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The spherical Whittaker Plancherel Theorem and the quantum non-periodic Toda Lattice

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Abstract. This paper is partially an exposition of the method of the proof of the continuous part of the general distributional Whittaker Plancherel Theorem in the special case of the spherical spectrum. It is also an explanation of how this result solves the quantum non-periodic Toda lattices. Combining the ideas involved in both of these results the paper also gives a new reduction of the calculation of spherical Whittaker functions to split groups over $\mathbb{R}$. It concludes with a new proof of an explicit functional equation which is used in the surjectivity result in the L^2 Plancherel Theorem and an explicit isomorphism theorem for the Whittaker Schwartz Space.

Mathematics Subject Classification 2020: 20E45, 22E30, 22E70, 43A85

Key Words and Phrases: Whittaker vector, Plancherel theorem, Inversion theorem

In memory of the accomplishments and the kindness of Joe Wolf

1. Introduction

Piatetski-Shapiro and his coworkers have amply demonstrated the importance of Whittaker Theory to the study of automorphic forms. The results in Chapter 15 in my book [Wal92] have been quoted often in the literature especially those in Section 15.10 which gives a formula in the spherical case. Unfortunately there are serious gaps in the proofs of the main results in the underpinning of that chapter as pointed out in [DK17] and the referees to our various attempts at filling the gaps. These problems were (finally) fixed in my papers [Wal24a] and [Wal24b]. This paper has a narrower scope and studies only the spherical case and some of its implications.

One of the main purposes of this paper is to give a complete, relatively self contained, proof of the Spherical Whittaker Plancherel Theorem for real reductive groups. This result is an important special case of the general theorem but it is unencumbered by the complications caused by discrete spectrum. It can be used as a help for the understanding the method used in [Wal24b] for the study of the continuous spectrum. However, in the above reference a full proof of the direct integral decomposition in the general case appears for the first time in [Wal24a]. In this paper both results can be found in the spherical case (Theorem 9.9) thus giving a somewhat more elementary proof of the much cited results in [Wal92, Section 15.10.3]. The analysis in this case leads to several new results about spherical Whittaker functions including one that generalizes a formula in [GW80] for rank 1 groups (also using the theory in [GW82; GW84; GW86]) which gives a formula for Spherical Whittaker functions in terms of those on split groups over $\mathbb{R}$ (Theorem 8.1). Also Appendix C contains a result of independent interest characterizing the restriction of the Whittaker Schwartz space to the identity component of an $\mathbb{R}$ split torus.

By relatively self-contained I mean based on three main results: The first is the Harish-Chandra Plancherel Formula for his Schwartz space for G/K (actually the inversion formula) as developed in the work of Helgason [Hel78; Hel94]. The second is the holomorphic continuation of the Jacquet Integral for minimal parabolic subgroups and its brilliant implication due to Raphaël Beuzart-Plessis [Beu20]. Thirdly, the work of [GW80; GW82; GW84; GW86] on Whittaker vectors and the quantum Toda lattice.

The applications of these results to the theory of automorphic forms inundate the literature (even though there are the above mentioned problems with the main reference). In this paper, I include as an example of an application, the solution to the quantum non-periodic Toda lattice (see Section 7 of this paper and [GW82; GW84; GW86] for more background on the subject). As it turns out, the ideas in Section 7 and the next play a role in the proof of the surjectivity assertion in the Plancherel Theorem. If you are a physicist or a mathematician who is not an expert in Representation Theory, I would recommend that you read Section 7 of this paper first.

2. Some notation

In this section we set the notation that will be used throughout the paper.

If G is a locally compact, unimodular group then when dg appears in a formula it stands for a choice of Haar measure. Let G be a real reductive group. In this paper there is no loss of generality to take G to be the identity component of a subgroup of $GL(n,\mathbb{R})$ that is the locus of zeros of a set of polynomials on $M_n(\mathbb{R})$ such that G is invariant under transpose ($g \mapsto g^T$). We will therefore assume these properties. We set $K = G \cap O(n)$ a maximal compact subgroup of G. We choose an Iwasawa decomposition of G given by $G = NAK$ with N a maximal unipotent subgroup (i.e. the elements of N are of the form $I + X$ with X nilpotent) of G, A a subgroup maximal among the subgroups of G contained in the set of symmetric positive definite matrices such that $aNa^{-1} \subset N$, $a \in A$. Let $\mathfrak{a} = \mathrm{Lie}(A)$, $\mathfrak{n} = \mathrm{Lie}(N)$ and $\mathfrak{g} = \mathrm{Lie}(G)$. Then $\mathrm{ad}(h)_{|\mathfrak{n}}$ for $h \in \mathfrak{a}$ simultaneously diagonalize yielding the roots Φ^+ of $\mathfrak{a}$ on $\mathfrak{n}$. The exponential map $\exp : \mathfrak{a} \to A$ is a Lie group isomorphism ($\mathfrak{a}$ is a group under addition) and we set $\log : A \to \mathfrak{a}$ equal to the inverse map. We define for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ (the complex valued linear functionals), $a \in A$, $a^\lambda = e^{\lambda(\log(a))}$. Define (as usual), for $h \in \mathfrak{a}$,

$$\rho(h) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}(h)_{|\mathfrak{n}}).$$

If $x, y \in \mathfrak{g}$ then set $\langle x, y\rangle = \mathrm{tr}(xy^T)$ and $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$. Let $m = \dim \mathfrak{n}$. On $\bigwedge^m \mathfrak{g}$ we put the inner product induced by $\langle \cdot, \cdot \rangle$. Let $\mathrm{Int}(g)$ be the transformation of $\mathfrak{g}$ given by $\mathrm{Int}(g)x = gxg^{-1}$. Define for $g \in G$, $|g|$ to be the operator norm of $\bigwedge^m \mathrm{Int}(g)$. Set A_G equal to the subgroup of elements of the center of G that are positive definite then $A_G \subset A$ and $KA_G[G,G] = G$. Define

$$\|kag\| = e^{\|\log a\|}\,|g|^{\frac{1}{2}}, \qquad k \in K,\ a \in A_G,\ g \in [G,G].$$

If $g \in G$ then g can be written in the form $g = k_1 a k_2$ with $a \in A$ such that $a^\alpha \geq 1$ for $\alpha \in \Phi^+$ and $k_1, k_2 \in K$. One can check that if g is of this form and if $g \in [G,G]$ then

$$|g| = a^{2\rho}.$$

It is easily seen that

$$\|x\| \geq 1, \quad \|xy\| \leq \|x\|\,\|y\|$$

and that the sets

$$\{x \in G \mid \|x\| \leq r\}$$

are compact for $r < \infty$. One has (cf. [Wal88b, Lemma 5.1.3]) which says that there exists d such that

$$\int_G |g|^{-1}\,(1+\log\|g\|)^{-d}\,\mathrm{d}g < \infty.$$

Here dg is a choice of Haar measure on G.

Let $M = \{k \in K \mid ka = ak, a \in A\}$.

We now recall the Harish-Chandra Schwartz space. If $f \in C^\infty(G)$, $x, y \in U(\mathfrak{g})$, $d \in \mathbb{R}$ then set

$$p_{x,y,d}(f) = \sup\{|R_x L_y f(g)|\,|g|^{\frac{1}{2}}\,(1+\log\|g\|)^d \mid g \in G\}.$$

Here if $X, Y \in \mathfrak{g}$ then

$$R_X f(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(g\exp(tX))_{|t=0}$$

$$\text{and} \quad L_Y f(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp(-tY)g)_{|t=0}$$

and since $[L_X, L_Y] = L_{[X,Y]}$ and $[R_X, R_Y] = R_{[X,Y]}$ the universal mapping property of the universal enveloping algebra allows us to define L_x and L_y for all $x, y \in U(\mathfrak{g})$.

Define $\mathcal{C}(G) = \{f \in C^\infty(G) \mid p_{x,y,d}(f) < \infty, x, y \in U(\mathfrak{g}), d \in \mathbb{R}\}$ endowed with the topology defined by the semi-norms $p_{x,y,d}$.

3. The Plancherel Theorem for $\mathcal{C}(G/K)$

The purpose of this section is to recall the basic harmonic analysis of G/K and extend Harish-Chandra's results to non K–finite Schwartz functions.

Note that the map $\theta : G \to G$ given by $\theta(g) = (g^T)^{-1}$ is an automorphism of G. Set $\overline{N} = \theta(N)$. We have the corresponding Iwasawa decomposition $G = \overline{N}AK$. Since the decomposition $g = \bar{n}ak$, $\bar{n} \in \overline{N}$, $a \in A$, $k \in K$, is unique we have the smooth functions $a(g) = a$, $k(g) = k$. If $f \in C^\infty(M\backslash K)$ define for $\nu \in \mathfrak{a}^*_{\mathbb{C}}$

$$f_\nu(g) = a(g)^{\nu-\rho} f(k(g)).$$

This defines f_ν as a C^∞ function on G. Define an action of G on $C^\infty(M\backslash K)$ by

$$(\pi_\nu(g)f)(k) = f_\nu(kg).$$

If we put the C^∞ topology on $C^\infty(M\backslash K)$ then $(\pi_\nu, C^\infty(M\backslash K))$ is a smooth Fréchet representation of G. We also put the L^2–inner product on $C^\infty(M\backslash K)$,

$$\langle u, w\rangle = \int_K u(k)\overline{w(k)}\,\mathrm{d}k,$$

and find that (cf. [Wal18, Lemma 8.3.1])

$$\langle \pi_\nu(g)u, w\rangle = \left\langle u, \pi_{-\bar{\nu}}(g^{-1})w\right\rangle$$

where $\bar{\nu}(h) = \overline{\nu(h)}$ for $h \in \mathfrak{a}$. In particular, if $\nu \in \mathfrak{a}^*$ then $(\pi_{i\nu}, L^2(M\backslash K))$ is a unitary representation. If $f \in C_c^\infty(G)$ then set

$$(\pi_\nu(f)u)(k) = \int_G u_\nu(kg)f(g)\,\mathrm{d}g.$$

If $\lambda \in \mathfrak{a}^*$ define $H_\lambda \in \mathfrak{a}$ by $\langle H_\lambda, h\rangle = \lambda(h)$ for $h \in \mathfrak{a}$. Define $(\lambda, \mu) = \langle H_\lambda, H_\mu\rangle$ and extend $(\,\cdot\,,\cdot\,)$ to $\mathfrak{a}^*_{\mathbb{C}}$ complex bilinearly. Harish-Chandra's c–function is defined by the formula

$$c(\nu) = \int_N a(n)^{\nu-\rho}\, \mathrm{d}n$$

where it converges and extended by meromorphic continuation. We leave, for the moment, the bi-invariant measure on N unnormalized. This integral converges absolutely for $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ such that

$$\operatorname{Re}(\nu, \alpha) < 0$$

for all $\alpha \in \Phi^+$ and uniformly in compacta in this set. Thus $c(\nu)$ defines a holomorphic function on this subset. It has a meromorphic continuation to all of $\mathfrak{a}^*_{\mathbb{C}}$. Indeed, Gindikin–Karpelevic derived an explicit formula for it (cf. [Hel94] or [Wal18, Section 8.10.18]) if we set for $\alpha \in \Phi_0^+ = \{\beta \in \Phi^+ \mid \frac{\beta}{2} \notin \Phi^+\}$,

$$c_\alpha(\nu) = \begin{cases} B\left(\frac{\dim \mathfrak{n}_\alpha}{2}, \frac{(\nu,\alpha)}{(\alpha,\alpha)}\right) & \text{if } 2\alpha \notin \Phi^+, \\ B\left(\frac{\dim \mathfrak{n}_\alpha}{2}, \frac{(\nu,\alpha)}{(\alpha,\alpha)}\right) B\left(\frac{\dim \mathfrak{n}_{2\alpha}}{2}, \frac{(\nu,\alpha)}{2(\alpha,\alpha)} + \frac{\dim \mathfrak{n}_\alpha + \dim \mathfrak{n}_{2\alpha}}{2}\right) & \text{if } 2\alpha \in \Phi^+. \end{cases}$$

Here $\mathfrak{n}_\alpha = \mathfrak{g}_\alpha$ (the α root space) for $\alpha \in \Phi^+$ and B is the classical Beta function. Then the Haar measure on N can be normalized so that

$$c(\nu) = \prod_{\alpha \in \Phi_0^+} c_\alpha(\nu).$$

From now on this will be our normalization. Set for $\nu \in \mathfrak{a}^*$

$$\mu(\nu) = \frac{1}{c(i\nu)c(-i\nu)} = \frac{1}{|c(i\nu)|^2}.$$

One can show using the formula for $c(\nu)$ and basic properties of the Γ–function that $\mu(\nu) \leq C(1 + \|\nu\|^r)$ for some r and $C > 0$.

We are now ready to develop the Plancherel theorem for G/K. We first recall a special case of the Harish-Chandra Plancherel Theorem (cf. [Wal92, Theorem 13.4.1]).

Theorem 3.1. *The measures on G and $\mathfrak{a}^*$ can be normalized so that if $\phi \in \mathcal{C}(K\backslash G/K)$ (that is $\phi(k_1 g k_2) = \phi(g)$, $k_1, k_2 \in K$) then*

$$\phi(g) = \int_{\mathfrak{a}^*} \langle \pi_{i\nu}(L_{g^{-1}}\phi)1, 1\rangle\, \mu(\nu)\, \mathrm{d}\nu.$$

There is a relatively elementary proof of this theorem due to Anker [Ank91] that specifically proves this result. Harish-Chandra proved much more. In particular,

Theorem 3.2 (cf. [Wal92, Theorem 13.3.2]). *Set for $f \in \mathcal{C}(K\backslash G/K), \nu \in \mathfrak{a}^*$*

$$T(f)(\nu) = \langle \pi_{i\nu}(f)1, 1\rangle\,.$$

Then $T(f) \in \mathcal{S}(\mathfrak{a}^*)^W$ and T defines a bijection $T : \mathcal{C}(K\backslash G/K) \to \mathcal{S}(\mathfrak{a}^*)^W$. Here if V is a finite dimensional vector space over $\mathbb{R}$ then $\mathcal{S}(V)$ stands for its Schwartz space.

We now recall an argument in Helgason [Hel94, Section III.1], in the proof of the following implication.

Theorem 3.3. *Let* $f \in \mathcal{C}(G/K)$ *then with the same normalizations as in the previous result we have*

$$f(g) = \int_{\mathfrak{a}^*} \langle \pi_{i\nu}(L_{g^{-1}}f)1, 1\rangle \, \mu(\nu) \, \mathrm{d}\nu.$$

Note. The point here is that f is not necessarily left K–finite (since under that condition the result is a special case of Harish-Chandra's Theorem).

Proof. Define for $g, x \in G$

$$\phi(g, x) = \int_K f(gkx) \, \mathrm{d}k.$$

This function for g fixed can be expressed as

$$\int_K (L_{g^{-1}}f)(kx) \, \mathrm{d}k.$$

Since the left and right translates of an element of $\mathcal{C}(G)$ are in $\mathcal{C}(G)$ and the K-isotypic components of elements are in $\mathcal{C}(G)$, if g is fixed the function $x \mapsto \phi(g, x)$ denoted $\phi(g, \cdot\,)$ is in $\mathcal{C}(G)$. Thus keeping g fixed we have

$$\phi(g, I) = \int_{\mathfrak{a}^*} \langle \pi_{i\nu}(\phi(g, \cdot\,))1, 1\rangle \, \mu(\nu) \, \mathrm{d}\nu.$$

Now, taking into account that $\int_G \int_K f(gkx) \, \langle \pi_{i\nu}(x)1, 1\rangle \, \mathrm{d}k \, \mathrm{d}x$ converges absolutely, we have

$$\begin{aligned}
\langle \pi_{i\nu}(\phi(g, \cdot\,))1, 1\rangle &= \int_G \phi(g, x) \, \langle \pi_{i\nu}(x)1, 1\rangle \, \mathrm{d}x = \int_G \int_K f(gkx) \, \langle \pi_{i\nu}(x)1, 1\rangle \, \mathrm{d}k \, \mathrm{d}x \\
&= \int_K \int_G f(gx) \left\langle \pi_{i\nu}(k^{-1}x)1, 1\right\rangle \, \mathrm{d}x \, \mathrm{d}k = \int_K \int_G L_{g^{-1}}f(x) \left\langle \pi_{i\nu}(k)^{-1}\pi_{i\nu}(x)1, 1\right\rangle \, \mathrm{d}x \, \mathrm{d}k \\
&= \int_K \int_G L_{g^{-1}}f(x) \, \langle \pi_{i\nu}(x)1, \pi_{i\nu}(k)1\rangle \, \mathrm{d}x \, \mathrm{d}k = \langle \pi_{i\nu}(L_{g^{-1}}f)1, 1\rangle .
\end{aligned}$$

Noting that $\phi(g, I) = f(g)$ completes the proof. □

Harish-Chandra's theorem says more in the K-finite case (cf. [Wal92, Theorem. 12.7.1]).

Theorem 3.4. *Let* $u \in C^\infty(M\backslash K)$ *be right* K*–finite and let* $\alpha \in \mathcal{S}(\mathfrak{a}^*)$ *then the function* f *defined by*

$$f(g) = \int_{\mathfrak{a}^*} \langle \pi_{i\nu}(g)1, u\rangle \, \alpha(\nu)\mu(\nu) \, \mathrm{d}\nu$$

is in $\mathcal{C}(K\backslash G)$.

4. The holomorphic continuation of Jacquet integrals and a theorem of Beuzart-Plessis

Retain the notation of the previous sections. If $\chi : N \to S^1$ is a unitary character of N then we consider for $u \in C^\infty(M\backslash K)$

$$J_{\chi,\nu}(u) = \int_N \chi(n)^{-1} u_\nu(n) \, \mathrm{d}n.$$

Note that if $\chi = 1$ and $u = 1$ then the integral is the one that defines the Harish-Chandra c-function. This implies that the integral defining $J_{\chi,\nu}$ converges absolutely if $\operatorname{Re}(\nu,\alpha) < 0$ for all $\alpha \in \Phi^+$. Let Δ be the set of elements of Φ^+ that appear in $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ (that is, the simple roots in Φ^+). Thus $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}] = \oplus_{\alpha\in\Delta} (\mathfrak{n}/[\mathfrak{n},\mathfrak{n}])_\alpha$. We say that χ is generic if its differential is non-zero on each of the spaces $\mathfrak{n}_\alpha$ with $\alpha \in \Delta$. The Jacquet integrals are the $J_{\chi,\nu}$ with χ generic. The holomorphic continuation of Jacquet integrals in this generality (non-K-finite u) was proved in [Wal88a] (cf. [Wal92, Theorem 15.6.7]).

Theorem 4.1. *If χ is generic and if $u \in C^\infty(M\backslash K)$ then $J_{\chi,\nu}(u)$ has a holomorphic continuation to $\mathfrak{a}^*_{\mathbb{C}}$, furthermore the map $\nu \mapsto J_{\chi,\nu}$ is a weakly holomorphic map of $\mathfrak{a}^*_{\mathbb{C}}$ to $C^\infty(M\backslash K)'$ (the continuous dual space). Finally, if $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ then $J_{\chi,\nu} \neq 0$.*

The following is a result of Beuzart-Plessis [Beu20, Appendix B.3.1].

Theorem 4.2. *If χ is a generic character of N then there exists $\varepsilon > 0$ such that*

$$\int_{\ker(\chi)} a(n)^{-(1-\varepsilon)\rho}\, \mathrm{d}n < \infty.$$

We will also need the following results (see [Wal24b, Theorem 43] for the full details of a proof of Proposition 4.4) whose proof is complicated and uses parts of our proof of the holomorphic continuation of the Jacquet integrals ([Wal88a]). We will just give an idea of why they are true.

Lemma 4.3. *Assume that χ is generic. There exists a continuous semi-norm, q, on $C^\infty(M\backslash K)$ such that*

$$|J_{\chi,\nu}(u)| \leq q(u)c(\operatorname{Re}\nu)$$

if $u \in C^\infty(M\backslash K)$ and $\operatorname{Re}(\nu,\alpha) < 0$ for all $\alpha \in \Phi^+$.

Proof. If ν satisfies the condition of the assertion then

$$J_{\chi,\nu}(u) = \int_N \chi(n)^{-1} u_\nu(n)\, \mathrm{d}n = \int_N \chi(n)^{-1} a(n)^{\nu-\rho} u(k(n))\, \mathrm{d}n$$

so defining $q(u) = \sup_{k\in K} |u(k)|$ we have

$$|J_{\chi,\nu}(u)| \leq q(u) \int_N a(n)^{\operatorname{Re}\nu-\rho}\, \mathrm{d}n = q(u)c(\operatorname{Re}\nu). \qquad \square$$

Proposition 4.4. *Assume that χ is generic. And let $0 < r < \infty$. There exists a continuous semi-norm, q_r, on $C^\infty(M\backslash K)$ and m_r such that*

$$|J_{\chi,\nu}(u)| \leq q_r(u)(1 + \|\operatorname{Im}\nu\|)^{m_r}$$

for $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ such that $0 > \operatorname{Re}(\nu,\alpha) > -r(\rho,\alpha)$, $\alpha \in \Phi^+$.

Proof. This is proved using an argument involving tensoring with finite dimensional representations and details from the shift argument used in the proof of the holomorphic continuation of $J_{\chi,\nu}(u)$ in [Wal92, Section 15.5]. $\square$

Proposition 4.5. *Assume that* χ *is generic. If* $0 < r < \infty$ *is fixed and if* $f \in \mathcal{C}(G/K)$ *then for each* $m \in \mathbb{R}$ *there exists* $C_{m,r}$ *such that*

$$|J_{\chi,i\nu+z\rho}(\pi_{i\nu}(f)1)| \leq C_{m,r}(1+\|\nu\|)^{-m}$$

for $\nu \in \mathfrak{a}^*$ *and* $0 \geq \operatorname{Re} z > -r$.

Proof. Note that if p is a continuous semi-norm on $C^\infty(M\backslash K)$ then there exists $k_p \in \mathbb{R}$ and a constant $L_p > 0$ such that

$$p(u) \leq L_p \left\|(1+C_K)^{k_p} u\right\|,$$

where C_K denotes the Casimir element of K. Thus Proposition 4.4 implies that

$$|J_{\chi,i\nu+z\rho}(\pi_{i\nu}(f)1)| \leq q_r(\pi_{i\nu}(f)1)(1+\|\nu\|)^{m_r} \leq L_{q_r} \left\|\pi_{i\nu}(L_{(1+C_K)^{k_{q_r}}} f)1\right\| (1+\|\nu\|)^{m_r}.$$

Now applying Lemma A.1 in Appendix A to $L_{(1+C_K)^{k_{q_r}}} f$ with $m = m_r + l$ completes the proof. □

By analogy with the Harish-Chandra Schwartz space we have the Whittaker Schwartz space, which we now recall. If $g \in G$ and $g = nak$ $n \in N$, $a \in A$, $k \in K$, then set $a_o(g) = a$. That is, $a_o(g) = a(\theta(g))^{-1}$. Define

$$C^\infty(N\backslash G;\chi) = \{f \in C^\infty(G) \mid f(ng) = \chi(n)f(g), n \in N, g \in G\}.$$

If $f \in C^\infty(N\backslash G;\chi)$, $x \in U(\mathfrak{g})$, $d \in \mathbb{R}$ then set

$$q_{x,d}(f) = \sup_{g\in G} a_o(g)^{-\rho}(1+\|\log a_o(g)\|)^d |R_x f(g)|.$$

Then $\mathcal{C}(N\backslash G;\chi)$ is the space of $f \in C^\infty(N\backslash G;\chi)$ such that $q_{x,d}(f) < \infty$ for all x, d.

The following observation will be used in Section 8.

Lemma 4.6. *If* $h \in \mathfrak{a}$ *and* $\alpha(h) < 0$ *for all* $\alpha \in \Phi^+$ *and if* $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ *is such that* $\operatorname{Re}(\nu,\alpha) < 0$ *for all* $\alpha \in \Phi^+$ *then*

$$\lim_{t\to+\infty} e^{-t(\nu+\rho)(h)} J_{\chi,\nu}(\pi_\nu(\exp th)1) = c(\nu).$$

Proof. We calculate

$$\begin{aligned}\int_N \chi(n)^{-1} a(n\exp{(th)})^{\nu-\rho}\,\mathrm{d}n &= \int_N \chi(n)^{-1} a(\exp{(th)}\exp(-th)n\exp{(th)})^{\nu-\rho}\,\mathrm{d}n\\ &= e^{t(\nu-\rho)(h)} \int_N \chi(n)^{-1} a(\exp{(-th)}\,n\exp{(th)})^{\nu-\rho}\,\mathrm{d}n\\ &= e^{t(\nu+\rho)(h)} \int_N \chi(\exp{(th)}\,n\exp{(-th)})^{-1} a(n)^{\nu-\rho}\,\mathrm{d}n.\end{aligned}$$

Now multiply by $e^{-t(\nu+\rho)(h)}$ and take the limit keeping in mind dominant convergence. □

5. The key formula

The main result in this section is the key to our proof of the spherical Whittaker inversion theorem.

In this section $f \in \mathcal{C}(G/K)$. Theorem 3.3 says that

$$f(g) = \int_{\mathfrak{a}^*} \langle \pi_{i\nu}(L_{g^{-1}}f)1, 1\rangle\,\mu(\nu)\,\mathrm{d}\nu = \int_{\mathfrak{a}^*} \langle \pi_{i\nu}(f)1, \pi_{i\nu}(g)1\rangle\,\mu(\nu)\,\mathrm{d}\nu.$$

Thus

$$\overline{f(g)} = \int_{\mathfrak{a}^*} \langle \pi_{i\nu}(g)1, \pi_{i\nu}(f)1\rangle\, \mu(\nu)\, \mathrm{d}\nu.$$

The following result is critical to our approach to the Whittaker theory.

Theorem 5.1. $\int_N \chi(n)\overline{f(ng)}\, \mathrm{d}n = \int_{\mathfrak{a}^*} J_{\chi^{-1},i\nu}(\pi_{i\nu}(g)1)\overline{J_{\chi^{-1},i\nu}(\pi_{i\nu}(f)1)}\mu(\nu)\, \mathrm{d}\nu.$

Proof. In this proof we will use the notation $u(\nu) = \pi_{i\nu}(f)1 \in C^\infty(M\backslash K)$. Using the non-compact model for the unitary principal series (cf. [Wal18, Section 8.4.7]) one has

$$\langle \pi_{i\nu}(g)1, \pi_{i\nu}(f)1\rangle = \int_N a(ng)^{i\nu-\rho} a(n)^{-i\nu-\rho}\overline{u(\nu)(k(n))}\, \mathrm{d}n.$$

Thus

$$\int_N \chi(n)\overline{f(ng)}\, \mathrm{d}n = \int_N \chi(n_1) \int_{\mathfrak{a}^*} \int_N a(nn_1g)^{i\nu-\rho} a(n)^{-i\nu-\rho}\overline{u(\nu)(k(n))}\, \mathrm{d}n\, \mu(\nu)\, \mathrm{d}\nu\, \mathrm{d}n_1.$$

We first deform the parameter and consider

$$\int_N \chi(n_1) \int_{\mathfrak{a}^*} \int_N a(nn_1g)^{i\nu-(1+z)\rho} a(n)^{-i\nu-(1+z)\rho}\overline{u(\nu)(k(n))}\, \mathrm{d}n\, \mu(\nu)\, \mathrm{d}\nu\, \mathrm{d}n_1$$

for $\operatorname{Re} z > 0$. If we put absolute values on all of the terms we have

$$\int_N \int_{\mathfrak{a}^*} \int_N a(nn_1g)^{-(1+\operatorname{Re} z)\rho} a(n)^{-(1+\operatorname{Re} z)\rho} \left|\overline{u(\nu)(k(n))}\right|\, \mathrm{d}n\, \mu(\nu)\, \mathrm{d}\nu\, \mathrm{d}n_1$$

and using Lemma A.1 in Appendix A (and the notation therein) we have $\left|\overline{u(\nu)(k(n))}\right| \le D_{1,l}(1+\|\nu\|)^{-l}$ with $0 < D_{1,l} < \infty$. Thus the integrand is dominated by

$$C_m a(nn_1g)^{-(1+\operatorname{Re} z)\rho} a(n)^{-(1+\operatorname{Re} z)\rho}(1+\|\nu\|)^{-m}$$

for any $m > 0$ since $\mu(\nu) \le B(1+\|\nu\|)^r$ for some r. So, if we take m to be greater than $\dim A$, then the integral converges for $\operatorname{Re} z > 0$. Noting that $a(ng)^{-\rho} \le C_\omega a(n)^{-\rho}$ if $g \in \omega$ (a compact set; see Lemma A.2), the integral of the absolute values is dominated by a multiple of

$$C_\omega^{(1+\operatorname{Re} z)} \int_N \int_N a(n_1)^{-(1+\operatorname{Re} z)\rho} a(n)^{-(1+\operatorname{Re} z)\rho}\, \mathrm{d}n\, \mathrm{d}n_1 < \infty.$$

We can therefore do the deformed integral in any order. We choose

$$\begin{aligned}\int_{\mathfrak{a}^*} \int_N \chi(n_1) a(n_1g)^{i\nu-(1+z)\rho}\, \mathrm{d}n_1 \int_N \chi(n)^{-1} a(n)^{-i\nu-(1+z)\rho}\overline{u(\nu)(k(n))}\mu(\nu)\, \mathrm{d}n\, \mathrm{d}\nu \\ = \int_{\mathfrak{a}^*} J_{\chi^{-1},i\nu-z\rho}(\pi_{i\nu-z\rho}(g)1)\overline{J_{\chi^{-1},i\nu-\bar{z}\rho}(\pi_{i\nu}(f)1)}\mu(\nu)\, \mathrm{d}\nu.\end{aligned}$$

We are left with taking the limit as $z \to 0$ under the integral sign. This will be done indirectly.

Let x_o be perpendicular to $\ker d\chi$ relative to $\langle \cdot, \cdot\rangle$ and choose x_o such that $\mathrm{d}\chi(x_o) = i$. We define

$$\tau_z(t) = \int_{\ker\chi} \int_{\mathfrak{a}^*} \int_N a(n\exp(tx_o)n_1g)^{i\nu-(1+z)\rho} a(n)^{-i\nu-(1+z)\rho}\overline{u(\nu)(k(n))}\, \mathrm{d}n\, \mu(\nu)\, \mathrm{d}\nu\, \mathrm{d}n_1$$

for $\operatorname{Re} z \ge 0$. Then Proposition A.3 in Appendix A imply that if ω is a compact subset of G then

$$|\tau_z(t)| \le C_\omega^{(1+\operatorname{Re} z)} B(1+|t|)^d$$

with C_ω, d and B finite. Fix $g \in G$. The estimate above, on τ_z, implies that we can define a family of tempered distributions on $\mathbb{R}$ by

$$T_z(\phi) = \int_{-\infty}^{\infty} \tau_z(t)\phi(t)\,\mathrm{d}t$$

for $\operatorname{Re} z \geq 0$. Dominated convergence implies that the map $z \mapsto T_z$ is a weakly continuous map of $\{z \mid \operatorname{Re} z \geq 0\}$ to $\mathcal{S}'(\mathbb{R})$ (the space of tempered distributions). Let $\mathcal{F}$ denote the usual Fourier transform, that is

$$\mathcal{F}(\phi)(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist}\phi(t)\,\mathrm{d}t,$$

which is a continuous linear isomorphism of $\mathcal{S}(\mathbb{R})$. If T is a tempered distribution we define (as usual) $\mathcal{F}(T) = T \circ \mathcal{F}$. If T is given by integration by an element of $L^1(\mathbb{R})$, i.e. $T(\phi) = \int_{-\infty}^{\infty} \tau(t)\phi(t)\,\mathrm{d}t$ with $\tau \in L^1(\mathbb{R})$, then

$$\mathcal{F}(T)(\phi) = \int_{-\infty}^{\infty} \mathcal{F}(\tau)(t)\phi(t)\,\mathrm{d}t$$

If $\operatorname{Re} z > 0$ then as we have seen above

$$n_1 \longmapsto \int_{\mathfrak{a}^*} \int_N a(nn_1g)^{i\nu-(1+z)\rho} a(n)^{-i\nu-(1+z)\rho} \overline{u(\nu)(k(n))}\,\mathrm{d}n\,\mu(\nu)\,\mathrm{d}\nu$$

defines an element of $L^1(N_1)$. So Fubini's theorem implies that $\tau_z \in L^1(\mathbb{R})$ if $\operatorname{Re} z > 0$. For $z = 0$ then

$$\tau_0(t) = \int_{\ker\chi} \int_{\mathfrak{a}^*} \langle \pi_{i\nu}(\exp(tx_o)ng)1, \pi_{i\nu}(f)1\rangle\,\mu(\nu)\,\mathrm{d}\nu\,\mathrm{d}n = \int_{\ker\chi} \overline{f(\exp(tx_o)ng)}\,\mathrm{d}n.$$

Since f, in particular, is in $\mathcal{C}(G)$, the function $n \mapsto \overline{f(ng)}$ on N is in $L^1(N)$. Thus $\tau_0 \in L^1(\mathbb{R})$. Let for $s, t \in \mathbb{R}$, $n \in \ker\chi$, $\chi_s(\exp(tx_o)n) = e^{its}$. Then $\chi_1 = \chi$ and χ_s is generic for $s \neq 0$. Proposition D.2 in Appendix D proves that the map

$$(\mathbb{R} - \{0\}) \times \mathfrak{a}^*_{\mathbb{C}} \times C^\infty(M\backslash K) \longrightarrow \mathbb{C}, \quad (t, \nu, \phi) \longmapsto J_{\chi_t,\nu}(\phi)$$

is continuous. Note that if $\operatorname{Re} z > 0$ and $s \neq 0$ then

$$\mathcal{F}(\tau_z)(s) = \int_{\mathfrak{a}^*} J_{\chi_s^{-1}, i\nu - z\rho}(\pi_{i\nu-z\rho}(g)1)\overline{J_{\chi_s^{-1}, i\nu-\bar{z}\rho}(\pi_{i\nu}(f)1)}\mu(\nu)\,\mathrm{d}\nu.$$

Also if $z = 0$ and $s \neq 0$ then

$$\mathcal{F}(\tau_0)(s) = \overline{\int_N \chi_s(n)^{-1} f(ng)\,\mathrm{d}n}.$$

Define for $s \neq 0$

$$\sigma_z(s) = \int_{\mathfrak{a}^*} J_{\chi_s^{-1}, i\nu - z\rho}(\pi_{i\nu-z\rho}(g)1)\overline{J_{\chi_s^{-1}, i\nu-\bar{z}\rho}(\pi_{i\nu}(f)1)}\mu(\nu)\,\mathrm{d}\nu$$

then Proposition 4.5 implies that σ_z is continuous in z for $s \neq 0$ and $\operatorname{Re} z \geq 0$. Furthermore, if $\operatorname{Re} z > 0$ and $s \neq 0$, then $\sigma_z(s) = \mathcal{F}(\tau_z)(s)$. Hence, if ϕ has support in $\mathbb{R} - \{0\}$,

$$\lim_{\substack{\operatorname{Re} z > 0 \\ z \to 0}} \mathcal{F}(\tau_z)(\phi) = \mathcal{F}(\tau_0)(\phi) = \int_{-\infty}^{\infty} \int_N \chi_s(n) f(ng)\,\mathrm{d}n\,\phi(s)\,\mathrm{d}s.$$

Also

$$\lim_{\substack{\operatorname{Re} z>0\\ z\to 0}} \int_{-\infty}^{\infty}\int_{\mathfrak{a}^*} J_{\chi_s^{-1},i\nu-z\rho}(\pi_{i\nu-z\rho}(g)1)\overline{J_{\chi_s^{-1},i\nu-\bar{z}\rho}(\pi_{i\nu}(f)1)}\mu(\nu)\,\mathrm{d}\nu\,\phi(s)\,\mathrm{d}s$$

$$= \int_{-\infty}^{\infty}\sigma_0(s)\phi(s)\,\mathrm{d}s = \int_{-\infty}^{\infty}\int_{\mathfrak{a}^*} J_{\chi_s^{-1},i\nu}(\pi_{i\nu}(g)1)\overline{J_{\chi^{-1},i\nu}(\pi_{i\nu}(f)1)}\mu(\nu)\,\mathrm{d}\nu\,\phi(s)\,\mathrm{d}s,$$

completing the proof. □

Using the same methods as in the proof of the above theorem one can prove:

Theorem 5.2. *Assume that χ is generic. Let u, α and f be defined as in Theorem 3.4 then*

$$\int_N \chi(n)^{-1}f(ng)\,\mathrm{d}n = \int_{\mathfrak{a}^*} J_{\chi,i\nu}(\pi_{i\nu}(g)1)\overline{J_{\chi,i\nu}(u)}\alpha(\nu)\mu(\nu)\,\mathrm{d}\nu.$$

Corollary 5.3. *If $\beta \in C_c^{\infty}(\mathfrak{a}^*)$ then*

$$\left(g \mapsto \int_{\mathfrak{a}^*} J_{\chi,i\nu}(\pi_{i\nu}(g)1)\beta(\nu)\mu(\nu)\,\mathrm{d}\nu\right) \in \mathcal{C}(N\backslash G/K,\chi).$$

Proof. Let ω be the support of β. If $\nu \in \omega$ let u_ν be a K–finite element of $C^\infty(M\backslash K)$ such that $J_{i\nu}(u_\nu) \neq 0$ (this exists since $J_\nu \neq 0$ for all ν). Let U_ν be an open set with compact closure in $\mathfrak{a}^*$ such that $J_{i\lambda}(u_\nu) \neq 0$ for $\lambda \in U_\nu$. Then the covering U_ν of ω has a finite refinement $U_{\nu_1}, \ldots, U_{\nu_m}$. Let $\alpha_1, \ldots, \alpha_r$ be a partition of unity subordinate to this covering of ω. Then for each i there exists $w_i \in C^\infty(M\backslash K)$ that is K-finite and W_i an open set containing the support of α_i such that $J_{\chi,i\lambda}(w_i) \neq 0$ for $\lambda \in W_i$. Define

$$\varphi_i(\lambda) = \frac{\alpha_i(\lambda)\beta(\lambda)}{\overline{J_{\chi,i\lambda}(w_i)}}.$$

Then $\varphi_i \in C_c^\infty(\mathfrak{a}^*)$ and

$$\beta(\lambda) = \sum_{i=1}^{r} \overline{J_{\chi,i\lambda}(w_i)}\varphi_i(\lambda).$$

Now apply the previous theorem and Theorem B.4. □

6. The spherical Whittaker Inversion Theorem

If χ is a generic character of N set $J_{\chi,\nu}$ equal to the corresponding Jacquet integral. We will need the following:

Lemma 6.1. *Assume χ is generic. Let $\psi \in \mathcal{C}(N\backslash G;\chi)$ and let $\varphi \in C_c^\infty(N)$ such that*

$$\int_N \chi(n)^{-1}\varphi(n)\,\mathrm{d}n = 1.$$

Set $f(nak) = \varphi(n)\psi(ak)$ for $n \in N$, $a \in A$, $k \in K$. Then $f \in \mathcal{C}(G)$ and if $u \in C^\infty(M\backslash K)$ then

$$J_{\chi^{-1},i\nu}(\pi_{i\nu}(f)u) = \int_{N\backslash G} J_{\chi^{-1},i\nu}(\pi_{i\nu}(g)u)\psi(g)\,\mathrm{d}g.$$

Proof. Corollary B.3 in Appendix B proves that $f \in \mathcal{C}(G)$. We calculate

$$\begin{aligned}\int_{N\backslash G} J_{\chi^{-1},i\nu}(\pi_{i\nu}(g)u)\psi(g)\,\mathrm{d}g &= \int_{A\times K} a^{-2\rho} J_{\chi^{-1},i\nu}(\pi_{i\nu}(ak)u)\psi(ak)\,\mathrm{d}a\,\mathrm{d}k \\ &= \int_N \chi(n)^{-1} \int_{A\times K} a^{-2\rho} J_{\chi^{-1},i\nu}(\pi_{i\nu}(ak)u)\varphi(n)\psi(ak)\,\mathrm{d}a\,\mathrm{d}k\,\mathrm{d}n \\ &= \int_{N\times A\times K} a^{-2\rho} J_{\chi^{-1},i\nu}(\pi_{i\nu}(nak)u)\varphi(n)\psi(ak)\,\mathrm{d}a\,\mathrm{d}k\,\mathrm{d}a \\ &= \int_G J_{\chi^{-1},i\nu}(\pi_{i\nu}(g)u)f(g)\,\mathrm{d}g.\end{aligned}$$

For each $\nu \in \mathfrak{a}^*$, $J_{\chi^{-1},i\nu}$ is a continuous functional on $C^\infty(M\backslash K)$. Also, since $J_{\chi,i\nu}$ is tame in the sense of [Wal92, Section 15.2.3] (here the following changes in the statement of the Lemma 15.2.3 should be made: χ must be assumed generic and λ is tame for all standard parabolic pairs (Q, A)), [Wal92, Theorem 15.2.5] implies that for each ν in $\mathfrak{a}^*$ there exist C and d such that if $u \in C^\infty(M\backslash K)$ then

$$|J_{\chi^{-1}.i\nu}(\pi_{i\nu}(g)1)| \le C\,|g|^{-\frac{1}{2}}\,(1 + \log\|g\|)^d.$$

This implies that

$$\int_G J_{\chi^{-1},i\nu}(\pi_{i\nu}(g)u)f(g)\,\mathrm{d}g = J_{\chi^{-1},i\nu}\left(\int_G f(g)\pi_{i\nu}(g)u\,\mathrm{d}g\right) = J_{\chi^{-1},i\nu}(\pi_{i\nu}(f)u). \qquad \square$$

In the above proof the following assertion was proved:

Lemma 6.2. *If* $u \in C^\infty(M\backslash K)$, $f \in \mathcal{C}(G)$ *and* $\nu \in \mathfrak{a}^*$ *then*

$$\int_G J_{\chi,i\nu}(\pi_{i\nu}(g)u)f(g)\,\mathrm{d}g = J_{\chi^{-1},i\nu}(\pi_{i\nu}(f)u).$$

If η is a character of N then define for $u \in \mathcal{C}(G)$, $u_\eta(g) = \int_N \eta(n)^{-1}u(ng)\,\mathrm{d}n$.

Theorem 6.3. *Let* $\psi \in \mathcal{C}(N\backslash G/K;\chi)$. *Set*

$$W_\chi(\nu,\psi) = \int_{N\backslash G} \overline{J_{\chi,i\nu}(\pi_{i\nu}(g)1)}\psi(g)\,\mathrm{d}g$$

then

$$\psi(g) = \int_{\mathfrak{a}^*} W_\chi(\nu,\psi)J_{\chi,i\nu}(\pi_{i\nu}(g)1)\mu(\nu)\,\mathrm{d}\nu.$$

Proof. Let f be as in Lemma 6.1 for ψ. We observe that

$$\overline{J_{\chi,i\nu}(\pi_{i\nu}(g)1)} = J_{\chi^{-1},-i\nu}(\pi_{-i\nu}(g)1).$$

Thus if f is defined as above for ψ then Lemma 6.1 implies that

$$W_\chi(\nu,\psi) = J_{\chi^{-1},-i\nu}(\pi_{i\nu}(f)1).$$

The spherical Plancherel Theorem implies that

$$f(g) = \int_{\mathfrak{a}^*} \langle \pi_{i\nu}(L_{g^{-1}}f)1, 1\rangle\,\mu(\nu)\,\mathrm{d}\nu = \int_{\mathfrak{a}^*} \langle \pi_{i\nu}(f)1, \pi_{i\nu}(g)1\rangle\,\mu(\nu)\,\mathrm{d}\nu$$

so

$$\bar{f}(g) = \overline{f(g)} = \int_{\mathfrak{a}^*} \langle \pi_{i\nu}(g)1, \pi_{i\nu}(f)1\rangle\,\mu(\nu)\,\mathrm{d}\nu.$$

Theorem 5.1 implies that

$$\overline{f}_{\chi^{-1}}(g) = \int_N \chi(n)\overline{f(ng)}\,\mathrm{d}n = \int_{\mathfrak{a}^*} J_{\chi^{-1},i\nu}(\pi_{i\nu}(g)1)\overline{J_{\chi^{-1},i\nu}(\pi_{i\nu}(f)1)}\mu(\nu)\,\mathrm{d}\nu.$$

Thus

$$\psi(g) = f_\chi(g) = \int_N \chi(n)^{-1}f(ng)\,\mathrm{d}n = \int_{\mathfrak{a}^*} \overline{J_{\chi^{-1},i\nu}(\pi_{i\nu}(g)1)}J_{\chi^{-1},i\nu}(\pi_{i\nu}(f)1)\mu(\nu)\,\mathrm{d}\nu.$$

The first part of this proof implies that

$$J_{\chi^{-1},i\nu}(\pi_{i\nu}(f)1) = W_\chi(-\nu,\psi).$$

Also,

$$\overline{J_{\chi^{-1},i\nu}(\pi_{i\nu}(g)1)} = J_{-i\nu}(\pi_{-i\nu}(g)1).$$

Hence

$$\psi(g) = \int_{\mathfrak{a}^*} W_\chi(-\nu,\psi)J_{-i\nu}(\pi_{-\nu}(g)1)\mu(\nu)\,\mathrm{d}\nu.$$

This proves the theorem since $\mu(\nu) = \mu(-\nu)$. □

Corollary 6.4. *With the notation as in the previous theorem, if* $f, h \in \mathcal{C}(N\backslash G/K;\chi)$ *then*

$$\int_{N\backslash G} f(g)\overline{h(g)}\,\mathrm{d}g = \int_{\mathfrak{a}^*} W_\chi(\nu,f)\overline{W_\chi(\nu,h)}\mu(\nu)\,\mathrm{d}\nu.$$

Proof. We calculate. The previous theorem implies that

$$\begin{aligned}\int_{N\backslash G} f(g)\overline{h(g)}\,\mathrm{d}g &= \int_{N\backslash G}\int_{\mathfrak{a}^*} W_\chi(\nu,f)J_{\chi,i\nu}(\pi_{i\nu}(g)1)\mu(\nu)\,\mathrm{d}\nu\overline{h(g)}\,\mathrm{d}g \\ &= \int_{\mathfrak{a}^*} W_\chi(\nu,f)\overline{\int_{N\backslash G} h(g)\overline{J_{\chi,i\nu}(\pi_{i\nu}(g)1)}\,\mathrm{d}g}\mu(\nu)\,\mathrm{d}\nu \\ &= \int_{\mathfrak{a}^*} W_\chi(\nu,f)\overline{W_\chi(\nu,h)}\mu(\nu)\,\mathrm{d}\nu.\end{aligned}$$

□

The above result implies that the map

$$W_\chi : \mathcal{C}(N\backslash G/K;\chi) \longrightarrow L^2(\mathfrak{a}^*,\mu(\nu)\,\mathrm{d}\nu)$$

extends to a continuous inner product preserving map of $L^2(N\backslash G/K,\chi)$ to $L^2(\mathfrak{a}^*,\mu(\nu)\,\mathrm{d}\nu)$. However it does not describe the image. This will be done in the last section, before the appendices, of this paper. The general version of the description of the image appears in [Wal24a]. The proof in this paper is more explicit and elementary.

7. The non-periodic Toda Lattice

The original non-periodic Toda Lattice is the Hamiltonian system with Hamiltonian

$$H(p,q) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} c_i^2 e^{2(q_i-q_{i+1})},$$

$c_i \in \mathbb{R} - \{0\}$ and $p, q \in \mathbb{R}^n$. Using the quantization rules (here Planck's constant is normalized to 1) $p_j \to i\frac{\partial}{\partial q_j}$ and $f(q) \to M_{f(q)}$ with $M_{f(q)}$ the operator given by multiplication by $f(q)$ on $C^\infty(\mathbb{R}^n)$, the quantum Hamiltonian is

$$\mathcal{H} = -\frac{1}{2}\Delta_q + \sum_{i=1}^{n-1} c_i^2 e^{2(q_i-q_{i+1})}$$

where

$$\Delta_q = \sum_{i=1}^{n} \frac{\partial^2}{\partial q_j^2}.$$

Consider the algebra, $\mathcal{A}$, of linear differential operators on $\mathbb{R}^n$ with coefficients in the algebra generated by $e^{(q_i - q_{i+1})}, i = 1, \dots, n-1$ and $q_i, i = 1, \dots, n$ over $\mathbb{C}$. We take as a domain for this algebra the space, $\mathcal{T}$, of $f \in C^\infty(\mathbb{R}^n)$ such that

$$t_{m,d,x}(f) = \sup_{q \in \mathbb{R}^n} e^{\sum_{j=1}^{n-1} m_i(q_i - q_{i+1})}(1 + \|q\|)^d |xf(q)| < \infty$$

with $m = (m_1, \dots, m_n), m_i$ and $d \in \mathbb{Z}_{\geq 0}$ and x is a constant coefficient differential operator on $\mathbb{R}^n$. Endow $\mathcal{T}$ with the topology induced by these semi-norms. This space is invariant under $\mathcal{A}$ and $\mathcal{A}$ acts continuously on it. In [GW82, Section 2] it was shown that the centralizer of $\mathcal{H}$ in $\mathcal{A}$ is a commutative algebra generated over $\mathbb{C}$ by n elements $D_1 = \sum_{i=1}^n \frac{\partial}{\partial q_j}, D_2 = \mathcal{H}, D_3, \dots, D_n$, with algebraically independent symbols (considering the D_i as differential operators on $\mathbb{R}^n$) $\sigma(D_1), \dots, \sigma(D_n)$ that are generators for the S_n invariant constant coefficient differential operators. A solution to the quantum Toda lattice is thus a family of functions $K_\nu(q)$, $\nu \in (\mathbb{R}^n)^*$ such that

$$\int_{(\mathbb{R}^n)^*} |K_\nu(q) f(q)| \, \mathrm{d}q < \infty, \qquad f \in \mathcal{T}$$

and

$$D_j K_\nu = \sigma(D_j)(\nu) K_\nu, \qquad j = 1, \dots, n. \tag{$*$}$$

One has the following inversion formula: There exists a non-negative function $\gamma(\nu)$ on $(\mathbb{R}^n)^*$ such that $|\gamma(\nu)| \leq C(1 + \|\nu\|)^r$ for some $r \in \mathbb{R}$ and all $\nu \in (\mathbb{R}^n)^*$ and such that if $f \in \mathcal{T}$ and if

$$\mathcal{K}(f)(\nu) = \int_{\mathbb{R}^n} f(q) \overline{K_\nu(q)} \, \mathrm{d}q$$

then

$$f(q) = \int_{(\mathbb{R}^n)^*} \mathcal{K}(f)(\nu) K_\nu(q) \gamma(\nu) \, \mathrm{d}\nu,$$

and if $f_1, f_2 \in \mathcal{T}$ then

$$\int_{\mathbb{R}^n} f_1(q) \overline{f_2(q)} \, \mathrm{d}q = \int_{(\mathbb{R}^n)^*} \mathcal{K}(f_1)(\nu) \overline{\mathcal{K}(f_2)(\nu)} \gamma(\nu) \, \mathrm{d}\nu.$$

Remark 7.1. Note there should also be a description of the Hilbert space completion of the space of functions $\mathcal{K}(f)(\nu)$ for $f \in \mathcal{T}$. It will be given in the last section of this paper.

We now return to the situation of the preceding sections. Let G and the notation be as in Section 2, so we assume that, in particular, $G \subset GL(n, \mathbb{R})$ for some n. In particular, $\mathfrak{g} = \mathrm{Lie}(G)$. Let C be the Casimir operator corresponding to the invariant form

$$B(X, Y) = \operatorname{tr} XY$$

for X, Y in $\mathfrak{g}$. That is, if $X_1, \dots, X_m$ is a basis of $\mathfrak{g}$ and $Y_1, \dots, Y_m$ are defined by $B(X_i, Y_j) = \delta_{ij}$ then

$$C = \sum X_i Y_i \in U(\mathfrak{g}).$$

Let $\theta, N, A, K, \mathfrak{n}, \mathfrak{a}, \mathfrak{k}, \Phi^+, \Delta$ be as before. Then $\mathfrak{n} = \sum_{\alpha\in\Phi^+} \mathfrak{n}_\alpha$ and let $X_{\alpha,j}, j = 1, \ldots, m_\alpha = \dim \mathfrak{n}_\alpha$ be an orthonormal basis of $\mathfrak{n}_\alpha$ relative to the inner product

$$\langle X, Y\rangle = -B(X, \theta Y).$$

We define the generalized quantum non-periodic Toda Lattices associated with G to be the operator on $C^\infty(A)$ given by

$$L_c = -\frac{\sum h_i^2}{2} + \sum_{\alpha\in\Delta} c_\alpha^2 a^{2\alpha}$$

with $c_\alpha \in \mathbb{R} - \{0\}$. For $G = GL(n, \mathbb{R})$ take A to be the group of diagonal $n \times n$ matrices with positive coefficients and N the group of upper triangular $n\times n$ matrices with ones on the main diagonal. Then identifying $\mathfrak{a}$ with $\mathbb{R}^n$ via the map

$$(x_1, \ldots, x_n) \longmapsto \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & x_n \end{bmatrix}$$

we have $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$ with $\alpha_i(x) = x_i - x_{i-1}$. Thus

$$L_c = -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{i=1}^{n-1} c_{\alpha_i}^2 e^{2(x_i - x_{i+1})}.$$

Set $K_\nu(x) = a^{-\rho} J_{i\nu}(\pi_{i\nu}(\exp x)1)$ for $x \in \mathfrak{a}$. Let $(\,\cdot\,,\cdot\,)$ also denote the complex bilinear extension of the dual form, $(\,\cdot\,,\cdot\,)$ of $B_{|\mathfrak{a}}$ to $\mathfrak{a}_{\mathbb{C}}^*$.

Proposition 7.2. *Let χ be a generic character of N. Set for $\alpha \in \Delta$*

$$c_\alpha^2 = -\sum_{j=1}^{m_\alpha} (d\chi(X_{\alpha,j}))^2 > 0.$$

If $\nu \in \mathfrak{a}^*$ and $a \in A$ then

$$L_c K_\nu(a) = \frac{\|\nu\|^2}{2} K_\nu(a).$$

Proof. We have for $\mu \in \mathfrak{a}_{\mathbb{C}}^*$

$$C J_\mu(\pi_\mu(g)1) = J_\mu(\pi_\mu(g) d\pi_\mu(C)1) = ((\mu, \mu) - (\rho, \rho)) J_\mu(\pi_\mu(g)1).$$

The proposition now follows directly from Lemma C.1 in Appendix C. □

Note that if $\mathcal{W}(\mathfrak{a})$ is as in Appendix B and if $\phi \in \mathcal{W}(\mathfrak{a})$ then the function $a^{-\rho}\phi$ is in the space $\mathcal{T}(\mathfrak{a})$ defined as follows: Define for $u \in C^\infty(\mathfrak{a})$ the semi-norm

$$t_{d,r,x}(u) = \sup_{h\in\mathfrak{a}} e^{\sum_{\alpha\in\Delta} r_\alpha \alpha(h)} (1 + \|h\|)^d\, |xu(h)|$$

with $r = (r_\alpha)_{\alpha\in\Delta}$, $r_\alpha, d \in \mathbb{Z}_{\geq 0}$ and x is a constant coefficient differential operator on $\mathfrak{a}$. $\mathcal{T}(\mathfrak{a})$ is the space of all u in $C^\infty(\mathfrak{a})$ such that $t_{d,r,x}(u) < \infty$ for all d, r, x endowed with the topology induced by these semi-norms. Then the map

$$\mu \longmapsto \left[h \mapsto e^{-\rho(h)}\mu(\exp h)\right]$$

defines a topological isomorphism of $\mathcal{W}(\mathfrak{a})$ onto $\mathcal{T}(\mathfrak{a})$ with inverse $u \mapsto (a \mapsto e^\rho u(\log a))$. The main results of the previous section can be stated in the following form.

Theorem 7.3. *If* $u \in \mathcal{T}(\mathfrak{a}), \nu \in \mathfrak{a}^*$ *set*

$$\mathcal{K}(u)(\nu) = \int_{\mathfrak{a}} u(h)\overline{K_\nu(h)}\,\mathrm{d}h$$

then

$$u(h) = \int_{\mathfrak{a}^*} K_\nu(h)\mathcal{K}(u)(\nu)\mu(\nu)\,\mathrm{d}\nu.$$

Furthermore, if $u, w \in \mathcal{T}(\mathfrak{a})$ *then*

$$\int_{\mathfrak{a}} u(h)\overline{w(h)}\,\mathrm{d}h = \int_{\mathfrak{a}^*} \mathcal{K}(u)(\nu)\overline{\mathcal{K}(w)(\nu)}\mu(\nu)\,\mathrm{d}\nu.$$

Proof. We note that if $u \in \mathcal{T}(\mathfrak{a})$ then $u(h) = e^{-\rho(h)}\phi(\exp h)$ with $\phi \in \mathcal{W} = \mathcal{C}(N\backslash G/K;\chi)_{|A}$. Thus if $\psi(nak) = \chi(n)\phi(a)$ then

$$\begin{aligned}\mathcal{K}(u)(\nu) &= \int_A a^{-\rho}\phi(a)a^{-\rho}\overline{J_{i\nu}(\pi_{i\nu}(a)1)}\,\mathrm{d}a = \int_A a^{-2\rho}\phi(a)\overline{J_{i\nu}(\pi_{i\nu}(a)1)}\,\mathrm{d}a\\ &= \int_{N\backslash G}\psi(g)\overline{J_{i\nu}(\pi_{i\nu}(g)1)}\,\mathrm{d}g = \mathcal{W}_\chi(\nu,\psi).\end{aligned}$$

The theorem says that

$$\psi(g) = \int_{\mathfrak{a}^*}\mathcal{W}_\chi(\nu,\psi)J_{i\nu}(\pi_{i\nu}(g)1)\mu(\nu)\,\mathrm{d}\nu.$$

So

$$\begin{aligned}u(h) = e^{-\rho(h)}\psi(\exp h) &= e^{-\rho(h)}\int_{\mathfrak{a}^*}\mathcal{K}(u)(\nu)J_{i\nu}(\pi_{i\nu}(\exp h)1)\mu(\nu)\,\mathrm{d}\nu\\ &= \int_{\mathfrak{a}^*}\mathcal{K}(u)(\nu)K_\nu(h)\mu(\nu)\,\mathrm{d}\nu.\end{aligned}$$

The above formulas lead to the second assertion of the theorem. □

The Hilbert space completion of the space of functions $\nu \mapsto \mathcal{K}(u)(\nu)$, $u \in \mathcal{T}(\mathfrak{a})$ relative to $\mu(\nu)\,\mathrm{d}\nu$ is given in Theorem 9.9.

The rest of this section involves recalling several results from [GW82; GW84; GW86] which are used in the proof of the integrability of the generalized Toda Lattices (that is, condition (∗) above).

Since $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ the Poincaré–Birkhoff–Witt Theorem implies that

$$U(\mathfrak{g}) = U(\mathfrak{n} \oplus \mathfrak{a}) \oplus U(\mathfrak{g})\mathfrak{k}.$$

Let p denote the projection of $U(\mathfrak{g})$ onto $U(\mathfrak{n} \oplus \mathfrak{a})$ corresponding to this direct sum decomposition. If $x \in U(\mathfrak{g})^{\mathfrak{k}}$ (the centralizer of $\mathfrak{k}$ in $U(\mathfrak{g})$) and if $y \in U(\mathfrak{g})$ then $y = p(y) + \sum u_iY_i$ and $x = p(x) + \sum w_iY_i$,with $Y_i \in \mathfrak{k}, u_i, w_i \in U(\mathfrak{g})$. Thus

$$\begin{aligned}yx &= p(y)x + \sum u_iY_ix = p(y)x + \sum u_ixY_i\\ &= p(y)p(x) + p(y)\sum w_iY_i + \sum u_ixY_i.\end{aligned}$$

Thus, $p(yx) = p(y)p(x)$. Consider the two-sided ideal in $U(\mathfrak{n} \oplus \mathfrak{a})$,

$$\mathcal{I} = U(\mathfrak{n} \oplus \mathfrak{a})[\mathfrak{n}, \mathfrak{n}].$$

Then

$$U(\mathfrak{n} \oplus \mathfrak{a})/\mathcal{I} \cong U(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] \oplus \mathfrak{a}).$$

Let $\mu : U(\mathfrak{n} \oplus \mathfrak{a}) \to U(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] \oplus \mathfrak{a})$ be the corresponding surjection and $q : U(\mathfrak{g})^{\mathfrak{k}} \to U(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] \oplus \mathfrak{a})$ be the corresponding homomorphism, that is, $q = \mu \circ p$. If $h \in \mathfrak{a}$ and

$f \in C^\infty(\mathfrak{a})$ then define $\partial_h f(x) = \frac{\mathrm{d}}{\mathrm{d}t} f(x+th)_{|t=0}$. If $\lambda \in \mathfrak{a}^*$ define $e_\lambda f = e^\lambda f$. Then $[\partial_h, e_\lambda] = \lambda(h)e_\lambda$. Let $\mathcal{A}$ be the algebra of operators on $C^\infty(\mathfrak{a})$ generated by ∂_h and e_α for $h \in \mathfrak{a}$ and $\alpha \in \Delta$. We define $\tau(h) = \partial_h$ and $\tau(x) = \mathrm{d}\chi(x)e_\alpha$ if $x \in (\mathfrak{n}/[\mathfrak{n},\mathfrak{n}])_\alpha$. Then $\tau \circ q$ defines a homomorphism of $U(\mathfrak{n}/[\mathfrak{n},\mathfrak{n}] \oplus \mathfrak{a})$ onto $\mathcal{A}$. In [GW82; GW84; GW86] we proved

Theorem 7.4. *The centralizer of* $\tau \circ q(C)$ *in* $\mathcal{A}$ *is* $\tau \circ q(U(\mathfrak{g})^{\mathfrak{k}})$.

This result implies that the centralizer of L_c in $\mathcal{A}$ is an algebra generated by $\dim A$ elements with algebraically independent symbols. The results of [GW82; GW84; GW86] imply that

Theorem 7.5. *If* $D \in \mathcal{A}$ *and* $[D, L_c] = 0$ *then* $DK_\nu = \sigma(D)(i\nu)K_\nu$.

Thus, in particular, all of the assertions above for the quantum non-periodic Toda lattice have been proved.

8. An implication

The purpose of this section is to prove a restriction theorem for spherical Whittaker functions which is noted indirectly in the case of real rank 1 in [GW80].

If V is a smooth Fréchet module set $Wh_\chi(V)$ equal to the space of all $\mu \in V'$ (the continuous dual) such that $\mu(nv) = \chi(n)\mu(v)$ for $n \in N, v \in V$. We now construct a split subgroup of G, $G_o = G_{o,\chi}$, such that $\mathfrak{a}$ is a Cartan subalgebra of $\mathrm{Lie}(G_o)$, $\theta(G_o) = G_o$, $(G_o \cap K)(G_o \cap A)(G_o \cap N)$ is an Iwasawa decomposition of G_o and $\chi_{|G_o \cap N}$ is a generic character of $G_o \cap N$.

Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ and, for each j, choose $e_j \in \mathfrak{g}_{\alpha_j}$ so that $\mathrm{d}\chi(e_j) \neq 0$, $-B(e_j, \theta e_j) = \frac{2}{(\alpha_j, \alpha_j)}$. Take $f_j = -\theta e_j$ and note that then $h_j = [e_j, f_j] \in \mathfrak{a}$ and $B(h_j, h) = \alpha_j(h)$ if $h \in \mathfrak{a}$. In particular this implies that $[h_j, e_j] = 2e_j$ and $[h_j, f_j] = -2f_j$ (this can be done since if $x \in \mathfrak{n}$ and $x \neq 0$ then $B(x, \theta x) < 0$). Then $h_i, \ldots, h_l$ is a basis of $[\mathfrak{g},\mathfrak{g}] \cap \mathfrak{a}$. Take $\mathfrak{g}_o$ to be the Lie algebra generated by the $e_i, f_i, i = 1, \ldots, l$ and $\mathfrak{a}$. One checks that if $A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}$ then

$$[h_i, e_j] = A_{ij} e_j, \qquad [h_i, f_j] = -A_{ij} f_j.$$

We also note that if $i \neq j$ then, since the α_i are simple, $[e_i, f_j] = 0$ and $A_{ij} \leq 0$. This implies, using the representation theory of $\mathrm{SL}(2,\mathbb{R})$, that if $i \neq j$ then

$$\mathrm{ad}(e_i)^{-A_{ij}+1} e_j = \mathrm{ad}(f_i)^{-A_{ij}+1} f_j = 0.$$

Thus Serre's variant [Ser01, Theorem in Appendix to Chapter VI] of Harish-Chandra's theorem constructing the semi-simple Lie algebras split over fields of characteristic 0 implies that $\mathfrak{g}_o$ has the desired properties.

Let G_o be the connected subgroup of G corresponding to $\mathfrak{g}_o$ and let $G_o = N_o A K_o$ be an Iwasawa decomposition of G_o with $K_o = G_o \cap K$ and $N_o = G_o \cap N$. We can now define the objects $J, c(\nu), \pi_\nu$, etc. for G_o. We will use a superscript for an object if it pertains to the group G or respectively G_o. That is J^G or J^{G_o}. Also to indicate the dependence on χ we will write $J^G_{\chi,\nu}$.

If χ is a generic character of N let $\mathrm{d}\chi_{|\mathfrak{n}_\alpha} = i\xi_\alpha$, with $\xi_\alpha \in \mathfrak{n}^*_\alpha$ for $\alpha \in \Delta$, as before. We choose η a character of N_o by the condition that $\mathrm{d}\eta(e_i) = i\,\|\xi_{\alpha_i}\|$.

For the proof of the next result we need to recall some well known facts about spherical representations (cf. [Wal92, Sections 11.2 and 11.3] for the generalization of the results to representations containing small K–types). Let γ^{G_o} denote the Harish-Chandra homomorphism of $U((\mathfrak{g}_o)_{\mathbb{C}})^{K_o}$ to $U(\mathfrak{a}_{\mathbb{C}})^{W(A)}$(see [Wal88b, Theorem 3.3.3]). We define for $\nu \in \mathfrak{a}_{\mathbb{C}}^*$

$$Y^\nu = U((\mathfrak{g}_o)_{\mathbb{C}}) \otimes_{U((\mathfrak{k}_o)_{\mathbb{C}})U((\mathfrak{g}_o)_{\mathbb{C}})^{K_o}} \mathbb{C}_\nu,$$

with $\mathbb{C}_\nu$ the $U((\mathfrak{k}_o)_{\mathbb{C}})U((\mathfrak{g}_o)_{\mathbb{C}})^{\mathfrak{k}_o}$ module, $\mathbb{C}$, with $(\mathfrak{k}_o)_{\mathbb{C}}$ acting by 0 and $U((\mathfrak{g}_o)_{\mathbb{C}})^{K_o}$ acting by $x \mapsto \nu(\gamma^{G_o}(x))$. Noting that if $x \in U((\mathfrak{g}_o)_{\mathbb{C}})^{\mathfrak{k}_o}$ then $\pi_\nu^{G_o}(x)1 = \nu(\gamma^{G_o}(x))$, we define

$$V_\nu : Y^\nu \longrightarrow I_\nu$$

by $V_\nu(x) = \pi_\nu^{G_o}(x)1$ for $x \in U((\mathfrak{g}_o)_{\mathbb{C}})$. Then since $c(\nu) \neq 0$ for $\operatorname{Re}\nu < 0$ we see that V_ν surjective for $\operatorname{Re}\nu < 0$. Since as representations of $\mathfrak{k}_o$–modules Y^ν and I_ν are isomorphic, for $\operatorname{Re}\nu < 0$ we write U_ν for V_ν^{-1}.

Theorem 8.1. *Let χ be a generic character of G and let η be as above for χ then for all $a \in \mathcal{A}$*

$$c^{G_o}(\nu)a^{-\rho^G} J_{\chi,\nu}^G(\pi_\nu^G(a)1) = c^G(\nu)a^{-\rho^{G_o}} J_{\eta,\nu}^{G_o}(\pi_\nu^{G_o}(a)1).$$

Proof. Fix ν and χ. Define $\Phi_\nu(h) = e^{-\rho^G(h)} J_{\chi,\nu}^G(\pi_\nu^G(\exp h)1)$ for $h \in \mathfrak{a}$. Taking $c_\alpha^2 = \|\xi_\alpha\|^2$ for $\alpha \in \Delta$ in the notation of the previous section, we have the operator L_c in the algebra $\mathcal{A}$. We have seen in the previous section (Theorem 7.5) that if $z \in \mathcal{A}^{L_c}$ (the centralizer of L_c) then

$$z\Phi_\nu(h) = \sigma(z)(\nu)\Phi_\nu(h).$$

Define for $g = nak$ with $n \in N_o, a \in A$ and $k \in K_o$

$$u_\nu(nak) = \eta(n)a^{\rho^{G_o}}\Phi_\nu(\log a).$$

If $x \in U(\mathfrak{g}_o)^{K_o}$ then $xu_\nu = \nu(\gamma^{G_o}(x))u$ and if $x \in \mathfrak{k}_o$ then $L_x u_\nu = 0$. Thus we have a homomorphism

$$T_\nu : Y^\nu \longrightarrow C^\infty(N_o \backslash G_o, \eta)$$

given by $T_\nu(x) = R_x u_\nu$ which induces a $(\mathfrak{g}_o, K_o)$-module homomorphism. Composing with the isomorphism U_ν of I_ν onto Y^ν for $\operatorname{Re}(\nu, \alpha) < 0$ for $\alpha \in \Phi^+$ one has a $(\mathfrak{g}_o, K_o)$-module homomorphism

$$S_\nu : I_\nu \longrightarrow T(Y^\nu)).$$

Define $\lambda_\nu \in I_\nu^* = C^\infty(M_o \backslash K_o)$ by

$$\lambda_\nu(f) = S_\nu(f)(e)$$

(e the identity of G_o) then if $X \in \mathfrak{n}_o$

$$\lambda_\nu(\mathrm{d}\pi_\nu^{G_o}(X)f) = \mathrm{d}\eta(X)\lambda_\nu(f).$$

Also if $f = U(x \otimes 1)$ and if $y \in U((\mathfrak{g}_o)_{\mathbb{C}})$ then

$$\lambda_\nu(\mathrm{d}\pi_\nu^{G_o}(y)f) = R_y\left(R_x u_\nu\right)(e).$$

Since there exist C and d such that

$$|R_x u_\nu(g)| \leq C\,\|g\|^d, \qquad g \in G_o$$

one sees that $\lambda_\nu \in (I_\nu)^*_{\mathrm{mod}}$ in the sense of [Wal92, Section 11.6.1]. Since an interpretation of the Casselman–Wallach theorem is that $(I_\nu)^*_{\mathrm{mod}}$ is exactly the restriction of the continuous dual of I_ν^∞ to I_ν (see [Wal92, Proposition 11.6.2]) we see that λ_ν extends to an element of $Wh_\eta(I_\nu^\infty)$, a one dimensional space with basis $J_{\eta,\nu}^{G_o}$ (Corollary 15.6.2 combined with Lemma 15.6.3 in [Wal92]). Thus if $\mathrm{Re}(\nu,\alpha) < 0$ for $\alpha \in \Phi^+$ then there exists $\beta(\nu)$ holomorphic in ν such that

$$\lambda_\nu = \beta(\nu) J_{\eta,\nu}^{G_o}.$$

This implies that if $\mathrm{Re}(\nu,\alpha) < 0$ for $\alpha \in \Phi^+$ then

$$u_\nu(g) = \beta(\nu) J_{\eta,\nu}^{G_o}(\pi_\nu(g)1).$$

Thus, if $\mathrm{Re}(\nu,\alpha) < 0$ for $\alpha \in \Phi^+$ then

$$a^{-\rho^G} J_{\chi,\nu}^G(\pi_\nu^G(a)1) = \beta(\nu) a^{-\rho^{G_o}} J_{\eta,\nu}^{G_o}(\pi_\nu^{G_o}(a)1).$$

If $(\nu,\alpha) < 0$ for all $\alpha \in \Delta$ then the limit formula in Lemma 4.6 implies that

$$c^G(\nu) = \beta(\nu) c^{G_o}(\nu).$$

The result now follows by analytic continuation. $\square$

9. The Spherical Whittaker Plancherel Space

This section contains the full Plancherel Theorem for $L^2(N\backslash G/K,\chi)$.

We begin this section with functional equations for J_ν. In this section, χ, a regular character of N, will be fixed. If $s \in W$ let $s^* \in K$ be such that $\mathrm{ad}(s^*)_{|\mathfrak{a}} = s$. We first recall the Kunze–Stein intertwining operators. For $s \in W$ we will denote by $d\bar{n}_s$ the quotient measure on $((s^*\overline{N}(s^*)^{-1} \cap \overline{N})\backslash\overline{N}$ relative to a choice of invariant measures on the groups $\overline{N}$ and $((s^*\overline{N}(s^*)^{-1} \cap \overline{N})$. If $\mathrm{Re}(\nu,\alpha) < 0$ for all $\alpha \in \Phi^+$ define

$$A_s(\nu) : C^\infty(M\backslash K) \longrightarrow C^\infty(M\backslash K)$$

by

$$A_s(\nu)\phi(k) = \int_{(s^*\overline{N}(s^*)^{-1}\cap\overline{N})\backslash\overline{N}} \phi_\nu(s^*\bar{n}_s k)\, d\bar{n}_s$$

for $k \in K$. Then $(\nu,\phi) \mapsto A_s(\nu)\phi$ is holomorphic in ν and continuous in ϕ and extends meromorphically to all $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. They are called intertwining operators because

$$A_s(\nu)\pi_\nu(g) = \pi_{s\nu}(g)A_s(\nu).$$

These results follow from [Wal92, Theorem 10.1.14] if we note that in the notation of [Wal92, Section 10.1] $A_s(\nu) = J_{s^*\overline{P}(s^*)^{-1}|\overline{P}} \circ \pi_\nu(s^*)$ (see also [Wal18, Sections 8.10 and 8.11]).

Proposition 9.1. *If $s \in W$ there exists a meromorphic function, $\delta_s(\nu)$ on $\mathfrak{a}_{\mathbb{C}}^*$ such that*

$$J_{s\nu} \circ A_s(\nu) = \delta_s(\nu) J_\nu.$$

Proof. We have observed in the last section that the dimension of the space $Wh_\chi(I_\nu^\infty)$ equals 1. Since both $J_{s\nu} \circ A_s(\nu)$ and J_ν are in $Wh_\chi(I_\nu^\infty)$ and $J_\nu \neq 0$ (see [Wal92, Theorem 15.6.7]) the functional $J_{s\nu} \circ A_s(\nu)$ is a multiple of J_ν, call it $\delta_s(\nu)$ where $A_s(\nu)$ is defined. The meromorphy of $\delta_s(\nu)$ follows from the fact that A_s is weakly meromorphic and J_ν is weakly holomorphic. $\square$

Corollary 9.2. *There exists a meromorphic function, $\xi_s(\nu)$ on $\mathfrak{a}^*_{\mathbb{C}}$ such that*

$$J_{s\nu}(\pi_{s\nu}(g)1) = \xi_s(\nu)J_\nu(\pi_\nu(g)1).$$

Proof. Define $c_s(\nu)$ by

$$A_s(\nu)1 = c_s(\nu)1.$$

We have

$$J_{s\nu}(A_s(\nu)\pi_\nu(g)1) = J_{s\nu}(\pi_{s\nu}(g)A_s(\nu)1) = c_s(\nu)J_{s\nu}(\pi_{s\nu}(g)1).$$

Now apply the proposition above. □

Note that if s_o is the longest element of $W(A)$ relative to Φ^+ then $c_{s_o}(\nu) = c(\nu)$ in the notation of the previous sections. The following is critical to our argument and is the spherical version of the Maas-Selberg relations in [Har18].

Theorem 9.3. $|\xi_s(i\nu)| = 1$ *for all $s \in W$ and $\nu \in \mathfrak{a}^*$.*

The proof will be based on the calculations in [GW80] of the analogues of the $\xi_s(\nu)$ in their context and will involve several auxiliary results. Let G_o and η be as in the previous section. Then in [GW80] $\xi_s(\nu)$ was calculated explicitly for G_o but for J calculated on principal series induced from P rather than $\overline{P}$. Let $P_o = P \cap G_o$ and $\overline{P}_o = \overline{P} \cap G_o$. Choose s^* in the normalizer of A in K_o such that $\mathrm{ad}(s^*)_{|\mathfrak{a}} = s$ and s_o is such that $s_o\Phi^+ = -\Phi^+$. In [GW80] we prove our formulas for

$$J_{\eta,P_o,\nu}(f) = \int_{N_o} \eta(n)^{-1} a_{P_o}(s_o^* n)^{\nu+\rho} f(k_{P_o}(s_o^* n))\, dn$$

for $\mathrm{Re}(\nu,\alpha) > 0$ all $\alpha \in \Phi^+$. Also, in the computed formulas they assume a normalization of η:

$$\left\| d\eta_{|(\mathfrak{g}_o)_\alpha} \right\| = \frac{\|\alpha\|}{\sqrt{2}}.$$

Clearly, this normalization can be obtained by composing with the action of an element of A. Indeed, if $a \in A$ then $(\eta^a(n) = \eta(ana^{-1}), a \in A, n \in N_o)$

$$\left(J_{\eta,P_o,\nu} \circ \pi_{P_o,\nu}(a)\right)\left(\pi_{P_o,\nu}(n) f\right) = \eta^a(n)\left(J_{\eta,P_o,\nu} \circ \pi_{P_o,\nu}(a)\right)(f)$$

and

$$\mathrm{d}\eta^a_{|(\mathfrak{g}_o)_\alpha} = a^\alpha \mathrm{d}\eta_{|(\mathfrak{g}_o)_\alpha}.$$

So their calculation of $\xi_{P_o,s}(\nu)$ is correct for any generic η.

The formula [GW80, Lemma 7.5 and Corollary 7.9] is

Theorem 9.4. *If $s \in W$ then*

$$\xi_{P_o,s}(\nu) = \prod_{\alpha \in \langle s \rangle} \frac{\Gamma\left(\frac{1}{2} + \frac{(\nu,\alpha)}{(\alpha,\alpha)}\right)}{\Gamma\left(\frac{1}{2} - \frac{(\nu,\alpha)}{(\alpha,\alpha)}\right)} 2^{2\frac{(\nu,\alpha)}{(\alpha,\alpha)}},$$

where $\langle s \rangle = \{\alpha \in \Phi^+ \mid s\alpha \notin \Phi^+\}$.

Notice that since $\Gamma(\bar{z}) = \overline{\Gamma(z)}$, $|\xi_{P_o,s}(i\nu)| = 1$ if $\nu \in \mathfrak{a}^*$. We will use a subscript P to give the analogues of the various objects given for representations induced from P. If $g \in G$ then $g \in NaK$ with a in A, and we set $a_P(g) = a$. Similarly, if $g \in \overline{N}uK$

with $u \in A$ then set $a_{\overline{P}}(g) = u$. Let (where they converge and in their meromorphic continuations)

$$c_P(\nu) = \int_{\overline{N}} a_P(\overline{n})^{\nu+\rho}\,\mathrm{d}\overline{n}$$

and

$$c_{\overline{P}}(\nu) = \int_N a_{\overline{P}}(n)^{\nu-\rho}\,\mathrm{d}n.$$

If $g = nak$ with $n \in N$, $a \in A$, $k \in K$ then $s_o^* g\,(s_o^*)^{-1} \in \overline{N} s_o(a) K$. Thus

$$a_P(g) = s_o(a_{\overline{P}}((s_o^*)^{-1}\, g s_o^*)).$$

This easily implies

Lemma 9.5.

(1) $c_P(\nu) = c_{\overline{P}}(s_o\nu)$.

(2) $J_{P,\nu}(\pi_{P,\nu}(a)1) = J_{\overline{P},s_o\nu}(\pi_{\overline{P},s_o\nu}(a)1)$.

We have proved in Theorem 8.1 that

$$c^{G_o}(\nu)a^{-\rho^G} J^G_{\chi,\nu}(\pi^G_\nu(a)1) = c^G(\nu)a^{-\rho^G_o} J^{G_o}_{\eta,\nu}(\pi^{G_o}_\nu(a)1).$$

The above lemma and the above theorem imply

Theorem 9.6. *If χ is a generic character of N and if $s \in W$ then*

$$J_{s\nu}(\pi_{s\nu}(g)1) = \xi_s(\nu) J_\nu(\pi_\nu(g)1)$$

and

$$\xi_s(\nu) = \frac{c^{G_o}(\nu)c^G(s\nu)}{c^{G_o}(s\nu)c^G(\nu)}\xi_{P_o,s_oss_o}(\nu).$$

We have seen that $|\xi_{P_o,s_oss_o}(i\nu)| = 1$. We also note that $\mu(s\nu) = \mu(\nu)$ for $s \in W$. Also by definition

$$|c(i\nu)| = \frac{1}{\sqrt{\mu(\nu)}}.$$

This implies that

$$\left|\frac{c^{G_o}(i\nu)c^G(is\nu)}{c^{G_o}(is\nu)c^G(i\nu)}\right| = 1,$$

proving Theorem 9.3.

Proposition 9.7. *Let $\phi \in C^\infty(\mathfrak{a}^*)$ satisfy the two conditions*

(1) *If $m \in \mathbb{R}_{\geq 0}$ then there exists C_m such that $|\phi(\nu)| \leq C_m(1 + \|\nu\|)^{-m}$.*

(2) *$\phi(s\nu) = \overline{\xi_s(i\nu)}\phi(\nu)$ for $s \in W$ and $\nu \in \mathfrak{a}^*$.*

Then $\int_{\mathfrak{a}^*} \phi(\nu) J_{i\nu}(\pi_{i\nu}(g)1)\mu(\nu)\,\mathrm{d}\nu = 0$ for all $g \in G$ implies that $\phi = 0$.

Proof. If $f \in \mathcal{C}(G)$ then we have seen that

$$\int_G J_{i\nu}(\pi_{i\nu}(g)1) f(g)\,\mathrm{d}g = J_{i\nu}(\pi_{i\nu}(f)1).$$

Thus if $f \in \mathcal{C}(G)$ then the hypotheses imply that

$$\int_{\mathfrak{a}^*} J_{i\nu}(\pi_{i\nu}(f)1)\phi(\nu)\mu(\nu)\,\mathrm{d}\nu = 0.$$

If f is an arbitrary element of $\mathcal{C}(K\backslash G/K)$ then $\pi_{i\nu}(f)1 = \beta(\nu)1$ with $\beta(\nu)$ a general element of $\mathcal{S}(\mathfrak{a}^*)^W$ by Theorem 3.1. This implies that

$$\int_{\mathfrak{a}^*} \beta(\nu) J_{i\nu}(1)\phi(\nu)\mu(\nu)\,\mathrm{d}\nu = 0$$

with $\beta \in \mathcal{S}(\mathfrak{a}^*)^W$. Let $\beta \in C_c^\infty((\mathfrak{a}^*)^+)$. Set (as usual) $(\mathfrak{a}^*)' = \bigcup_{s\in W} s(\mathfrak{a}^*)^+$ and defining $\beta(s\nu) = \beta(\nu)$ for $s \in W$ and $\nu \in (\mathfrak{a}^*)^+$ this defines an element of $C_c^\infty((\mathfrak{a}^*)')^W \subset \mathcal{S}(\mathfrak{a}^*)^W$. Our hypothesis implies that

$$\sum_{s\in W} \int_{U_{\nu_o}} \beta(\nu) J_{is\nu}(1)\phi(s\nu)\mu(\nu)\,\mathrm{d}\nu = 0.$$

Using the functional equations we have

$$\sum_{s\in W} \int_{U_{\nu_o}} |\xi_s(i\nu)|^2 \beta(\nu) J_{i\nu}(1)\phi(\nu)\mu(\nu)\,\mathrm{d}\nu = 0$$

for all $\beta \in C_c^\infty((\mathfrak{a}^*)^+)$. This implies that

$$\left(\sum_{s\in W} |\xi_s(i\nu)|^2\right) J_{i\nu}(1)\phi(\nu)\mu(\nu) = 0$$

for $\nu \in (\mathfrak{a}^*)^+$. Since $(\sum_{s\in W} |\xi_s(i\nu)|^2) J_{i\nu}(1)\mu(\nu)$ is real analytic, and not identically zero this implies that $\phi(\nu) = 0$ on an open dense subset of $(\mathfrak{a}^*)^+$. Thus $\phi_{|(\mathfrak{a}^*)^+} = 0$. The assumed functional equation for ϕ now implies that $\phi = 0$. □

Let $\mathcal{R}_w(\mathfrak{a}^*)$ be the space of all elements in $C^\infty(\mathfrak{a}^*)$ satisfying the two conditions in the preceding theorem. Let H_w be the Hilbert space completion of $\mathcal{R}_w(\mathfrak{a}^*)$ relative to the measure $\mu(\nu)\,\mathrm{d}\nu$.

Lemma 9.8. *The map $f \mapsto f_{|(\mathfrak{a}^*)^+}$ defines a unitary isomorphism from H_w onto*

$$L^2((\mathfrak{a}^*)^+, |W|\mu(\nu)\,\mathrm{d}\nu).$$

Proof. Observe that $C_c^\infty((\mathfrak{a}^*)^+)$ is dense in $L^2((\mathfrak{a}^*)^+, |W|\mu(\nu)\,\mathrm{d}\nu)$. If $f \in C_c^\infty((\mathfrak{a}^*)^+)$ then defining, for $\nu \in s(\mathfrak{a}^*)^+, L(f)(\nu) = \overline{\xi_s(i\nu)} f(s^{-1}\nu)$ one has $L(C_c^\infty((\mathfrak{a}^*)^+)) \subset \mathcal{S}_w(\mathfrak{a}^*)$. This implies that the map $f \mapsto f_{|(\mathfrak{a}^*)^+}$ of H_w to $L^2((\mathfrak{a}^*)^+, |W|\mu(\nu)\,\mathrm{d}\nu)$ has dense image. Since it is clearly unitary it is surjective. □

We finally have

Theorem 9.9. *If $f \in \mathcal{C}(N\backslash G/K;\chi)$ then let $W(\nu, f)$ be as in Theorem 6.3. Write $W(f)(\nu) = W(\nu, f)$. Then $f \mapsto W(f)_{|(\mathfrak{a}^*)^+}$ extends to an isomorphism of the Hilbert spaces $L^2(N\backslash G/K, \chi)$ and $L^2((\mathfrak{a}^*)^+, |W|\mu(\nu)\,\mathrm{d}\nu)$.*

Proof. We have seen that if $\alpha \in C_c^\infty((\mathfrak{a}^*)^+)$ then

$$f(g) = \int_{\mathfrak{a}^*} L(\alpha)(\nu) \overline{J_{i\nu}(\pi_{i\nu}(g))}\mu(\nu)\,\mathrm{d}\nu$$

defines an element of $\mathcal{C}(N\backslash G/K, \chi)$ (Corollary 5.3). Now $W(f) \in \mathcal{R}_w(\mathfrak{a}^*)$ and Theorem 6.3 implies that

$$\int_{\mathfrak{a}^*} W(f)(\nu) J_{i\nu}(\pi_{i\nu}(g)1)\mu(\nu)\,\mathrm{d}\nu = f(g), \qquad g \in G.$$

Thus

$$\int_{\mathfrak{a}^*} (W(f)(\nu) - L(\alpha))\, J_{i\nu}(\pi_{i\nu}(g)1)\mu(\nu)\,\mathrm{d}\nu = 0, \qquad g \in G.$$

Hence Proposition 9.7 implies $W(f) = L(\alpha)$. □

A. Some inequalities

The purpose of this appendix is to prove some estimates that will be used in the body of the paper. The notation is as in Section 2.

Lemma A.1. *Let $f \in \mathcal{C}(G/K)$ then for each $l \geq 0$ and D a constant coefficient differential operator on $\mathfrak{a}^*$ there exists $B_{D,l}$ such that*

$$|D(\pi_{i\nu}(f)1)(k)| \leq B_{D,l}(1 + \|\nu\|)^{-l}, \qquad \nu \in \mathfrak{a}^*, k \in K.$$

Proof. By definition

$$(\pi_{i\nu}(f)1)\,(k) = \int_G f(g)a(kg)^{i\nu-\rho}\,\mathrm{d}g = \int_G f(k^{-1}g)a(g)^{i\nu-\rho}\,\mathrm{d}g.$$

Up to normalization of measures one has the standard integration formula (cf. [Wal18, Section 7.7.4])

$$\int_G \varphi(g)\,\mathrm{d}g = \int_{\overline{N}\times A\times K} a^{2\rho}\varphi(\overline{n}ak)\,\mathrm{d}\overline{n}\,\mathrm{d}a\,\mathrm{d}k.$$

Thus

$$(\pi_{i\nu}(f)1)\,(k) = \int_{\overline{N}\times A} a^{2\rho} f(k^{-1}\overline{n}a)a^{i\nu-\rho}\,\mathrm{d}\overline{n}\,\mathrm{d}a = \int_{\overline{N}\times A} a^{\rho} f(k^{-1}\overline{n}a)a^{i\nu}\,\mathrm{d}\overline{n}\,\mathrm{d}a.$$

The map $\varphi \mapsto [h \mapsto e^{\rho(h)} \int_{\overline{N}} \varphi(\overline{n}\exp h)\,\mathrm{d}\overline{n} = \varphi^{\overline{P}}(h)]$ is a continuous map of $\mathcal{C}(G/K)$ to $\mathcal{S}(\mathfrak{a})$ (cf. [Wal88b, Theorem 7.2.1]). The map $k \mapsto L_k f$ is a continuous map from K to $\mathcal{C}(G/K)$ and the Fourier transform, $\mathcal{F}$, is a continuous map from $\mathcal{S}(\mathfrak{a})$ to $\mathcal{S}(\mathfrak{a}^*)$. So, if q is a continuous semi-norm on $\mathcal{S}(\mathfrak{a}^*)$ then $q(\mathcal{F}((R_k f)^{\overline{P}})) \leq B_q$. Thus, if $q_{D,l}(u) = \sup_{\nu\in\mathfrak{a}^*}(1 + \|\nu\|)^l\,|Du(\nu)|$ for $l \geq 0$ and D a constant coefficient differential operator on $\mathfrak{a}^*$ then, since K is compact,

$$\max q_{D,l}(\mathcal{F}((R_k f)^{\overline{P}})) \leq B_{D_l,q_l}.$$

This implies that

$$\left|D\mathcal{F}((R_k f)^{\overline{P}})(\nu)\right| \leq B_{D,q_l}(1 + \|\nu\|)^{-l}.$$

Unraveling the above we have

$$(\pi_{i\nu}(f)1)\,(k) = \mathcal{F}((R_k f)^{\overline{P}})(-\nu)$$

proving the Lemma. □

Lemma A.2. *If $x, g \in G$ then $a(xg)^{-\rho} \leq |g|^{\frac{1}{2}}\, a(x)^{-\rho}$.*

Proof. Let (as in Section 2) $E = \bigwedge^m \mathfrak{g}$ ($m = \dim \mathfrak{n}$) and let $u_o \in \bigwedge^m \bar{\mathfrak{n}}$ be a unit vector. If $g \in G$ with $g = \bar{n}a(g)k$ with $\bar{n} \in \overline{N}$ and $k \in K$ then $\|\bigwedge^m g^{-1}u_o\| = \|\bigwedge^m k^{-1} \wedge^m a(g)^{-1} \bigwedge^m \bar{n}^{-1}u_o\| = a(g)^{2\rho}$. This implies that if $x, g \in G$ then

$$a(x)^{2\rho} = \left\| \bigwedge^m g \bigwedge^m g^{-1} \bigwedge^m x^{-1} v_o \right\| \le |g|\, a(xg)^{2\rho}$$

hence

$$a(xg)^{-\rho} \le |g|^{\frac{1}{2}}\, a(x)^{-\rho}. \qquad \square$$

Fix, χ, a generic character of N. Let $x_o \in \mathfrak{n}$ be orthogonal to $[\mathfrak{n}, \mathfrak{n}]$ and such that $d\chi(x_o) = i$.

Proposition A.3. *Let $f \in \mathcal{C}(G/K)$ and if $\nu \in \mathfrak{a}^*$ then set $u(\nu) = \pi_{i\nu}(f)1 \in C^\infty(M\backslash K)$. If ω is a compact subset of G then there exist constants $U, D, C_\omega, d < \infty$ such that if $g \in \omega$ then for* $\operatorname{Re} z \ge 0$

$$\left| \int_{\ker \chi} \int_{\mathfrak{a}^*} \int_N 1_{i\nu - z\rho}(n_1 \exp(tx_o)\, ng)\overline{u(\nu)_{i\nu - \bar{z}\rho}(n_1)}\, \mathrm{d}n_1\, \mu(\nu)\, \mathrm{d}\nu\, \mathrm{d}n \right| \\ \le C_\omega^{1+\operatorname{Re} z} DU(1+|t|)^d.$$

Proof. We put the absolute values inside the integration. We are estimating

$$\int_{\ker \chi} \int_{\mathfrak{a}^*} \int_N a(n_1 n(\exp(tx_o))g)^{-(1+\operatorname{Re} z)\rho} a(n_1)^{-(1+\operatorname{Re} z)\rho}\, |u(\nu)(k(n_1))|\, \mathrm{d}n_1\, \mu(\nu)\, \mathrm{d}\nu\, \mathrm{d}n$$

(denote this expression by I). Note that the formula for $c(\nu)$ and standard facts about the Beta function imply

$$\mu(\nu) \le B(1 + \|\nu\|)^r$$

for some r and all $\nu \in \mathfrak{a}^*$. Lemma A.1 implies that there exists a constant L_m such that

$$|u(\nu)| \le L_m(1 + \|\nu\|)^{-r-m}$$

with m arbitrary. We take m to be such that $m > \dim \mathfrak{a}$. Thus

$$I \le U \int_{\ker \chi} \int_N a(n_1 n(\exp(tx_o))g)^{-(1+\operatorname{Re} z)\rho} a(n_1)^{-(1+\operatorname{Re} z)\rho}\, \mathrm{d}n_1\, \mathrm{d}n$$

with

$$U = BL_m \int_{\mathfrak{a}^*} (1 + \|\nu\|)^{-m}\, \mathrm{d}\nu < \infty.$$

Also, Lemma A.2 implies that

$$a(n_1 n(\exp tx_o)g)^{-(1+\operatorname{Re} z)\rho} \le |g|^{\frac{1+\operatorname{Re} z}{2}}\, a(n_1 n(\exp tx_o))^{-(1+\operatorname{Re} z)\rho} \\ \le |g|^{\frac{1+\operatorname{Re} z}{2}}\, a(n_1 n(\exp tx_o))^{-\rho} \le |\exp tx_o|^{\frac{1}{2}}\, |g|^{\frac{1+\operatorname{Re} z}{2}}\, a(n_1 n)^{-\rho}.$$

Since, $\bigwedge^m \mathrm{ad}(\exp tx_o)$ is a polynomial in t with values in $\mathrm{End}(\bigwedge^m \mathfrak{g})$, there exists a constant $Q < \infty$ such that $|\bigwedge^m \mathrm{ad}(\exp tx_o)| \le Q(1+|t|)^d$. Setting $C_\omega = \max_{g\in\omega} |g|^{\frac{1}{2}}$

$$\begin{aligned} I &\le UC_\omega^{\frac{1+\mathrm{Re}\,z}{2}} Q(1+|t|)^d \int_{\ker\chi}\int_N a(n_1 n)^{-(1+\mathrm{Re}\,z)\rho} a(n_1)^{-(1+\mathrm{Re}\,z)\rho}\, \mathrm{d}n_1\, \mathrm{d}n \\ &\le UC_\omega^{\frac{1+\mathrm{Re}\,z}{2}} Q(1+|t|)^d \int_{\ker\chi}\int_N a(n_1 n)^{-\rho} a(n_1)^{-\rho}\, \mathrm{d}n_1\, \mathrm{d}n \\ &= UC_\omega^{\frac{1+\mathrm{Re}\,z}{2}} \left(Q(1+|t|)^d\right)^{\frac{1+\mathrm{Re}\,z}{2}} \int_{\ker\chi} \Xi(n)\, \mathrm{d}n. \end{aligned}$$

As in the proof of Lemma A.2 we have $a(g)^\rho = \|\bigwedge^m \mathrm{ad}(g)^{-1} v_o\|^{\frac{1}{2}} \le |g^{-1}|^{\frac{1}{2}}$. Also,

$$\Xi(x) \le N\,|x|^{-\frac{1}{2}}\,(1+\log|x|)^s$$

for some $s, N < \infty$ (cf. [Wal88b, Theorem 5.5.3]). Hence

$$\Xi(x) = \Xi(x^{-1}) \le N\left|x^{-1}\right|^{-\frac{1}{2}}\left(1+\log\left|x^{-1}\right|\right)^s \le N_\varepsilon \left|x^{-1}\right|^{-\frac{1}{2}+\varepsilon} \le N_\varepsilon a(x)^{-(1-2\varepsilon)\rho}$$

for each $\varepsilon > 0$. Theorem 4.2 says that if ε is sufficiently small

$$\int_{\ker\chi} a(n)^{-(1-2\varepsilon)\rho}\, \mathrm{d}n < \infty.$$

□

B. The restriction of $\mathcal{C}(N\backslash G/K;\chi)$ to A

The purpose of this appendix is to give a complete description of $\mathcal{C}(N\backslash G/K;\chi)_{|A}$. The notation will be as in the previous sections. In particular, $\Delta = \{\alpha_1,\dots,\alpha_l\}$ is the set of simple roots of Φ^+.

Lemma B.1. *Let χ be a generic unitary character of N. If $m = (m_1,\dots,m_l)$, $m_i, d \in \mathbb{Z}_{\ge 0}$ then there exists a continuous semi-norm, $q_{m,d}$, on $\mathcal{C}(N\backslash G;\chi)$ such that if $f \in \mathcal{C}(N\backslash G;\chi)$ then*

$$|f(\exp(h)k)| \le q_{m,d}(f) e^{\rho(h)} e^{-\sum m_i\alpha_i(h)} (1+\|h\|)^{-d}$$

for $h \in \mathfrak{a}$.

Proof. Let $F = \{i \mid m_i > 0\}$ and let $x_1,\dots,x_n$ be a basis of $\mathfrak{g}$. If $X \in \mathfrak{g}$ and if $k \in K$ then we can write $\mathrm{ad}(k)X = \sum a_i(k,X)x_i$. Note that there exists C such that

$$|a_i(k,X)| \le C\,\|X\|$$

for all $k \in K$. Let X_i be an element of $\mathfrak{n}_{\alpha_i}$ such that $\mathrm{d}\chi(X_i) = z_i \ne 0$. Then

$$\begin{aligned} f(\exp(h)k) = z_i^{-1} L_{X_i} f(\exp(h)k) &= z_i^{-1} \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f(\exp(tX_i)\exp(h)k) \\ &= z_i^{-1} \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f(\exp(h)\exp(t\,\mathrm{ad}(\exp(-h))X_i)k) \\ &= z_i^{-1} \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f(\exp(h)\exp(te^{-\alpha_i(h)}X_i)k) \\ &= z_i^{-1} \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f(\exp(h)\exp(te^{-\alpha_i(h)}\,\mathrm{ad}(k^{-1})X_i)k) \\ &= e^{-\alpha_i(h)} z_i^{-1} \sum a_j(k^{-1},X_i) R_{X_j} f(\exp(h)k). \end{aligned}$$

Iterating this argument yields an expression

$$f(\exp(h)k) = e^{-\sum_{i\in F} m_i\alpha_i(h)} Z(k) f(ak)$$

with Z a smooth function from K to $L = U^{\sum_{i\in F} m_i}(\mathfrak{g})$ with $U^j(\mathfrak{g})$, the standard filtration. If we choose a basis of L, $y_1, \ldots, y_r$ then we have

$$Z(k) = \sum b_i(k) y_i$$

with b_i continuous functions on K. Let $C_j = \max_{k\in K} |b_j(k)|$. We have

$$\begin{aligned} |f(\exp(h)k)| &\leq e^{-\sum_{i\in F} m_i\alpha_i(h)} \sum_j C_j \, |y_j f(\exp(h)k)| \\ &\leq e^{-\sum_{i\in F} m_i\alpha_i(h)} (1 + \|h\|)^{-d} e^{\rho_o(h)} \sum_j C_j q_{d,y_j}(f). \end{aligned}$$

The last inequality is a consequence of the definition of $\mathcal{C}(N\backslash G;\chi)$. □

The following lemma also appears in [Wal24b, Lemma 24].

Lemma B.2. *Let $\psi \in C^\infty(G)$ be expressed in the form*

$$\psi(nak) = \sum_{i=1}^{r}\sum_{j=1}^{s} \phi_i(n) f_{ij}(a) \gamma_j(k)$$

for $n \in N$, $a \in A$, $k \in K$ with $r, s < \infty$, $\phi_i \in C_c^\infty(N)$, $\gamma_j \in C^\infty(K)$ and $f_{ij} \in C^\infty(A)$ satisfying the following condition: if $m = (m_1, \ldots, m_l)$ with $m_i, d \in \mathbb{Z}_{\geq 0}$ and $x \in U(\mathfrak{a})$ then there exists $C_{ij,m,d,x}$ such that

$$|x f_{ij}(a)| \leq C_{ij,m,d,x} a^\rho a^{-m_1\alpha_1 - \cdots - m_l\alpha_l} (1 + \|\log a\|)^{-d}.$$

Then for $d \in \mathbb{Z}_{\geq 0}$ there exists B_d such that

$$|\psi(g)| \leq B_d \, |g|^{-\frac{1}{2}} (1 + \log \|g\|)^{-d}.$$

Also, if $x, y \in U(\mathfrak{g})$ then $L_x R_y \psi$ is of the same form.

Proof. In the proof we may assume that $r, s = 1$ so take

$$\psi(nak) = \phi(n) f(a) \gamma(k).$$

Let ω be the support of ϕ. Let $c_1 \geq 1$ be such that

$$\max_{n\in\omega}\{\|n\|, \|n^{-1}\|\} \leq c_1, \qquad \min_{n\in\omega}\{\|n\|, \|n^{-1}\|\} \geq c_1^{-1}.$$

Note that $|n|^{\frac{1}{2}} = \|n\|$ and $|k| = \|k\| = 1$. We have for $n \in \omega$, $a \in A$, $k \in K$

$$|nak| = |na| \leq c_1 \, |a|$$

and

$$|a| = \left|n^{-1} nak\right| \leq c_1 \, |nak| \, .$$

By the same argument we have the same inequalities for $\|\ldots\|$. If $h \in \mathfrak{a}$ let $s \in W(A)$ be such that $\alpha(sh) \geq 0, \alpha \in \Phi^+$. Then $|a|^{\frac{1}{2}} = \exp(sh)^\rho = a^{s^{-1}\rho}$ so if s_o is the element of $W(A)$ such that $s_o\Phi^+ = -\Phi^+$ then

$$|nak|^{-\frac{1}{2}} \leq c_1 \, |a|^{-\frac{1}{2}} = c_1 a^{s^{-1}s_o\rho}.$$

Also note that $s^{-1}s_o\rho = \rho - \sum_{i=1}^{l} u_i\alpha_i$ with $u_i \in \mathbb{Z}_{\geq 0}$. We also leave it to the reader to check that there exists $c_2 > 0$ such that $\|a\| \leq e^{c_2\|\log a\|}$ for $a \in A$. Thus $(1+\log\|a\|) \geq c_3(1+\|\log a\|)$. With these observations in place we have

$$\begin{aligned}|\psi(nak)| &\leq \left(\sup_{n\in\omega, k\in K} |\phi(n)\gamma(k)|\right)|f(a)| = c_4\,|f(a)| \\ &\qquad \leq c_4 C_{1,1,u,d}\, a^{\rho - \sum u_i\alpha_i}(1+\|\log a\|)^{-d} = c_4 C_{1,1,u,d}\,|a|^{-\frac{1}{2}}\, c_3^{-d}(1+\log\|a\|)^{d}.\end{aligned}$$

Thus if we take the maxima of the $C_{1,1,u,d}$ for the $s \in W(A)$ and incorporate the constants that appear in the estimates at the beginning of the proof we have

$$|\psi(g)| \leq C_d\,|g|^{-\frac{1}{2}}\,(1+\log\|g\|)^{-d}$$

as asserted.

To complete the proof of the lemma we now consider the derivatives. It is enough to show that $R_X\psi$ and $L_X\psi$ are of the same form for $X \in \mathfrak{g}$. We start with R_X. Again it is enough to show that if $s, t = 1$ then $R_X\psi$ is of the form indicated in the statement of the lemma. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ such that $X_1, \ldots, X_r \in \mathfrak{n}$ with $[h, X_i] = \beta_i(h)X_i$, $h \in \mathfrak{a}$, $X_{r+1}, \ldots, X_{r+l} \in \mathfrak{a}$, $X_{r+l+1}, \ldots, X_n \in \mathrm{Lie}(K)$. Then

$$\mathrm{ad}(k)X = \sum c_i(k, X)X_i.$$

We have

$$\begin{aligned}R_X\psi(nak) &= \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\psi(nak\exp(tX)) = \frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\psi(na\exp(t\,\mathrm{ad}(k)X)k) \\ &= \sum_{i=1}^{n} c_i(k,X)\frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\psi(na\exp(tX_i)\,k) \\ &= \sum_{i=1}^{r} c_i(k,X)a^{\beta_i}\left(R_{X_i}\phi(n)\right)f(a)\gamma(k) + \sum_{i=r+1}^{r+l} c_i(k,X)\phi(n)\left(R_{X_i}f(a)\right)\gamma(k) \\ &\qquad + \sum_{i=r+l+1}^{n} c_i(k,X)\phi(n)f(a)\left(L_{X_i}\gamma(k)\right)\end{aligned}$$

which is easily seen to be of the right form.

To handle the left derivative we consider a different basis $Y_i = X_i$, $i = 1, \ldots, r+l$, $Y_{r+l+1}, \ldots, Y_{r+l+m}$ a basis of $\mathrm{Lie}(M)$ and $Y_{r+l+m+i} = \theta X_i, i = 1, \ldots, r$. Then

$$\mathrm{ad}(n^{-1})X = \sum \mathrm{d}_i(n, X)Y_i$$

so

$$\begin{aligned}L_X\psi(nak) &= -\sum_{i=1}^{n} \mathrm{d}_i(n,X)\frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\psi(n\exp(tY_i)\,ak) \\ = &-\sum_{i=1}^{r} \mathrm{d}_i(n,X)\left(R_{i_i}\phi(n)\right)f(a)\gamma(k) + \sum_{i=r+1}^{r+l} d_i(n,X)\phi(n)L_{Y_i}f(a)\gamma(k) \\ &+ \sum_{i=r+l+1}^{r+l+m} \mathrm{d}_i(n,X)\phi(n)f(a)L_{Y_i}\gamma(k) - \sum_{i=r+l+m+1}^{n} d_i(n,X)\frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\psi(n\,(\exp tY_i)\,ak).\end{aligned}$$

All but the last term are of the right form so we will show that it is also. Set $\mu = r+l+m$ then

$$\exp(tY_{\mu+i})a = a\exp(t\,\mathrm{ad}(a)^{-1}Y_{\mu+i}) = a\exp(ta^{\beta_i}Y_{\mu+i}).$$

So we are looking at

$$-\sum_{i=1}^{r} \mathrm{d}_{\mu+i}(n,X)a^{\beta_i}\frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\psi(na\exp\left(tY_{\mu+i}\right)k).$$

Now $Y_{\mu+i}+X_i = Z_i \in \mathrm{Lie}(K)$. Thus $Y_{\mu+i} = Z_i - X_i$. Thus

$$\begin{aligned}
&-\sum_{i=1}^{r} \mathrm{d}_{\mu+i}(n,X)a^{\beta_i}\frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\psi(na\exp\left(tY_{\mu+i}\right)k)\\
&\quad=\sum_{i=1}^{r} \mathrm{d}_{\mu+i}(n,X)a^{\beta_i}\frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\psi(na\exp(tX_i)k)-\sum_{i=1}^{r} \mathrm{d}_{i}(n,X)a^{\beta_i}\frac{\mathrm{d}}{\mathrm{d}t_{|t=0}}\psi(na(\exp tZ_I)k)\\
&\quad=\sum_{i=1}^{r} \mathrm{d}_{\mu+i}(n,X)a^{2\beta_i}R_{X_i}\phi(n)f(a)\gamma(k)-\sum_{i=1}^{r} \mathrm{d}_{\mu+i}(n,X)\phi(n)f(a)L_{Z_i}\gamma(k).
\end{aligned}$$

This completes the proof. ☐

Corollary B.3. *If $f \in \mathcal{C}(N\backslash G;\chi)$ is right K-finite and if $\phi \in C_c^\infty(N)$ then the function on G defined by*

$$\psi(nak) = \phi(n)f(a)$$

is in $\mathcal{C}(G)$.

Proof. Let $V = \mathrm{span}_{\mathbb{C}}\{R_k f \mid k \in K\}$. Then $\dim V < \infty$. Let $v_1,\dots,v_m$ be a basis of V then $R_k f = \sum \gamma_i(k)v_i$. Thus

$$f(ak) = \sum v_i(a)\gamma_i(k).$$

Since, $v_i \in \mathcal{C}(N\backslash G;\chi)$, $R_x v_{i|A}$ satisfies the inequalities for all $x \in U(\mathfrak{a})$. The result is now a direct consequence of the definition of $\mathcal{C}(G)$ and Lemma B.2. ☐

Theorem B.4. *If $\psi \in \mathcal{C}(G)$ then $\psi_\chi(g) = \int_N \chi(n)^{-1}\psi(ng)\,\mathrm{d}n$ defines an element of $\mathcal{C}(N\backslash G;\chi)$.*

Proof. The Harish-Chandra spherical function $\Xi(g) = \langle \pi_0(g)1,1\rangle$ satisfies

$$\Xi(g) \geq |g|^{-\frac{1}{2}}$$

(cf. [Wal88b, Theorem 4.5.3]). Thus since

$$|\psi(na)| \leq C_d\,|na|^{-1/2}\,(1+\log\|na\|)^{-d}$$

for all $d \geq 0$ we have

$$|\psi_\chi(ak)| \leq C_d \int_N |na|^{-\frac{1}{2}}\,(1+\log\|an\|)^{-d}\,\mathrm{d}n \leq C_d \int_N \Xi(na)(1+\log\|na\|)^{-d}\,\mathrm{d}n.$$

Also $\|na\| \geq \|a\|$ (since $\bigwedge^m \mathfrak{g}$ has an orthonormal basis with $a \in A$ acting diagonally and $n \in N$ action with upper triangular matrix with 1's on the main diagonal) and

$$\|n\| = \left\|naa^{-1}\right\| \leq \left\|a^{-1}\right\|\|an\| = \|a\|\,\|an\| \leq \|an\|^2.$$

Thus

$$|\psi_\chi(ak)| \leq C_{d+r}(1+\log\|a\|)^{-r}\int_N \Xi(na)(1+\log\|na\|)^{-d}\,\mathrm{d}n.$$

Now in the proof of [Wal88b, Theorem 7.2.1] we have seen that there exists d such that

$$a^{-\rho}\int_N \Xi(na)(1+\log\|na\|)^{-d}\,\mathrm{d}n \le B < \infty.$$

Since $(R_x\psi)_\chi = R_x(\psi_\chi)$ the theorem now follows from the definition of $\mathcal{C}(N\backslash G;\chi)$. □

If $f \in C^\infty(\mathfrak{a})$ define for $m=(m_1,\dots,m_l)\in\mathbb{Z}^l_{\ge 0}, d\in\mathbb{Z}_{\ge 0}$ and x a constant coefficient differential operator on $\mathfrak{a}$

$$w_{m,d,x}(f)=\sup_{h\in\mathfrak{a}} e^{-\rho(h)}e^{\sum m_i\alpha_i(h)}(1+\|h\|)^d\,|xf(h)|\,.$$

Then we set $\mathcal{W}(\mathfrak{a})$ equal to the space of all $f\in C^\infty(\mathfrak{a})$ such that $w_{m,d,x}(f)<\infty$ for all $m=(m_1,\dots,m_l)\in\mathbb{Z}^l_{\ge 0}$, $d\in\mathbb{Z}_{\ge 0}$ and x, endowed with the topology determined by these semi-norms.

Theorem B.5. *Assume that χ is generic. If $\psi\in C^\infty(N\backslash G/K;\chi)$ set $T(\psi)(h)=\psi(\exp h)$ for $h\in\mathfrak{a}$. Then $T(\mathcal{C}(N\backslash G/K;\chi))\subset\mathcal{W}(\mathfrak{a})$ and T defines a continuous isomorphism of $\mathcal{C}(N\backslash G/K;\chi)$ onto $\mathcal{W}(\mathfrak{a})$.*

Proof. Lemma B.1 implies that $T(\mathcal{C}(N\backslash G/K;\chi))\subset\mathcal{W}(\mathfrak{a})$. Using the definitions of the semi-norms defining the topologies of $\mathcal{C}(N\backslash G/K;\chi)$ and $\mathcal{W}(\mathfrak{a})$ the continuity of T follows. T is injective since $G=NAK$. Let $\phi\in C^\infty_c(N)$ be such that

$$\int_N \chi(n)^{-1}\phi(n)\,\mathrm{d}n=1.$$

If $f\in\mathcal{W}(\mathfrak{a})$ set $\psi(nak)=\phi(n)f(a)$. Lemma B.2 implies that $\psi\in\mathcal{C}(G/K)$. Then $\psi_\chi\in\mathcal{C}(N\backslash G/K;\chi)$ and $T(\psi_\chi)=f$. Thus the map is surjective. The open mapping theorem now implies that T^{-1} is continuous. □

C. The Whittaker radial component of the Casimir operator

Let χ be a unitary character of N. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. In this appendix we will calculate the differential operator on A corresponding to the Casimir operator of G on $C^\infty(N\backslash G/K;\chi)$.

Let $X_{\alpha,i}, i=1,\dots,m_\alpha$ be a basis of $\mathfrak{n}_\alpha$ such that $B(X_{\alpha,i},\theta X_{\alpha,j})=-\delta_{ij}$. Note, before we start calculating, that this implies that

$$[X_{\alpha,i},\theta X_{\alpha,j}]=-\delta_{ij}h_\alpha+m_{ij}$$

with $m_{ij}\in\mathfrak{m}$ and $m_{ij}=0$ if $i=j$. Let $h_1,\dots,h_l$ be an orthonormal basis of $\mathfrak{a}$ relative to $\langle\cdot,\cdot\rangle$. Then the Casimir operator of G relative to B is

$$C=-\sum_{\alpha\in\Phi^+}\sum_{j=1}^{m_\alpha}(X_{\alpha,j}\theta X_{\alpha,j}+\theta X_{\alpha,j}X_{\alpha,j})+C_{\mathfrak{m}}+\sum_{i=1}^{l}h_i^2,$$

where $C_{\mathfrak{m}}$ is the Casimir operator corresponding to $B_{|\mathfrak{m}}$. Let $f\in C^\infty(N\backslash G/K,\chi)$. We wish to calculate $R_Cf(a)$ for $a\in A$. We observe that $X_{\alpha,j}+\theta X_{\alpha,j}\in\mathrm{Lie}(K)$ and

$$(X_{\alpha,j}+\theta X_{\alpha,j})^2=X_{\alpha,j}^2+\theta X_{\alpha,j}^2+X_{\alpha,j}\theta X_{\alpha,j}+\theta X_{\alpha,j}X_{\alpha,j}.$$

Thus

$$R_{X_{\alpha,j}\theta X_{\alpha,j}+\theta X_{\alpha,j}X_{\alpha,j}}f=-R_{X^2_{\alpha,j}}f-R_{\theta X^2_{\alpha,j}}f.$$

Also,

$$R_{\theta X_{\alpha,j}}R_{\theta X_{\alpha,j}}f = -R_{\theta X_{\alpha,j}}R_{X_{\alpha,j}}f = -R_{h_\alpha}f - R_{X_{\alpha,j}}R_{\theta X_{\alpha,j}}f = -R_{h_\alpha}f + R^2_{X_{\alpha,j}}f,$$
$$\left(R_{X^2_{\alpha,j}}f\right)(a) = a^{2\alpha}\left(L_{X^2_{\alpha,j}}f\right)(a) = \mathrm{d}\chi(X_{\alpha,j})^2a^{2\alpha}f(a)$$

and

$$R_{C_{\mathfrak{m}}}f = 0.$$

The upshot is that

$$R_Cf(a) = 2\sum_{\alpha\in\Phi^+}\sum_{j=1}^{m_\alpha}\mathrm{d}\chi(X_{\alpha,j})^2a^{2\alpha}f(a) - \sum_{\alpha\in\Phi^+}m_\alpha h_\alpha f(a) + \sum_{i=1}^{l}h_i^2f(a).$$

If $\alpha\notin\Delta$ then $\mathrm{d}\chi(X_{\alpha,i}) = 0$ and $\sum_{\alpha\in\Phi^+}m_\alpha h_\alpha = 2h_\rho$, so

$$R_Cf(a) = 2\sum_{\alpha\in\Delta}\sum_{j=1}^{m_\alpha}\mathrm{d}\chi(X_{\alpha,j})^2a^{2\alpha}f(a) - 2h_\rho f(a) + \sum_{i=1}^{l}h_i^2f(a).$$

Noting that $\mathrm{d}\chi = i\xi_\chi$ with $\xi_\chi\in\mathfrak{n}^*$ thus if we set $\xi_{\chi,\alpha} = \xi_{\chi|\mathfrak{n}_\alpha}$

$$\sum_{j=1}^{m_\alpha}\mathrm{d}\chi(X_{\alpha,j})^2 = -\left\|\xi_{\chi,\alpha}\right\|^2.$$

We also have

$$a^\rho\sum_{i=1}^{l}h_i^2a^{-\rho} = (\rho,\rho) - 2h_\rho + \sum_{i=1}^{l}h_i^2.$$

We have derived

Lemma C.1. *If $f\in C^\infty(N\backslash G/K;\chi)$ and $a\in A$ then*

$$(C + (\rho,\rho))f(a) = a^\rho\left(\sum_{i=1}^{l}h_i^2 - 2\sum_{\alpha\in\Delta}\left\|\xi_{\chi,\alpha}\right\|^2a^{2\alpha}\right)a^{-\rho}f(a).$$

D. A continuity result used in the proof of Theorem 5.1

Let the notation be as in the beginning of Section 5. Set $Z = \bigoplus_{\alpha\in\Delta}\mathfrak{n}_\alpha$ and let U denote the open subset of Z consisting of the elements $\sum_{\alpha\in\Delta}x_\alpha$ with $x_\alpha\in\mathfrak{n}_\alpha$ and $x_\alpha\neq 0$.

Lemma D.1. *If $x = \sum_{\alpha\in\Delta}x_\alpha\in U$ then there exist $c_\alpha\in\mathbb{R}$ such that if $y = \sum_{\alpha\in\Delta}c_\alpha\theta x_\alpha$ and if $h = [x,y]$ then $h\in\mathfrak{a}$ and $[h,x] = 2x$ and $[h,y] = -2y$.*

Proof. If $\beta\in\Delta$ let $h_\beta\in\mathfrak{a}$ be defined by $B(h_\beta,w) = \beta(w)$, $w\in\mathfrak{a}$. By the definition of Δ, $[x_\alpha,\theta x_\beta] = 0$ if $\alpha\neq\beta$ and $\alpha,\beta\in\Delta$. Thus if $c_\alpha\in\mathbb{R}$ then

$$\left[x,\sum_{\alpha\in\Delta}c_\alpha\theta x_\alpha\right] = \sum_{\alpha\in\Delta}c_\alpha[x_\alpha,\theta x_\alpha].$$

Note that $\theta[x_\alpha, \theta x_\alpha] = [\theta x_\alpha, x_\alpha] = -[x_\alpha, \theta x_\alpha]$ and thus $[x_\alpha, \theta x_\alpha] \in \mathfrak{a}$. If $u = \sum_{\alpha\in\Delta} c_\alpha [x_\alpha, \theta x_\alpha]$ then

$$\begin{aligned}\beta(u) &= \sum_{\alpha\in\Delta} c_\alpha \beta([x_\alpha, \theta x_\alpha]) = \sum_{\alpha\in\Delta} c_\alpha B([x_\alpha, \theta x_\alpha], h_\beta)\\ &= -\sum_{\alpha\in\Delta} c_\alpha B(x_\alpha, [h_\beta, \theta x_\alpha]) = \sum \alpha(h_\beta) c_\alpha B(x_\alpha, \theta x_\alpha).\end{aligned}$$

The matrix

$$[A_{\beta,\alpha}] = [\alpha(h_\beta) B(x_\alpha, \theta x_\alpha)]$$

is non-singular thus we can solve for c_α, $\alpha \in \Delta$ such that

$$\sum_{\alpha\in\Delta} c_\alpha \alpha(h_\beta) B(x_\alpha, \theta x_\alpha) = 2$$

for all $\beta \in \Delta$. □

Proposition D.2. *Let χ be a unitary generic character of N then there exists $x \in U$ such that*

$$d\chi(u) = -iB(x, \theta u).$$

Let h be as in the previous lemma for x. If χ_t is as in the proof of Theorem 5.1 and if $t \neq 0$ then

$$J_{\chi_t,\nu} = e^{-(\rho+\nu)(h)(\frac{\log|t|}{2})} J_{\chi_{sgn(t)},\nu} \circ \pi_\nu\left(\exp\left(\log\frac{|t|}{2} h\right)\right).$$

In particular, the map $(t, \nu, \phi) \mapsto J_{\chi_t,\nu}(\phi)$ from $(\mathbb{R} - \{0\}) \times \mathfrak{a}^*_{\mathbb{C}} \times C^\infty(M\backslash K) \to \mathbb{C}$ is continuous.

Proof. The first assertion is obvious since the bilinear form on Z given by

$$(x, u) \longmapsto -B(x, \theta u), \qquad u \in \mathfrak{n}$$

is positive definite. We now prove the second formula for $\mathrm{Re}(\nu, \alpha) < 0$ for all $\alpha \in \Phi^+$ so assume that ν has this property. Recall that $x_o \in Z$ is chosen such that $d\chi(x_o) = i$. Since $d\chi(x) = -B(x, \theta x)$ we may take $x_o = \frac{-x}{B(x,\theta x)}$ thus

$$\chi_t(\exp X) = \exp\left(\frac{-itB(x, \theta X)}{B(x, \theta x)}\right).$$

If η is a generic unitary character of N then

$$\begin{aligned}J_{\eta,\nu}(\pi_\nu(\exp(sh))\phi) &= \int_N \eta(n)^{-1} \phi_\nu(n \exp(sh))\, dn\\ &= \int_N \eta(n)^{-1} \phi_\nu(\exp(sh)\exp(-sh) n \exp(sh))\, dn\\ &= e^{s(\nu-\rho)(h)} e^{2s\rho(h)} \int_N \eta(\exp(sh) n \exp(-sh))^{-1} \phi_\nu(n)\, dn\\ &= e^{s(\nu+\rho)(h)} \int_N \eta(\exp(sh) n \exp(-sh))^{-1} \phi_\nu(n)\, dn.\end{aligned}$$

Now note that

$$\eta(\exp(sh)\exp X \exp(-sh)) = e^{d\eta(\mathrm{ad}(\exp(sh))X)}.$$

So, applying this to $\eta = \chi_{\pm 1}$, we have

$$\begin{aligned}\exp(d\chi_{\pm}(\operatorname{ad}(\exp(sh))X)) &= \exp\left(\frac{\mp iB(x, \theta \operatorname{ad}(\exp(sh))X)}{B(x, \theta x)}\right) \\ &= \exp\left(\frac{\mp iB(\operatorname{ad}(\exp(sh))x, \theta X)}{B(x, \theta x)}\right) = \exp\left(\frac{\mp ie^{2s}B(x, \theta X)}{B(x, \theta x)}\right) = \chi_{\pm e^{2s}}(\exp X).\end{aligned}$$

Thus, if $\operatorname{Re}(\nu, \alpha) < 0$ for all $\alpha \in \Phi^+$ then

$$J_{\chi_\pm, \nu} \circ \pi_\nu(\exp(sh)) = e^{s(\nu+\rho)(h)} J_{\chi_t, \nu}$$

with $t = \pm e^{2s}(X)$. This formula is true for all $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ by analytic continuation. This implies the proposition. ☐

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