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Conformal Lie 2-algebras and conformal omni-Lie algebras

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Abstract. The notions of conformal Lie 2-algebras and conformal omni-Lie algebras are introduced. It is proved that the category of conformal Lie 2-algebras and the category of 2-term conformal L_∞-algebras are equivalent. We construct conformal Lie 2-algebras from conformal omni-Lie algebras and Leibniz conformal algebras.

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1. Introduction

The algebraic theory of Lie 2-algebras, which constitutes a categorification of Lie algebras, was extensively developed by Baez and Crans [BC04]. As a truncated version of L_∞-algebras, Lie 2-algebras replace the underlying vector space with a 2-term chain complex of vector spaces (a 2-vector space) and the Jacobi identity with a natural transformation called the Jacobiator, subject to specific coherence laws. Baez and Crans established an equivalence between the category of Lie 2-algebras and that of 2-term L_∞-algebras. For further developments in this area, we refer to [LL20; LSZ14; SL13; SL16].

Omni-Lie algebras, introduced by Weinstein [Wei00], represent a central example of Lie 2-algebras. They arise as linearizations of Courant algebroids [LWX97], and since every Courant algebroid induces a Lie 2-algebra, omni-Lie algebras naturally fall within this framework. Subsequent research has extended their study to various generalizations, including omni-Lie algebroids and omni-Lie 2-algebras, as explored in [CL10; KW01; SLZ11]. More recently, [ZL14] investigated the Dirac structures of omni-Lie superalgebras.

In this paper, we consider whether a categorification of Lie conformal algebras or vertex Lie algebras exists. We answer this affirmatively by introducing the notion of a conformal Lie 2-algebra, which is a Lie conformal algebra object in the category of $\mathbb{C}[\partial]$-modules internal to 2-vector spaces. We also define 2-term conformal L_∞-algebras and prove an equivalence between the category of conformal Lie 2-algebras and that of 2-term conformal L_∞-algebras.

The second part of the paper is devoted to constructing examples of conformal Lie 2-algebras. In Section 4.3, we introduce conformal omni-Lie algebras and use them to produce conformal Lie 2-algebras. Moreover, in the final part of Section 4, we provide a construction of conformal Lie 2-algebras from Leibniz conformal algebras.

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To illustrate this construction, we also provide explicit examples, as not every Leibniz conformal algebra arises from an omni-Lie algebra.

The paper is organized as follows. In Section 2, we recall background material on Lie conformal algebras and Leibniz conformal algebras. Section 3 introduces conformal Lie 2-algebras and 2-term conformal L_∞-algebras, and establishes the categorical equivalence between them. Section 4 examines special cases of conformal Lie 2-algebras, such as skeletal and strict types. We also define the conformal omni-Lie algebra $\mathcal{E}$ associated with a Lie conformal algebra and a representation. Finally, we construct conformal Lie 2-algebras from conformal omni-Lie algebras and Leibniz conformal algebras.

Throughout this paper, we denote by $\mathbb{C}$ the field of complex numbers. Let M be a vector space. The space of polynomials of λ with coefficients in M is denoted by $M[\lambda]$.

2. Preliminaries

In this section, we will recall some facts and definitions about Lie conformal algebras and conformal Leibniz algebras, see [BKV99; DK98; GT14; HB20; Kac98; LTW13; Lib08; Zha15] for more details.

A *conformal algebra* R is a $\mathbb{C}[\partial]$-module with a λ-product $\cdot_\lambda\cdot$ which defines a $\mathbb{C}$-linear map from $A\otimes A\to A[\lambda]$ satisfying the conformal sesquilinearity condition

$$(\partial x)_\lambda y=-\lambda x_\lambda y,\quad x_\lambda \partial y=(\partial+\lambda)x_\lambda y, \tag{1}$$

for all $x,y\in A$.

Definition 2.1. A *Lie conformal algebra* R is a conformal algebra with the $\mathbb{C}$-bilinear map $[\cdot_\lambda\cdot]:R\times R\to R[\lambda]$ satisfying

$$[x_\lambda y]=-[y_{-\partial-\lambda}x], \qquad \text{(skew-symmetry)} \tag{2}$$

$$[x_\lambda[y_\mu z]]=[[x_\lambda y]_{\lambda+\mu}z]+[y_\mu[x_\lambda z]], \qquad \text{(Jacobi identity)} \tag{3}$$

for all $x,y,z\in R$.

Using the skew-symmetry condition, we can rewrite the Jacobi identity as the following identity:

$$[[x_\lambda y]_{\lambda+\mu}z]=[x_\lambda[y_\mu z]]+[[x_\lambda z]_{-\partial-\mu}y]. \tag{4}$$

Definition 2.2. A left *Leibniz conformal algebra* $(R,[\cdot_\lambda\cdot])$ is a conformal algebra satisfying the following left Leibniz identity

$$x\circ_\lambda(y\circ_\mu z)=(x\circ_\lambda y)\circ_{\lambda+\mu}z+y\circ_\mu(x\circ_\lambda z). \tag{5}$$

Definition 2.3. A right *Leibniz conformal algebra* $(R,[\cdot_\lambda\cdot])$ is a conformal algebra satisfying the following right Leibniz identity

$$(x\circ_\lambda y)\circ_{\lambda+\mu}z=x\circ_\lambda(y\circ_\mu z)+(x\circ_\lambda z)\circ_{-\partial-\mu}y. \tag{6}$$

Remark 2.4. Using the equivalent identity (4), we can see a Lie conformal algebra as both a left and a right Leibniz conformal algebra. However, it should be noted that the converse is not true as the general Leibniz conformal algebra does not meet the requirement of skew-symmetry.

Definition 2.5. A *left module* M over a Lie conformal algebra R is a $\mathbb{C}[\partial]$-module endowed with a map $R \times M \to M[\lambda]$, $(x, v) \mapsto x \triangleright_\lambda v$, satisfying the following axioms:

$$(\partial x) \triangleright_\lambda v = -\lambda x \triangleright_\lambda v, \quad x \triangleright_\lambda (\partial v) = (\partial + \lambda) x \triangleright_\lambda v, \tag{7}$$

$$[x_\lambda y] \triangleright_{\lambda+\mu} v = x \triangleright_\lambda (y \triangleright_\mu v) - y \triangleright_\mu (x \triangleright_\lambda v), \tag{8}$$

for all $x, y \in R, v \in M$. Similarly, a *right module* M over a Lie conformal algebra R is a $\mathbb{C}[\partial]$-module endowed with a bilinear map $M \times R \to M[\lambda]$, $(v, x) \mapsto v \triangleleft_\lambda x$, satisfying the following axioms:

$$(\partial v) \triangleleft_\lambda x = -\lambda v \triangleleft_\lambda x, \quad v \triangleleft_\lambda (\partial x) = (\partial + \lambda) v \triangleleft_\lambda x, \tag{9}$$

$$v \triangleleft_\mu [x_\lambda y] = (v \triangleleft_\mu x) \triangleleft_{\lambda+\mu} y - (v \triangleleft_\mu y) \triangleleft_{-\partial-\lambda} x. \tag{10}$$

A right R-module become a left R-module if we define $x \triangleright_\lambda v$ to be $-v \triangleleft_{-\partial-\lambda} x$.

Definition 2.6 ([Zha15])**.** A *module* or *representation* M over a Leibniz conformal algebra $(R, [\cdot_\lambda \cdot])$ is a $\mathbb{C}[\partial]$-module endowed with two $\mathbb{C}$-bilinear maps $R \times M \to M[\lambda]$, $(x, v) \mapsto x \triangleright_\lambda v$ and $M \times R \to M[\lambda]$, $(v, x) \mapsto v \triangleleft_\lambda x$ satisfying the following axioms ($a, b \in R, v \in M$):

$$(\partial x) \triangleright_\lambda v = -\lambda x \triangleright_\lambda v, \quad x \triangleright_\lambda (\partial v) = (\partial + \lambda) x \triangleright_\lambda v, \tag{11}$$

$$(\partial v) \triangleleft_\lambda x = -\lambda v \triangleleft_\lambda x, \quad v \triangleleft_\lambda (\partial x) = (\partial + \lambda) v \triangleleft_\lambda x, \tag{12}$$

$$x \triangleright_\lambda (y \triangleright_\mu v) = (x \circ_\lambda y) \triangleright_{\lambda+\mu} v + y \triangleright_\mu (x \triangleright_\lambda v), \tag{13}$$

$$x \triangleright_\lambda (v \triangleleft_\mu y) = (x \triangleright_\lambda v) \triangleleft_{\lambda+\mu} y + v \triangleleft_\mu (x \circ_\lambda y), \tag{14}$$

$$v \triangleleft_\lambda (x \circ_\mu y) = (x \triangleleft_\lambda x) \triangleleft_{\lambda+\mu} y + x \triangleright_\mu (v \triangleleft_\lambda y). \tag{15}$$

We denote it by $(M, \triangleright_\lambda, \triangleleft_\lambda)$.

Next, we give some examples of Lie conformal algebras.

Example 2.7. Let $\mathfrak{g}$ be a Lie algebra. Let $\mathrm{Cur}\,\mathfrak{g} := \mathbb{C}[\partial] \otimes \mathfrak{g}$ be the free $\mathbb{C}[\partial]$-module. Then $\mathrm{Cur}\,\mathfrak{g}$ is a Lie conformal algebra, called current Lie conformal algebra, with λ-bracket given by:

$$[(f(\partial) \otimes x)_\lambda (g(\partial) \otimes y)] := f(-\lambda) g(\lambda + \partial) \otimes [x, y],$$

for all $f(\partial), g(\partial) \in \mathbb{C}[\partial], x, y \in \mathfrak{g}$.

The Virasoro Lie conformal algebra Vir is the simplest nontrivial example of Lie conformal algebras. It is defined by

$$\mathrm{Vir} = \mathbb{C}[\partial] L, \quad [L_\lambda L] = (\partial + 2\lambda) L.$$

Let M and N be $\mathbb{C}[\partial]$-modules. A conformal linear map f from M to N is a $\mathbb{C}$-linear map $f_\lambda : M \to \mathbb{C}[\lambda] \otimes N$ such that $f_\lambda \partial = (\partial + \lambda) f_\lambda$. The category of $\mathbb{C}[\partial]$-modules with conformal linear maps as morphisms is denoted by Vect^∂.

Let $\mathrm{Chom}(M, N)$ denote the set of conformal linear maps from M to N. Then $\mathrm{Chom}(M, N)$ is a $\mathbb{C}[\partial]$-module via:

$$\partial f_\lambda = -\lambda f_\lambda.$$

The composition $fg : L \to N$ of conformal linear maps $f : M \to N$ and $g : L \to M$ is given by $(f_\lambda g)_{\lambda+\mu} = f_\lambda g_\mu$. If M is a finitely generated $\mathbb{C}[\partial]$-module, then $\mathrm{Cend}(M) := \mathrm{Chom}(M, M)$ is an associative conformal algebra with respect to the above composition. Thus, $\mathrm{Cend}(M)$ becomes a Lie conformal algebra, which is denoted as $\mathrm{Cend}(M)$, with respect to the following λ-bracket

$$[f_\lambda g]_\mu = f_\lambda g_{\mu-\lambda} - g_{\mu-\lambda} f_\lambda,$$

equivalently,

$$[f_\lambda g] = f_\lambda g - g_{-\partial-\lambda} f.$$

All $\mathbb{C}[\partial]$-modules in this paper are assumed to be finitely generated.

A homomorphism between two Lie conformal algebras $(R, [\cdot_\lambda \cdot])$ and $(R', [\cdot_\lambda \cdot]')$ is a conformal linear map $f : R \to R'$ such that

$$f_\lambda \partial(x) = (\partial + \lambda) f_\lambda(x), \quad f([x_\lambda y]) = [f(x)_\lambda f(y)]',$$

for all $x, y \in R$.

An n-cochain ($n \in \mathbb{Z}_{\geq 0}$) of a Lie conformal algebra R with coefficients in a module M is a $\mathbb{C}$-linear map

$$\begin{aligned} \gamma : \qquad R^{\otimes n} &\longrightarrow M[\lambda_1, \dots, \lambda_{n-1}], \\ (x_1, \dots, x_n) &\longmapsto \gamma_{\lambda_1,\dots,\lambda_{n-1}}(x_1, \dots, x_n) = \gamma(x_{1\lambda_1} \dots x_{n-1\lambda_{n-1}} x_n), \end{aligned}$$

where $M[\lambda_1, \dots, \lambda_{n-1}]$ denotes the space of polynomials with coefficients in M, satisfying the following conditions:

(1) Conformal linearity:

$$\gamma_{\lambda_1,\dots,\lambda_{n-1}}(x_1, \dots, \partial x_i, \dots, x_n) = -\lambda_i \gamma_{\lambda_1,\dots,\lambda_{n-1}}(x_1, \dots, x_i, \dots, x_n).$$

(2) Skew-symmetry:

$$\gamma(x_1, \dots, x_{i+1}, \partial(x_i), \dots, x_n) = -\gamma(x_1, \dots, x_i, \partial(x_{i+1}), \dots, x_n).$$

Let $R^{\otimes 0} = \mathbb{C}$ so that a 0-cochain is an element $v \in M$ and $(\delta v)_\lambda x = x_\lambda v$. Define a differential δ of a cochain R by

$$\begin{aligned} &(\delta\gamma)_{\lambda_1,\dots,\lambda_n}(x_1, \dots, x_{n+1}) \\ &= \sum_{i=1}^{n+1} (-1)^{i+1} x_i \triangleright_{\lambda_i} \gamma_{\lambda_1,\dots,\hat{\lambda}_i,\dots,\lambda_n}(x_1, \dots, \widehat{x}_i, \dots, x_{n+1}) \\ &\quad + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \gamma_{\lambda_i+\lambda_j,\lambda_1,\dots,\hat{\lambda}_i,\dots\hat{\lambda}_j,\dots,\lambda_n}([x_{i\lambda_i} x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_{n+1}). \end{aligned}$$

It is proved that $\delta^2 = 0$ [BKV99]. Therefore $\{C^*(R, M), \delta\}$ is indeed a cochain complex, whose cohomology is called the cohomology of the Lie conformal algebra R with coefficients in the representation M.

Using the skew-symmetry, we can also write the differential δ as follows:

$$
\begin{aligned}
&(\delta\gamma)_{\lambda_1,\dots,\lambda_n}(x_1,\dots,x_{n+1})\\
&=\sum_{i=1}^{n}(-1)^{i+1}x_i \triangleright_{\lambda_i} \gamma_{\lambda_1,\dots,\hat{\lambda}_i,\dots,\lambda_n}(x_1,\dots,\widehat{x}_i,\dots,x_{n+1})\\
&\quad+(-1)^{n+1}\gamma_{\lambda_1,\dots,\lambda_{n-1}}(x_1,\dots,x_n)\triangleleft_{\lambda_1+\cdots+\lambda_n}x_{n+1}\\
&\quad+\sum_{1\le i<j\le n+1}(-1)^{i}\gamma_{\lambda_1,\dots,\hat{\lambda}_i,\dots,\lambda_i+\lambda_j,\dots,\lambda_n}(x_1,\dots,x_{j-1},[x_{i\lambda_i}x_j],x_{j+1},\dots,x_{n+1}).
\end{aligned}
$$

This formula was first appeared in [Zha15] as the cohomology of a Leibniz conformal algebras.

Remark 2.8. We use a slightly different notation as in [BKV99; DK98] where they use $\gamma_{\lambda_1,\dots,\hat{\lambda}_i,\dots,\lambda_{n+1}}$ to denote the map γ, but in fact the index λ_{n+1} always does't appear in the calculation of the cohomology theory of a Lie conformal algebra, so we use $\gamma_{\lambda_1,\dots,\hat{\lambda}_i,\dots,\lambda_n}$ instead of it.

3. Conformal Lie 2-algebras and 2-term conformal L_∞-algebras

In this section, we introduced the concept of conformal Lie 2-algebras and 2-term conformal L_∞-algebras. It is proved that the category of conformal Lie 2-algebras and the category of 2-term conformal L_∞-algebras are equivalent.

3.1. Conformal Lie 2-algebras

Let's denote the category of $\mathbb{C}[\partial]$-modules or ∂-vector spaces by Vect^∂.

Definition 3.1. A *conformal 2-vector space* is a category in Vect^∂.

More precisely, a conformal 2-vector space $\mathcal{V}$ is a category with a $\mathbb{C}[\partial]$-module $\mathcal{V}_0$ and a $\mathbb{C}[\partial]$-module $\mathcal{V}_1$. The category has source and target maps $s,t\colon \mathcal{V}_1\to\mathcal{V}_0$, an identity-assigning map $i\colon \mathcal{V}_0\to\mathcal{V}_1$, and a composition map $\circ\colon \mathcal{V}_1\times_{\mathcal{V}_0}\mathcal{V}_1\to\mathcal{V}_1$. These maps are all conformal linear maps. A morphism f from source x to target y is denoted by $f\colon x\to y$, where $s(f)=x$ and $t(f)=y$. The notation $i(x)$ is also written as 1_x.

Conformal 2-vector spaces are in one-to-one correspondence with 2-term complexes of $\mathbb{C}[\partial]$-modules. A 2-term complex of $\mathbb{C}[\partial]$-modules is a pair of $\mathbb{C}[\partial]$-modules $\mathcal{C}_1$ and $\mathcal{C}_0$ with a differential between them: $\mathcal{C}_1\xrightarrow{\mathrm{d}}\mathcal{C}_0$. Given a conformal 2-vector space $\mathcal{V}$, $\mathrm{Ker}(s)\xrightarrow{t}\mathcal{V}_0$ is a 2-term complex of $\mathbb{C}[\partial]$-modules. Conversely, any 2-term complex of $\mathbb{C}[\partial]$-modules $\mathcal{V}_1\xrightarrow{\mathrm{d}}\mathcal{V}_0$ gives rise to a conformal 2-vector space. In this conformal 2-vector space, the set of objects is $\mathcal{C}_0$ and the set of morphisms is $\mathcal{C}_0\oplus\mathcal{C}_1$. The source map s is given by $s(x,h)=x$, and the target map t is given by $t(x,h)=x+\mathrm{d}h$, where $x\in\mathcal{V}_0$ and $h\in\mathcal{V}_1$. The conformal 2-vector space associated to the 2-term complex

$\mathcal{V}_1 \xrightarrow{\mathrm{d}} \mathcal{V}_0$ is denoted by $\mathcal{V}$:

$$\mathcal{V} = \begin{tikzcd} \mathcal{V}_1 := \mathcal{C}_0 \oplus \mathcal{C}_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ \mathcal{V}_0 := \mathcal{C}_0. \end{tikzcd} \tag{16}$$

Definition 3.2. A *conformal Lie 2-algebra* consists of a conformal 2-vector space $\mathcal{L}$ equipped with

- a skew-symmetric conformal sesquilinear functor, the *bracket*, $[\cdot_\lambda \cdot] \colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}[\lambda]$,
- a conformal sesquilinear natural isomorphism, the *Jacobiator*,

$$J_{x_{\lambda_1} y_{\lambda_2} z} \colon [x_{\lambda_1}[y_{\lambda_2} z]] \longrightarrow [[x_{\lambda_1} y]_{\lambda_1+\lambda_2} z] + [y_{\lambda_2}[x_{\lambda_1} z]],$$

such that the following *Jacobiator identity* is satisfied

$$\begin{aligned} &J_{x_{\lambda_1} y_{\lambda_2} [z_{\lambda_3} t]} \Big(1 + [y_{\lambda_2} J_{x_{\lambda_1} z_{\lambda_3} t}]\Big) \Big(J_{[x_{\lambda_1} y]_{\lambda_1+\lambda_2} z_{\lambda_3} t} + J_{y_{\lambda_2} [x_{\lambda_1} z]_{\lambda_1+\lambda_3} t} + J_{y_{\lambda_2} z_{\lambda_3} [x_{\lambda_1} t]}\Big) \\ &= [x_{\lambda_1} J_{y_{\lambda_2} z_{\lambda_3} t}] \Big(J_{x_{\lambda_1} [y_{\lambda_2} z]_{\lambda_2+\lambda_3} t} + J_{x_{\lambda_1} z_{\lambda_3} [y_{\lambda_2} t]}\Big) \Big([J_{x_{\lambda_1} y_{\lambda_2} z \lambda_1+\lambda_2+\lambda_3} t] + 1 + [z_{\lambda_3} J_{x_{\lambda_1} y_{\lambda_2} t}]\Big). \end{aligned} \tag{17}$$

We represent the Jacobiator identity in the following commutative diagram, which illustrates the relationship between two methods of utilizing the Jacobiator to rebracket the expression $[x_{\lambda_1}[y_{\lambda_2}[z_{\lambda_3} t]]]$:

$$\begin{tikzcd}[column sep=tiny] & {[x_{\lambda_1}[y_{\lambda_2}[z_{\lambda_3} t]]]} \arrow[dl, "{[x_{\lambda_1} J_{y_{\lambda_2} z_{\lambda_3} t}]}"'] \arrow[dr, "{J_{x_{\lambda_1} y_{\lambda_2} [z_{\lambda_3} t]}}"] & \\ {[x_{\lambda_1}[[y_{\lambda_2} z]_{\lambda_2+\lambda_3} t]] + [x_{\lambda_1}[z_{\lambda_3}[y_{\lambda_2} t]]]} \arrow[d, "{J_{x_{\lambda_1} [y_{\lambda_2} z]_{\lambda_2+\lambda_3} t} + J_{x_{\lambda_1} z_{\lambda_3} [y_{\lambda_2} t]}}"] & & {[[x_{\lambda_1} y]_{\lambda_1+\lambda_2}[z_{\lambda_3} t]] + [y_{\lambda_2}[x_{\lambda_1}[z_{\lambda_3} t]]]} \arrow[d, "{1 + [y_{\lambda_2} J_{x_{\lambda_1} z_{\lambda_3} t}]}"'] \\ \begin{array}{l} {[[x_{\lambda_1}[y_{\lambda_2} z]]_{\lambda_1+\lambda_2+\lambda_3} t] + [[y_{\lambda_2} z]_{\lambda_2+\lambda_3}[x_{\lambda_1} t]]} \\ {+ [[x_{\lambda_1} z]_{\lambda_1+\lambda_3}[y_{\lambda_2} t]] + [z_{\lambda_3}[x_{\lambda_1}[y_{\lambda_2} t]]]} \end{array} \arrow[dr, "{[J_{x_{\lambda_1} y_{\lambda_2} z \lambda_1+\lambda_2+\lambda_3} t] + 1 + [z_{\lambda_3} J_{x_{\lambda_1} y_{\lambda_2} t}]}"'] & & \begin{array}{l} {[[x_{\lambda_1} y]_{\lambda_1+\lambda_2}[z_{\lambda_3} t]] + [y_{\lambda_2}[[x_{\lambda_1} z]_{\lambda_1+\lambda_3} t]]} \\ {+ [y_{\lambda_2}[z_{\lambda_3}[x_{\lambda_1} t]]]} \end{array} \arrow[dl, "{J_{[x_{\lambda_1} y]_{\lambda_1+\lambda_2} z_{\lambda_3} t} + J_{y_{\lambda_2} [x_{\lambda_1} z]_{\lambda_1+\lambda_3} t} + J_{y_{\lambda_2} z_{\lambda_3} [x_{\lambda_1} t]}}"] \\ & P = Q & \end{tikzcd}$$

where P and Q are given by

$$\begin{aligned} P = {} & [[[x_{\lambda_1} y]_{\lambda_1+\lambda_2} z]_{\lambda_1+\lambda_2+\lambda_3} t] + [[y_{\lambda_2}[x_{\lambda_1} z]]_{\lambda_1+\lambda_2+\lambda_3} t] + [[y_{\lambda_2} z]_{\lambda_2+\lambda_3}[x_{\lambda_1} t]] \\ & + [[x_{\lambda_1} z]_{\lambda_1+\lambda_3}[y_{\lambda_2} t]] + \big[z_{\lambda_3}[[x_{\lambda_1} y]_{\lambda_1+\lambda_2} t]\big] + [z_{\lambda_3}[y_{\lambda_2}[x_{\lambda_1} t]]] \qquad = Q. \end{aligned}$$

If we drop the skew-symmetric condition for the bracket, we obtain the concept of conformal Leibniz 2-algebras.

Definition 3.3. A *conformal Leibniz 2-algebra* consists of a conformal 2-vector space $\mathcal{L}$ equipped with a conformal sesquilinear functor $[\cdot_\lambda \cdot] \colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}[\lambda]$ and a conformal sesquilinear natural isomorphism

$$J_{x_{\lambda_1} y_{\lambda_2} z} \colon [x_{\lambda_1}[y_{\lambda_2} z]] \longrightarrow [[x_{\lambda_1} y]_{\lambda_1+\lambda_2} z] + [y_{\lambda_2}[x_{\lambda_1} z]],$$

such that the above (17) is satisfied.

In fact, all the results in the following of this section can be easily generalized to the realm of conformal Leibniz 2-algebra, but we omit the details.

Definition 3.4. Given two conformal Lie 2-algebras $(\mathcal{L}, [\cdot_\lambda \cdot], J)$ and $(\mathcal{L}', [\cdot_\lambda \cdot]', J')$, a conformal Lie 2-algebra morphism $F : \mathcal{L} \to \mathcal{L}'$ consists of:

- a conformal functor (F^0, F^1) from the underlying conformal 2-vector space of $\mathcal{L}$ to that of $\mathcal{L}'$;
- a natural transformation
$$F^2_\lambda(x, y) : [F^0(x)_\lambda F^0(y)]' \longrightarrow F^0([x_\lambda y])$$
such that the following diagram commutes:

$$\begin{tikzcd}
{[F^0(x)_\lambda[F^0(y)_\mu F^0(z)]']'} \arrow[rr, "{[1,F^2_\mu(y,z)]'}"] \arrow[d, "J_{F^0(x)_\lambda F^0(y)_\mu F^0(z)}"'] & & {[F^0(x)_\lambda F^0([y_\mu z])]'} \arrow[d, "{F^2_\lambda(x,[y_\mu z])}"] \\
{[[F^0(x)_\lambda F^0(y)]'_{\lambda+\mu} F^0(z)]' + [F^0(y)_\mu[F^0(x)_\lambda F^0(z)]']'} \arrow[d, "{[F^2_\lambda(x,y),1]'+[1,F^2_\lambda(x,z)]'}"'] & & {F^0([x_\lambda[y_\mu z]])} \arrow[d, "F^0 J_{x_\lambda y_\mu z}"] \\
{[F^0([x_\lambda y])_{\lambda+\mu} F^0(z)]' + [F^0(y)_\mu F^0[x_\lambda z]]'} \arrow[rr, "{F^2_{\lambda+\mu}([x_\lambda y],z)+F^2_\mu(y,[x_\lambda z])}"] & & {F^0([[x_\lambda y]_{\lambda+\mu} z]) + F^0([y_\mu[x_\lambda z]]).}
\end{tikzcd}$$

The identity morphism $\mathrm{id}_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ is associated with the identity functor and an identity natural transformation $(\mathrm{id}_{\mathcal{L}})_2$. The composition of a pair of conformal Lie 2-algebra morphisms, denoted by $G \circ F$, is defined as follows: let $\mathcal{L}$, $\mathcal{L}'$ and $\mathcal{L}''$ be conformal Lie 2-algebras. The functor $((G \circ F)^0, (G \circ F)^1)$ is obtained by composing (G^0, G^1) and (F^0, F^1), while $(G \circ F)^2$ is defined as the following composition:

$$\begin{tikzcd}
{[G^0 \circ F^0(x)_\lambda G^0 \circ F^0(y)]''} \arrow[dr, "{(G\circ F)^2_\lambda(x,y)}"] \arrow[dd, "{G^2_\lambda(F^0(x),F^0(y))}"'] & \\
& {G^0 \circ F^0[x_\lambda y].} \\
{G^0[F^0(x)_\lambda F^0(y)]'} \arrow[ur, "{G^1(F^2_\lambda(x,y))}"'] &
\end{tikzcd}$$

It is easy to see that there is a category *LieC2Alg* with conformal Lie 2-algebras as objects and conformal Lie 2-algebra morphisms as morphisms.

3.2. 2-term conformal L_∞-algebras

Now we introduced the concept of 2-term conformal L_∞-algebras. For general theory of L_∞-algebras with applications, see [BHR10; Kon03; LS93; RW98; SS85].

Definition 3.5. A 2-term conformal L_∞-algebra $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ is a complex consisting of the following data:

- two $\mathbb{C}[\partial]$-modules $\mathcal{V}_0$ and $\mathcal{V}_1$ together with a conformal linear map $\mathrm{d}\colon V_1 \to V_0$,
- a conformal sesquilinear map $l^2_\lambda \colon \mathcal{V}_i \times \mathcal{V}_j \to \mathcal{V}_{i+j}[\lambda]$, where $0 \le i + j \le 1$,
- a conformal sesquilinear map $l^3_{\lambda_1,\lambda_2} \colon \mathcal{V}_0 \times \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_1[\lambda_1, \lambda_2]$.

These maps satisfy the following conditions:

(a) $l^2_\lambda(x,y)+l^2_{-\partial-\lambda}(y,x)=0$,

(b) $l^2_\lambda(x,h)+l^2_{-\partial-\lambda}(h,x)=0$,

(c) $l^2_\lambda(h,k)=0$,

(d) $l^3_{\lambda_1,\lambda_2}(x,y,z)$ is totally skew symmetric,

(e) $\mathrm{d}(l^2_\lambda(x,h))=l^2_\lambda(x,\mathrm{d}h)$,

(f) $l^2_\lambda(\mathrm{d}h,k)=l^2_\lambda(h,\mathrm{d}k)$,

(g) $\mathrm{d}(l^3_{\lambda_1,\lambda_2}(x,y,z))=l^2_{\lambda_1}(x,l^2_{\lambda_2}(y,z))-l^2_{\lambda_1+\lambda_2}(l^2_{\lambda_1}(x,y),z)-l^2_{\lambda_2}(y,l^2_{\lambda_1}(x,z))$,

(h) $l^3_{\lambda_1,\lambda_2}(x,y,\mathrm{d}h)=l^2_{\lambda_1}(x,l^2_{\lambda_2}(y,h))-l^2_{\lambda_1+\lambda_2}(l^2_{\lambda_1}(x,y),h)-l^2_{\lambda_2}(y,l^2_{\lambda_1}(x,h))$,

(i) $$\begin{aligned}&\delta l^3_{\lambda_1,\lambda_2,\lambda_3}(x,y,z,t)\\&=l^2_{\lambda_1}(x,l^3_{\lambda_2,\lambda_3}(y,z,t))-l^2_{\lambda_2}(y,l^3_{\lambda_1,\lambda_3}(x,z,t))+l^2_{\lambda_3}(z,l^3_{\lambda_1,\lambda_2}(x,y,t))\\&\quad+l^2_{\lambda_1+\lambda_2+\lambda_3}(l^3_{\lambda_1,\lambda_2}(x,y,z),t)-l^3_{\lambda_1+\lambda_2,\lambda_3}(l^2_{\lambda_1}(x,y),z,t)\\&\quad-l^3_{\lambda_2,\lambda_1+\lambda_3}(y,l^2_{\lambda_1}(x,z),t)-l^3_{\lambda_2,\lambda_3}(y,z,l^2_{\lambda_1}(x,t))+l^3_{\lambda_1,\lambda_2+\lambda_3}(x,l^2_{\lambda_2}(y,z),t)\\&\quad+l^3_{\lambda_1,\lambda_3}(x,z,l^2_{\lambda_2}(y,t))-l^3_{\lambda_1,\lambda_2}(x,y,l^2_{\lambda_3}(z,t))\\&=0,\end{aligned}$$

for all $x,y,z,t\in\mathcal{V}_0$ and $h,k\in\mathcal{V}_1$.

Definition 3.6. Let $(\mathcal{V};\mathrm{d},l^2_\lambda,l^3_{\lambda_1,\lambda_2})$ and $(\mathcal{V}';\mathrm{d}',l'^2_\lambda,l'^3_{\lambda_1,\lambda_2})$ be two 2-term conformal L_∞-algebras. A CL_∞-morphism $f=(f^0,f^1,f^2)$ from $\mathcal{V}$ to $\mathcal{V}'$ consists of conformal linear maps $f^0:\mathcal{V}_0\to\mathcal{V}'_0$, $f^1:\mathcal{V}_1\to\mathcal{V}'_1$ and $f^2:\mathcal{V}_0\times\mathcal{V}_0\to\mathcal{V}'_1[\lambda]$, such that the following equalities hold for all $x,y,z\in\mathcal{V}_0, a\in\mathcal{V}_1$,

(i) $f^0\mathrm{d}=\mathrm{d}'f^1$,

(ii) $f^0l^2_\lambda(x,y)-l'^2_\lambda(f^0(x),f^0(y))=\mathrm{d}'f^2_\lambda(x,y)$,

(iii) $f^1l^2_\lambda(x,a)-l'^2_\lambda(f^0(x),f^1(a))=f^2_\lambda(x,\mathrm{d}a)$,

(iv) $$\begin{aligned}&f^1(l^3_{\lambda_1,\lambda_2}(x,y,z))-l'^3_{\lambda_1,\lambda_2}(f^0(x),f^0(y),f^0(z))\\&=f^2_{\lambda_1}(x,l^2_{\lambda_2}(y,z))-f^2_{\lambda_1+\lambda_2}(l^2_{\lambda_1}(x,y),z)-f^2_{\lambda_2}(y,l^2_{\lambda_1}(x,z))\\&\quad+l'^2_{\lambda_1}(f^0(x),f^2_{\lambda_2}(y,z))-l'^2_{\lambda_1+\lambda_2}(f^2_{\lambda_1}(x,y),f^0(z))-l'^2_{\lambda_2}(f^0(y),f^2_{\lambda_1}(x,z)).\end{aligned}$$

If $f^2=0$, the CL_∞-morphisms f is called a strict CL_∞-morphisms.

Let $f:\mathcal{V}\to\mathcal{V}'$ and $g:\mathcal{V}'\to\mathcal{V}''$ be two CL_∞-morphisms, then their composition $g\circ f:\mathcal{V}\to\mathcal{V}''$ is a CL_∞-morphism defined as $(g\circ f)^0=g^0\circ f^0$, $(g\circ f)^1=g^1\circ f^1$ and

$$(g\circ f)^2_\lambda(x,y)=g^2_\lambda(f^0(x)f^0(y))+g^1(f^2_\lambda(x,y)).$$

The identity CL_∞-morphism $1_\mathcal{V}:\mathcal{V}\to\mathcal{V}$ has the identity chain map together with $(1_\mathcal{V})_2=0$.

There is a category $2CL_\infty$ with 2-term conformal L_∞-algebras as objects and CL_∞-morphisms as morphisms.

3.3. Equivalence

Now we establish the equivalence between the category of conformal Lie 2-algebras and 2-term conformal L_∞-algebras.

Theorem 3.7. *The categories $2CL_\infty$ and LieC2Alg are equivalent.*

Proof. First, we show how to construct a conformal Lie 2-algebra from a 2-term conformal L_∞-algebra.

Let $\mathcal{V} = (\mathcal{V}_1 \xrightarrow{\mathrm{d}} \mathcal{V}_0, l^2_\lambda, l^3_{\lambda_1,\lambda_2})$ be a 2-term conformal L_∞-algebra, we consider the conformal 2-vector space $\mathcal{L} = (\mathcal{V}_0 \oplus \mathcal{V}_1 \rightrightarrows \mathcal{V}_0)$ given by (16), that is $\mathcal{L}$ has objects $\mathcal{L}_0 = \mathcal{V}_0$, $\mathcal{L}_1 = \mathcal{V}_0 \oplus \mathcal{V}_1$ and morphisms $f\colon x \to y$ in $\mathcal{L}_1$ by $f = (x, h)$ where $x \in \mathcal{V}_0$ and $h \in \mathcal{V}_1$. The source, target, and identity-assigning maps in $\mathcal{L}$ are given by

$$\begin{aligned} s(f) &= s(x,h) = x, \\ t(f) &= t(x,h) = x + \mathrm{d}h, \\ i(x) &= (x, 0), \end{aligned}$$

and we have $t(f) - s(f) = \mathrm{d}h$.

Now we define a skew-symmetric conformal sesquilinear functor on $\mathcal{L}$ by

$$[(x,h)_\lambda (y,k)] = (l^2_\lambda(x,y), l^2_\lambda(x,k) + l^2_\lambda(h,y) + l^2_\lambda(\mathrm{d}h,k)),$$

for all $(x,h),(y,k) \in \mathcal{V}_0 \oplus \mathcal{V}_1$. The Jacobiator is given as follows

$$J_{x_{\lambda_1} y_{\lambda_2} z} := ([x_{\lambda_1}[y_{\lambda_2} z]],\ l^3_{\lambda_1,\lambda_2}(x,y,z)),$$

where $x, y, z \in \mathcal{V}_0$. Then by Condition (g), we have $J_{x_{\lambda_1} y_{\lambda_2} z}$ is a morphism from source $[x_{\lambda_1}[y_{\lambda_2} z]]$ to target $[[x_{\lambda_1} y]_{\lambda_1+\lambda_2} z] + [y_{\lambda_2}[x_{\lambda_1} z]]$.

Next we show that $J_{x_{\lambda_1} y_{\lambda_2} z}$ is a natural isomorphism. We only check naturality in the third variable, the other two cases are similar. Let $f\colon z \to z'$. Then, $J_{x_{\lambda_1} y_{\lambda_2} z}$ is natural in z if the following diagram commutes:

$$\begin{tikzcd} {[x_{\lambda_1}[y_{\lambda_2} z]]} \arrow[r, "{[[1_x, 1_y], f]}"] \arrow[d, "J_{x_{\lambda_1} y_{\lambda_2} z}"'] & {[x_{\lambda_1}[y_{\lambda_2} z']]} \arrow[d, "J_{x_{\lambda_1} y_{\lambda_2} z'}"] \\ {[[x_{\lambda_1} y]_{\lambda_1+\lambda_2} z] + [y_{\lambda_2}[x_{\lambda_1} z]]} \arrow[r, "{[1_x,[1_y,f]]+[[1_x,f],1_y]}"] & {[[x_{\lambda_1} y]_{\lambda_1+\lambda_2} z'] + [y_{\lambda_2}[x_{\lambda_1} z']]} \end{tikzcd}$$

Thus we only need to show

$$\begin{aligned} &([x_{\lambda_1}[y_{\lambda_2} z]], l^3_{\lambda_1,\lambda_2}(x,y,z') + [x_{\lambda_1}[y_{\lambda_2} h]]) \\ &\qquad = ([x_{\lambda_1}[y_{\lambda_2} z]], [x_{\lambda_1} y]_{\lambda_1+\lambda_2} h]] + [y_{\lambda_2}[x_{\lambda_1} h]] + l^3_{\lambda_1,\lambda_2}(x,y,z)). \end{aligned}$$

This holds by condition (h) in Definition 3.5.

From condition (i) in Definition 3.5, we have

$$\begin{aligned} &l^2_{\lambda_1}(x, l^3_{\lambda_2,\lambda_3}(y,z,t)) - l^2_{\lambda_2}(y, l^3_{\lambda_1,\lambda_3}(x,z,t)) + l^2_{\lambda_3}(z, l^3_{\lambda_1,\lambda_2}(x,y,t)) \\ &\quad + l^2_{\lambda_1+\lambda_2+\lambda_3}(l^3_{\lambda_1,\lambda_2}(x,y,z),t) - l^3_{\lambda_1+\lambda_2,\lambda_3}(l^2_{\lambda_1}(x,y),z,t) \\ &\quad - l^3_{\lambda_2,\lambda_1+\lambda_3}(y, l^2_{\lambda_1}(x,z),t) - l^3_{\lambda_2,\lambda_3}(y,z,l^2_{\lambda_1}(x,t)) + l^3_{\lambda_1,\lambda_2+\lambda_3}(x, l^2_{\lambda_2}(y,z),t) \\ &\qquad + l^3_{\lambda_1,\lambda_3}(x,z,l^2_{\lambda_2}(y,t)) - l^3_{\lambda_1,\lambda_2}(x,y,l^2_{\lambda_3}(z,t)) = 0. \end{aligned}$$

By the skew-symmetry of $l^2_\lambda = [\cdot_\lambda \cdot]$, we obtain

$$\begin{aligned} & l^3_{\lambda_1,\lambda_2}(x,y,[z_{\lambda_3}t]) + [y_{\lambda_2} l^3_{\lambda_1,\lambda_3}(x,z,t))] \\ & \qquad + l^3_{\lambda_1+\lambda_2,\lambda_3}([x_{\lambda_1}y],z,t) + l^3_{\lambda_2,\lambda_1+\lambda_3}(y,[x_{\lambda_1}z],t) + l^3_{\lambda_2,\lambda_3}(y,z,[x_{\lambda_1}t]) \\ & = [x_{\lambda_1} l^3_{\lambda_2,\lambda_3}(y,z,t)] + [z_{\lambda_3} l^3_{\lambda_1,\lambda_2}(x,y,t)] \\ & \qquad + [l^3_{\lambda_1,\lambda_2}(x,y,z)_{\lambda_1+\lambda_2+\lambda_3}t) + l^3_{\lambda_1,\lambda_2+\lambda_3}(x,[y_{\lambda_2}z],t) + l^3_{\lambda_1,\lambda_3}(x,z,[y_{\lambda_2}t]). \end{aligned}$$

This is equivalent to Jacobiator identity (17) in Definition 3.2. Thus from a 2-term conformal L_∞-algebra, we obtain a conformal Lie 2-algebra.

For any CL_∞-morphism $f = (f^0, f^1, f^2)$ from $\mathcal{V}$ to $\mathcal{V}'$, we construct a conformal Lie 2-algebra morphism $F = T(f)$ from $L = T(\mathcal{V})$ to $L' = T(\mathcal{V}')$ as follows.

Let $F^0 = f^0$, $F^1 = f^0 \oplus f^1$, and F^2 be given by

$$F^2_\lambda(x,y) = ([f^0(x)_\lambda f^0(y)], f^2_\lambda(x,y)).$$

Then $F^2(x,y)$ is a natural isomorphism from $[F^0(x)_\lambda F^0(y)]$ to $F^0[x_\lambda y]$, and $F = (F^0, F^1, F^2)$ is a conformal Lie 2-algebra morphism from $\mathcal{L}$ to $\mathcal{L}'$.

One can also deduce that T preserves the identity CL_∞-morphisms and the composition of CL_∞-morphisms. Thus, we obtain a functor from $2CL_\infty$ to *LieC2Alg*.

Conversely, given a conformal Lie 2-algebra $\mathcal{L}$, we define l^2_λ and $l^3_{\lambda_1,\lambda_2}$ on the 2-term complex $\mathcal{L}_1 \supseteq \ker(s) = \mathcal{V}_1 \xrightarrow{\mathrm{d}} \mathcal{V}_0 = \mathcal{L}_0$ by

- $l_1 h = t(h)$ for $h \in \mathcal{V}_1 \subseteq \mathcal{L}_1$,
- $l^2_\lambda(x,y) = [x_\lambda y]$ for $x, y \in \mathcal{V}_0 = \mathcal{L}_0$,
- $l^2_\lambda(x,h) = [1_{x\lambda} h]$, for $x \in \mathcal{V}_0 = \mathcal{L}_0$ and $h \in \mathcal{V}_1 \subseteq \mathcal{L}_1$,
- $l^2_\lambda(h,k) = 0$ for $h, k \in \mathcal{V}_1 \subseteq \mathcal{L}_1$,
- $l^3_{\lambda_1,\lambda_2}(x,y,z) = p_1 J_{x_{\lambda_1} y_{\lambda_2} z}$ for $x, y, z \in \mathcal{V}_0 = \mathcal{L}_0$, where $p_1 : \mathcal{L}_1 = \mathcal{V}_0 \oplus \mathcal{V}_1 \to \mathcal{V}_1$ is the projection.

Then one can verify that $(\mathcal{V}_1 \xrightarrow{\mathrm{d}} \mathcal{V}_0, l^2_\lambda, l^3_{\lambda_1,\lambda_2})$ is a 2-term conformal L_∞-algebra.

Let $F = (F^0, F^1, F^2) : \mathcal{L} \to \mathcal{L}'$ be a conformal Lie 2-algebra morphism, and $S(\mathcal{L}) = \mathcal{V}$, $S(\mathcal{L}') = \mathcal{V}'$. Define $S(F) = f = (f^0, f^1, f^2)$ as follows. Let $f^0 = F^0$, $f^1 = F^1|_{V_1 = \mathrm{Ker}(s)}$ and define f^2 by

$$f^2_\lambda(x,y) = F^2_\lambda(x,y) - i(s(F^2_\lambda(x,y))).$$

It is easy to see that f is a $2CL_\infty$-algebra morphism. Furthermore, S also preserves the identity morphisms and the composition of morphisms. Thus, we obtain a functor from *LieC2Alg* to CL_∞.

One show that there are natural isomorphisms $\alpha : T \circ S \to 1_{LieC2Alg}$ and $\beta : S \circ T \to 1_{2CL_\infty}$. This complete the proof. $\square$

4. Construction of conformal Lie 2-algebras

In this section, we present concrete examples and special cases of conformal Lie 2-algebras. These include Lie conformal algebras with 3-cocycles, crossed modules of Lie conformal algebras, string Lie conformal algebras, and conformal omni-Lie algebras. Finally, we demonstrate that conformal Lie 2-algebras can be derived from any Leibniz conformal algebra.

4.1. Skeletal conformal Lie 2-algebras

In the case of a 2-term conformal L_∞-algebra with $\mathrm{d} = 0$, it is referred to as *skeletal*. From conditions (a) and (g), it can be deduced that $\mathcal{V}_0$ constitutes a Lie conformal algebra. Furthermore, conditions (b) and (h) indicate that $\mathcal{V}_1$ serves as a representation of $\mathcal{V}_0$ through the action established by $x \triangleright_\lambda h := l^2_\lambda(x,h)$. Moreover, condition (i) can be expressed as a 3-cocycle condition in the Lie conformal algebra cohomology of $\mathcal{V}_0$ with respect to $\mathcal{V}_1$ as its values.

Proposition 4.1. *Skeletal 2-term conformal L_∞-algebras are in one-to-one correspondence with quadruples $(R, M, \triangleright_\lambda, l^3_{\lambda_1,\lambda_2})$ where R is a Lie conformal algebra, M is a $\mathbb{C}[\partial]$-module, $\triangleright_\lambda$ is an action of R on M and $l^3_{\lambda_1,\lambda_2}$ is a 3-cocycle on R with coefficients in M.*

Let M be a $\mathbb{C}[\partial]$-module. A conformal bilinear form on M is a $\mathbb{C}$-bilinear map $\langle \cdot , \cdot \rangle_\lambda : M \times M \to \mathbb{C}[\lambda]$ such that

$$\langle \partial v, w\rangle_\lambda = -\lambda \langle v, w\rangle_\lambda = -\langle v, \partial w\rangle_\lambda, \qquad \text{for all } v, w \in M.$$

The conformal bilinear form is symmetric if $\langle v, w\rangle_\lambda = \langle w, v\rangle_{-\lambda}$ for all $v, w \in V$. The conformal bilinear form in a Lie conformal algebra R is called invariant if

$$\langle [x_\mu y], z\rangle_\lambda = \langle x, [y_{\lambda-\partial} z]\rangle_\mu = -\langle x, [z_{-\lambda} y]\rangle_\mu,$$

for all $x, y, z \in R$.

Example 4.2. Given a Lie conformal algebra $(R, [\cdot_\lambda \cdot])$ with symmetric invariant conformal bilinear form $\langle \cdot , \cdot \rangle_\lambda$, we construct a 2-term conformal L_∞-algebra as follows. Let $\mathcal{V}_1 = \mathbb{C}, \mathcal{V}_0 = R, \mathrm{d} = 0$, and define $l^2_\lambda, l^3_{\lambda_1,\lambda_2}$ by

$$l^2_\lambda(x,y) = [x_\lambda y], \quad l^2_\lambda(x,h) = 0, \quad l^3_{\lambda_1,\lambda_2}(x,y,z) = \langle [x_{\lambda_1} y], z\rangle_{\lambda_2}, \tag{18}$$

where $x, y, z \in R, h \in \mathbb{C}$. All the conditions in the Definition 3.5 are satisfied. Thus we obtain a 2-term conformal L_∞-algebra $(\mathbb{C} \xrightarrow{0} R, l^2_\lambda, l^3_{\lambda_1,\lambda_2})$. We call it the *string conformal Lie 2-algebra.*

4.2. Strict conformal Lie 2-algebras

Another kind of 2-term conformal L_∞-algebra is called *strict* if $l^3_{\lambda_1,\lambda_2} = 0$. These conformal Lie 2-algebras can be characterized using crossed modules of Lie conformal algebras.

Definition 4.3. Let $(R, [\cdot , \cdot]_R)$ and $(H, [\cdot , \cdot]_H)$ be two Lie conformal algebras. A crossed module of Lie conformal algebras is a homomorphism of Lie conformal algebras $\varphi : H \to R$ together with an action of R on H, denoted by $x \triangleright_\lambda h$, such that

$$\varphi(x \triangleright_\lambda h) = [x_\lambda \varphi(h)]_R, \quad \varphi(h) \triangleright_\lambda k = [h_\lambda k]_H,$$

for all $h, k \in H, x \in R$.

Proposition 4.4. *There is an one-to-one correspondence between strict 2-term conformal L_∞-algebras and crossed modules of Lie conformal algebras.*

Proof. Let $\mathcal{V}_1 \xrightarrow{\mathrm{d}} \mathcal{V}_0$ be a 2-term conformal L_∞-algebra with $l^3_{\lambda_1,\lambda_2} = 0$. We construct Lie conformal algebras on $R = \mathcal{V}_0$ and $H = \mathcal{V}_1$ as follows. The bracket on R and H are defined by

$$[h_\lambda k]_H := l^2_\lambda(\mathrm{d}h, k), \quad \forall\, h, k \in H = \mathcal{V}_1,$$
$$[x_\lambda y]_R := l^2_\lambda(x, y), \quad \forall\, x, y \in R = \mathcal{V}_0.$$

By condition (a) and (g) in Definition 3.5, it is easy to see that $[\cdot_\lambda \cdot]_R$ satisfies the Jacobi identity. By (h), we have

$$\begin{aligned}
&[h_\lambda[k_\mu l]] - [[h_\lambda k]_{\lambda+\mu} l] - [k_\mu[h,_\lambda l]\\
&\qquad = l^2_\lambda(\mathrm{d}h, l^2_\mu(\mathrm{d}k, l)) - l^2_{\lambda+\mu}(\mathrm{d}l^2_\lambda(\mathrm{d}h, k), l) - l^2_\mu(\mathrm{d}k, l^2_\lambda(\mathrm{d}h, l))\\
&\qquad = l^2_\lambda(\mathrm{d}h, l^2_\mu(\mathrm{d}k, l)) - l^2_{\lambda+\mu}(l^2_\lambda(\mathrm{d}h, \mathrm{d}k), l) - l^2_\mu(\mathrm{d}k, l^2_\lambda(\mathrm{d}h, l))\\
&\qquad = 0.
\end{aligned}$$

Thus $[\cdot_\lambda \cdot]_H$ satisfies the Jacobi identity. Now let $\varphi = \mathrm{d}$, then by (e), we have

$$\varphi([h_\lambda k]_H) = \mathrm{d}(l^2_\lambda(\mathrm{d}h, k)) = l^2_\lambda(\mathrm{d}h, \mathrm{d}k) = [\varphi(h)_\lambda \varphi(k)]_R,$$

which implies that φ is a homomorphism of Lie conformal algebras.

Now define the maps of $\triangleright_\lambda : R \times H \to H[\lambda]$ by

$$x \triangleright_\lambda h := l^2_\lambda(x, h), \quad \forall\, x \in R, h \in H,$$

which is an action of R on H and it is easy to check that

$$\varphi(x \triangleright_\lambda h) = \mathrm{d}(l^2_\lambda(x, h)) = l^2_\lambda(x, \mathrm{d}h) = [x_\lambda \varphi(h)]_R,$$
$$\varphi(h) \triangleright_\lambda k = l^2_\lambda(\mathrm{d}h, k) = [h_\lambda k]_H.$$

Therefore, we obtain a crossed module of Lie conformal algebras.

Conversely, a crossed module of Lie conformal algebras gives rise to a 2-term conformal L_∞-algebra with $\mathrm{d} = \varphi$, $\mathcal{V}_0 = R$ and $\mathcal{V}_1 = H$, where the brackets are given by

$$l^2_\lambda(x, y) := [x_\lambda y]_R, \quad \forall\, x, y \in R,$$
$$l^2_\lambda(x, h) := x \triangleright_\lambda h, \quad \forall\, h \in H,$$
$$l^2_\lambda(h, k) := 0.$$

The crossed module conditions give various conditions for 2-term conformal L_∞-algebras with $l^3_{\lambda_1,\lambda_2} = 0$. □

Given a crossed module (R, H, φ), we consider $\mathfrak{g} = \ker(\varphi)$ and $M = \ker(\varphi)$. The Lie conformal algebra structure on H induces a Lie conformal algebra structure on $\mathfrak{g}$. Moreover, the action of H on R induces an action of $\mathfrak{g}$ on M. Hence a crossed module yields an exact sequence

$$0 \longrightarrow M \xrightarrow{i} R \xrightarrow{\varphi} H \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0,$$

where $\pi : H \to \mathfrak{g}$ is the projection map. Two crossed modules (R, H, φ) and (R', H', φ') over $\mathfrak{g}$ with kernel M are said to be equivalent if there is a morphism of crossed modules $(R, H, \varphi) \to (R', H', \varphi')$ which induces identity maps on $\mathfrak{g}$ and M. Let $\mathrm{Cross}(\mathfrak{g}, M)$ be the equivalence classes of such crossed modules.

Proposition 4.5. *The equivalence classes of crossed modules are in one-to-one correspondence with the elements of third cohomology group of $\mathfrak{g}$ with coefficients in M.*

Proof. Given a crossed module extension

$$0 \longrightarrow M \xrightarrow{i} R \xrightarrow{\varphi} H \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0.$$

Choose $\mathbb{C}$-linear sections $s : \mathfrak{g} \to H$ with $\pi s = \mathrm{id}$ and $q : \mathrm{Im}(\partial) \to R$ with $\partial q = \mathrm{id}$. Since π is a Lie conformal algebra homomorphism, for all $x, y \in \mathfrak{g}$, we have

$$\pi([s(x)_\lambda s(y)] - s[x_\lambda y]) = [\pi s(x)_\lambda \pi s(y)] - \pi s[x_\lambda y] = [x_\lambda y] - [x_\lambda y] = 0.$$

This shows that $[s(x)_\lambda s(y)] - s[x_\lambda y] \in \ker(\pi) = \mathrm{Im}(\partial)$. Take

$$g_\lambda(x, y) = q([s(x)_\lambda s(y)] - s[x_\lambda y]) \in R.$$

Define a map $\theta_{\mathcal{E},s,q} : \mathfrak{g}^{\otimes 3} \to M[\lambda]$ by

$$\begin{aligned}\theta_{\mathcal{E},s,q}(x, y, z) = [s(x)_\lambda g_\mu(y, z)] - [g_\lambda(x, y)_{\lambda+\mu} s(z)] - [s(y)_\nu g_\lambda(x, z)] \\ - g_{\lambda+\mu}([x_\lambda y], z) + g_\lambda(x, [y_\mu z]) + g_{-\partial-\mu}([x_\lambda z], y).\end{aligned}$$

Since ∂ is a map of H-modules, it follows that $\partial(\theta_{\mathcal{E},s,q}(x, y, z)) = 0$. Thus we obtain $\theta_{\mathcal{E},s,q}(x, y, z) \in \ker(\partial) = M$. Therefore $\theta_{\mathcal{E},s,q} : \mathfrak{g}^{\otimes 3} \to M[\lambda]$ is well defined. Moreover, the map $\theta_{\mathcal{E},s,q}$ is a 3-cocycle ($\delta(\theta_{\mathcal{E},s,q}) = 0$) in the cohomology of $\mathfrak{g}$ with coefficients in M. It follows that equivalence classes of crossed modules gives the same cohomology 3-cocycle classes. The proof is similar as the Lie algebra case, see Wagemann [Wag06] for more details. □

Let R be a Lie conformal algebra, recall that a map $D \in \mathrm{Cend}(R)$ is called a conformal derivation [GT14] if

$$D_\lambda[x_\mu y] = [(D_\lambda x)_{\lambda+\mu} y] + [x_\mu (D_\lambda y)].$$

For example, the adjoint action $(\mathrm{ad}\, x)_\lambda$ defined by

$$(\mathrm{ad}\, x)_\lambda(y) = [x_\lambda y]$$

from R to itself is a conformal derivation. This is called the inner conformal derivation.

Denote by $\mathrm{Der}(R)$ the vector space spanned by all conformal derivations. We find that $\mathrm{Der}(R)$ becomes a Lie conformal algebra under the bracket

$$[D_\lambda D'] = D_\lambda D' - D'_{-\partial-\lambda} D,$$

equivalently,

$$[D_\lambda D']_{\lambda'} = D_\lambda D'_{\lambda'-\lambda} - D'_{\lambda'-\lambda} D_\lambda,$$

where D, D' are conformal derivations of R.

For an operation ω on R such that $\omega_\lambda : R \times R \to R[\lambda]$, we define the adjoint operator $\mathrm{ad}_\omega : R \to \mathrm{Cend}(R)$ by

$$\mathrm{ad}_{\omega_\lambda}(x)(y) = \omega_\lambda(x, y) \in R, \quad \forall\, x, y \in R.$$

Then the graph of the adjoint operator

$$\mathcal{F}_\omega := \{\mathrm{ad}_\omega\, x + x \mid \forall\, x \in R\} \subset \mathcal{E} := \mathrm{Cend}(R) \oplus R$$

is a subspace of $\mathcal{E}$.

Proposition 4.6. *D is a conformal derivation of R if and only if $\mathcal{F}_\omega$ is an invariant subspace of D, that is $D \circ \mathcal{F}_\omega \subseteq \mathcal{F}_\omega$, where the action D on $\mathcal{F}_\omega$ is defined by*

$$D_\lambda \circ (\mathrm{ad}_{\omega_\mu}(x) + x) = [D_\lambda \, \mathrm{ad}_{\omega_\mu}(x)] + D_\lambda x, \tag{19}$$

for all $\mathrm{ad}_{\omega_\mu}(x) + x \in \mathcal{F}_{\omega_\mu}$.

Proof. The right hand side of (19) is contained in $\mathcal{F}_{\omega_\mu}$ if and only if

$$[D_\lambda \, \mathrm{ad}_{\omega_\mu}(x)] = \mathrm{ad}_{\omega_\mu}(D_\lambda x),$$

this is equivalent to

$$\begin{aligned} D_\lambda \, \mathrm{ad}_{\omega_\mu}(x)(y) - \mathrm{ad}_{\omega_\mu}(x)D(y) - \mathrm{ad}_{\omega_\mu}(D_\lambda x)(y) &= D_\lambda[x_\mu y] - [x_\mu (D_\lambda y)] - [(D_\lambda x)_{\lambda+\mu} y] \\ &= 0. \end{aligned}$$

Thus D is a conformal derivation if and only if $D \circ \mathcal{F}_{\omega_\lambda} \subseteq \mathcal{F}_{\omega_\lambda}$. □

We call the set of $D \in \mathrm{Cend}(R)$ such that $D \circ \mathcal{F}_{\omega_\lambda} \subseteq \mathcal{F}_\omega$ the normalizer of $\mathcal{F}_\omega$, which is denoted by $N(\mathcal{F}_\omega)$.

Proposition 4.7. *Let D and D' be two conformal derivations. Then $[D_\lambda D']$ is also a conformal derivation. Thus we have* $\mathrm{Der}(R) = N(\mathcal{F}_\omega)$ *is a conformal Lie subalgebra of* $\mathrm{Cend}(R)$.

Proof. Let $D \circ \mathcal{F}_\omega \subseteq \mathcal{F}_\omega$ and $D' \circ \mathcal{F}_\omega \subseteq \mathcal{F}_\omega$, then

$$[D_\lambda \, \mathrm{ad}_{\omega_\mu}(x)] = \mathrm{ad}_{\omega_\mu}(D_\lambda x), \quad [D'_{\lambda'} \, \mathrm{ad}_{\omega_\mu}(x)] = \mathrm{ad}_{\omega_\mu}(D'_{\lambda'} x).$$

By the Jacobi identity of $\mathrm{Cend}(R)$, we have

$$\begin{aligned} [[D_\lambda D']_{\lambda+\lambda'} \, \mathrm{ad}_{\omega_\mu}(x)] &= [D_\lambda [D'_{\lambda'} \, \mathrm{ad}_{\omega_\mu}(x)]] - [D'_{\lambda'} [D_\lambda \, \mathrm{ad}_{\omega_\mu}(x)] \\ &= [D_\lambda \, \mathrm{ad}_{\omega_\mu}(D'_{\lambda'} x)] - [D'_{\lambda'} \, \mathrm{ad}_{\omega_\mu}(D_\lambda x)] \\ &= \mathrm{ad}_{\omega_\mu}(D_\lambda D'_{\lambda'} x) - \mathrm{ad}_{\omega_\mu}(D'_{\lambda'} D_\lambda x) \\ &= \mathrm{ad}_{\omega_\mu}([D_\lambda D']_{\lambda+\lambda'} x). \end{aligned}$$

This is equivalent to $[D_\lambda D'] \circ \mathcal{F}_\omega \subseteq \mathcal{F}_\omega$, so the bracket in $\mathrm{Der}(R)$ is closed. Thus $\mathrm{Der}(R)$ is a Lie conformal algebra as a conformal Lie subalgebra of $\mathrm{Cend}(R)$. □

Proposition 4.8. *Let R be a Lie conformal algebra,* $\mathrm{Der}(R)$ *and* $\mathrm{Inn}(R)$ *be the set of their conformal derivations and inner conformal derivations. Then we obtain a crossed module $i : \mathrm{Inn}(R) \to \mathrm{Der}(R)$, with* $\mathrm{Der}(R)$ *acting* $\mathrm{Inn}(R)$ *by* $D \triangleright_\lambda \mathrm{ad}_x = \mathrm{ad}_{D_\lambda x}$.

4.3. Conformal omni-Lie algebras

The notion of omni-Lie algebras was generalized to omni-Lie superalgebras in [ZL14]. In this subsection, we introduce the concept of conformal omni-Lie algebras and construct 2-term conformal L_∞-algebras from them.

Let R be a Lie conformal algebra and M be a left R-module. We define an operation $\circ$ on $\mathcal{E} := R \oplus M$ by

$$(x + u) \circ_\lambda (y + v) = [x_\lambda y] + x \triangleright_\lambda v, \tag{20}$$

for all $x, y \in R$ and $u, v \in M$. Then we have

Proposition 4.9. *$(\mathcal{E}, \circ_\lambda)$ is a Leibniz conformal algebra.*

Proof. We have to check the Leibniz identity for the operation $\circ_\lambda$. Let $e_1 = x+u, e_2 = y+v, e_3 = z+w$. Then we have

$$\begin{aligned}
&\{e_1 \circ_\lambda e_2\} \circ_{\lambda+\mu} e_3 - e_1 \circ_\lambda \{e_2 \circ_\mu e_3\} - e_2 \circ_\mu \{e_1 \circ_\lambda e_3\} \\
&= ([x_\lambda y] + x \triangleright_\lambda v) \circ_{\lambda+\mu} (z+w) - (x+u) \circ_\lambda ([y_\mu z] + y \triangleright_\mu w) \\
&\quad - (y+v) \circ_\mu ([x \triangleright_\lambda w] + z \triangleright_\lambda w) \\
&= [[x_\lambda y]_{\lambda+\mu} z] - [x_\lambda [y_\mu z]] - [y_\mu [x_\lambda z]] \\
&\quad + [x_\lambda y] \triangleright_{\lambda+\mu} w - x \triangleright_\lambda (y \triangleright_\mu w) - y \triangleright_\mu (x \triangleright_\lambda w) \\
&= 0.
\end{aligned}$$

The right hand side of the above equation is zero because R is a Lie conformal algebra acting on M. □

We call this type of Leibniz conformal algebra structure on $\mathcal{E}$ the *hemisemidirect product* of R with M as in [KW01], denote it by $R \ltimes_H M$. Note that the operation $\circ_\lambda$ is not skew-symmetry. The skew-symmetrized bracket of it is:

$$[\![(x+u)_\lambda (y+v)]\!] := [x_\lambda y] + \frac{1}{2}\left(x \triangleright_\lambda v - y \triangleright_{-\partial-\lambda} u\right). \tag{21}$$

This is called *demisemidirect product* of R with M, denoted by $R \ltimes_D M$.

For $e_1 = x+u, e_2 = y+v, e_3 = z+w \in \mathcal{E}$, denote by

$$\widetilde{J}(e_1, e_2, e_3) := [\![e_{1\lambda_1} [\![e_{2\lambda_2} e_3]\!]]\!] - [\![e_{2\lambda_2} [\![e_{1\lambda_1} e_3]\!]]\!] - [\![[\![e_{1\lambda_1} e_2]\!]_{\lambda_1+\lambda_2} e_3]\!].$$

Then $\widetilde{J}(e_1, e_2, e_3) \neq 0$. For example, let $e_1 = x, e_2 = y, e_3 = w$, then

$$\widetilde{J}(x, y, z) = -\frac{1}{4}[x_{\lambda_1} y] \triangleright_{\lambda_1+\lambda_2} w.$$

Thus $R \ltimes_D M$ can not to be a Lie conformal algebra but a Lie conformal 2-algebra, see Theorem 4.10 as follows.

Now, for a Lie conformal algebra R and a left R-module M, let

$$\mathcal{V}_0 = R \ltimes_D M, \quad \mathcal{V}_1 = M, \quad \mathrm{d} = i : M \hookrightarrow R \ltimes_D M,$$

where i is the inclusion map and define

$$l^2_\lambda = [\![\cdot_\lambda \cdot]\!], \quad l^3_{\lambda_1,\lambda_2} = \widetilde{J}.$$

Theorem 4.10. *Let R be a Lie conformal algebra and M a left R-module. Then we obtain a nontrivial 2-term conformal L_∞-algebra $(M \xhookrightarrow{i} R \ltimes_D M, l^2_\lambda, l^3_{\lambda_1,\lambda_2})$.*

Proof. It can be checked that various conditions in Definition 3.5 hold. First we have

$$\begin{aligned}
&[\![(x+u)_\lambda (y+v)]\!] + [\![(y+v)_{-\partial-\lambda} (x+u)]\!] \\
&= [x_\lambda y] + \frac{1}{2}\left(x \triangleright_\lambda v - y_{-\partial-\lambda} u\right) + [y_{-\partial-\lambda} x] + \frac{1}{2}\left(y_{-\partial-\lambda} u - x \triangleright_\lambda v\right) \\
&= [x_\lambda y] + [y_{-\partial-\lambda} x] + \frac{1}{2}\left(x \triangleright_\lambda v - y_{-\partial-\lambda} u\right) + \frac{1}{2}\left(y_{-\partial-\lambda} u - x \triangleright_\lambda v\right) \\
&= 0.
\end{aligned}$$

Thus condition (a) holds.

Let $e_1 = x, e_2 = y, e_3 = z$ where $x, y, z \in R$ and $e_4 = t \in M$, then we have

$$\begin{aligned}
&[\![x_{\lambda_1} l^3_{\lambda_2,\lambda_3}(y,z,t)]\!] - [\![y_{\lambda_2} l^3_{\lambda_1,\lambda_3}(x,z,t)]\!] \\
&\quad + [\![z_{\lambda_3} l^3_{\lambda_1,\lambda_2}(x,y,t)]\!] + [\![l^3_{\lambda_1,\lambda_2}(x,y,z)_{\lambda_1+\lambda_2+\lambda_3} t]\!] - l^3_{\lambda_1+\lambda_2,\lambda_3}([\![x_{\lambda_1} y]\!], z, t) \\
&\quad - l^3_{\lambda_2,\lambda_1+\lambda_3}(y, [\![x_{\lambda_1} z]\!], t) - l^3_{\lambda_2,\lambda_3}(y, z, [\![x_{\lambda_1} t]\!]) \\
&\quad + l^3_{\lambda_1,\lambda_2+\lambda_3}(x, [\![y_{\lambda_2} z]\!], t) + l^3_{\lambda_1,\lambda_3}(x, z, [\![y_{\lambda_2} t]\!]) - l^3_{\lambda_1,\lambda_2}(x, y, [\![z_{\lambda_3} t]\!]) \\
&= -\tfrac{1}{8} x \triangleright_{\lambda_1} ([y_{\lambda_2} z]_{\lambda_2+\lambda_3} \triangleright t) + \tfrac{1}{8} y \triangleright_{\lambda_2} ([x_{\lambda_1} z]_{\lambda_1+\lambda_3} \triangleright t) \\
&\quad - \tfrac{1}{8} z \triangleright_{\lambda_3} ([x,y] \triangleright_{\lambda_1+\lambda_2} t) + 0 + \tfrac{1}{4}[[x_{\lambda_1} y]_{\lambda_1+\lambda_2} z] \triangleright_{\lambda_1+\lambda_2+\lambda_3} t \\
&\quad - \tfrac{1}{4}[[x_{\lambda_1} z]_{\lambda_1+\lambda_3} y] \triangleright_{\lambda_1+\lambda_2+\lambda_3} t + \tfrac{1}{4}[[y_{\lambda_2} z]_{\lambda_2+\lambda_3} x] \triangleright_{\lambda_1+\lambda_2+\lambda_3} t \\
&\quad + \tfrac{1}{8}[y_{\lambda_2} z] \triangleright_{\lambda_2+\lambda_3} (x \triangleright_{\lambda_1} t) - \tfrac{1}{8}[x_{\lambda_1} z] \triangleright_{\lambda_1+\lambda_3} (y \triangleright_{\lambda_2} t) + \tfrac{1}{8}[x_{\lambda_1} y] \triangleright_{\lambda_1+\lambda_2} (z \triangleright_{\lambda_3} t) \\
&= -\tfrac{3}{8}([x_{\lambda_1}[y_{\lambda_2} z]] - [[x_{\lambda_1} y]_{\lambda_1+\lambda_2} z] - [y_{\lambda_2}[x_{\lambda_1} z]]) \triangleright_{\lambda_1+\lambda_2+\lambda_3} t \\
&= 0.
\end{aligned}$$

Thus condition (i) holds. The other conditions can be checked similarly. □

Furthermore, we define a symmetric bilinear form on $\mathcal{E}$ with values in $M[\lambda]$ by

$$\langle x+u, y+v \rangle_\lambda := \tfrac{1}{2}(x \triangleright_\lambda v + y \triangleright_\lambda u). \tag{22}$$

The triple $(\mathcal{E}, [\![\cdot_\lambda \cdot]\!], \langle \cdot, \cdot \rangle_\lambda)$ is called a *conformal omni-Lie algebra.*

When $R = \mathrm{Cend}(M)$, all possible Lie conformal algebra structures on M can be characterized by means of the conformal omni-Lie algebras.

For an operation $\omega_\lambda : M \times M \to M[\lambda]$, we define the adjoint operator

$$\mathrm{ad}_{\omega_\lambda} : M\, \mathrm{Cend}(M), \quad \mathrm{ad}_{\omega_\lambda}(x)(y) = \omega_\lambda(x,y) \in M[\lambda],$$

where $x, y \in M$. Then the graph of the adjoint operator:

$$\mathcal{F}_\omega = \{\mathrm{ad}_\omega x + x \mid \forall\, x \in M\} \subset \mathcal{E} = \mathrm{Cend}(M) \oplus M$$

is a subspace of $\mathcal{E}$. Denote $\mathcal{F}_\omega^\perp$ the orthogonal complement of $\mathcal{F}_\omega$ in $\mathcal{E}$ with respect to the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{E}$ given in (22).

Proposition 4.11. *With the above notations, (M, ω_λ) is a Lie conformal algebra if and only if its graph $\mathcal{F}_\omega$ is maximal isotropic, i.e. $\mathcal{F}_\omega = \mathcal{F}_\omega^\perp$, and is closed with respect to the bracket $[\![\cdot_\lambda \cdot]\!]$.*

Proof. If ω_λ is skew symmetric, i.e. $\omega_\lambda(x,y) + \omega_{-\partial-\lambda}(y,x) = 0$, then

$$\begin{aligned}
\langle \mathrm{ad}_{\omega_\lambda}(x) + x, \mathrm{ad}_{\omega_{-\partial-\lambda}}(y) + y \rangle_\lambda &= \tfrac{1}{2}(\mathrm{ad}_{\omega_\lambda}(x) y + \mathrm{ad}_{\omega_{-\partial-\lambda}}(y) x) \\
&= \tfrac{1}{2}(\omega_\lambda(x,y) + \omega_{-\partial-\lambda}(y,x)).
\end{aligned}$$

This means that ω_λ is skew-symmetric if and only if its graph is isotropic, i.e. $\mathcal{F}_\omega \subseteq \mathcal{F}_\omega^\perp$. Moreover, by dimension analysis, we have $\mathcal{F}_\omega$ is maximal isotropic.

Next let $[x_\lambda y] := \omega_\lambda(x, y)$, we shall check that the Jacobi identity on M is satisfied if and only if $\mathcal{F}_\omega$ is closed under bracket (21) on $\mathcal{E}$. In fact,

$$\begin{aligned}
[\![\mathrm{ad}_{\omega_\lambda}(x) + x, \mathrm{ad}_{\omega_\mu}(y) + y]\!] &= [\mathrm{ad}_{\omega_\lambda}(x), \mathrm{ad}_{\omega_\mu}(y)] + \tfrac{1}{2}(\mathrm{ad}_{\omega_\lambda}(x)y - \mathrm{ad}_{\omega_{-\partial-\mu}}(y)x) \\
&= [\mathrm{ad}_{\omega_\lambda}(x), \mathrm{ad}_{\omega_\mu}(y)] + \tfrac{1}{2}(\omega_\lambda(x, y) - \omega_{-\partial-\lambda}(y, x)) \\
&= [\mathrm{ad}_{\omega_\lambda}(x), \mathrm{ad}_{\omega_\mu}(y)] + \omega_\lambda(x, y).
\end{aligned}$$

Thus this bracket is closed if and only if

$$[\mathrm{ad}_{\omega_\lambda}(x), \mathrm{ad}_{\omega_\mu}(y)] = \mathrm{ad}_{\omega_{\lambda+\mu}}(\omega_\lambda(x, y)).$$

In this case, for all $z \in M$, we have

$$\begin{aligned}
&[\mathrm{ad}_{\omega_\lambda}(x), \mathrm{ad}_{\omega_\mu}(y)](z) - \mathrm{ad}_{\omega_{\lambda+\mu}}(\omega_\lambda(x, y))(z) \\
&= \mathrm{ad}_{\omega_\lambda}(x)\,\mathrm{ad}_{\omega_\mu}(y)(z) - \mathrm{ad}_{\omega_\mu}(y)\,\mathrm{ad}_{\omega_\lambda}(x)(z) - \mathrm{ad}_{\omega_{\lambda+\mu}}(\omega_\lambda(x, y))(z) \\
&= \mathrm{ad}_{\omega_\lambda}(x)\omega_\mu(y, z) - \mathrm{ad}_{\omega_\mu}(y)\omega_\lambda(x, z) - \omega_{\lambda+\mu}(\omega_\lambda(x, y), z) \\
&= \omega_\lambda(x, \omega_\mu(y, z)) - \omega_\mu(y, \omega_\lambda(x, z)) - \omega_{\lambda+\mu}(\omega_\lambda(x, y), z) \\
&= [x_\lambda[y_\mu z]] - [y_\mu[x_\lambda z]] - [[x_\lambda y]_{\lambda+\mu} z] \\
&= 0.
\end{aligned}$$

This is exactly the Jacobi identity on M. $\square$

We define a Dirac structure of $\mathrm{Cend}(M)\oplus M$ to be any maximal isotropic subspace $L \subseteq \mathrm{Cend}(M)\oplus M$ which is closed under the bracket operation. Then we obtain that (M, ω_λ) is a Lie conformal algebra if and only if $\mathcal{F}_{\omega_\lambda}$ is a Dirac structure of the conformal omni-Lie algebra $R\oplus M$.

For a general characterization for all Dirac structures of $\mathcal{E}$, we adapt the theory of characteristic pairs developed in [Liu00]. One can prove the following result.

Proposition 4.12. *There is a one-to-one correspondence between Dirac structures of the conformal omni-Lie algebra* $(\mathrm{Cend}(M) \oplus M, [\![\cdot_\lambda\cdot]\!], \langle\cdot,\cdot\rangle_\lambda)$ *and Lie conformal algebra structures on subspaces of* M*.*

4.4. Skew-symmetrization of Leibniz conformal algebras

From above subsection, we have seen that a conformal omni-Lie algebra $\mathcal{E} = R \ltimes_D M$ is in fact the skew-symmetrization of Leibniz conformal algebra $R \ltimes_H M$ and M is the left center of $\mathcal{E}$. This observation prompts the following question: Can we derive a conformal Lie 2-algebra from any given Leibniz conformal algebra? In the following subsection, we provide a positive answer to this inquiry.

Let L be a Leibniz conformal algebra. We define the *Leibniz kernel* $\mathrm{Ker}(L)$ of L to be the set of elements spanned by

$$\{x \circ_\lambda y + y \circ_{-\partial-\lambda} x \mid \forall\, x, y \in L\}.$$

Note that if L is a Lie conformal algebra, then $\mathrm{Ker}(L) = 0$, otherwise if L is a non-Lie Leibniz conformal algebra, then $\mathrm{Ker}(L) \neq 0$.

The *left center* $Z^l(L)$ of L is defined by

$$Z^l(L) = \{t \in L \mid t \circ_\lambda x = 0, \forall\, x \in L\}.$$

It is easy to see that $\mathrm{Ker}(L)$ and $Z^l(L)$ are ideals of L and the quotient algebras $L/\mathrm{Ker}(L)$ and $L/Z^l(L)$ are Lie conformal algebras.

Proposition 4.13. *The Leibniz kernel* $\mathrm{Ker}(L)$ *is contained in the left center* $Z^l(L)$*. Thus for any non-Lie Leibniz conformal algebra, the set* $Z^l(L)$ *is not empty.*

Proof. Let $x \circ_\lambda y + y \circ_{-\partial-\lambda} x \in \mathrm{Ker}(L)$. Then we get

$$\begin{aligned}
&\big(x \circ_\lambda y + y \circ_{-\partial-\lambda} x\big) \circ_{\lambda+\mu} z \\
&\qquad = x \circ_\lambda (y \circ_\mu z) - y \circ_\mu (x \circ_\lambda z) + y \circ_\mu (x \circ_\lambda z) - x \circ_\lambda (y \circ_\mu z) \\
&\qquad = 0,
\end{aligned}$$

for all $z \in L$. Thus $x \circ_\lambda y + y \circ_{-\partial-\lambda} x \in Z^l(L)$. Therefore $\mathrm{Ker}(L)$ is contained in $Z^l(L)$. □

For a Leibniz conformal algebra $(L, \circ_\lambda)$, since the operation $\circ_\lambda$ is not skew-symmetry, we introduce the following skew-symmetric bracket on L by

$$[\![x_\lambda y]\!] = \tfrac{1}{2}\left(x \circ_\lambda y - y \circ_{-\partial-\lambda} x\right), \quad \forall\, x, y \in L, \tag{23}$$

and let $\widetilde{J}$ be given by

$$\widetilde{J}(x,y,z) := \; [\![[\![x_{\lambda_1} y]\!]_{\lambda_1+\lambda_2} z]\!] - [\![x_{\lambda_1}[\![y_{\lambda_2} z]\!]]\!] - [\![y_{\lambda_2}[\![x_{\lambda_1} z]\!]]\!]. \tag{24}$$

Now it is easy to prove that

Proposition 4.14. *If* $t \in Z^l(L)$*, i.e.* $t \circ x = 0$ *for all* $x \in L$*, then we have*

$$[\![x_\lambda t]\!] = \tfrac{1}{2} x \circ_\lambda t, \quad \widetilde{J}(x,y,t) = -\tfrac{1}{4}(x \circ_{\lambda_1} y) \circ_{\lambda_1+\lambda_2} t.$$

At last, for any non-Lie Leibniz conformal algebra L, we construct nontrivial conformal Lie 2-algebras as follows. Let

$$\mathcal{V}_0 = L, \quad \mathcal{V}_1 = \mathrm{Z}^l(L), \quad \mathrm{d} = i : \mathrm{Z}^l(L) \hookrightarrow L, \quad l^2_\lambda = [\![\,\cdot_\lambda\cdot\,]\!], \quad l^3_{\lambda_1,\lambda_2} = \widetilde{J}.$$

Theorem 4.15. *With the above notations, from a Leibniz conformal algebra* $(L, \circ_\lambda)$*, we derive a nontrivial 2-term conformal* L_∞*-algebra* $(Z^l(L) \overset{\mathrm{d}}{\hookrightarrow} L,\, l^2_\lambda,\, l^3_{\lambda_1,\lambda_2})$.

Proof. By definition of d, l^2_λ and $l^3_{\lambda_1,\lambda_2}$, it is easy to see that conditions (a)–(h) hold. For condition (i), we verify the case of $x, y, z \in L$ and $t \in Z^l(L)$ as follows:

$$\begin{aligned}
&[\![x_{\lambda_1} l^3_{\lambda_2,\lambda_3}(y,z,t)]\!] - [\![y_{\lambda_2} l^3_{\lambda_1,\lambda_3}(x,z,t)]\!] \\
&\quad + [\![z_{\lambda_3} l^3_{\lambda_1,\lambda_2}(x,y,t)]\!] + [\![l^3_{\lambda_1,\lambda_2}(x,y,z)_{\lambda_1+\lambda_2+\lambda_3} t]\!] - l^3_{\lambda_1+\lambda_2,\lambda_3}([\![x_{\lambda_1} y]\!], z, t) \\
&\quad - l^3_{\lambda_2,\lambda_1+\lambda_3}(y, [\![x_{\lambda_1} z]\!], t) - l^3_{\lambda_2,\lambda_3}(y, z, [\![x_{\lambda_1} t]\!]) \\
&\quad + l^3_{\lambda_1,\lambda_2+\lambda_3}(x, [\![y_{\lambda_2} z]\!], t) + l^3_{\lambda_1,\lambda_3}(x, z, [\![y_{\lambda_2} t]\!]) - l^3_{\lambda_1,\lambda_2}(x, y, [\![z_{\lambda_3} t]\!])
\end{aligned}$$

$$
\begin{aligned}
&= -\tfrac{1}{4}\Big([\![x_{\lambda_1} ((y \circ z) \circ_{\lambda_2+\lambda_3} t)]\!] - [\![y_{\lambda_2} ((x \circ_{\lambda_1} z) \circ_{\lambda_1+\lambda_3} t)]\!] \\
&\qquad + [\![z_{\lambda_3} ((x \circ_{\lambda_1} y) \circ_{\lambda_1+\lambda_2} t)]\!] - 0 - ([\![x_{\lambda_1} y]\!] \circ_{\lambda_1+\lambda_2} z) \circ_{\lambda_1+\lambda_2+\lambda_3} t \\
&\qquad + ([\![x_{\lambda_1} z]\!] \circ_{\lambda_1+\lambda_3} y) \circ_{\lambda_1+\lambda_2+\lambda_3} t - (y \circ_{\lambda_2} z) \circ_{\lambda_2+\lambda_3} [\![x_{\lambda_1} t]\!] \\
&\qquad - (x \circ_{\lambda_1} [\![y_{\lambda_2} z]\!]) \circ_{\lambda_1+\lambda_2+\lambda_3} t + (x \circ_{\lambda_1} z) \circ_{\lambda_1+\lambda_3} [\![y_{\lambda_2} t]\!] \\
&\qquad - (x \circ_{\lambda_1} y) \circ_{\lambda_1+\lambda_2} [\![z_{\lambda_3} t]\!] \Big) \\
&= -\tfrac{3}{8}\big(x \circ_{\lambda_1} (y \circ_{\lambda_2} z) - (x \circ_{\lambda_1} y) \circ_{\lambda_1+\lambda_2} z - y \circ_{\lambda_2} (x \circ_{\lambda_1} z) \big) \circ_{\lambda_1+\lambda_2+\lambda_3} t \\
&= 0.
\end{aligned}
$$

The other cases can be checked similarly. The proof is completed. □

It is natural to introduce the general notions of L_∞-conformal algebras and $Leib_\infty$-conformal algebras. In fact, an initial version of this paper appeared in arXiv: 2301.06903. Since then, several articles developing the theory presented in this manuscript have been appeared, such as those in [AW24; DS23; SD23]. We encourage interested readers to consult these references for further developments in the field.

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