (b_m)_{m\geq 1}\big)$ such that*

$$\overline{\lim_{m\to\infty}} \sqrt[m]{|a_m|} \leq 1, \quad \overline{\lim_{m\to\infty}} \sqrt[m]{|b_m|} \leq 1.$$

The alluded bijection is given by

$$f(u,v) = a_0 + \sum_{m\geq 1} \left(a_m \frac{f_m(u^2+v^2-1, 2u)}{(u^2+(v+1)^2)^m} + b_m \frac{m h_m(u^2+v^2-1, 2u)}{(u^2+(v+1)^2)^m} \right)$$

for all $(u,v) \in \mathcal{H}$.

In particular

$$\{1\} \cup \left\{ \frac{f_m(u^2+v^2-1, 2u)}{(u^2+(v+1)^2)^m}, \ \frac{m h_m(u^2+v^2-1, 2u)}{(u^2+(v+1)^2)^m} : m \geq 1 \right\}$$

is a set of linearly independent analytic harmonic functions on $\mathcal{H}$.

Proof. The bilinear map $w(z) = i\frac{1+z}{1-z}$ is a holomorphic diffeomorphism of $\mathcal{D}$ onto $\mathcal{H}$ and its inverse is given by $z(w) = \frac{w-i}{w+i}$.

If $(a_m)_{m\geq 0} \in S$, let $F(z) = \sum_{m\geq 0} a_m z^m$ be the corresponding holomorphic function in $\mathcal{D}$ given in Theorem 4.5 and define

$$G(w) = F(z(w)) = \sum_{m\geq 0} a_m \Big(\frac{w-i}{w+i} \Big)^m, \quad w \in \mathcal{H}.$$

If a_m, b_m are real for all $m \geq 0$, then we can compute the real and imaginary parts of $G(w)$. If $w = u + iv$, then

$$\frac{w-i}{w+i} = \frac{w\overline{w} - i(w+\overline{w}) - 1}{w\overline{w} - i(w-\overline{w}) + 1} = \frac{u^2+v^2-1}{u^2+(v+1)^2} - i\frac{2u}{u^2+(v+1)^2}$$

and

$$\left(\frac{w-i}{w+i}\right)^m = \sum_{j\geq 0}\binom{m}{j}\left(\frac{u^2+v^2-1}{u^2+(v+1)^2}\right)^{m-j}\left(\frac{-2iu}{u^2+(v+1)^2}\right)^j.$$

Hence:

$$\begin{aligned}\mathrm{Re}\Big(\frac{w-i}{w+i}\Big)^m &= \frac{1}{(u^2+(v+1)^2)^m}\sum_{j\geq 0}(-1)^j\binom{m}{2j}\Big(u^2+v^2-1\Big)^{m-2j}(2u)^{2j}\\ &= \frac{f_m(u^2+v^2-1,2u)}{(u^2+(v+1)^2)^m},\end{aligned}$$

$$\begin{aligned}\mathrm{Im}\Big(\frac{w-i}{w+i}\Big)^m &= \frac{-1}{(u^2+(v+1)^2)^m}\sum_{j\geq 0}(-1)^j\binom{m}{2j+1}\Big(u^2+v^2-1\Big)^{m-2j-1}(2u)^{2j+1}\\ &= -\frac{mh_m(u^2+v^2-1,2u)}{(u^2+(v+1)^2)^m}.\end{aligned}$$

Therefore

$$f(u,v) = a_0 + \sum_{m\geq 1}\left(a_m\frac{f_m(u^2+v^2-1,2u)}{(u^2+(v+1)^2)^m} + b_m\frac{mh_m(u^2+v^2-1,2u)}{(u^2+(v+1)^2)^m}\right)$$

is analytic and harmonic on $\mathcal{H}$, because $G(w)$ is holomorphic on $\mathcal{H}$.

Finally, assume

$$\left(\sum_{m=0}^{M} a_m\frac{f_m(u^2+v^2-1,2u)}{(u^2+(v+1)^2)^m} + b_m\frac{mh_m(u^2+v^2-1,2u)}{(u^2+(v+1)^2)^m}\right) = 0$$

and let $a = a_0, a_1, \dots, a_M, 0, \dots,$ $b = b_1, b_2 \dots, b_M, 0, \dots.$ Then $(a,b)\in S$. Hence $(a,b)=0$. The theorem is proved. □

We move on to find the Taylor expansion of $G(w)$ at $w=i$.

Lemma 4.7. *Let $g(w)=(w+i)^{-m}$. Then $g^{(j)} = (-1)^j(m)_j(w+i)^{-(m+j)}$ and*

$$\begin{aligned}(w+i)^{-m} &= (-1)^m\sum_{j\geq 0}\Big(\frac{i}{2}\Big)^{m+j}\binom{m+j-1}{m-1}(w-i)^j,\\ G(w)=\Big(\frac{w-i}{w+i}\Big)^m &= (-1)^m\sum_{j\geq 0}\Big(\frac{i}{2}\Big)^{m+j}\binom{m+j-1}{m-1}(w-i)^{m+j}.\end{aligned}$$

The coefficient $(m)_j$ is the Pochhammer symbol for $m(m+1)\cdots(m+j-1)$ if $j\geq 1$, and $(m)_0=1$. The radius of convergence of the Taylor series of $G(w)$ is $R=1/2$.

Proof. The first assertion follows by induction on $j\geq 0$. In fact $g^(0)=1$. Suppose $g^{(j)} = (-1)^j(m)_j(w+i)^{-(m+j)}$ for $j\geq 0$. Then

$$g^{(j+1)} = -(-1)^j(m)_j(m+j)(w+i)^{-(m+j+1)} = (-1)^{j+1}(m)_{j+1}(w+i)^{-(m+j+1)}.$$

Hence

$$
\begin{aligned}
(w+i)^{-m} &= \sum_{j\geq 0} \frac{(-1)^j (m)_j (2i)^{-(m+j)}}{j!}(w-i)^j \\
&= (-1)^m \sum_{j\geq 0} \frac{(-1)^j (m)_j (2i)^{-(m+j)}}{j!}(w-i)^j \\
&= (-1)^m \sum_{j\geq 0} \left(\frac{i}{2}\right)^{m+j} \binom{m+j-1}{m-1}(w-i)^j.
\end{aligned}
$$

The Taylor series expansion of $G(w)$ is now obvious. Now we estimate the radius of convergence of the Taylor series expansion of $G(w)$:

$$
\lim_{j\to\infty} \sqrt[m+j]{2^{-m-j}\binom{m+j-1}{m-1}} \leq \frac{1}{2}\lim_{j\to\infty} \sqrt[m+j]{\frac{(m+j-1)^{m-1}}{(m-1)!}} = \frac{1}{2}. \qquad \square
$$

Theorem 4.8. *The linear space $H_0(\mathcal{H}) = \{f \in H(\mathcal{H}) : \Delta f = 0\}$ of all harmonic analytic functions on the upper half plane is in a one to one correspondence with the complex vector space S of all pairs of sequences of complex numbers $\big((a_m)_{m\geq 0}, (b_m)_{m\geq 1}\big)$ such that*

$$
\overline{\lim_{m\to\infty}} \sqrt[m]{|a_m|} \leq 1 \qquad \text{and} \qquad \overline{\lim_{m\to\infty}} \sqrt[m]{|b_m|} \leq 1.
$$

The alluded bijection is given by

$$
f(u,v) = a_0 + \sum_{m\geq 1} \left(a_m f_m(u,v-1) + b_m m h_m(u,v-1)\right)
$$

for all $f \in H_0(\mathcal{H})$. The power series being absolutely convergent in the polydisc $P = \{(u,v) \in \mathbb{R}^2 : |u|, |v-1| < 1/2\}$.

Proof. If $f \in H_0(\mathcal{H})$ is real valued, then it is the real part of a holomorphic function on $\mathcal{H}$ whose Taylor series expansion at $w = i$ has radius of convergence $R \geq 1$. Hence the Taylor series expansion of f at $(0,1)$ is absolutely convergent in P. Associating monomials of total degree m we get a homogeneous harmonic polynomial of degree m. From Proposition 3.4 we obtain

$$
f(u,v) = a_0 + \sum_{m\geq 1} \left(a_m f_m(u,v-1) + b_m m h_m(u,v-1)\right)
$$

for all $(u,v) \in P$, because the Taylor series expansion of f at $(0,1)$ is absolutely convergent in P. In the general case, we have a function $f + ih \in H_0(\mathcal{H})$ where f and h are real valued. Hence

$$
\begin{aligned}
f(u,v) &= a_0 + \sum_{m\geq 1} \left(a_m f_m(u,v-1) + b_m m h_m(u,v-1)\right) \\
h(u,v) &= a'_0 + \sum_{m\geq 1} \left(a'_m f_m(u,v-1) + b'_m m h_m(u,v-1)\right).
\end{aligned}
$$

If we define $\alpha_m = a_m + ia'_m$ and $\beta_m = b_m + ib'_m$ we get

$$
(f+ih)(u,v) = \alpha_0 + \sum_{m\geq 1} \left(\alpha_m f_m(u,v-1) + \beta_m m h_m(u,v-1)\right),
$$

for all $(u,v) \in P$ and $\big((\alpha_m)_{m\geq 0}, (\beta_m)_{m\geq 1}\big) \in S$ because S is a vector space. Hence we have a linear map $\mathfrak{S} : H_0(\mathcal{H}) \to S$ given by $\mathfrak{S}(f) = \big((a_m)_{m\geq 0}, (b_m)_{m\geq 1}\big)$. It is one to one by the principle of analytic continuation (see [Die60, (9.4.2) p. 202]).

To prove surjectivity take $\big((a_m)_{m\geq 0}, (b_m)_{m\geq 1}\big) \in S$ and consider

$$F(w) = \sum_{m\geq 0} a_m \Big(\frac{w-i}{w+i}\Big)^m, \quad H(w) = \sum_{m\geq 1} b_m \Big(\frac{w-i}{w+i}\Big)^m.$$

A key observation is the following: both functions are holomorphic in $\mathcal{H}$ because by hypothesis the corresponding functions $F(z(w)) = \sum_{m\geq 0} a_m z^m$ and $H(z(w)) = \sum_{m\geq 1} b_m z^m$ are holomorphic in the open unit disc $\mathcal{D}$. Hence $\operatorname{Re} F(w)$, $\operatorname{Im} F(w)$, $\operatorname{Re} H(w)$ and $\operatorname{Im} H(w)$ are analytic in the upper half plane $\mathcal{H}$.

Suppose first that the coefficients a_m, b_m are real valued and compute the real and imaginary parts of $F(w), H(w)$: From Lemma 4.7 we obtain

$$\begin{aligned}
\operatorname{Re} F(w) &= \sum_{m\geq 0} a_m (-1)^m \operatorname{Re} \sum_{j\geq 0} \Big(\frac{i}{2}\Big)^{m+j} \binom{m+j-1}{m-1} (w-i)^{m+j} \\
&= \sum_{m\geq 0} a_m (-1)^m \operatorname{Re} \sum_{j\geq 0} 2^{-m-j} \binom{m+j-1}{m-1} (iu-v+1)^{m+j} \\
&= \sum_{m\geq 0} a_m \sum_{j\geq 0} (-1)^j 2^{-m-j} \binom{m+j-1}{m-1} \operatorname{Re} \sum_{k\geq 0} \binom{m+j}{k} (-iu)^k (v-1)^{m+j-k} \\
&= \sum_{m\geq 0} a_m \sum_{j\geq 0} (-1)^j 2^{-m-j} \binom{m+j-1}{m-1} \sum_{k\geq 0} \binom{m+j}{2k} (-)^k u^{2k} (v-1)^{m+j-2k} \\
&= \sum_{m\geq 0} a_m \sum_{j\geq 0} (-1)^j 2^{-m-j} \binom{m+j-1}{m-1} f_{m+j}(u, v-1).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\operatorname{Im} F(w) &= -\sum_{m\geq 1} a_m \sum_{j\geq 0} (-1)^j 2^{-m-j} \binom{m+j-1}{m-1} (m+j) h_{m+j}(u, v-1), \\
\operatorname{Re} H(w) &= \sum_{m\geq 1} b_m \sum_{j\geq 0} (-1)^j 2^{-m-j} \binom{m+j-1}{m-1} f_{m+j}(u, v-1), \\
\operatorname{Im} H(w) &= \sum_{m\geq 1} b_m \sum_{j\geq 0} (-1)^j 2^{-m-j} \binom{m+j-1}{m-1} (m+j) h_{m+j}(u, v-1).
\end{aligned}$$

The Taylor series expansion of $\operatorname{Re} F(w)$ at $(0,1)$ is absolutely convergent in P, as it was proved before, and it is

$$\begin{aligned}\operatorname{Re} F(w) &= \sum_{m\geq 0} a_m \sum_{j\geq 0}(-1)^j 2^{-m-j}\binom{m+j-1}{m-1} f_{m+j}(u,v-1)\\ &= a_0 + \sum_{m\geq 1} a_m \sum_{p\geq m}(-1)^{p-m}2^{-p}\binom{p-1}{m-1} f_p(u,v-1)\\ &= a_0 + \sum_{p\geq 1}\left(\sum_{m=1}^{p} a_m(-1)^{p-m}2^{-p}\binom{p-1}{m-1}\right) f_p(u,v-1)\\ &= a_0' + \sum_{p\geq 1} a_p' f_p(u,v-1).\end{aligned}$$

By what was proved at the beginning of the proof we have $\overline{\lim}_{p\to\infty}\sqrt[p]{|a_p'|}\leq 1$, and

$$a_p' = \sum_{m=1}^{p} a_m(-1)^{p-m}2^{-p}\binom{p-1}{m-1}. \tag{32}$$

The infinite matrix of the system of linear equations (32) is lower triangular with diagonal elements $(2^{-1},2^{-2},2^{-3},\dots)$. Hence given a sequence of complex numbers $(a_p')_{p\geq 1}$ there is a unique sequence $(a_m)_{m\geq 1}$ solution of (32). Moreover

$$\overline{\lim_{p\to\infty}}\sqrt[p]{|a_p'|}\leq 1 \quad \text{if and only if} \quad \overline{\lim_{m\to\infty}}\sqrt[m]{|a_m'|}\leq 1.$$

Thus, if we start with a sequence of real numbers $(a_p')_p$ satisfying $\overline{\lim}_{p\to\infty}\sqrt[p]{|a_p'|}\leq 1$, the function

$$\operatorname{Re} F(w) = \operatorname{Re}\sum_{m\geq 0} a_m\left(\frac{w-i}{w+i}\right)^m = a_0' + \sum_{p\geq 1} a_p' f_p(u,v-1),$$

with coefficients $(a_m)_{m\geq 0}$ defined by the system of linear equations (32), is harmonic and analytic in $\mathcal{H}$.

Similarly, we obtain

$$\operatorname{Im} F(w) = \operatorname{Im}\sum_{m\geq 1} a_m\left(\frac{w-i}{w+i}\right)^m = -\sum_{p\geq 1} a_p'\, p h_p(u,v-1),$$

where $(a_m)_{m\geq 1}$ is the unique solution of the following system of linear equations:

$$a_p' = -\sum_{m=1}^{p} a_m(-1)^{p-m}2^{-p}\binom{p-1}{m-1}.$$

Also, we get

$$\operatorname{Re} H(w) = \operatorname{Im}\sum_{m\geq 1} b_m\left(\frac{w-i}{w+i}\right)^m = \sum_{p\geq 1} b_p' f_p(u,v-1),$$

where $(b_m)_{m\geq 1}$ is the unique solution of the system of linear equations:

$$b_p' = \sum_{m=1}^{p} b_m(-1)^{p-m}2^{-p}\binom{p-1}{m-1}.$$

Finally, we obtain

$$\operatorname{Im} H(w) = \operatorname{Im}\sum_{m\geq 1} b_m\left(\frac{w-i}{w+i}\right)^m = \sum_{p\geq 1} b_p'\, p h_p(u,v-1),$$

where $(b_m)_{m\geq 1}$ is the unique solution of the system of linear equations:

$$b'_p = \sum_{m=1}^{p} b_m (-1)^{p-m} 2^{-p} \binom{p-1}{m-1}.$$

We have proved that, given a pair of real valued sequences $\Big((a'_p)_{p\geq 0}, (b'_p)_{p\geq 1}\Big) \in S$, the functions

$$f(u,v) = a'_0 + \sum_{p\geq 1} a'_p \Big(f_p(u,v-1) + p h_p(u,v-1)\Big),$$
$$h(u,v) = b'_0 + \sum_{p\geq 1} b'_p \Big(f_p(u,v-1) + p h_p(u,v-1)\Big)$$

are harmonic and real analytic in $\mathcal{H}$ and $\mathfrak{S}(f+h) = \Big((a'_p)_{p\geq 0}, (b'_p)_{p\geq 1}\Big)$, proving the surjectivity assertion in this particular case. The general case is reduced to this one by decomposing both sequences in their real and imaginary parts. This completes the proof of the theorem. □

5. Modular Harmonic Functions on the Upper Half Plane

In the proof of Theorem 4.5 we established a linear bijection ρ from the vector space S_1 of all sequences of complex numbers $(a_m)_{m\geq 0}$ such that $\overline{\lim}_{m\to\infty} \sqrt[m]{|a_m|} \leq 1$ onto the vector space of all holomorphic functions $f(w)$ on $\mathcal{H}$ given y

$$\rho\Big((a_m)_{m\geq 0}\Big) = \sum_{m\geq 0} a_m \left(\frac{w-i}{w+i}\right)^m.$$

We move on to characterize some linear subspaces $H(\mathcal{H})^\Gamma$ of all holomorphic functions on the upper half plane which are left invariant, or transform according to a representation, under the left action of a subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$, in terms of $S_1^\Gamma = \rho^{-1}\left(H(\mathcal{H})^\Gamma\right)$.

Lemma 5.1. *For all $m \in \mathbb{N}$ and $|y| < 1$ the following identity holds*

$$\left(\sum_{k\geq 0} y^k\right)^m = \sum_{k\geq 0} \binom{k+m-1}{m-1} y^k.$$

Let

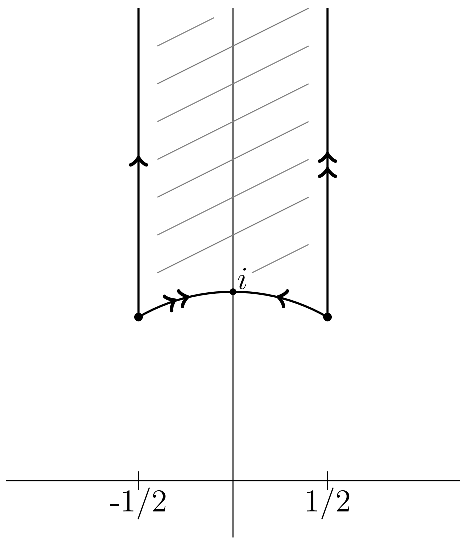
$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then it is well known that $\{\gamma_1, \gamma_2\}$ is a set of generators of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Therefore

$$H(\mathcal{H})^\Gamma = \{f \in H(\mathcal{H}) : f(\gamma_1 \cdot w) = f(\gamma_2 \cdot w) = f(w), \text{ for all } w \in \mathcal{H}\}.$$

Let $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2)$ and $\mathcal{H} = G/K$ the upper half plane. Let Γ be a discrete subgroup of G. Then $\Gamma\backslash\mathcal{H}$ is an interesting object. Let us call a *fundamental domain* $\mathcal{F}$ for Γ in $\mathcal{H}$ a subset of $\mathcal{H}$ which contains a representative for each orbit of Γ in $\mathcal{H}$ and such that if two points $w, w' \in \mathcal{F}$ lie in the same orbit, then they lie on the boundary of $\mathcal{F}$. If $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ it is well known (see [Sch74, Chapter I]) that a fundamental domain is

$$\mathcal{F} = \left\{ w = u + vi : -\frac{1}{2} \le u \le \frac{1}{2},\ u^2 + v^2 \ge 1,\ u > 0 \right\}.$$



The generators γ_1, γ_2 of $\mathrm{SL}(2,\mathbb{Z})$ satisfy the key relations: $\gamma_2^2 = -I$ and $(\gamma_2\gamma_1)^3 = -I$. Since $-I$ acts as the identiy transformation on $\mathcal{H}$ we are really interested in $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$ generated by the images of γ_1, γ_2 in $\mathrm{PSL}(2,\mathbb{Z})$ satisfying the relations: $\gamma_2^2 = I$ and $(\gamma_2\gamma_1)^3 = I$. Hence $\Gamma/\{\pm I\} = \mathbb{Z}_2 * \mathbb{Z}_3$ is the free product of the cyclic groups $\mathbb{Z}_2 = (\gamma_2)$ and $\mathbb{Z}_3 = (\gamma_2\gamma_1)$, (see [Sch74, Chapter I]).

We say that two points in the boundary of the fundamental domain $\mathcal{F}$ are Γ-equivalent if one is transformed into the other by an element of Γ.

Proposition 5.2 (See [Sch74, Chapter I])**.** *The Γ-equivalent classes of $\partial\mathcal{F}$ are:*

$$\left\{-\frac{1}{2} + vi, \frac{1}{2} + vi\right\}_{v \ge \sqrt{3}/2} \quad \textit{and} \quad \{-u + vi, u + vi\}_{0 \le u \le 1/2, u^2+v^2=1}.$$

Let us consider the cyclic subgroup $\mathbb{Z}_4 = \langle\gamma_2\rangle$ of Γ generated by γ_2. In the next proposition we give a good characterization of the $\mathbb{Z}_4$-modular holomorphic functions in the upper half plane.

Proposition 5.3. *If f is a holomorphic function on $\mathcal{H}$ and we put*

$$f(w) = a_0 + \sum_{m \ge 1} a_m \left(\frac{w-i}{w+i}\right)^m \qquad \textit{with} \qquad \overline{\lim_{m\to\infty}} \sqrt[m]{|a_m|} \le 1,$$

then $f(\gamma_2 \cdot w) = f(w)$ for all $w \in \mathcal{H}$ if and only if $a_m = 0$ for all odd $m \ge 1$.

Proof. By definition $\gamma_2 \cdot w = -1/w$. Hence, for $m \ge 1$ we have

$$\frac{\gamma_2 \cdot w - i}{\gamma_2 \cdot w + i} = -\frac{w-i}{w+i}.$$

Then we have

$$f(\gamma_2 \cdot w) = a_0 + \sum_{m \ge 1} a_m (-1)^m \left(\frac{w-i}{w+i}\right)^m.$$

Hence $f(\gamma_2 \cdot w) = f(w)$ for all $w \in \mathcal{H}$ if and only if $a_m = 0$ for all m odd. ☐

Let us consider the cyclic subgroup $\mathbb{Z}_2 = \langle \gamma_2^2 \rangle$ of Γ generated by $\gamma_2^2 = -I$. In the next proposition we give a good characterization of the holomorphic functions f in the upper half plane such that $f(\gamma \cdot w) = \chi(\gamma) f(w)$ for all $w \in \mathcal{H}, \gamma \in \mathbb{Z}_2$. Here χ denotes the nontrivial character of $\mathbb{Z}_2$.

Proposition 5.4. *If f is a holomorphic function on $\mathcal{H}$ and we put*

$$f(w) = a_0 + \sum_{m \geq 1} a_m \left(\frac{w-i}{w+i} \right)^m \qquad \textit{with} \qquad \overline{\lim_{m\to\infty}} \sqrt[m]{|a_m|} \leq 1,$$

then $f(\gamma_2^2 \cdot w) = -f(w)$ for all $w \in \mathcal{H}$ if and only if $a_m = 0$ for all even $m \geq 0$.

Proposition 5.5. *Let us consider the entire functions $e_n(w) = e^{2\pi n i w}$, $n \in \mathbb{Z}$. Then $e_n(\gamma_1 \cdot w) = e_n(w)$ for all $w \in \mathbb{C}$, and*

$$e_n(w) = \sum_{j \geq 0} c_j^{(n)} \left(\frac{w-i}{w+i} \right)^j, \quad \textit{for all} \quad w \in \mathcal{H}, \tag{33}$$

where the sequence $(c_j^{(n)})_{j \geq 0}$ is defined inductively by the following system of linear equations: $c_0^{(n)} = e^{-2\pi n}$, $c_1^{(n)} = e^{-2\pi n}(-4\pi n)$ and

$$\sum_{j=1}^{r} c_j^{(n)} (-1)^{r-j} \binom{r-1}{j-1} = e^{-2\pi n} \frac{(-4\pi n)^r}{r!} \tag{34}$$

for all $r \geq 1$. Since $e_n(w)$ is holomorphic on $\mathcal{H}$ it follows that $\overline{\lim}_{j\to\infty} \sqrt[m]{\left| c_j^{(n)} \right|} \leq 1$. Moreover, we have the following explicit formula:

$$c_j^{(n)} = e^{-2\pi n} \sum_{k=1}^{j} \binom{j-1}{k-1} \frac{(-4\pi n)^k}{k!}, \quad \textit{for all} \quad j \geq 1. \tag{35}$$

Proof. From (33) we get

$$e_n(w) = c_0^{(n)} + \sum_{j \geq 1} c_j^{(n)} \left(\frac{w-i}{w+i} \right)^j = c_0^{(n)} + \sum_{r \geq 1} \sum_{j=1}^{r} c_j^{(n)} \binom{r-1}{j-1} (-1)^{r-j} \left(\frac{1}{2i} \right)^r (w-i)^r.$$

Since

$$e_n(w) = e^{2\pi n i w} = e^{2\pi n i (w-i+i)} = e^{-2\pi n} e^{2\pi n i (w-i)} = e^{-2\pi n} \sum_{r \geq 0} \frac{\big(2\pi n i (w-i)\big)^r}{r!}$$

we obtain

$$c_0^{(n)} + \sum_{r \geq 1} \sum_{j=1}^{r} c_j^{(n)} \binom{r-1}{j-1} (-1)^{r-j} \left(\frac{1}{2i} \right)^r (w-i)^r = e^{-2\pi n} \sum_{r \geq 0} \frac{\big(2\pi n i (w-i)\big)^r}{r!}$$

or equivalently $c_0^{(n)} = e^{-2\pi n}$ and for all $r \geq 1$

$$\sum_{j=1}^{r} c_j^{(n)} (-1)^{r-j} \binom{r-1}{j-1} = e^{-2\pi n} (2i)^r \frac{(2\pi n i)^r}{r!} = e^{-2\pi n} \frac{(-4\pi n)^r}{r!}.$$

We now observe that the coefficient matrix of this infinite system of inhomogeneous linear equations is a lower triangular matrix with ones in the diagonal. Therefore it has a unique solution $(c_j^{(n)})_{j \geq 1}$.

Now we will check the explicit formula for $c_j^{(n)}$:

$$e^{-2\pi n}\sum_{j=1}^{r}\sum_{k=1}^{j}(-1)^{r-j}\binom{r-1}{j-1}\binom{j-1}{k-1}\frac{(-4\pi n)^k}{k!}$$
$$= e^{-2\pi n}\sum_{k=1}^{r}\frac{(-4\pi n)^k}{k!}\sum_{j=k}^{r}(-1)^{r-j}\binom{r-1}{j-1}\binom{j-1}{k-1} = e^{-2\pi n}\frac{(-4\pi n)^r}{r!},$$

because

$$\sum_{j=k}^{r-1}(-1)^{r-j}\binom{r-1}{j-1}\binom{j-1}{k-1} = \sum_{j=k}^{r-1}\frac{(-1)^{r-j}(r-1)!}{(r-j)!(j-k)!}$$
$$= \frac{(r-1)!}{(r-k)!}\sum_{j=k}^{r-1}(-1)^{r-j}\binom{r-k}{j-k} = 0.$$

The lemma is thus proved. □

The statement of the next proposition is due to A. Grünbaum and is a significant contribution. He discovered the three term recurrence relation (37) by using software of computing algebra, see [Grü25].

Proposition 5.6. *Let $(c_k(x))_{k\geq 0}$ be the sequence of polynomials of degree k defined by the following system of infinite linear equations: $c_0 = 1$ and*

$$\sum_{k=1}^{r} c_k(x)(-1)^{r-k}\binom{r-1}{k-1} = \frac{(-4x)^r}{r!}, \tag{36}$$

for all $r \geq 1$. Then the sequence $(c_k(x))_{k\geq 0}$ satisfies the following three term recurrence relation

$$-\frac{k-1}{4}c_{k-1}(x) + \frac{k}{2}c_k(x) - \frac{k+1}{4}c_{k+1}(x) = xc_k(x).$$

If we replace x by $x/4$ we get

$$-(k-1)c_{k-1}\left(\frac{x}{4}\right) + 2kc_k\left(\frac{x}{4}\right) - (k+1)c_{k+1}\left(\frac{x}{4}\right) = xc_k\left(\frac{x}{4}\right) \tag{37}$$

which is the three term recurrence relation

$$(k+1)L_{k+1}^{(-1)}(x) - (2k-x)L_k^{(-1)}(x) + (k-1)L_{k-1}^{(-1)}(x) = 0$$

satisfy by the generalized Laguerre polynomials $L_k^{(\alpha)}(x)$ with parameter $\alpha = -1$. Therefore $c_k(x) = L_k^{(-1)}(4x)$.

Proof. Since the explicit formula (35) holds for all n by putting $x = 4\pi n$ we get

$$c_j(x) = \sum_{k=1}^{j}\binom{j-1}{k-1}\frac{(-4x)^k}{k!}.$$

Now we start checking the three term recurrence relation (37):

$$
\begin{aligned}
&-(j-1)c_{j-1}\Big(\frac{x}{4}\Big)+(2j-x)c_j\Big(\frac{x}{4}\Big)-(j+1)c_{j+1}\Big(\frac{x}{4}\Big)\\
&=-(j-1)\sum_{k=1}^{j-1}\binom{j-2}{k-1}\frac{(-x)^k}{k!}+2j\sum_{k=1}^{j}\binom{j-1}{k-1}\frac{(-x)^k}{k!}+\sum_{k=1}^{j}\binom{j-1}{k-1}\frac{(-x)^{k+1}}{k!}\\
&\quad-(j+1)\sum_{k=1}^{j+1}\binom{j}{k-1}\frac{(-x)^k}{k!}=\sum_{k=1}^{j-1}\frac{(-x)^k}{k!}\bigg(-(j-1)\binom{j-2}{k-1}\\
&\quad+2j\binom{j-1}{k-1}+\binom{j-1}{k-2}k-(j+1)\binom{j}{k-1}\bigg)+2\frac{(-x)^j}{(j-1)!}+\frac{(-x)^j}{(j-2)!}\\
&\quad+\frac{(-x)^{j+1}}{j!}-(j+1)\frac{(-x)^j}{(j-1)!}-\frac{(-x)^{j+1}}{j!}\\
&=\sum_{k=1}^{j-1}\frac{(-x)^k}{k!}\bigg(-(j-1)\binom{j-2}{k-1}+2j\binom{j-1}{k-1}+\binom{j-1}{k-2}k-(j+1)\binom{j}{k-1}\bigg)\\
&=0
\end{aligned}
$$

because

$$
\begin{aligned}
&-(j-1)\binom{j-2}{k-1}+2j\binom{j-1}{k-1}+\binom{j-1}{k-2}k-(j+1)\binom{j}{k-1}\\
&=\frac{(j-1)!}{(k-1)!(j-k+1)!}\big(-(j-k+1)(j-k)+2j(j-k+1)+k(k-1)-(j+1)j\big)\\
&=0.
\end{aligned}
$$

□

Corollary 5.7. *The sequence $(c_j^{(n)})_{j\geq 0}$ defined in (34) in terms of the sequence of polynomials $c_k(x)$ introduced in (36) is given by $c_j^{(n)}=e^{-2\pi n}c_j(\pi n)$. Therefore*

$$-(j-1)c_{j-1}^{(n)}+2jc_j^{(n)}-(j+1)c_{j+1}^{(n)}=4\pi nc_j^{(n)}.$$

Let us consider the cyclic subgroup $\mathbb{Z}=\langle\gamma_1\rangle$ of Γ generated by γ_1. In the next proposition we give a large family of $\mathbb{Z}$-modular holomorphic functions in the upper half plane.

Proposition 5.8. *Let us consider the formal power series $\sum_{m\geq 0}a_me_m(w)$. The series is absolutely convergent for all $w\in\mathcal{H}$ if and only if $\overline{\lim}_{m\to\infty}\sqrt[m]{|a_m|}\leq 1$. Moreover $f(w)=\sum_{m\geq 0}a_me_m(w)$ is holomorphic on $\mathcal{H}$ and $f(\gamma_1\cdot w)=f(w)$ for all $w\in\mathcal{H}$.*

Proof. We have $|e^{2\pi miw}|=e^{-2\pi mv}$ where $w=u+vi$. Hence for all $v>0$

$$\sum_{m\geq 0}|a_me_m(w)|=\sum_{m\geq 0}|a_m|e^{-2\pi mv}\leq\sum_{m\geq 0}|a_m|<\infty$$

if $\overline{\lim}_{m\to\infty}\sqrt[m]{|a_m|}\leq 1$.

Conversely, if $\frac{1}{R}=\overline{\lim}_{m\to\infty}\sqrt[m]{|a_m|}>1$, then $\sum_{m\geq 0}|a_m|e^{-2\pi mv}=\infty$ for all $0<v<-\frac{1}{2\pi}\log R$. Contradiction.

Since the last assertions are now obvious the proposition is proved. □

Lemma 5.9. *The following asymptotic upper bound for $L_j^{(-1)}(x)$ holds for all $x > 0$, $j > 1$:*

$$\left|L_j^{(-1)}(x)\right| \leq \frac{je^{x/2}\sqrt{jx}}{x}.$$

Proof. This follows from the following asymptotic upper bound of $L_j^{(-1)}(x)$ in terms of the Bessel function of first kind $J_1(x)$. For $x > 0$ we have

$$\left|L_j^{(-1)}(x)\right| \leq \frac{je^{x/2}}{x} J_1\left(2\sqrt{jx}\right)$$

see [Sze75, Chapter VIII]. Now from the integral representation

$$J_1(x) = \frac{x}{2\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} \sqrt{1-t^2}e^{ixt}\,\mathrm{d}t$$

taken from [Sze75, (1.71.6) p. 15], we obtain

$$\left|J_1(x)\right| \leq \frac{x}{\pi} \int_{-1}^{1} \sqrt{(1-t^2}\,\mathrm{d}t = \frac{x}{2}.$$

Hence

$$\left|L_j^{(-1)}(x)\right| \leq \frac{je^{x/2}}{x} J_1\left(2\sqrt{jx}\right) \leq \frac{je^{x/2}\sqrt{jx}}{x}.$$

The lemma is proved. □

Theorem 5.10. *Let us consider the holomorphic function $f(w) = \sum_{m\geq 0} a_m e_m(w)$ on $\mathcal{H}$, where*

$$\overline{\lim_{m\to\infty}} \sqrt[m]{|a_m|} \leq 1.$$

Then the double series $\sum_{m,j\geq 0} a_m c_j^{(m)} \left(\frac{w-i}{w+i}\right)^m$ is absolutely convergent for all $w \in \mathcal{H}$, or $\sum_{m,j\geq 0} a_m c_j^{(m)} z^m$ is absolutely convergent for all $|z| < 1$. Therefore

$$f(w) = \sum_{j\geq 0}\left(\sum_{m\geq 0} a_m c_j^{(m)}\right)\left(\frac{w-i}{w+i}\right)^j$$

and $f(\gamma_1 \cdot w) = f(w)$ for all $w \in \mathcal{H}$.

Proof. To prove the first assertion it is enough to recall the biholomorphic correspondence $w(z) = i\frac{1+z}{1-z}, z \in \mathcal{D}$ and $z(w) = \frac{w-i}{w+i}, w \in \mathcal{H}$ between the unit disc and the upper half plane, introduced in the proof of Theorem 4.6.

From Proposition 5.6 and Corollary 5.7 we obtain $c_j^{(m)} = e^{-2\pi m} L_j^{(-1)}(4\pi m)$. By using the previous Lemma we know that $\left|L_j^{(-1)}(4\pi m)\right| \leq \frac{j\sqrt{j}e^{2\pi m}}{2\sqrt{\pi m}}$ for $m > 0$ and $j > 1$. Hence

$$\sum_{m,j\geq 0} |a_m c_j^{(m)} z^j| = |a_0| + \sum_{m>0} |a_m| e^{-2\pi m} + \sum_{m>0} 4\pi m |a_m| e^{-2\pi m} |z| + \sum_{m>0}\sum_{j>1} |a_m c_j^{(m)} z^j|.$$

We have

$$\sum_{m>0}\sum_{j>1}|a_m c_j^{(m)} z^m| = \sum_{m>0}\sum_{j>1}|a_m|e^{-2\pi m}\Big|L_j^{(-1)}(4\pi m)\Big||z|^j$$
$$\leq \sum_{m>0}\sum_{j>1}\frac{|a_m|}{2\sqrt{\pi m}} j\sqrt{j}|z|^j = \sum_{m>0}\frac{|a_m|}{2\sqrt{\pi m}}\sum_{j\geq 1} j\sqrt{j}|z|^j < \infty,$$

since $\sum_{j>1} j\sqrt{j}|z|^j < \infty$ because the radius of convergence is one, and $\sum_{m>0}\frac{|a_m|}{2\sqrt{\pi m}} < \infty$ because

$$\overline{\lim_{m\to\infty}}\ \sqrt[m]{\frac{|a_m|}{2\sqrt{\pi m}}} = \overline{\lim_{m\to\infty}}\ \sqrt[m]{|a_m|}\lim_{m\to\infty}\ \sqrt[m]{\frac{1}{2\sqrt{\pi m}}} = \overline{\lim_{m\to\infty}}\ \sqrt[m]{|a_m|} \leq 1$$

by hypothesis. Moreover $\sum_{m\geq 0}|a_m|e^{-2\pi m} < \infty$ and $\sum_{m>0} 4\pi m|a_m|e^{-2\pi m}|z| < \infty$. This completes the proof. □

Let V be a finite dimensional complex vector space and take $T \in \mathrm{End}(V)$. Let $\mathbb{Z} = \langle\gamma_1\rangle$ and consider the representation $\chi : \mathbb{Z} \to \mathrm{GL}(V)$ defined by $\chi(\gamma_1) = \exp(2\pi miT)$. Set the operator valued entire function $E_m(w) = \exp(2\pi miwT)$. Then

$$F(\gamma_1\cdot w) = \exp(2\pi mi(w+1)T) = \exp(2\pi miT + 2\pi miwT) = \chi(\gamma_1)F(w).$$

Proposition 5.11. *If* $\overline{\lim}_{m\to\infty}\sqrt[m]{|a_m|} \leq 1$ *the power series* $\sum_{m\geq 0} a_m E_m(w)$ *is absolutely convergent for all* $w \in \mathcal{H}$*, in any operator norm. Hence the operator valued function* $F(w) = \sum_{m\geq 0} a_m E_m(w)$ *is holomorphic on* $\mathcal{H}$*. Conversely* $\sum_{m\geq 0} a_m E_m(w)$ *is not absolutely convergent in norm if* $\overline{\lim}_{m\to\infty}\sqrt[m]{|a_m|} > 1$.

Proof. Choose an operator norm $\|\cdot\|$ in $\mathrm{End}(V)$. If $w = u + vi$ we have

$$\|\exp(2\pi miwT)\| \leq \exp\Big(|2\pi miw|\|T\|\Big) = \exp\Big(|2\pi miw|\Big)^{\|T\|}$$
$$= \exp(2\pi miu - 2\pi mv)^{\|T\|} = \exp(-2\pi mv)^{\|T\|}.$$

Therefore, for all $v > 0$

$$\sum_{m\geq 0}\|a_m E_m(w)\| = \sum_{m\geq 0}|a_m|\Big(\exp(-2\pi v\|T\|)\Big)^m < \infty$$

if $\overline{\lim}_{m\to\infty}\sqrt[m]{|a_m|} \leq 1$.

We now start to prove the converse assertion: If λ is an eigenvalue of T, then $\|T\| \geq |\lambda|$, and for $w = iv$, $v > 0$ we have

$$\sum_{m\geq 0}|a_m|\|\exp(2\pi miwT)\| = \sum_{m\geq 0}|a_m|\exp(-2\pi mv\,\mathrm{Re}\,\lambda)$$
$$= \sum_{m\geq 0}|a_m|\Big(\exp(-2\pi v\,\mathrm{Re}\,\lambda)\Big)^m = \infty$$

if $\frac{1}{R} = \overline{\lim}_{m\to\infty}\sqrt[m]{|a_m|} > 1$ and $2\pi v\,\mathrm{Re}\,\lambda < -\log R$. The proposition is proved. □

In the literature, there are many deep results on elliptic modular functions including instances of holomorphic modular functions on the upper half plane which are invariant by large subgroups Γ of $\mathrm{SL}(2,\mathbb{Z})$. For example in [Thr53, Section 30] a holomorphic function $\mu(w)$ on $\mathcal{H}$ is constructed, which satisfies $\mu(\phi \cdot w) = \mu(w)$ for all $\phi \in \Phi$, $w \in \mathcal{H}$. Here Φ is the subgroup of $\mathrm{SL}(2,\mathbb{Z})$ generated by

$$\phi_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \phi_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Φ is a proper subgroup of $\mathrm{SL}(2,\mathbb{Z})$ because it contains only hyperbolic and parabolic elements.

Also the elliptic modular function J is holomorphic on the upper half plane and it is invariant for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. It is a fundamental object in the theory of modular forms. The J-function parameterizes isomorphism classes of elliptic curves over $\mathbb{C}$.

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