Journal
of Lie
Theory

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Journal of Lie Theory
Online first, 13 pages.
https://doi.org/10.5802/jolt.1428
Published online: August 25, 2026

JoLT is a member of Centre Mersenne
https://www.centre-mersenne.org/
e-ISSN: 2981-2917

Journal of Lie Theory
Volume **0** (0) 1–13

Homological finiteness of principal series representations of $\mathrm{GL}_n(\mathbb{R})$

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Communicated by Toshiyuki Kobayashi

Abstract. Let $G = \mathrm{GL}_n(\mathbb{R})$ be the general linear group over $\mathbb{R}$ and let B denote the Borel subgroup of G consisting of nonsingular upper triangular matrices. For every principal series representation π of G, we show that the Schwartz homology $\mathrm{H}_i(B;\pi)$ is finite-dimensional for every $i \in \mathbb{Z}$. The proof is based on a simple computation modulo the theory of Nash manifolds. Moreover, by applying Casselman's subrepresentation theorem, we show that the conclusion still holds with the principal series representation π replaced by a Casselman–Wallach representation.

Mathematics Subject Classification 2020: 22E46

Key Words and Phrases: Homological finiteness, Principal series representations, Schwartz sections, General linear groups

1. Introduction and main result

Let $G = \mathrm{GL}_n(\mathbb{R})(n \geq 1)$ be the real general linear group. Let $B = AN$ be the subgroup of upper triangular matrices, where A denotes the Levi component of B, which consists of nonsingular diagonal matrices and N denotes the unipotent radical of B. Fix a character χ of A and view it as a character of B in the usual way. Define

$$I_\chi = {}^n\mathrm{Ind}_B^G(\chi) \quad \text{(normalized smooth induction)},$$

which is called a principal series representation of G.

Let $\mathfrak{n}_\mathbb{C}$ be the complexified Lie algebra of the unipotent group N. The following result is due to Casselman.

Theorem A. *The $\mathfrak{n}_\mathbb{C}$-homologies of the principal series representation I_χ are finite-dimensional, namely,*

$$\dim_\mathbb{C} \mathrm{H}_i(\mathfrak{n}_\mathbb{C}, I_\chi) < \infty, \quad \textit{for every } i \in \mathbb{Z}.$$

Here $\mathrm{H}_\bullet(\mathfrak{n}_\mathbb{C}, \star)$ denotes the Lie algebra homology of $\mathfrak{n}_\mathbb{C}$. Theorem A follows from two classical results of Casselman, namely, the finiteness of $\mathfrak{n}_\mathbb{C}$-homologies of Harish-Chandra modules [CO78, Corollary 2.4] and Casselman's comparison theorem [HT98; LLY21]. It should be pointed out that for every Casselman–Wallach representation V [Cas89] of G, the $\mathfrak{n}_\mathbb{C}$-homologies $\mathrm{H}_\bullet(\mathfrak{n}_\mathbb{C}, V)$ are finite-dimensional. However, as every Casselman–Wallach representation can be embedded into a generalized principal series representation (one induced from a finite-dimensional representation of the minimal parabolic) by Casselman's subrepresentation theorem, we are mainly concerned with the (generalized) principal series representations as a first step.

The first author was partially supported by National Natural Science Foundation of China (No. 12301035) and the basic research program of Jiangsu (No. BK20221057).

For applications to the representation theory of real reductive groups, it is important to study these homology groups in the setting of group homology, say, $\mathrm{H}_i(N; I_\chi)$. There are several kinds of group homology in the literature, see for example, [Bla85; BW80]. For our purpose, we choose to work in the setting of Schwartz homology, which was defined and systematically studied by Chen and Sun in [CS21].

From now on, unless otherwise stated, the notation $\mathrm{H}_i(G; V)$ stands for the Schwartz homology of the representation V, which is naturally a locally convex topological vector space, for every smooth Fréchet representation V of moderate growth of G. We refer the reader to [CS21] for more details about the Schwartz homology. Now we can state the main result of this article.

Theorem B. *Let $G = \mathrm{GL}_n(\mathbb{R})$ and $B = AN$ be the subgroup of upper triangular matrices. For every character χ of A, the Schwartz homologies $\mathrm{H}_\bullet(B; I_\chi)$ of the principal series representation $I_\chi = {}^n\mathrm{Ind}_B^G(\chi)$ of G are finite-dimensional, namely,*

$$\dim_{\mathbb{C}} \mathrm{H}_i(B; I_\chi) < \infty, \quad \textit{for every } i \in \mathbb{Z}.$$

Remark C.

(a) By [CS21, Theorem 7.7], $\mathrm{H}_i(N; I_\chi) \simeq \mathrm{H}_i(\mathfrak{n}_{\mathbb{C}}, I_\chi)$, which by Theorem A, is finite-dimensional. The $\mathfrak{n}_{\mathbb{C}}$-homologies, or equivalently, the Schwartz homology $\mathrm{H}_\bullet(N; \star)$ of representations of the unipotent group N were extensively studied in the literature, see for example, [CO75; Col84; HS83; Kos61]. However, to the best of our knowledge, less is known about the Schwartz homologies $\mathrm{H}_\bullet(B; \star)$ of the Borel subgroup B.

(b) It should be pointed out that for almost every χ, $\mathrm{H}_i(B; I_\chi)$ vanishes for every $i \in \mathbb{Z}$. We sketch it here for the convenience of the reader. By [CS21, Theorem 7.7], $\mathrm{H}_i(B; I_\chi) \simeq \mathrm{H}_i(\mathfrak{b}_{\mathbb{C}}, B \cap K; I_\chi)$, here $\mathfrak{b}_{\mathbb{C}}$ denotes the complexified Lie algebra of the group B and K is the subgroup consisting of orthogonal matrices. Let $R(\mathfrak{b}_{\mathbb{C}}, B \cap K)$ be the Hecke algebra (cf. [KV95, Chapter I]) of the pair $(\mathfrak{b}_{\mathbb{C}}, B \cap K)$. Then by some variation of Shapiro's lemma ([KV95, Theorem 2.119]) and Mackey's isomorphism ([KV95, Theorem 2.103]), one has

$$\begin{aligned} \mathrm{H}_i(\mathfrak{b}_{\mathbb{C}}, B \cap K; I_\chi) &\simeq \mathrm{H}_i\left(\mathfrak{g}_{\mathbb{C}}, K; R(\mathfrak{g}_{\mathbb{C}}, K) \bigotimes_{R(\mathfrak{b}_{\mathbb{C}}, B\cap K)} I_\chi\right) \\ &\simeq \mathrm{H}_i\left(\mathfrak{g}_{\mathbb{C}}, K; (R(\mathfrak{g}_{\mathbb{C}}, K) \bigotimes_{R(\mathfrak{b}_{\mathbb{C}}, B\cap K)} \mathbb{C}) \bigotimes_{\mathbb{C}} I_\chi\right) \\ &\simeq \mathrm{Tor}_i^{\mathfrak{g}_{\mathbb{C}}, K}\left(R(\mathfrak{g}_{\mathbb{C}}, K) \bigotimes_{R(\mathfrak{b}_{\mathbb{C}}, B\cap K)} \mathbb{C}, I_\chi\right), \end{aligned}$$

which vanishes by [BW80, Chapter I, Theorem 5.3], if the infinitesimal character of I_χ differs from that of $(R(\mathfrak{g}_{\mathbb{C}}, K) \bigotimes_{R(\mathfrak{b}_{\mathbb{C}}, B\cap K)} \mathbb{C})^*$. Here $\mathbb{C}$ denotes the trivial $(\mathfrak{b}_{\mathbb{C}}, B \cap K)$-module.

(c) Our approach to the proof of Theorem B is to use tempered vector bundles and Schwartz sections, which were established in [CS21]. More explicitly, we will show that the principal series representations I_χ can be interpreted as Schwartz sections of some line bundle over Nash manifolds. Then the homological finiteness result about Schwartz sections of G-equivariant tempered vector bundles in [BC21] can be applied to derive the main theorem of this article.

As mentioned earlier, every Casselman–Wallach representation of G can be embedded into a generalized principal series representation. With Theorem B and a little bit more efforts, we can prove the following stronger result.

Corollary D. *Let $G = \mathrm{GL}_n(\mathbb{R})$ and B be the subgroup of upper triangular matrices. Then the Schwartz homologies $\mathrm{H}_\bullet(B;V)$ are finite-dimensional for every Casselman–Wallach representation V of G.*

Remark E.

(a) As suggested by the referee, the finiteness of the Schwartz homologies $\mathrm{H}_\bullet(B;V)$ for every Casselman–Wallach representation V can be derived from that of $\mathrm{H}_\bullet(N;V)$ and the Hochschild–Serre spectral sequence $\mathrm{H}_\bullet(A;\mathrm{H}_\bullet(N;V))$. However, it should be noted that the proof for $\mathrm{H}_\bullet(N;V)$ is indirect, that is, the finiteness is proved for Harish-Chandra modules at first and then by applying Casselman's comparison theorem and [CS21, Theorem 7.7]. In contrast to this, our proof for $\mathrm{H}_\bullet(B;I_\chi)$ is more direct and based on a simple computation modulo the theory of Nash manifolds, which may have some independent interest.

(b) It should be pointed out that Corollary D holds for a general real reductive group G, with the upper triangular Borel subgroup B replaced by a minimal parabolic subgroup P of G. As will be clear later, the arguments in this article can be slightly modified for the general case, see Remark 5.8 for more details.

It is well known that homology groups of representations are closely related to the automatic continuous property of linear functionals on their underlying representation spaces, see for example, [Aiz+16, Theorem A] or [CS21, Corollary 4.8].

Corollary F. *With the notations as in Corollary D, the subspace of V generated by $b.v - v$, where $b \in B, v \in V$, is closed in V and every B-invariant linear functional on V is automatically continuous.*

This article is organized as follows: in the next section, we will collect knowledges about tempered vector bundles and Schwartz sections. Along the way, we will bring the main tools that will be used in the proof of Theorem B. In Section 3, we recall the Bruhat decomposition and Iwasawa decomposition for $G = \mathrm{GL}_n(\mathbb{R})$, which have some independent interest and are also indispensable components of our proof. Then we introduce some notions from representations of Lie algebra in Section 4 and finally with these preparations, we prove Theorem B and Corollaries D and F in the last section.

2. Homological finiteness of Schwartz sections

In this section, let G be an almost linear Nash group and we will recall a homological finiteness result about Schwartz sections of some G-equivariant tempered vector bundles. For the notions of Nash manifold, Nash maps and Nash groups, we refer the reader to [Sun15].

2.1. Tempered vector bundles and Schwartz sections

In this subsection, we recall briefly the notions of tempered vector bundles and Schwartz sections. For more details, we refer the reader to [CS21, Sections 2 and 6].

Let M be a Nash manifold and let $\mathcal{E}$ be a Fréchet bundle over M, namely, a topological vector bundle over M such that all the fibres are Fréchet spaces. A tempered vector bundle is a triple $(M, \mathcal{E}, \mathcal{T}_{\mathcal{E}})$, where M is a Nash manifold, $\mathcal{E}$ is a Fréchet bundle over M and $\mathcal{T}_{\mathcal{E}}$ is a tempered structure on $\mathcal{E}$. A tempered structure is like the smooth structure on a smooth manifold, which requires the transition maps between various local charts to be tempered maps. When the tempered structure is clear or canonical, we will omit it from the notations and simply write $(M, \mathcal{E})$ for the tempered vector bundle.

Tempered vector bundles arise naturally from smooth Fréchet representations with moderate growth. Recall that a representation V of G is said to be of moderate growth, if for every continuous seminorm $|\cdot|_1$ on V, there is a positive Nash function f on G and a continuous seminorm $|\cdot|_2$ on V such that

$$|g.v|_1 \leq f(g)|v|_2, \quad \text{for all } g \in G, v \in V.$$

Here and as usual, we do not distinguish a representation with its underlying vector space. We need the following criterion of moderate growth property for characters.

Proposition 2.1. *A character χ of G is of moderate growth if and only if $\chi|_{\mathcal{U}_G}$ is unitary, where $\mathcal{U}_G$ denotes the unipotent radical of G.*

Proof. This is a special case of [CS21, Proposition 7.10]. □

Let H be a Nash subgroup of G and let V be a smooth Fréchet representation of H of moderate growth. Then $(H\backslash G, H\backslash(G \times V))$ is canonically a tempered vector bundle. Here and as usual, $H\backslash(G \times V)$ denotes the orbit space of the action

$$H \times (G \times V) \longrightarrow G \times V, \quad (h, (g, v)) \longmapsto (hg, h.v).$$

Schwartz sections of tempered vector bundles were defined analogously to that of Nash vector bundles, which were studied by Aizenbud and Gourevitch in [AG08, Section 5]. Roughly speaking, they were defined locally as Schwartz functions and then by extension by zero and by summation to get the global ones. Let $(M, \mathcal{E})$ be a tempered vector bundle. Denote by $\Gamma(M, \mathcal{E})$ the space of Schwartz sections of $\mathcal{E}$ over M, which is naturally a Fréchet space.

2.2. Schwartz sections and Schwartz inductions

Let H be a Nash subgroup of G and let V be a smooth Fréchet representation of H of moderate growth. Let $\mathcal{S}(G, V)$ be the space of Schwartz functions on G with values in V. For the notion of Schwartz functions on Nash manifolds, we refer the reader to [AG08]. Define

$$\Phi : \mathcal{S}(G, V) \longrightarrow \operatorname{Ind}_H^G V$$
$$\varphi \longmapsto \left(g \mapsto \int_H h.\varphi(h^{-1}g)\mathrm{d}h \right),$$

where $\mathrm{d}h$ is a left invariant Haar measure on H and $\operatorname{Ind}_H^G V$ denotes the unnormalized smooth induction. Then the Schwartz induced representation $\operatorname{ind}_H^G V$ is defined as the image of the map Φ, equipped with the quotient topology of the domain. Schwartz induced representation was first defined and studied by du Cloux, see [Du 91, Section 2].

It turns out that the Schwartz induced representation can be interpreted in terms of Schwartz sections.

Proposition 2.2 ([CS21, Proposition 6.7])**.** *Let H be a Nash subgroup of an almost linear Nash group G. For every smooth Fréchet representation V of H of moderate growth,*

$$\Gamma(H\backslash G, H\backslash(G \times V)) \cong \operatorname{ind}_H^G V$$

as representations of G.

For the G-actions on the space of Schwartz sections $\Gamma(H\backslash G, H\backslash(G \times V))$, we refer the reader to [CS21, Proposition 6.3]. The following result enables us to connect the smooth inductions with Schwartz inductions.

Proposition 2.3. *Let V be a smooth Fréchet representation of H of moderate growth. If $H\backslash G$ is compact, then $\operatorname{ind}_H^G V = \operatorname{Ind}_H^G V$.*

For a proof, see [Du 91, Remark 2.1.4] or [LS13, Lemma 3.1].

2.3. Homological finiteness theorem

Recall that a representation V of G is said to be homologically finite if the Schwartz homology $\mathrm{H}_i(G; V)$ is finite-dimensional for every $i \in \mathbb{Z}$. It follows easily that finite-dimensional representations are always homologically finite. In this section we will recall a general homological finiteness result in [BC21], which is crucial to the proof of our main theorem.

Let X be a G-Nash manifold, namely, a Nash manifold that carries a Nash action $G \times X \to X$. Let E be a G-equivariant tempered vector bundle over X, namely, a tempered vector bundle over X together with a tempered bundle action $G \times \mathsf{E} \to \mathsf{E}$. Then the space of Schwartz sections $\Gamma^\varsigma(X, \mathsf{E})$ of the tempered vector bundle E over X carries a natural action of G, which as a representation of G, is smooth Fréchet and of moderate growth, see [CS21, Proposition 6.3].

For every $x \in X$, let G_x denote its stabilizer in G, and let E_x denote the fibre of E at x. Write

$$\mathrm{N}_x := \frac{\mathrm{T}_x(X)}{\mathrm{T}_x(G.x)} \otimes_{\mathbb{R}} \mathbb{C} \qquad (\mathrm{T}_x \text{ stands for the tangent space})$$

for the complexified normal space, and write

$$\mathrm{N}_x^* := \text{the dual space of } \mathrm{N}_x,$$

which is the complexified conormal space.

Theorem 2.4 ([BC21, Theorem 1.5]). *Let (X, E) be a G-equivariant tempered vector bundle such that G acts on X with finitely many orbits. Assume that all the fibres of E are finite-dimensional and for every $x \in X$, the trivial representation of G_x does not occur as a subquotient of*

$$\mathsf{E}_x \otimes \mathrm{Sym}^k(\mathrm{N}_x^*) \otimes \delta_{G/G_x},$$

for sufficiently large k. Then $\Gamma^{\varsigma}(X, \mathsf{E})$ is homologically finite as a representation of G.

Here Sym^k indicates the kth symmetric power and δ_G denotes the modular character of the group G, while $\delta_{G/G_x} := (\delta_G)|_{G_x} \cdot \delta_{G_x}^{-1}$. Later in Section 5, we will realize the principal series representation of the general linear group over $\mathbb{R}$ as the Schwartz sections of some G-equivariant line bundle and verify that the conditions in Theorem 2.4 are satisfied in this special situation.

3. Bruhat decomposition and Iwasawa decomposition

The results in this section hold for general real reductive groups, however, for simplicity of notations, we restrict ourselves to the case of $G = \mathrm{GL}_n(\mathbb{R})$. Let $K = \mathrm{O}(n)$ be the subgroup of orthogonal matrices in G, which is a maximal compact subgroup of G. Let A (resp. N) be the subgroup of G consisting of diagonal matrices (resp. upper triangular unipotent matrices). Let $B = AN$ be the Borel subgroup of G.

3.1. Weyl group and Bruhat decomposition

The Weyl group W of G with respect to A is defined as

$$W := W(G, A) := N_K(A)/Z_K(A),$$

the normalizer $N_K(A)$ modulo the centralizer $Z_K(A)$ of A in K. The Weyl group plays an important role in the representation theory of real reductive groups, see for example [Vog81].

In the general linear group case, the Weyl group has a simple form. Let S_n denote the nth symmetric group. There exists a natural embedding

$$\iota : S_n \hookrightarrow K, \quad \sigma \longmapsto \sum_i E_{\sigma(i),i},$$

where $E_{i,j}$ denotes the usual fundamental matrix whose ith row and jth column is one and other entries are zero. We will identify S_n as a subgroup of K through this embedding. It is easy to see that $S_n \subseteq N_K(A)$. Moreover,

Proposition 3.1. *The homomorphism*

$$S_n \longrightarrow W, \quad \sigma \longmapsto \sigma Z_K(A)$$

induced by $S_n \hookrightarrow N_K(A)$, is an isomorphism of groups.

This result is elementary and well known. We will identify the Weyl group W with S_n through this isomorphism in this article.

Let $X = B\backslash G$ be the flag manifold, which carries a natural action of G by right multiplication.

Proposition 3.2. *The action (by right multiplication) of B on the flag manifold X has finitely many orbits. More explicitly, the natural map*

$$S_n \longrightarrow B\backslash G/B, \quad \sigma \longmapsto B\sigma B$$

is bijective.

For a proof of this proposition, see [Kna02, Theorem 7.40]. This bijective map is called the Bruhat decomposition of G. We will use this result in the proof of our main theorem.

3.2. Iwasawa decomposition

We also need some topological property of the flag manifold $X = B\backslash G$.

Proposition 3.3. *The multiplication map*

$$K \times A^\circ \times N \longrightarrow G, \quad (k, a, n) \longmapsto kan$$

is a diffeomorphism between smooth manifolds, where A° denotes the identity connected component of A.

This is called the Iwasawa decomposition of G. For a proof, see for example [Kna02, Theorem 6.46].

Corollary 3.4. *Equipped with the quotient topology from G, the flag manifold X is compact.*

Proof. Let $\pi : G \to B\backslash G$ be the quotient map. It follows from Proposition 3.3 that $X = \pi(K)$, which must be compact, since K is compact. □

4. Some preliminaries from Lie algebra

In this section, we recall some basic notions from the theory of Lie algebras. Recall that $G = \mathrm{GL}_n(\mathbb{R})$ and $B = AN$ is the subgroup of nonsingular upper triangular matrices, where A denotes the Levi component of B which consists of diagonal matrices and N denotes the unipotent radical of B. Let N^- denote the transpose of the unipotent subgroup N. Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{a}, \mathfrak{n}, \mathfrak{n}^-$ be the Lie algebras of the corresponding Lie groups. As usual, a subscript $\mathbb{C}$ means complexification. For example, $\mathfrak{g}_\mathbb{C}$ denotes the complexification of the Lie algebra $\mathfrak{g}$.

Lie groups have a natural action on their complexified Lie algebras, known as the adjoint action, usually denoted by Ad. It turns out that the adjoint action of the subgroup A of G on the Lie algebra $\mathfrak{g}_\mathbb{C}$ is completely reducible. Namely,

$$\mathfrak{g}_\mathbb{C} = \mathfrak{a}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta} (\mathfrak{g}_\mathbb{C})_\alpha,$$

where $(\mathfrak{g}_\mathbb{C})_\alpha$ corresponds to a nontrivial character (one-dimensional representation) of A and α is the differential of the corresponding character, which lies in the dual $\mathfrak{a}^*$ of the real vector space $\mathfrak{a}$, for every $\alpha \in \Delta$. Fix the standard identification $\mathfrak{a} \simeq \mathbb{R}^n$ and let $e_i \in \mathfrak{a}^*$ be the ith projection map for every $1 \leq i \leq n$, then $\Delta = \{e_i - e_j \mid i \neq j\}$. The elements in Δ are called roots of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{a}_\mathbb{C}$. The above decomposition is called the root space decomposition of $\mathfrak{g}_\mathbb{C}$ relative to $\mathfrak{a}_\mathbb{C}$.

Using the ordered projections $e_1, \ldots, e_n$, one can define a partial order, the lexicographical order on $\mathfrak{a}^*$. Under this order, $\Delta^+ := \{\alpha \in \Delta \mid \alpha > 0\} = \{e_i - e_j \mid i < j\}$ and

$$\mathfrak{n}_{\mathbb{C}} = \bigoplus_{\alpha \in \Delta^+} (\mathfrak{g}_{\mathbb{C}})_\alpha.$$

Definition 4.1. For a finite-dimensional completely reducible continuous representation (π, V) of A, we call the irreducible summands (must be one-dimensional) of the differential representation $(\mathrm{d}\pi, V)$ of $\mathfrak{a}_{\mathbb{C}}$ on V the weights of the representation π.

Remark 4.2. It seems that for the definition of weights it suffices to assume that the differential representation $(\mathrm{d}\pi, V)$ is completely reducible. However, as pointed out by the referee, since $A = A^\circ \times (A \cap K)$ and $A \cap K$ is a finite abelian group, it follows easily that (π, V) is completely reducible if and only if so is the $\mathfrak{a}_{\mathbb{C}}$-module $(\mathrm{d}\pi, V)$.

Here and as usual, we do not distinguish one-dimensional representation of Lie groups (resp. of Lie algebras) with the corresponding character (resp. a linear functional). For example, the weights of the adjoint representation $(\mathrm{Ad}, \mathfrak{n}_{\mathbb{C}})$ of A are exactly Δ^+. The multiset of weights (counted with multiplicity) of a finite-dimensional completely reducible representation π is denoted by $W(\pi)$.

Proposition 4.3. *Let (π, V) be a completely reducible finite-dimensional representation of A with the multiset of weights $W(\pi)$. Assume that a character χ of A occurs as a subquotient in V, then the differential $\mathrm{d}\chi \in W(\pi)$.*

Proof. Passing to the differential, we see that $\mathrm{d}\chi$ must also occur as a subquotient in V. Since as a representation of $\mathfrak{a}_{\mathbb{C}}$, V is completely reducible. It follows directly that $\mathrm{d}\chi \in W(\pi)$. □

The following results are obvious.

Proposition 4.4. *Let (π_i, V_i) $(i = 1, 2)$ be finite-dimensional completely reducible representations of A with the multiset of weights $W(\pi_i)$. Then $V_1 \otimes V_2$ is also completely reducible and $W(\pi_1 \otimes \pi_2) = \{\alpha + \beta \mid \alpha \in W(\pi_1), \beta \in W(\pi_2)\}$.*

Proposition 4.5. *Let (π, V) be a finite-dimensional completely reducible representations of A with the multiset of weights $W(\pi)$. Then π^* is also completely reducible and $W(\pi^*) = -W(\pi)$, here π^* denotes the contragradient representation of π.*

5. Proof of the main result

With the preparations in the previous sections, now we can prove the main results of this article.

5.1. The principal series case

This subsection is devoted to the proof of Theorem B. The notations are as in the Introduction. Recall that χ is a character of A, which is viewed as a character of B in the usual way. Denote $X = B\backslash G$, which is canonically a Nash manifold by [Sun15, Proposition 1.2]. Let $\mathsf{E} = B\backslash(G \times \mathbb{C})$ be the orbit space of the action

$$B \times (G \times \mathbb{C}) \longrightarrow G \times \mathbb{C}, \quad (b, (g, z)) \longmapsto (bg, \chi(b)\delta_B^{1/2}(b)z).$$

Since $\chi \otimes \delta_B^{1/2}$ is trivial on N, it follows from Proposition 2.1 that (X, E) is a tempered vector bundle. Let $p : \mathsf{E} \to X$ be the natural projection which is obviously G-equivariant. Here G acts on E and X by right translations.

Put $\widetilde{\chi} = \chi \otimes \delta_B^{1/2}$ for simplicity. Define

$$\begin{aligned} \Psi : \Gamma(X, \mathsf{E}) &\longrightarrow \operatorname{ind}_B^G \widetilde{\chi} \\ s &\longmapsto f_s, \end{aligned}$$

where f_s is determined by the equation $s(Bg) = B(g, f_s(g))$ for every $g \in G$ and ind denotes the Schwartz induction. By Proposition 2.2, the map Ψ is a G-equivariant topological linear isomorphism. By Corollary 3.4 and Proposition 2.3,

$$\operatorname{ind}_B^G \widetilde{\chi} = \operatorname{Ind}_B^G \widetilde{\chi} = I_\chi,$$

where the principal series representation I_χ was defined in the Introduction.

Note that

$$\mathrm{H}_i(B; I_\chi) \simeq \mathrm{H}_i(B; \operatorname{ind}_B^G \widetilde{\chi}) \simeq \mathrm{H}_i(B; \Gamma(X, \mathsf{E})),$$

thus it suffices to prove that $\dim_{\mathbb{C}} \mathrm{H}_i(B; \Gamma(X, \mathsf{E})) < \infty$, for every $i \in \mathbb{Z}$, namely, $\Gamma(X, \mathsf{E})$ is homologically finite as a representation of B. Since all the fibres of E (line bundle) are one-dimensional, by Proposition 3.2 and Theorem 2.4, the proof of Theorem B is reduced to the following proposition.

Proposition 5.1. *For every $x \in X$, the trivial representation of B_x does not occur as a subquotient of*

$$\mathsf{E}_x \otimes \operatorname{Sym}^k(\mathrm{N}_x^*) \otimes \delta_{B/B_x},$$

for sufficiently large k.

The rest of this section is devoted to the proof of Proposition 5.1. For $\sigma \in S_n$, we let B_σ denote the stabilizer of $B\sigma \in X$ and similarly, E_σ denote the fibre of E at $B\sigma$ and N_σ denote the complexified normal space at $B\sigma$.

Proposition 5.2. *For every $\sigma \in S_n$, $B_\sigma = \sigma^{-1}B\sigma \cap B$. As representations of B_σ, $\mathsf{E}_\sigma = \widetilde{\chi}^\sigma$ and*

$$\mathrm{N}_\sigma = \frac{\mathfrak{g}_{\mathbb{C}}}{\mathfrak{b}_{\mathbb{C}} + \sigma^{-1}\mathfrak{b}_{\mathbb{C}}\sigma},$$

where $\widetilde{\chi}^\sigma(b) = \widetilde{\chi}(\sigma b \sigma^{-1})$ for $b \in B_\sigma$.

Proof. This proposition follows by direct verifications from the definitions. We omit the details. □

By Proposition 3.2, to prove Proposition 5.1, it suffices to show that for every $\sigma \in S_n$, the trivial representation of B_σ does not occur as a subquotient of

$$\widetilde{\chi}^\sigma \otimes \operatorname{Sym}^k(\mathrm{N}_\sigma^*) \otimes \delta_{B/B_\sigma},$$

for sufficiently large k. Note that $A \subseteq B_\sigma$, Proposition 5.1 is further reduced to

Proposition 5.3. *For every $\sigma \in S_n$, the character $(\widetilde{\chi}^\sigma \otimes \delta_{B/B_\sigma})^{-1}$ of A does not occur as a subquotient of* $\mathrm{Sym}^k(\mathrm{N}_\sigma^*)$*, for sufficiently large k.*

Remark 5.4.

(a) For a more concrete description of the differential of the character $(\widetilde{\chi}^\sigma \otimes \delta_{B/B_\sigma})^{-1}$, we refer the reader to [Che23, Equation 13].

(b) Proposition 5.3 essentially follows from the statement that every weight in $S(\mathfrak{n}_\mathbb{C}) = \bigoplus_{k\in\mathbb{N}} \mathrm{Sym}^k(\mathfrak{n}_\mathbb{C})$ has finite multiplicity. This is familiar in the context of representation theory of semisimple Lie algebras. Actually, our arguments in the following provide a proof of this statement.

For the simplicity of notations, we prove a slightly more general statement, namely, for every $\sigma \in S_n$ and every character μ of A, μ does not occur as a subquotient of $\mathrm{Sym}^k(\mathrm{N}_\sigma^*)$, for sufficiently large k.

By Proposition 5.2, the presentation $(\mathrm{Ad}, \mathrm{N}_\sigma)$ of A is completely reducible and

$$\mathrm{N}_\sigma^* \subseteq (\mathfrak{g}_\mathbb{C}/\mathfrak{b}_\mathbb{C})^* \simeq (\mathfrak{n}_\mathbb{C}^-)^*,$$

here

$$\mathfrak{n}_\mathbb{C}^- = \bigoplus_{\alpha\in-\Delta^+} (\mathfrak{g}_\mathbb{C})_\alpha.$$

By Proposition 4.5, $W((\mathfrak{n}_\mathbb{C}^-)^*) = -W(\mathfrak{n}_\mathbb{C}^-) = \Delta^+$, thus $W(\mathrm{N}_\sigma^*) \subseteq \Delta^+$.

For every $\sigma \in S_n$, by Proposition 4.4, we conclude that

$$\mathrm{d}\mu \notin W(\mathrm{Sym}^k(\mathrm{N}_\sigma^*)),$$

for k sufficiently large. Actually, let

$$\Pi^+ := \{e_i - e_{i+1} \,|\, 1 \le i \le n-1\}$$

be the simple roots in Δ^+. If $\mathrm{d}\mu \notin W(\mathrm{Sym}^k(\mathrm{N}_\sigma^*))$ (for example, $\mathrm{Im}(\mu) \nsubseteq \mathbb{R}$) for every $k \in \mathbb{N}$, there is nothing to prove. On the other hand, if

$$\mathrm{d}\mu \in W(\mathrm{Sym}^{k_0}(\mathrm{N}_\sigma^*))$$

for some k_0, put

$$\mathrm{d}\mu = \sum_{\alpha\in\Pi^+} n_\alpha \alpha,$$

and define $M := \sum_{\alpha\in\Pi^+} n_\alpha$. It is easy to see that

$$\mathrm{d}\mu \notin W(\mathrm{Sym}^k(\mathrm{N}_\sigma^*))$$

for every $k > M$. Now it follows from Proposition 4.3 that the character μ of A does not occur as a subquotient of $\mathrm{Sym}^k(\mathrm{N}_\sigma^*)$, for sufficiently large k. Take $\mu = (\widetilde{\chi}^\sigma \otimes \delta_{B/B_\sigma})^{-1}$ finishing the proof of Proposition 5.3 and the proof of Theorem B is completed.

5.2. Principal series to Casselman–Wallach representations

Now we are ready to prove Corollaries D and F. Firstly, we need to generalize Theorem B to the generalized principal series case.

Proposition 5.5. *Let $G=\mathrm{GL}_n(\mathbb{R})$ and $B=AN$ be the subgroup of upper triangular matrices. For every finite-dimensional representation F of B, the Schwartz homologies $\mathrm{H}_\bullet(B;\mathrm{Ind}_B^G(F))$ of the generalized principal series representation $\mathrm{Ind}_B^G(F)$ of G are finite-dimenisonal.*

Proof. Since the Lie algebra $\mathfrak{n}$ of N is nilpotent, the representation F of B has a finite filtration whose consecutive subquotients carry trivial actions of N. By [CS21, Corollary 7.8] and the exactness of the induction functor, it suffices to prove the special case that F carries a trivial action of the group N. As a representation of A, F has a finite filtration whose consecutive subquotients are characters of A. By the same reason, it suffices to assume that F is a character of A. This was already established in Theorem B. □

Let V be a Casselman–Wallach representation of G. By Casselman's subrepresentation theorem, there exists an exact sequence (cf. [HT98, Proposition 3])

$$0\longrightarrow V\xrightarrow{f_0} I_1\xrightarrow{f_1} I_2\xrightarrow{f_2}\cdots,$$

where all I_i are generalized principal series representations of G. For every $i\geq 1$, one has short exact sequences

$$0\longrightarrow \mathrm{Ker}(f_i)\longrightarrow I_i\longrightarrow \mathrm{Im}(f_i)(=\mathrm{Ker}(f_{i+1}))\longrightarrow 0,$$

and the associated long exact sequences of Schwartz homologies (cf. [CS21, Corollary 7.8])

$$\cdots\longrightarrow \mathrm{H}_{p+1}(B;\mathrm{Ker}(f_{i+1}))\longrightarrow \mathrm{H}_p(B;\mathrm{Ker}(f_i))\longrightarrow \mathrm{H}_p(B;I_i)\longrightarrow\cdots.$$

Lemma 5.6. *For every $i\geq 1$ and every $p\in\mathbb{Z}$, if $\mathrm{H}_{p+1}(B;\mathrm{Ker}(f_{i+1}))$ is finite-dimensional, then so is $\mathrm{H}_p(B;\mathrm{Ker}(f_i))$.*

Proof. It follows directly from Proposition 5.5 and the long exact sequence of Schwartz homologies. □

Lemma 5.7. *For every $i\geq 1$ and every $p\in\mathbb{Z}$, the Schwartz homology $\mathrm{H}_p(B;\mathrm{Ker}(f_i))$ is finite-dimensional.*

Proof. By Lemma 5.6, it suffices to show that $\mathrm{H}_{p+k}(B;\mathrm{Ker}(f_{i+k}))$ is finite-dimensional for sufficiently large k, which is obvious since $\mathrm{H}_{p+k}(B;\mathrm{Ker}(f_{i+k}))$ vanishes for k sufficiently large (cf. the proof of [CS21, Theorem 7.7]). □

Now Corollary D follows from Lemma 5.7, since $V\simeq\mathrm{Ker}(f_1)$.

By [CS21, Proposition 5.13], $\mathrm{H}_0(B;V)=V_B$, which is finite-dimensional by Corollary D, here V_B denotes the coinvariant space of V. Now it follows from [BW80, Chapter IX, Lemma 3.4] that V_B is Hausdorff, or equivalently, the subspace generated by $b.v-v$ of V is closed, where $b\in B, v\in V$. Every B-invariant linear functional on V factors through V_B, which is a finite-dimensional Hausdorff Fréchet space, thus must be continuous. This completes the proof of Corollary F.

Note that by [CS21, Theorem 7.7]

$$\mathrm{H}_0(B;V)\simeq \mathrm{H}_0(\mathfrak{b}_{\mathbb{C}},B\cap K;V)\simeq \mathrm{H}_0(\mathfrak{b}_{\mathbb{C}},V)^{B\cap K}\subseteq \mathrm{H}_0(\mathfrak{b}_{\mathbb{C}},V).$$

By [Aiz+16, Theorem A], $\mathrm{H}_0(\mathfrak{b}_{\mathbb{C}}, V)$ is finite-dimensional, thus so is $\mathrm{H}_0(B; V)$. Therefore Corollary F also follows from [Aiz+16, Theorem A].

Remark 5.8. The results in Sections 3 and 4 hold for general real reductive groups and the arguments in Section 5 can be slightly modified for the general case. We conclude that the main result, namely, Corollary D holds for Casselman–Wallach representations of a general real reductive group. We state it here for the convenience of future reference. Let G be a real reductive group and P be a minimal parabolic subgroup of G. Let V be a Casselman–Wallach representation of G. Then

$$\dim_{\mathbb{C}} \mathrm{H}_i(P; V) < \infty, \quad \text{for every } i \in \mathbb{Z},$$

here $\mathrm{H}_\bullet(P; \star)$ stands for the Schwartz homology.

Acknowledgements

The authors would like to thank the referee for the valuable comments and helpful suggestions which improve the manuscript a lot.

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Received February 17, 2025
Accepted August 8, 2025
Published online August 25, 2026