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Abstract. In this paper, we investigate non-abelian extensions and inducibility of pairs of automorphisms of Lie triple systems. First, we introduce non-abelian cohomology groups and classify non-abelian extensions in terms of non-abelian cohomology groups. Next, we characterize non-abelian extensions using Maurer–Cartan elements. Furthermore, we explore the inducibility of pairs of automorphisms and derive the analog Wells exact sequences under the circumstance of Lie triple systems. Finally, we state the previous results under the context of abelian extensions of Lie triple systems.

Mathematics Subject Classification 2020: 17A30, 17B62, 17B38

Key Words and Phrases: Lie triple system, non-abelian extension, Maurer–Cartan element, inducibility, Wells exact sequence

1. Introduction

Lie triple systems originated from Cartan's studies of Riemannian geometry, in which he employed his classification of the real simple Lie algebras to classify symmetric spaces. The role played by Lie triple systems in the theory of symmetric spaces is like that of Lie algebras in the theory of Lie groups. Lie triple systems were investigated from the algebraic point of view by Jacobson in [Jac49] and studied later by Lister in [Lis52]. Subsequently, Lie triple systems have gained broad attention and a number of works have been gained, see [CLS23; MCL14; YBB21] and their references.

Extensions of some mathematical object are useful to understand the underlying structure. The non-abelian extension is a relatively general one among various extensions (e.g. central extensions, abelian extensions, non-abelian extensions, etc.). Non-abelian extensions were first developed by Eilenberg and Maclane [EM47], which led to the low dimensional non-abelian cohomology groups. Then numerous works have been devoted to non-abelian extensions of various kinds of algebras, such as Lie algebras, associative algebras, Lie groups, pre-Lie algebras, Leibniz algebras, 3-Lie algebras, Rota–Baxter Lie algebras, conformal algebras and Rota–Baxter Leibniz algebras, see [Fré14; Gou18; GH23; HZ22; IKL08; LSW18; MSW23; MÖW25; MDH23; Nee07; SMT18] and their references. But little is known about the non-abelian extension of Lie triple systems. This was our first motivation for writing the present paper.

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Our second motivation is to explore the inducibility of pairs of Lie triple system automorphisms. The inducibility of a pair of automorphisms has close relationship with extensions of algebras. Such study originated from extensions of abstract groups [Wel71]. This theme was subsequently extended to various algebraic structures, such as Lie algebras, Rota–Baxter Lie algebras, Rota–Baxter groups, associative conformal algebras and Lie–Yamaguti algebras, see [BS17; DR23; GMM24; HH20; HZ22; JL10; MDH23; PSY10; Wel71] and references therein. As byproducts, the Wells short exact sequences were derived for various kinds of algebras [DR23; GMM24; GH23; HH20; HZ22; JL10; MDH23], which connected the relative automorphism groups and the non-abelian second cohomology groups. Motived by these results and combined with our study of non-abelian extension of Lie triple systems, naturally we study inducibility of pairs of Lie triple system automorphisms and derive the Wells short exact sequences in the context of non-abelian extensions of Lie triple systems.

The paper is organized as follows. In Section 2, we recall basic knowledge of Lie triple systems. In Section 3, we investigate non-abelian extensions and classified the non-abelian extensions using non-abelian 3-cocycles. In Section 4, we characterize non-abelian extensions in terms of Maurer–Cartan elements. In Section 5, we address the inducibility of pairs of Lie triple system automorphisms and give some equivalent conditions In Section 6, we consider the analogue of the Wells short exact sequences under the case of Lie triple systems. In Section 7, we explain the previous results in the context of abelian extensions of Lie triple systems.

Throughout the paper, let k be a field. Unless otherwise specified, all vector spaces and algebras are over k.

2. Preliminary on Lie triple systems

A Lie triple system is a vector space $\mathfrak{g}$ together with a trilinear map $[\ ,\ ,\ ] : \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ satisfying

$$[x_1,x_1,x_2]=0, \tag{1}$$

$$[x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]=0, \tag{2}$$

$$[x_1,x_2,[y_1,y_2,y_3]]=[[x_1,x_2,y_1],y_2,y_3]+[y_1,[x_1,x_2,y_2],y_3]+[y_1,y_2,[x_1,x_2,y_3]], \tag{3}$$

for all $x_i, y_i\in\mathfrak{g}$ $(i=1,2,3)$.

Eqs. (1) and (2) yield that

$$[x_1,x_2,x_3]+[x_2,x_1,x_3]=0. \tag{4}$$

A subspace M of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$ if $[M,\mathfrak{g},\mathfrak{g}]\subseteq M$ and $[\mathfrak{g},\mathfrak{g},M]\subseteq M$. An ideal M of $\mathfrak{g}$ is said to be an abelian ideal of $\mathfrak{g}$ if $[M,M,\mathfrak{g}]=[\mathfrak{g},M,M]=0$.

Example 2.1. Let $\mathbb{R}^3$ be a 3-dimensional Euclidean space equipped with the standard inner product $(\cdot,\cdot)$. Define $[\ ,\ ,\ ]:\mathbb{R}^3\otimes\mathbb{R}^3\otimes\mathbb{R}^3\to\mathbb{R}^3$ by $[x,y,z]=(y,z)x-(x,z)y$. Then $(\mathfrak{g},[\ ,\ ,\ ])$ is a Lie triple system. Explicitly, assume that $\{e_1,e_2,e_3\}$ is an orthonormal basis of 3-dimensional Euclidean space $\mathbb{R}^3$, that is,

$$(e_i,e_j)=\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } j\neq i\end{cases}\quad \text{for } i,j=1,2,3.$$

Then $[e_i,e_j,e_k]=\delta_{jk}e_i-\delta_{ik}e_j$, $(i,j,k=1,2,3)$.

A representation of a Lie triple system $\mathfrak{g}$ consists of a vector space V together with a bilinear map $\theta : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{gl}(V)$ satisfying

$$\begin{aligned}\theta(x_3, x_4)\theta(x_1, x_2) - \theta(x_2, x_4)\theta(x_1, x_3)&\\ - \theta(x_1, [x_2, x_3, x_4]) + D_\theta(x_2, x_3)\theta(x_1, x_4) &= 0, \end{aligned}\tag{5}$$

$$\begin{aligned}\theta(x_3, x_4)D_\theta(x_1, x_2) - D_\theta(x_1, x_2)\theta(x_3, x_4)&\\ + \theta([x_1, x_2, x_3], x_4) + \theta(x_3, [x_1, x_2, x_4]) &= 0,\end{aligned}\tag{6}$$

for any $x_i \in \mathfrak{g}$ $(i = 1, 2, 3, 4)$, where

$$D_\theta(x_1, x_2) = \theta(x_2, x_1) - \theta(x_1, x_2).\tag{7}$$

One can refer to [Har61; HP02; Lis52; Yam58; Yam60; Zha14] for more information on Lie triple systems and representations.

Let $\mathfrak{g}$ be a Lie triple system and (V, θ) be a representation of it. Denote the $(2n+1)$-cochains group of $\mathfrak{g}$ with coefficients in V by $C^{2n+1}(\mathfrak{g}, V)$, where $f \in C^{2n+1}(\mathfrak{g}, V)$ is a multilinear map $f : \mathfrak{g} \times \cdots \times \mathfrak{g} \to V$ satisfying

$$f(x_1, \ldots, x_{2n-2}, x, x, y) = 0\tag{8}$$

and

$$f(x_1, \ldots, x_{2n-2}, x, y, z) + f(x_1, \ldots, x_{2n-2}, y, z, x) + f(x_1, \ldots, x_{2n-2}, z, x, y) = 0\tag{9}$$

for all $x_i, x, y, z \in \mathfrak{g}$ $(i = 1, \ldots, 2n-2)$ and $C^1(\mathfrak{g}, V) = \mathrm{Hom}(\mathfrak{g}, V)$.

The Yamaguti coboundary operator $\delta : C^{2n-1}(\mathfrak{g}, V) \to C^{2n+1}(\mathfrak{g}, V)$ is given by

$$\begin{aligned}&(\delta f)(x_1, \ldots, x_{2n+1})\\ &\quad= \theta(x_{2n}, x_{2n+1})f(x_1, \ldots, x_{2n-1}) - \theta(x_{2n-1}, x_{2n+1})f(x_1, \ldots, x_{2n-2}, x_{2n})\\ &\qquad+ \sum_{i=1}^{n}(-1)^{i+n}D_\theta(x_{2i-1}, x_{2i})f(x_1, \ldots, x_{2i-2}, x_{2i+1}, \ldots, x_{2n+1})\\ &\qquad+ \sum_{i=1}^{n}\sum_{j=2i+1}^{2n+1}(-1)^{i+n+1}f(x_1, \ldots, x_{2i-2}, x_{2i+1}, \ldots, x_{j-1}, [x_{2i-1}, x_{2i}, x_j], \ldots, x_{2n+1})\end{aligned}$$

for any $x_1, x_2, \ldots, x_{2n+1} \in \mathfrak{g}$. Denote the set of all $(2n-1)$-cocycles and $(2n-1)$-coboundaries respectively by $Z^{2n-1}(\mathfrak{g}, V)$ and $B^{2n-1}(\mathfrak{g}, V)$. Thus, we can define the $(2n-1)$-cohomology group by

$$H^{2n-1}(\mathfrak{g}, V) = \begin{cases} Z^1(A, V), & n = 1,\\ Z^{2n-1}(A, V)/B^{2n-1}(A, V), & n > 1.\end{cases}$$

See [KT04; Zha14] for more details on Yamaguti cohomology.

3. Non-abelian extensions of Lie triple systems

In this section, we are devoted to considering non-abelian extensions and non-abelian 3-cocycles of Lie triple systems.

Definition 3.1. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie triple systems. A non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$ is a Lie triple system $\widehat{\mathfrak{g}}$, which fits into a short exact sequence of Lie triple systems

$$\mathcal{E} : 0 \longrightarrow \mathfrak{h} \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \longrightarrow 0.$$

When $\mathfrak{h}$ is an abelian ideal of $\widehat{\mathfrak{g}}$, the extension $\mathcal{E}$ is called an abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$. Denote an extension as above simply by $\widehat{\mathfrak{g}}$ or $\mathcal{E}$. A section of p is a linear map $s : \mathfrak{g} \to \widehat{\mathfrak{g}}$ such that $ps = I_{\mathfrak{g}}$.

Definition 3.2. Let $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ be two non-abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$. They are said to be equivalent if there is an isomorphism of Lie triple systems $f : \widehat{\mathfrak{g}}_1 \to \widehat{\mathfrak{g}}_2$ such that the following commutative diagram holds:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r, "i_1"] \arrow[d, equal] & \widehat{\mathfrak{g}}_1 \arrow[r, "p_1"] \arrow[d, "f"] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathfrak{h} \arrow[r, "i_2"] & \widehat{\mathfrak{g}}_2 \arrow[r, "p_2"] & \mathfrak{g} \arrow[r] & 0. \end{tikzcd} \tag{10}$$

Denote by $\mathcal{E}_{nab}(\mathfrak{g}, \mathfrak{h})$ the set of all non-abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$.

Next, we define a non-abelian cohomology group and show that the non-abelian extensions are classified by the non-abelian cohomology groups.

Definition 3.3. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie triple systems. A non-abelian 3-cocycle on $\mathfrak{g}$ with values in $\mathfrak{h}$ is a triple (ω, θ, ρ) of maps such that $\omega : \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{h}$ is trilinear, $\theta : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ is bilinear and $\rho : \mathfrak{g} \to \mathrm{Hom}(\mathfrak{h} \otimes \mathfrak{h}, \mathfrak{h})$ is linear, and the following identities are satisfied for all $x_i, y_i, x, y, z, w \in \mathfrak{g}$ $(i = 1, 2, 3)$, $a, b, c \in \mathfrak{h}$,

$$\omega(x, x, y) = 0, \quad \omega(x, y, z) + \omega(y, z, x) + \omega(z, x, y) = 0, \tag{11}$$

$$\begin{aligned} &D_\theta(x_1, x_2)\omega(y_1, y_2, y_3) + \omega(x_1, x_2, [y_1, y_2, y_3]_{\mathfrak{g}}) \\ &\quad = \omega([x_1, x_2, y_1]_{\mathfrak{g}}, y_2, y_3) + \theta(y_2, y_3)\omega(x_1, x_2, y_1) + \omega(y_1, [x_1, x_2, y_2]_{\mathfrak{g}}, y_3) \\ &\qquad - \theta(y_1, y_3)\omega(x_1, x_2, y_2) + \omega(y_1, y_2, [x_1, x_2, y_3]_{\mathfrak{g}}) + D_\theta(y_1, y_2)\omega(x_1, x_2, y_3), \end{aligned} \tag{12}$$

$$\begin{aligned} &D_\theta(x, y)\theta(z, w)a - \theta(z, w)D_\theta(x, y)a \\ &\qquad = \theta([x, y, z]_{\mathfrak{g}}, w)a + \theta(z, [x, y, w]_{\mathfrak{g}})a \\ &\qquad\quad - D_\rho(w)(\omega(x, y, z), a) - \rho(z)(a, \omega(x, y, w)), \end{aligned} \tag{13}$$

$$\begin{aligned} &\theta(x, [y, z, w]_{\mathfrak{g}})a - \rho(x)(a, \omega(y, z, w)) \\ &\qquad = \theta(z, w)\theta(x, y)a - \theta(y, w)\theta(x, z)a + D_\theta(y, z)\theta(x, w)a, \end{aligned} \tag{14}$$

$$\begin{aligned} D_\theta(x, y)\rho(z)(a, b) &= \rho(z)(D_\theta(x, y)a, b) + \rho([x, y, z]_{\mathfrak{g}})(a, b) \\ &\quad + [\omega(x, y, z), a, b]_{\mathfrak{h}} + \rho(z)(a, D_\theta(x, y)b), \end{aligned} \tag{15}$$

$$D_\theta(y, z)\rho(x)(a, b) = \rho(x)(a, D_\theta(y, z)b) - \rho(z)(\theta(x, y)a, b) + \rho(y)(\theta(x, z)a, b), \tag{16}$$

$$\theta(y, z)(\rho(x)(a, b)) = \rho(x)(a, \theta(y, z)b) - D_\rho(z)(\theta(x, y)a, b) - \rho(y)(b, \theta(x, z)a), \tag{17}$$

$$D_\theta(x,y)([a,b,c]_{\mathfrak{h}}) = [D_\theta(x,y)a,b,c]_{\mathfrak{h}} + [a,D_\theta(x,y)b,c]_{\mathfrak{h}} + [a,b,D_\theta(x,y)c]_{\mathfrak{h}}, \tag{18}$$

$$\rho(x)(a,\rho(y)(b,c)) = \rho(y)(\rho(x)(a,b),c) + \rho(y)(b,\rho(x)(a,c)) \tag{19}$$

$$[a,b,\theta(x,y)c]_{\mathfrak{h}} = \theta(x,y)[a,b,c]_{\mathfrak{h}} - D_\rho(y)(D_\rho(x)(a,b),c) - \rho(x)(c,D_\rho(y)(a,b)), \tag{20}$$

$$[a,b,\rho(x)(c,d)]_{\mathfrak{h}} = \rho(x)([a,b,c]_{\mathfrak{h}},d) - [c,D_\rho(x)(a,b),d]_{\mathfrak{h}} + \rho(x)(c,[a,b,d]_{\mathfrak{h}}), \tag{21}$$

$$\rho(x)(a,[b,c,d]_{\mathfrak{h}}) = [\rho(x)(a,b),c,d]_{\mathfrak{h}} + [b,\rho(x)(a,c),d]_{\mathfrak{h}} + [b,c,\rho(x)(a,d)]_{\mathfrak{h}}, \tag{22}$$

where

$$D_\theta(x,y)a - \theta(y,x)a + \theta(x,y)a = 0, \quad \rho(x)(a,b) + D_\rho(x)(a,b) - \rho(x)(b,a) = 0. \tag{23}$$

Example 3.4. Let $\mathfrak{g}$ be a 2-dimensional Lie triple system with a basis $\{e_1,e_2\}$, whose non-zero multiplication $[\ ,\ ,\ ]_{\mathfrak{g}}$ is given by $[e_1,e_2,e_2]_{\mathfrak{g}} = e_1$, $[e_2,e_1,e_2]_{\mathfrak{g}} = -e_1$. Assume that $\mathfrak{h}$ is a 2-dimensional Lie triple system with a basis $\{\varepsilon_1,\varepsilon_2\}$, whose non-zero multiplication $[\ ,\ ,\ ]_{\mathfrak{h}}$ is given by $[\varepsilon_1,\varepsilon_2,\varepsilon_2]_{\mathfrak{h}} = \varepsilon_1$, $[\varepsilon_2,\varepsilon_1,\varepsilon_2]_{\mathfrak{h}} = -\varepsilon_1$. Define a trilinear map $\omega : \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{h}$, bilinear maps $\theta, D_\theta : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ and linear maps $\rho, D_\rho : \mathfrak{g} \to \mathrm{Hom}(\mathfrak{h}\otimes\mathfrak{h},\mathfrak{h})$ respectively by

$$\rho(e_2)(\varepsilon_i,\varepsilon_j) = \begin{cases} b\varepsilon_1, & j=i=2, \quad \forall\, b\in k,\\ 0, & \text{otherwise,}\end{cases}$$

$$\rho(e_1)(\varepsilon_j,\varepsilon_i) = \begin{cases} a\varepsilon_1, & j=i=2, \quad \forall\, a\in k,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\theta(x,y) = D_\theta(x,y) = D_\rho(x) = D_\rho(y) = 0, \quad \omega(e_1,e_2,e_2) = -a\varepsilon_1 = -\omega(e_2,e_1,e_2),$$

$$\begin{aligned}\omega(e_1,e_1,e_2) &= \omega(e_1,e_1,e_1) = \omega(e_2,e_2,e_1)\\ &= \omega(e_2,e_2,e_2) = \omega(e_1,e_2,e_1) = \omega(e_2,e_1,e_1) = 0.\end{aligned}$$

Then (ω,θ,ρ) is a non-abelian 3-cocycle on $\mathfrak{g}$ with values in $\mathfrak{h}$.

Definition 3.5. Let $(\omega_1,\theta_1,\rho_1)$ and $(\omega_2,\theta_2,\rho_2)$ be two non-abelian 3-cocycles on $\mathfrak{g}$ with values in $\mathfrak{h}$. They are said to be equivalent if there exists a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that for all $x,y,z\in\mathfrak{g}$ and $a,b\in\mathfrak{h}$, the following equalities hold:

$$\begin{aligned}&\omega_1(x,y,z) - \omega_2(x,y,z)\\ &= \theta_2(x,z)\varphi(y) - D_{\theta_2}(x,y)\varphi(z) + \rho_2(x)(\varphi(y),\varphi(z)) - \theta_2(y,z)\varphi(x)\\ &\quad + D_{\rho_2}(z)(\varphi(x),\varphi(y)) - \rho_2(y)(\varphi(x),\varphi(z)) - [\varphi(x),\varphi(y),\varphi(z)]_{\mathfrak{h}} + \varphi([x,y,z]_{\mathfrak{g}}),\end{aligned} \tag{24}$$

$$\theta_1(x,y)a - \theta_2(x,y)a = \rho_2(x)(a,\varphi(y)) - D_{\rho_2}(y)(a,\varphi(x)) + [a,\varphi(x),\varphi(y)]_{\mathfrak{h}}, \tag{25}$$

$$\rho_1(x)(a,b) - \rho_2(x)(a,b) = [a,\varphi(x),b]_{\mathfrak{h}}. \tag{26}$$

Using Eqs. (23), (25) and (26), we get

$$D_{\theta_1}(x,y)a - D_{\theta_2}(x,y)a = \rho_2(y)(\varphi(x),a) - \rho_2(x)(\varphi(y),a) + [\varphi(x),\varphi(y),a]_{\mathfrak{h}}, \tag{27}$$

and

$$D_{\rho_1}(x)(a,b) - D_{\rho_2}(x)(a,b) = [b,a,\varphi(x)]_{\mathfrak{h}}. \tag{28}$$

Denote the set of all non-abelian 3-cocycles on $\mathfrak{g}$ with values in $\mathfrak{h}$ by $Z^3_{nab}(\mathfrak{g},\mathfrak{h})$. The equivalent class of a non-abelian 3-cocycle (ω,θ,ρ) is denoted by $[(\omega,\theta,\rho)]$. The quotient of $Z^3_{nab}(\mathfrak{g},\mathfrak{h})$ by the above equivalence relation is denoted by $H^3_{nab}(\mathfrak{g},\mathfrak{h})$, which is called non-abelian cohomology group of $\mathfrak{g}$ with coefficients in $\mathfrak{h}$.

Using the above notations, we get

Proposition 3.6. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie triple systems. Assume that $\omega:\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{h}$ is a trilinear map, $\theta:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{gl}(\mathfrak{h})$ is a bilinear map and $\rho:\mathfrak{g}\to\mathrm{Hom}(\mathfrak{h}\otimes\mathfrak{h},\mathfrak{h})$ is a linear map. Define a trilinear map $[\ ,\ ,\ ]_{(\omega,\theta,\rho)}$ on $\mathfrak{g}\oplus\mathfrak{h}$ by*

$$\begin{aligned}[x+a,y+b,z+c]_{(\omega,\theta,\rho)} \\ &= [x,y,z]_{\mathfrak{g}}+\omega(x,y,z)+D_\theta(x,y)c+\theta(y,z)a-\theta(x,z)b \\ &\quad + D_\rho(z)(a,b)+\rho(x)(b,c)-\rho(y)(a,c)+[a,b,c]_{\mathfrak{h}} \end{aligned}\tag{29}$$

for all $x,y,z\in\mathfrak{g}$ and $a,b,c\in\mathfrak{h}$. Then $(\mathfrak{g}\oplus\mathfrak{h},[\ ,\ ,\ ]_{(\omega,\theta,\rho)})$ is a Lie triple system if and only if the triple (ω,θ,ρ) is a non-abelian 3-cocycle. Denote this Lie triple system $(\mathfrak{g}\oplus\mathfrak{h},[\ ,\ ,\ ]_{(\omega,\theta,\rho)})$ simply by $\mathfrak{g}\oplus_{(\omega,\theta,\rho)}\mathfrak{h}$.

Proof. $(\mathfrak{g}\oplus\mathfrak{h},[\ ,\ ,\ ]_{(\omega,\theta,\rho)})$ is a Lie triple system if and only if Eqs. (1)–(3) hold for $[\ ,\ ,\ ]_{(\omega,\theta,\rho)}$. In fact, it is easy to check that Eqs. (1)–(2) hold for $[\ ,\ ,\ ]_{(\omega,\theta,\rho)}$ if and only if (11), (23) hold. For Eq. (3), we discuss it for the following cases: for all $x_1,x_2,y_1,y_2,y_3\in\mathfrak{g}\oplus\mathfrak{h}$,

(I) when all of the five elements x_1,x_2,y_1,y_2,y_3 belongs to $\mathfrak{g}$, (3) holds if and only if (12) holds.

(II) when x_1,x_2,y_1,y_2,y_3 equal to:

- (i) x,y,z,a,w or x,y,a,z,w or x,y,z,w,a respectively, (3) holds if and only if (13) holds.
- (ii) x,a,y,z,w or a,x,y,z,w respectively, (3) holds for if and only if (14) holds.
- (iii) x,y,z,a,b or x,y,a,z,b or x,y,a,b,z respectively, (3) holds if and only if (15) holds.
- (iv) a,x,y,z,b or x,a,y,z,b respectively, (3) holds if and only if (16) holds.
- (v) x,a,b,y,z or a,x,b,y,z or x,a,y,b,z or a,x,y,b,z respectively, (3) holds if and only if (17) holds.
- (vi) a,b,x,y,z, (3) holds if and only if (15), (17) hold.
- (vii) x,y,a,b,c, (3) holds if and only if (18) holds.
- (viii) x,a,y,b,c or a,x,y,b,c or x,a,b,c,y or a,x,b,c,y or a,x,b,y,c or x,a,b,y,c, (3) holds if and only if (19) holds.
- (ix) a,b,x,c,y or a,b,c,x,y or a,b,x,y,c, (3) holds if and only if (20) holds.
- (x) a,b,c,d,x or a,b,c,x,d or a,b,x,c,d, (3) holds if and only if (21) holds.
- (xi) a,x,b,c,d or x,a,b,c,d, (3) holds if and only if (22) holds,

where $x,y,z,w\in\mathfrak{g}$ and $a,b,c,d\in\mathfrak{h}$ in all the cases of (i)–(xi).

This completes the proof. □

Let $\mathcal{E}:0\to\mathfrak{h}\xrightarrow{i}\hat{\mathfrak{g}}\xrightarrow{p}\mathfrak{g}\to 0$ be a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$ with a section s of p. Define $\omega_s:\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{h}$, $\theta_s:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{gl}(\mathfrak{h})$, $\rho_s:\mathfrak{g}\to\mathrm{Hom}(\mathfrak{h}\otimes\mathfrak{h},\mathfrak{h})$ respectively by

$$\omega_s(x,y,z) = [s(x), s(y), s(z)]_{\hat{\mathfrak{g}}} - s[x,y,z]_{\mathfrak{g}}, \tag{30}$$

$$\theta_s(x,y)a = [a, s(x), s(y)]_{\hat{\mathfrak{g}}}, \quad \rho_s(x)(a,b) = [s(x), a, b]_{\hat{\mathfrak{g}}}. \tag{31}$$

By Eq. (31), we have

$$D_{\theta_s}(x,y)a = [s(x), s(y), a]_{\hat{\mathfrak{g}}}, \quad D_{\rho_s}(x)(a,b) = [a, b, s(x)]_{\hat{\mathfrak{g}}} \tag{32}$$

for any $x, y, z \in \mathfrak{g}, a, b \in \mathfrak{h}$.

By direct computations, we have

Proposition 3.7. *With the above notions, $(\omega_s, \theta_s, \rho_s)$ is a non-abelian 3-cocycle on $\mathfrak{g}$ with values in $\mathfrak{h}$. We call it the non-abelian 3-cocycle corresponding to the extension $\mathcal{E}$ induced by s. Naturally, $(\mathfrak{g} \oplus \mathfrak{h}, [\ ,\ ,\ ]_{(\omega_s,\theta_s,\rho_s)})$ is a Lie triple system. Denote this Lie triple system simply by $\mathfrak{g} \oplus_{(\omega_s,\theta_s,\rho_s)} \mathfrak{h}$.*

In the following, we denote $(\omega_s, \theta_s, \rho_s)$ by (ω, θ, ρ) without ambiguity.

Remark 3.8. When $\mathcal{E}$ is an abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$, the map ρ_s given by (31) becomes to zero. Then $(\mathfrak{h}, \theta_s)$ is a representation of $\mathfrak{g}$ [Zha14]. Moreover, ω_s is a 3-cocycle of $\mathfrak{g}$ with coefficients in the representation $(\mathfrak{h}, \theta_s)$. We call it the 3-cocycle corresponding to the abelian extension $\mathcal{E}$ induced by s. We always denote ω_s simply by ω if there is no confusion.

Lemma 3.9. *Let $\mathcal{E} : 0 \to \mathfrak{h} \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0$ be a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$ with different sections s_1, s_2 of p. Assume that $(\omega_i, \theta_i, \rho_i)$ is the non-abelian 3-cocycle corresponding to the extension $\mathcal{E}$ induced by s_i (i=1,2). Then $(\omega_1, \theta_1, \rho_1)$ and $(\omega_2, \theta_2, \rho_2)$ are equivalent, that is, the equivalent classes of non-abelian 3-cocycles corresponding to a non-abelian extension induced by a section are independent on the choice of sections.*

Proof. Let $\hat{\mathfrak{g}}$ be a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$. Assume that s_1 and s_2 are two different sections of p, $(\omega_1, \theta_1, \rho_1)$ and $(\omega_2, \theta_2, \rho_2)$ are the corresponding non-abelian 3-cocycles. Define a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ by $\varphi(x) = s_2(x) - s_1(x)$. Since $p\varphi(x) = ps_2(x) - ps_1(x) = 0$, φ is well-defined. By Eqs. (30)–(32), we get

$$\begin{aligned}
&w_1(x,y,z) = [s_1(x), s_1(y), s_1(z)]_{\hat{\mathfrak{g}}} - s_1[x,y,z]_{\mathfrak{g}} \\
&= [s_2(x) - \varphi(x), s_2(y) - \varphi(y), s_2(z) - \varphi(z)]_{\hat{\mathfrak{g}}} - (s_2[x,y,z]_{\mathfrak{g}} - \varphi([x,y,z]_{\mathfrak{g}})) \\
&= [s_2(x), s_2(y), s_2(z)]_{\hat{\mathfrak{g}}} - [s_2(x), \varphi(y), s_2(z)]_{\hat{\mathfrak{g}}} - [s_2(x), s_2(y), \varphi(z)]_{\hat{\mathfrak{g}}} \\
&\quad + [s_2(x), \varphi(y), \varphi(z)]_{\hat{\mathfrak{g}}} - [\varphi(x), s_2(y), s_2(z)]_{\hat{\mathfrak{g}}} + [\varphi(x), \varphi(y), s_2(z)]_{\hat{\mathfrak{g}}} \\
&\quad + [\varphi(x), s_2(y), \varphi(z)]_{\hat{\mathfrak{g}}} - [\varphi(x), \varphi(y), \varphi(z)]_{\hat{\mathfrak{g}}} - s_2[x,y,z]_{\mathfrak{g}} + \varphi([x,y,z]_{\mathfrak{g}}) \\
&= w_2(x,y,z) + \theta_2(x,z)\varphi(y) - D_{\theta_2}(x,y)\varphi(z) + \rho_2(x)(\varphi(y), \varphi(z)) - \theta_2(y,z)\varphi(x) \\
&\quad + D_{\rho_2}(z)(\varphi(x), \varphi(y)) - \rho_2(y)(\varphi(x), \varphi(z)) - [\varphi(x), \varphi(y), \varphi(z)]_{\hat{\mathfrak{g}}} + \varphi[x,y,z]_{\mathfrak{g}},
\end{aligned}$$

which yields that Eq. (24) holds. Similarly, Eqs. (25) and (26) hold. This finishes the proof. $\square$

According to Proposition 3.6 and Proposition 3.7, given a non-abelian extension $\mathcal{E}: 0 \to \mathfrak{h} \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0$ of $\mathfrak{g}$ by $\mathfrak{h}$ with a section s of p, we have a non-abelian 3-cocycle $(\omega_s, \theta_s, \rho_s)$ and a Lie triple system $\mathfrak{g} \oplus_{(\omega_s,\theta_s,\rho_s)} \mathfrak{h}$, which yields that

$$\mathcal{E}_{(\omega_s,\theta_s,\rho_s)}: 0 \longrightarrow \mathfrak{h} \xrightarrow{i} \mathfrak{g} \oplus_{(\omega_s,\theta_s,\rho_s)} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

is a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$. Since any element $\widehat{w} \in \widehat{\mathfrak{g}}$ can be written as $\widehat{w} = a + s(x)$ with $a \in \mathfrak{h}, x \in \mathfrak{g}$, define a linear map

$$f: \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \oplus_{(\omega_s,\theta_s,\rho_s)} \mathfrak{h},\ f(\widehat{w}) = f(a + s(x)) = a + x.$$

It is easy to check that f is an isomorphism of Lie triple systems such that the following commutative diagram holds:

$$\begin{tikzcd}
\mathcal{E}: 0 \arrow[r] & \mathfrak{h} \arrow[r, "i"] \arrow[d, equal] & \widehat{\mathfrak{g}} \arrow[r, "p"] \arrow[d, "f"'] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{E}_{(\omega_s)}: 0 \arrow[r] & \mathfrak{h} \arrow[r, "i"] & \mathfrak{g} \oplus_{(\omega_s,\theta_s,\rho_s)} \mathfrak{h} \arrow[r, "\pi"] & \mathfrak{g} \arrow[r] & 0,
\end{tikzcd}$$

which indicates that the non-abelian extensions $\mathcal{E}$ and $\mathcal{E}_{(\omega_s,\theta_s,\rho_s)}$ of $\mathfrak{g}$ by $\mathfrak{h}$ are equivalent. On the other hand, if (ω, θ, ρ) is a non-abelian 3-cocycle on $\mathfrak{g}$ with values in $\mathfrak{h}$, there is a Lie triple system $\mathfrak{g} \oplus_{(\omega,\theta,\rho)} \mathfrak{h}$, which yields the following non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$:

$$\mathcal{E}_{(\omega,\theta,\rho)}: 0 \longrightarrow \mathfrak{h} \xrightarrow{i} \mathfrak{g} \oplus_{(\omega,\theta,\rho)} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0,$$

where i is the inclusion and π is the projection.

In the sequel, we characterize the relationship between non-abelian cohomology groups and non-abelian extensions in the following.

Proposition 3.10. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie triple systems. Then the equivalent classes of non-abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$ are classified by the non-abelian cohomology group, that is, $\mathcal{E}_{nab}(\mathfrak{g}, \mathfrak{h}) \simeq H^3_{nab}(\mathfrak{g}, \mathfrak{h})$.*

Proof. Define a linear map

$$\Theta : \mathcal{E}_{nab}(\mathfrak{g}, \mathfrak{h}) \longrightarrow H^3_{nab}(\mathfrak{g}, \mathfrak{h}),$$

where Θ assigns an equivalent class of non-abelian extensions to the class of corresponding non-abelian 3-cocycles. First, we check that Θ is well-defined. Assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are two equivalent non-abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$ via the linear map f, that is, the commutative diagram (10) holds. Let $s_1 : \mathfrak{g} \to \widehat{\mathfrak{g}}_1$ be a section of p_1. Then $p_2 f s_1 = p_1 s_1 = I_{\mathfrak{g}}$, which follows that $s_2 = f s_1$ is a section of p_2. Let $(\omega_1, \theta_1, \rho_1)$ and $(\omega_2, \theta_2, \rho_2)$ be two non-abelian 3-cocycles induced by the sections s_1, s_2 respectively. Then we have,

$$\begin{aligned}
\theta_1(x, y)a = f(\theta_1(x, y)a) &= f([a, s_1(x), s_1(y)]_{\hat{\mathfrak{g}}_1}) \\
&= [f(a), f s_1(x), f s_1(y)]_{\hat{\mathfrak{g}}_2} \\
&= [a, s_2(x), s_2(y)]_{\hat{\mathfrak{g}}_2} \\
&= \theta_2(x, y)a.
\end{aligned}$$

By the same token, we have

$$\omega_1(x, y, z) = \omega_2(x, y, z), \quad \rho_1(x)(a, b) = \rho_2(x)(a, b).$$

Thus, $(\omega_1,\theta_1,\rho_1)=(\omega_2,\theta_2,\rho_2)$, which means that Θ is well-defined.

Next, we verify that Θ is injective. Indeed, suppose that $\Theta([\mathcal{E}_1])=[(\omega_1,\theta_1,\rho_1)]$ and $\Theta([\mathcal{E}_2])=[(\omega_2,\theta_2,\rho_2)]$. If the equivalent classes $[(\omega_1,\theta_1,\rho_1)]=[(\omega_2,\theta_2,\rho_2)]$, we obtain that the non-abelian 3-cocycles $(\omega_1,\theta_1,\rho_1)$ and $(\omega_2,\theta_2,\rho_2)$ are equivalent via the linear map $\varphi:\mathfrak{g}\to\mathfrak{h}$, satisfying Eqs. (24)–(26). Define a linear map $f:\mathfrak{g}\oplus_{(\omega_1,\theta_1,\rho_1)}\mathfrak{h}\to\mathfrak{g}\oplus_{(\omega_2,\theta_2,\rho_2)}\mathfrak{h}$ by

$$f(x+a)=x-\varphi(x)+a,\quad \forall\, x\in\mathfrak{g}, a\in\mathfrak{h}.$$

According to Eq. (29), we get for all $x,y,z\in\mathfrak{g}, a,b,c\in\mathfrak{h}$,

$$\begin{aligned}
&f([x+a,y+b,z+c]_{(\omega_1,\theta_1,\rho_1)})\\
&\quad=f([x,y,z]_{\mathfrak{g}}+\omega_1(x,y,z)+D_{\theta_1}(x,y)c+\theta_1(y,z)a-\theta_1(x,z)b\\
&\qquad+D_{\rho_1}(z)(a,b)+\rho_1(x)(b,c)-\rho_1(y)(a,c)+[a,b,c]_{\mathfrak{h}})\\
&\quad=[x,y,z]_{\mathfrak{g}}-\varphi([x,y,z]_{\mathfrak{g}})+\omega_1(x,y,z)+D_{\theta_1}(x,y)c+\theta_1(y,z)a-\theta_1(x,z)b\\
&\qquad+D_{\rho_1}(z)(a,b)+\rho_1(x)(b,c)-\rho_1(y)(a,c)+[a,b,c]_{\mathfrak{h}},
\end{aligned}$$

and

$$\begin{aligned}
&[f(x+a),f(y+b),f(z+c)]_{(\omega_2,\theta_2,\rho_2)}\\
&=[x-\varphi(x)+a,y-\varphi(y)+b,z-\varphi(z)+c]_{(\omega_2,\theta_2,\rho_2)}\\
&=[x,y,z]_{\mathfrak{g}}+\omega_2(x,y,z)+D_{\theta_2}(x,y)(c-\varphi(z))+\theta_2(y,z)(a-\varphi(x))-\theta_2(x,z)(b-\varphi(y))\\
&\quad+T_2(z)(a-\varphi(x),b-\varphi(y))+\rho_2(x)(b-\varphi(y),c-\varphi(z))-\rho_2(y)(a-\varphi(x),c-\varphi(z))\\
&\quad+[a-\varphi(x),b-\varphi(y),c-\varphi(z)]_{\mathfrak{h}})\\
&=[x,y,z]_{\mathfrak{g}}+\omega_2(x,y,z)+D_{\theta_2}(x,y)(c-\varphi(z))+\theta_2(y,z)(a-\varphi(x))-\theta_2(x,z)(b-\varphi(y))\\
&\quad+D_{\rho_2}(z)(\varphi(x),\varphi(y))-D_{\rho_2}(z)(a,\varphi(y))-D_{\rho_2}(z)(\varphi(x),b)+D_{\rho_2}(z)(a,b)\\
&\quad+\rho_2(x)(\varphi(y),\varphi(z))-\rho_2(x)(\varphi(y),c)-\rho_2(x)(b,\varphi(z))+\rho_2(x)(b,c)-\rho_2(y)(\varphi(x),\varphi(z))\\
&\quad+\rho_2(y)(\varphi(x),c)+\rho_2(y)(a,\varphi(z))-\rho_2(y)(a,c)-[\varphi(x),\varphi(y),\varphi(z)]_{\mathfrak{h}}+[\varphi(x),\varphi(y),c]_{\mathfrak{h}}\\
&\quad+[\varphi(x),b,\varphi(z)]_{\mathfrak{h}}-[\varphi(x),b,c]_{\mathfrak{h}}+[a,\varphi(y),\varphi(z)]_{\mathfrak{h}}-[a,b,\varphi(z)]_{\mathfrak{h}}-[a,\varphi(y),c]_{\mathfrak{h}}+[a,b,c]_{\mathfrak{h}}.
\end{aligned}$$

In view of Eqs. (24)–(28), we have

$$f([x+a,y+b,z+c]_{(\omega_1,\theta_1,\rho_1)})=[f(x+a),f(y+b),f(z+c)]_{(\omega_2,\theta_2,\rho_2)}.$$

Hence, f is an isomorphism of Lie triple systems. Clearly, the following commutative diagram holds:

$$\begin{tikzcd}
\mathcal{E}_{(\omega_1,\theta_1,\rho_1)}:0 \arrow[r] & \mathfrak{h} \arrow[r, "i"] \arrow[d, equal] & \mathfrak{g}\oplus_{(\omega_1,\theta_1,\rho_1)}\mathfrak{h} \arrow[r, "\pi"] \arrow[d, "f"'] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0\\
\mathcal{E}_{(\omega_2,\theta_2,\rho_2)}:0 \arrow[r] & \mathfrak{h} \arrow[r, "i"] & \mathfrak{g}\oplus_{(\omega_2,\theta_2,\rho_2)}\mathfrak{h} \arrow[r, "\pi"] & \mathfrak{g} \arrow[r] & 0.
\end{tikzcd} \tag{33}$$

Thus $\mathcal{E}_{(\omega_1,\theta_1,\rho_1)}$ and $\mathcal{E}_{(\omega_2,\theta_2,\rho_2)}$ are equivalent non-abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$, which means that $[\mathcal{E}_{(\omega_1,\theta_1,\rho_1)}]=[\mathcal{E}_{(\omega_2,\theta_2,\rho_2)}]$. Thus, Θ is injective.

Finally, we claim that Θ is surjective. For any equivalent class of non-abelian 3-cocycles $[(\omega)]$, by Proposition 3.6, there is a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$:

$$\mathcal{E}_{(\omega)}:0\longrightarrow\mathfrak{h}\xrightarrow{i}\mathfrak{g}\oplus_{(\omega,\theta,\rho)}\mathfrak{h}\xrightarrow{\pi}\mathfrak{g}\longrightarrow 0.$$

Therefore, $\Theta([\mathcal{E}_{(\omega,\theta,\rho)}])=[(\omega,\theta,\rho)]$, which follows that Θ is surjective. In all, Θ is bijective. The proof is completed. $\square$

4. Non-abelian extensions in terms of Maurer Cartan elements

In this section, we classify non-abelian extensions using Maurer Cartan elements. We start with recalling the Maurer–Cartan elements.

Let $(L=\bigoplus_i L_i, [\ ,\ ], d)$ be a differential graded Lie algebra. The set $\mathrm{MC}(L)$ of Maurer–Cartan elements of $(L, [\ ,\ ], d)$ is defined by

$$\mathrm{MC}(L)=\left\{\eta\in L_1 \,\middle|\, d\eta+\frac{1}{2}[\eta,\eta]=0\right\}.$$

Moreover, $\eta_0,\eta_1\in \mathrm{MC}(L)$ are called gauge equivalent if and only if there exists an element $\varphi\in L_0$ such that

$$\eta_1=e^{ad_\varphi}\eta_0-\frac{e^{ad_\varphi}-1}{ad_\varphi}d\varphi.$$

Let $\mathfrak{g}$ be a vector space. Consider the graded vector space $C^*(\mathfrak{g},\mathfrak{g})=\bigoplus_{n\geq 0}C^n(\mathfrak{g},\mathfrak{g})$, where $C^n(\mathfrak{g},\mathfrak{g})$ is the set of linear maps $f\in \mathrm{Hom}(L\otimes\cdots\otimes L\otimes\mathfrak{g},\mathfrak{g})$, where $L=\mathfrak{g}\otimes\mathfrak{g}$, satisfying Eqs. (8) and (9) for all $X_i\in L$ $(i=1,2,\dots,n-1)$, $x,y,z\in\mathfrak{g}$. The degree of elements in $C^n(\mathfrak{g},\mathfrak{g})$ is n. Define

$$[f,g]_{Lts}=(-1)^{mn}i_f(g)-i_g(f),\quad \forall\, f\in C^n(\mathfrak{g},\mathfrak{g}),\ g\in C^m(\mathfrak{g},\mathfrak{g}), \tag{34}$$

where $i_f(g)\in C^{n+m}(\mathfrak{g},\mathfrak{g})$ is given by

$$i_f(g)=\sum_{k=1}^{n+1}(-1)^{m(k-1)}\sum_{\sigma\in sh(k-1,m)}(-1)^\sigma f\circ_k^\sigma g, \tag{35}$$

where σ is a permutation in $(k-1,m)$-shuffle and $f\circ_k^\sigma g$ is defined for $k=n+1$ by

$$f\circ_{n+1}^\sigma g(X_1,X_2,\dots,X_{n+m},z)=f(X_{\sigma(1)},\dots,X_{\sigma(n)},g(X_{\sigma(n+1)},\dots,X_{\sigma(n+m)},z)) \tag{36}$$

and for $1\leq k\leq n$ by

$$\begin{aligned}
&f\circ_k^\sigma g(X_1,\dots,X_{n+m},z)\\
&=f(X_{\sigma(1)},\dots,X_{\sigma(k-1)},g(X_{\sigma(k)},\dots,X_{\sigma(n+m-1)},x_{k+m}),y_{k+m},X_{k+m+1},\dots,X_{m+n},z)\\
&\quad+f(X_{\sigma(1)},\dots,X_{\sigma(k-1)},x_{k+m},g(X_{\sigma(k)},\dots,X_{\sigma(k+m-1)},y_{k+m}),X_{k+m+1},\dots,X_{m+n},z)
\end{aligned} \tag{37}$$

for all $X_i\in L$ $(i=1,2,\dots,n+m)$ and $z\in\mathfrak{g}$.

Proposition 4.1 ([Cht+23]). *With the above notations, $(C^*(\mathfrak{g},\mathfrak{g}),[\ ,\ ]_{Lts})$ is a graded Lie algebra. $\pi\in C^1(\mathfrak{g},\mathfrak{g})=\mathrm{Hom}(\wedge^2\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g})$ defines a Lie triple system structure on $\mathfrak{g}$ if and only if $[\ ,\ ]_{Lts}=0$, i.e., π is a Maurer–Cartan element of the graded Lie algebra $(C^*(\mathfrak{g},\mathfrak{g}),[\ ,\ ]_{Lts})$. Furthermore, $(C^*(\mathfrak{g},\mathfrak{g}),[\ ,\ ]_{Lts},d_\pi)$ is a differential graded Lie algebra, where d_π is given by*

$$d_\pi(f)=(-1)^{n-1}[\pi,f]_{Lts},\quad \forall\, f\in C^{n-1}(\mathfrak{g},\mathfrak{g}). \tag{38}$$

Let $(\mathfrak{g},\pi_\mathfrak{g})$ and $(\mathfrak{h},\pi_\mathfrak{h})$ be two Lie triple systems. Then $(\mathfrak{g}\oplus\mathfrak{h},\pi_{\mathfrak{g}\oplus\mathfrak{h}})$ is a Lie triple system, where $\pi_{\mathfrak{g}\oplus\mathfrak{h}}$ is defined by

$$\pi_{\mathfrak{g}\oplus\mathfrak{h}}(x+a,y+b,z+c)=\pi_\mathfrak{g}(x,y,z)+\pi_\mathfrak{h}(a,b,c)$$

for all $x,y,z\in\mathfrak{g},a,b,c\in\mathfrak{h}$.

In view of Proposition 4.1, $(C^*(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{g}\oplus\mathfrak{h}),[\ ,\ ]_{Lts},d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}})$ is a differential graded Lie algebra. Define $C^n_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})\subset C^n(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$ by

$$C^n(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})=C^n_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})\oplus C^n(\mathfrak{h},\mathfrak{h}).$$

Denote by $C_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})=\bigoplus_n C^n_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$.

Similar to the case of 3-Lie algebras [SMT18], we have

Proposition 4.2. *With the above notations, $(C_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}),[\ ,\ ]_{Lts},d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}})$ is a differential graded Lie subalgebra of $(C^*(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{g}\oplus\mathfrak{h}),[\ ,\ ]_{Lts},d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}})$. Moreover, its degree 0 part $C^0_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})=\mathrm{Hom}(\mathfrak{h},\mathfrak{h})$ is abelian.*

Proposition 4.3. *The following conditions are equivalent:*

(i) *$(\mathfrak{g}\oplus\mathfrak{h},[\ ,\ ,\ ]_{(\omega,\theta,\rho)})$ is a Lie triple system, which is a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$;*

(ii) *ϖ is a Maurer–Cartan element of the differential graded Lie algebra $(C_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}),[\ ,\ ]_{Lts},d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}})$, where*

$$\begin{aligned}\varpi(x+a,y+b,z+c)=\omega(x,y,z)+D_\theta(x,y)c+\theta(y,z)a-\theta(x,z)b\\+D_\rho(z)(a,b)+\rho(x)(b,c)-\rho(y)(a,c),\end{aligned}$$

for all $x,y,z\in\mathfrak{g},a,b,c\in\mathfrak{h}$.

Proof. According to the definition of Maurer–Cartan element, ϖ is a Maurer–Cartan element of the differential graded Lie algebra $(C_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}),[\ ,\ ]_{Lts},d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}})$ if and only if

$$d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varpi+\frac{1}{2}[\varpi,\varpi]_{Lts}=0.$$

Thus,

$$d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varpi+\frac{1}{2}[\varpi,\varpi]_{Lts}=[\pi_{\mathfrak{g}\oplus\mathfrak{h}},\varpi]_{Lts}+\frac{1}{2}[\varpi,\varpi]_{Lts}=\frac{1}{2}[\pi_{\mathfrak{g}\oplus\mathfrak{h}}+\varpi,\pi_{\mathfrak{g}\oplus\mathfrak{h}}+\varpi]=0,$$

which implies that $\pi_{\mathfrak{g}\oplus\mathfrak{h}}+\varpi$ is a Maurer–Cartan element of the differential graded Lie algebra $(C^*(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{g}\oplus\mathfrak{h}),[\ ,\ ]_{Lts},d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}})$. Combining Proposition 4.1, $(\mathfrak{g}\oplus\mathfrak{h},[\ ,\ ,\ ]_{(\omega,\theta,\rho)})$ is a Lie triple system and ϖ is a Maurer–Cartan element of $(C_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}),[\ ,\ ]_{Lts},d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}})$. □

Proposition 4.4. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie triple systems. Then the equivalent classes of non-abelian extensions $\mathfrak{g}$ by $\mathfrak{h}$ are one-to-one correspond to the gauge equivalence classes of Maurer–Cartan elements in the differential graded Lie algebra $(C_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}),[\ ,\ ]_{Lts},d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}})$.*

Proof. Let ϖ,ϖ' be two Maurer–Cartan elements of the differential graded Lie algebra $(C_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}),[\ ,\ ]_{Lts},d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}})$, and ϖ,ϖ' are gauge equivalent, there is a linear map $\varphi \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ such that

$$\begin{aligned}
\overline{\omega}' &= e^{ad_\varphi}\overline{\omega} - \frac{e^{ad_\varphi}-1}{ad_\varphi} d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varphi \\
&= \left(id + ad_\varphi + \frac{1}{2!}ad_\varphi^2 + \cdots + + \frac{1}{n!}ad_\varphi^n + \dots \right)\overline{\omega} \\
&\quad - \left(id + ad_\varphi + \frac{1}{2!}ad_\varphi + \cdots + \frac{1}{n!}ad_\varphi^{n-1} + \dots \right) d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varphi.
\end{aligned}$$

By Eqs. (29), (34)–(38), we obtain for all $w_i = x_i + a_i \in \mathfrak{g}\oplus\mathfrak{h}$ $(i = 1, 2, 3)$,

$$\begin{aligned}
ad_\varphi\overline{\omega} &= [\varphi, \overline{\omega}]_{Lts}(w_1, w_2, w_3) \\
&= -\overline{\omega}(\varphi(w_1), w_2, w_3) - \overline{\omega}(w_1, \varphi(w_2), w_3) - \overline{\omega}(w_1, w_2, \varphi(w_3)) \\
&= -\theta(x_2, x_3)\varphi(x_1) - D_\rho(x_3)(\varphi(x_1), a_2) + \rho(x_2)(\varphi(x_1), a_3) \\
&\quad + \theta(x_1, x_3)\varphi(x_2) - D_\rho(x_3)(a_1, \varphi(x_2)) - \rho(x_1)(\varphi(x_2), a_3) \\
&\quad - D_\theta(x_1, x_2)\varphi(x_3) - \rho(x_1)(a_2, \varphi(x_3)) + \rho(x_2)(a_1, \varphi(x_3)),
\end{aligned}$$

$$\begin{aligned}
ad_\varphi^2\overline{\omega} &= [\varphi, [\varphi, \overline{\omega}]_{Lts}]_{Lts}(w_1, w_2, w_3) \\
&= -[\varphi, \overline{\omega}]_{Lts}(\varphi(w_1), w_2, w_3) - [\varphi, \overline{\omega}]_{Lts}(w_1, \varphi(w_2), w_3) - [\varphi, \overline{\omega}]_{Lts}(w_1, w_2, \varphi(w_3)) \\
&= D_\rho(x_3)(\varphi(x_1), \varphi(x_2)) - \rho(x_2)(\varphi(x_1), \varphi(x_3)) + D_\rho(x_3)(\varphi(x_1), \varphi(x_2)) \\
&\quad + \rho(x_1)(\varphi(x_2), \varphi(x_3)) - \rho(x_2)(\varphi(x_1), \varphi(x_3)) + \rho(x_1)(\varphi(x_2), \varphi(x_3)) \\
&= 2D_\rho(x_3)(\varphi(x_1), \varphi(x_2)) - 2\rho(x_2)(\varphi(x_1), \varphi(x_3)) + 2\rho(x_1)(\varphi(x_2), \varphi(x_3)),
\end{aligned}$$

$$\begin{aligned}
d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varphi(w_1, w_2, w_3) &= [\pi_{\mathfrak{g}\oplus\mathfrak{h}}, \varphi]_{Lts}(w_1, w_2, w_3) \\
&= \pi_{\mathfrak{g}\oplus\mathfrak{h}}(\varphi(w_1), w_2, w_3) + \pi_{\mathfrak{g}\oplus\mathfrak{h}}(w_1, \varphi(w_2), w_3) \\
&\quad + \pi_{\mathfrak{g}\oplus\mathfrak{h}}(w_1, w_2, \varphi(w_3)) - \varphi(\pi_{\mathfrak{g}\oplus\mathfrak{h}}(w_1, w_2, w_3) \\
&= [\varphi(x_1), a_2, a_3]_{\mathfrak{h}} + [a_1, \varphi(x_2), a_3]_{\mathfrak{h}} + [a_1, a_2, \varphi(x_3)]_{\mathfrak{h}} - \varphi([x_1, x_2, x_3]_{\mathfrak{g}}),
\end{aligned}$$

$$\begin{aligned}
ad_\varphi(d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varphi) &= [\varphi, d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varphi]_{Lts}(w_1, w_2, w_3) \\
&= -d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varphi(\varphi(w_1), w_2, w_3) - d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varphi(w_1, \varphi(w_2), w_3) - d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varphi(w_1, w_2, \varphi(w_3)) \\
&= [\varphi(x_1), \varphi(x_2), a_3]_{\mathfrak{h}} + [\varphi(x_1), a_2\varphi(x_3)]_{\mathfrak{h}} + [\varphi(x_1), \varphi(x_2), a_3]_{\mathfrak{h}} \\
&\quad + [a_1, \varphi(x_2), \varphi(x_3)]_{\mathfrak{h}} - [\varphi(x_1), a_2, \varphi(x_3)]_{\mathfrak{h}} + [a_1, \varphi(x_2), \varphi(x_3)]_{\mathfrak{h}} \\
&= 2[a_1, \varphi(x_2), \varphi(x_3)]_{\mathfrak{h}} + 2[\varphi(x_1), a_2, \varphi(x_3)]_{\mathfrak{h}} + 2[\varphi(x_1), \varphi(x_2), a_3]_{\mathfrak{h}},
\end{aligned}$$

$$\begin{aligned}
ad_\varphi^2(d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varphi) &= [\varphi, [\varphi, d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}]_{Lts}]_{Lts}(w_1, w_2, w_3) \\
&= -[\varphi, d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}]_{Lts}(\varphi(x_1), w_2, w_3) - [\varphi, d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}]_{Lts}(w_1, \varphi(x_2), w_3) \\
&\quad - [\varphi, d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}]_{Lts}(\varphi(x_1), w_2, \varphi(x_3)) \\
&= 6[\varphi(x_1), \varphi(x_2), \varphi(x_3)]_{\mathfrak{h}},
\end{aligned}$$

and

$$ad_\varphi^n\overline{\omega} = 0, \quad ad_\varphi^n(d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\overline{\omega}) = 0, \quad \forall\, n \geq 3.$$

Therefore, Maurer–Cartan elements ϖ' and ϖ are gauge equivalent if and only if

$$\begin{aligned}\varpi' = \varpi + [\varphi, \pi]_{Lts} + \frac{1}{2!}[\varphi, [\varphi, \pi]_{Lts}]_{Lts}\\ - \left(d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varphi + \frac{1}{2!}[\varphi, d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varphi]_{Lts} + \frac{1}{3!}[\varphi, [\varphi, d_{\pi_{\mathfrak{g}\oplus\mathfrak{h}}}\varphi]_{Lts}]_{Lts}\right).\end{aligned}$$

Hence, two Maurer–Cartan elements ϖ and ϖ' are equivalent if and only if Eqs. (24)–(26) hold. The proof is completed. □

5. Inducibility of pairs of Lie triple system automorphisms

In this section, we study inducibility of pairs of Lie triple system automorphisms and characterize them by equivalent conditions.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie triple systems, and

$$0 \longrightarrow \mathfrak{h} \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \longrightarrow 0,$$

be a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$ with a section s of p. For any automorphism $\gamma \in \mathrm{Aut}(\widehat{\mathfrak{g}})$, then $\gamma \in \mathrm{Aut}(\mathfrak{h})$ if and only if $\gamma(\mathfrak{h}) \subseteq \mathfrak{h}$. Denote $\mathrm{Aut}_{\mathfrak{h}}(\widehat{\mathfrak{g}}) = \{\gamma \in \mathrm{Aut}(\widehat{\mathfrak{g}}) \mid \gamma(\mathfrak{h}) = \mathfrak{h}\}$. We define a linear map $\bar{\gamma} : \mathfrak{g} \to \mathfrak{g}$ by

$$\bar{\gamma}(x) = p\gamma s(x), \quad \forall\, x \in \mathfrak{g}.$$

Assume that s_1 and s_2 are two distinct sections of p, since $ps_1(x) - ps_2(x) = 0$, $s_1(x) - s_2(x) \in \mathrm{Ker}\, p \cong \mathfrak{h}$, which indicates that $\gamma(s_1(x) - s_2(x)) \in \mathfrak{h}$. Thus, $p\gamma s_1(x) = p\gamma s_2(x)$, which yields that $\bar{\gamma}$ is independent on the choice of sections.

For all $x, y, z \in \mathfrak{g}$, since $p|_{\mathfrak{h}} = 0$,

$$\begin{aligned}\bar{\gamma}([x, y, z]_{\mathfrak{g}}) &= p\gamma(s[x, y, z]_{\mathfrak{g}})\\ &= p\gamma([s(x), s(y), s(z)]_{\widehat{\mathfrak{g}}} - \omega(x, y, z))\\ &= p\gamma([s(x), s(y), s(z)]_{\widehat{\mathfrak{g}}})\\ &= [p\gamma s(x), p\gamma s(y), p\gamma s(z)]_{\mathfrak{g}}\\ &= [\bar{\gamma}(x), \bar{\gamma}(y), \bar{\gamma}(z)]_{\mathfrak{g}}),\end{aligned}$$

which yields that $\bar{\gamma}$ is a homomorphism of Lie triple systems. It is easy to check that $\bar{\gamma}$ is bijective. Thus, $\bar{\gamma} \in \mathrm{Aut}(\mathfrak{g})$. Then we can define a group homomorphism

$$\lambda : \mathrm{Aut}_{\mathfrak{h}}(\widehat{\mathfrak{g}}) \longrightarrow \mathrm{Aut}(\mathfrak{g}) \times \mathrm{Aut}(\mathfrak{h}), \quad \lambda(\gamma) = (\bar{\gamma}, \gamma|_{\mathfrak{h}}).$$

Definition 5.1. A pair $(\alpha, \beta) \in \mathrm{Aut}(\mathfrak{g}) \times \mathrm{Aut}(\mathfrak{h})$ is said to be inducible if (α, β) is an image of λ.

In the following, we investigate when is a pair (α, β) inducible.

Theorem 5.2. *Let $0 \to \mathfrak{h} \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0$ be a non-abelian extension of the Lie triple system $\mathfrak{g}$ by $\mathfrak{h}$ with a section s of p and (ω, θ, ρ) be the corresponding non-abelian 3-cocycle induced by s. A pair $(\alpha, \beta) \in \mathrm{Aut}(\mathfrak{g}) \times \mathrm{Aut}(\mathfrak{h})$ is inducible if and only if* *there is a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ satisfying the following conditions:*

$$\begin{aligned}&\beta(\theta(x,y)a) - \theta(\alpha(x),\alpha(y))\beta(a)\\&\qquad = [\beta(a),\varphi(x),\varphi(y)]_{\mathfrak{h}} - D_\rho(\alpha(y))(\beta(a),\varphi(x)) + \rho(\alpha(x))(\beta(a),\varphi(y)),\end{aligned} \tag{39}$$

$$\beta(\rho(x)(a,b)) - \rho(\alpha(x))(\beta(a),\beta(b)) = [\beta(a),\varphi(x),\beta(b)]_{\mathfrak{h}}, \tag{40}$$

$$\begin{aligned}&\beta\omega(x,y,z) - \omega(\alpha(x),\alpha(y),\alpha(z))\\&\quad = D_\rho(\alpha(z))(\varphi(x),\varphi(y)) - \rho(\alpha(y))(\varphi(x),\varphi(z)) - \theta(\alpha(y),\alpha(z))\varphi(x)\\&\qquad + \rho(\alpha(x))(\varphi(y),\varphi(z)) + \theta(\alpha(x),\alpha(z))\varphi(y) - D_\theta(\alpha(x),\alpha(y))\varphi(z)\\&\qquad + \varphi([x,y,z]_{\mathfrak{g}}) - [\varphi(x),\varphi(y),\varphi(z)]_{\mathfrak{h}},\end{aligned} \tag{41}$$

for all $x,y,z \in \mathfrak{g}$ and $a,b \in \mathfrak{h}$.

Proof. Suppose that $(\alpha,\beta) \in \mathrm{Aut}(\mathfrak{g}) \times \mathrm{Aut}(\mathfrak{h})$ is inducible, then there is an automorphism $\gamma \in \mathrm{Aut}_{\mathfrak{h}}(\widehat{\mathfrak{g}})$ such that $\gamma|_{\mathfrak{h}} = \beta$ and $p\gamma s = \alpha$. For all $x \in \mathfrak{g}$, since s is a section of p, that is, $ps = I_{\mathfrak{g}}$,

$$p(\gamma s - s\alpha)(x) = \alpha(x) - \alpha(x) = 0,$$

which implies that $(\gamma s - s\alpha)(x) \in \mathrm{Ker}\, p \cong \mathfrak{h}$. So we can define a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ by

$$\varphi(x) = (s\alpha - \gamma s)(x), \quad \forall\, x \in \mathfrak{g}.$$

For $x, y \in \mathfrak{g}, a \in \mathfrak{h}$, using Eqs. (31)–(32), we get

$$\begin{aligned}&\beta(\theta(x,y)a) - \theta(\alpha(x),\alpha(y))\beta(a)\\&\quad = \beta[a,s(x),s(y)]_{\hat{\mathfrak{g}}} - [\beta(a),s\alpha(x),s\alpha(y)]_{\hat{\mathfrak{g}}}\\&\quad = [\beta(a),\beta s(x),\beta s(y)]_{\hat{\mathfrak{g}}} - [\beta(a),s\alpha(x),s\alpha(y)]_{\hat{\mathfrak{g}}}\\&\quad = [\beta(a),\beta s(x) - s\alpha(x),\beta s(y)]_{\hat{\mathfrak{g}}} + [\beta(a),s\alpha(x),\beta s(y)]_{\hat{\mathfrak{g}}} - [\beta(a),s\alpha(x),s\alpha(y)]_{\hat{\mathfrak{g}}}\\&\quad = [\beta(a),\varphi(x),\beta s(y)]_{\hat{\mathfrak{g}}} + [\beta(a),s\alpha(x),\varphi(y)]_{\hat{\mathfrak{g}}}\\&\quad = [\beta(a),\varphi(x),\beta s(y) - s\alpha(y)]_{\hat{\mathfrak{g}}} + [\beta(a),\varphi(x),s\alpha(y)]_{\hat{\mathfrak{g}}} + [\beta(a),s\alpha(x),\varphi(y)]_{\hat{\mathfrak{g}}}\\&\quad = [\beta(a),\varphi(x),\varphi(y)]_{\hat{\mathfrak{g}}} + [\beta(a),\varphi(x),s\alpha(y)]_{\hat{\mathfrak{g}}} - [s\alpha(x),\beta(a),\varphi(y)]_{\hat{\mathfrak{g}}}\\&\quad = [\beta(a),\varphi(x),\varphi(y)]_{\mathfrak{h}} + D_\rho(\alpha(y))(\beta(a),\varphi(x)) - \rho(\alpha(x))(\beta(a),\varphi(y)),\end{aligned}$$

that is, Eq. (39) holds. Similarly, Eqs. (40) and (41) hold.

Conversely, suppose that $(\alpha,\beta) \in \mathrm{Aut}(\mathfrak{g}) \times \mathrm{Aut}(\mathfrak{h})$ and there is a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ satisfying Eqs. (39)–(41). Due to s being a section of p, then all $\widehat{w} \in \widehat{\mathfrak{g}}$ can be written as $\widehat{w} = a + s(x)$ for some $a \in \mathfrak{h}, x \in \mathfrak{g}$. Define a linear map $\gamma : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ by

$$\gamma(\widehat{w}) = \gamma(a + s(x)) = \beta(a) - \varphi(x) + s\alpha(x).$$

If $\gamma(\widehat{w}) = 0$, then $s\alpha(x) = 0$ and $\beta(a) + \varphi(x) = 0$. In view of s and α being injective, we get $x = 0$, it follows that $a = 0$. Thus, $\widehat{w} = a + s(x) = 0$, that is γ is injective. For any $\widehat{w} = a + s(x) \in \widehat{\mathfrak{g}}$,

$$\gamma(\beta^{-1}(a) + \beta^{-1}\varphi\alpha^{-1}(x) + s\alpha^{-1}(x)) = a + s(x) = \widehat{w},$$

which yields that γ is surjective. In all, γ is bijective.

Next we show that γ is a homomorphism of Lie triple systems. In fact, for all $\widehat{w}_i = a_i + s(x_i) \in \widehat{\mathfrak{g}}$ $(i = 1, 2, 3)$,

$$
\begin{aligned}
&[\gamma(\widehat{w}_1), \gamma(\widehat{w}_2), \gamma(\widehat{w}_3)]_{\widehat{\mathfrak{g}}} \\
&= [\beta(a_1) - \varphi(x_1) + s\alpha(x_1), \beta(a_2) - \varphi(x_2) + s\alpha(x_2), \beta(a_3) - \varphi(x_3) + s\alpha(x_3)]_{\widehat{\mathfrak{g}}} \\
&= [\beta(a_1), \beta(a_2), \beta(a_3)]_{\widehat{\mathfrak{g}}} - [\beta(a_1), \beta(a_2), \varphi(x_3)]_{\widehat{\mathfrak{g}}} + [\beta(a_1), \beta(a_2), s\alpha(x_3)]_{\widehat{\mathfrak{g}}} \\
&\quad - [\beta(a_1), \varphi(x_2), \beta(a_3)]_{\widehat{\mathfrak{g}}} + [\beta(a_1), \varphi(x_2), \varphi(x_3)]_{\widehat{\mathfrak{g}}} - [\beta(a_1), \varphi(x_2), s\alpha(x_3)]_{\widehat{\mathfrak{g}}} \\
&\quad + [\beta(a_1), s\alpha(x_2), \beta(a_3)]_{\widehat{\mathfrak{g}}} - [\beta(a_1), s\alpha(x_2), \varphi(x_3)]_{\widehat{\mathfrak{g}}} + [\beta(a_1), s\alpha(x_2), s\alpha(x_3)]_{\widehat{\mathfrak{g}}} \\
&\quad - [\varphi(x_1), \beta(a_2), \beta(a_3)]_{\widehat{\mathfrak{g}}} + [\varphi(x_1), \beta(a_2), \varphi(x_3)]_{\widehat{\mathfrak{g}}} - [\varphi(x_1), \beta(a_2), s\alpha(x_3)]_{\widehat{\mathfrak{g}}} \\
&\quad + [\varphi(x_1), \varphi(x_2), \beta(a_3)]_{\widehat{\mathfrak{g}}} - [\varphi(x_1), \varphi(x_2), \varphi(x_3)]_{\widehat{\mathfrak{g}}} + [\varphi(x_1), \varphi(x_2), s\alpha(x_3)]_{\widehat{\mathfrak{g}}} \\
&\quad - [\varphi(x_1), s\alpha(x_2), \beta(a_3)]_{\widehat{\mathfrak{g}}} + [\varphi(x_1), s\alpha(x_2), \varphi(x_3)]_{\widehat{\mathfrak{g}}} - [\varphi(x_1), s\alpha(x_2), s\alpha(x_3)]_{\widehat{\mathfrak{g}}} \\
&\quad + [s\alpha(x_1), \beta(a_2), \beta(a_3)]_{\widehat{\mathfrak{g}}} - [s\alpha(x_1), \beta(a_2), \varphi(x_3)]_{\widehat{\mathfrak{g}}} + [s\alpha(x_1), \beta(a_2), s\alpha(x_3)]_{\widehat{\mathfrak{g}}} \\
&\quad - [s\alpha(x_1), \varphi(x_2), \beta(a_3)]_{\widehat{\mathfrak{g}}} + [s\alpha(x_1), \varphi(x_2), \varphi(x_3)]_{\widehat{\mathfrak{g}}} - [s\alpha(x_1), \varphi(x_2), s\alpha(x_3)]_{\widehat{\mathfrak{g}}} \\
&\quad + [s\alpha(x_1), s\alpha(x_2), \beta(a_3)]_{\widehat{\mathfrak{g}}} - [s\alpha(x_1), s\alpha(x_2), \varphi(x_3)]_{\widehat{\mathfrak{g}}} + [s\alpha(x_1), s\alpha(x_2), s\alpha(x_3)]_{\widehat{\mathfrak{g}}} \\
&= \beta([a_1, a_2, a_3]_{\widehat{\mathfrak{g}}}) - [\beta(a_1), \beta(a_2), \varphi(x_3)]_{\widehat{\mathfrak{g}}} + D_\rho(\alpha(x_3))(\beta(a_1), \beta(a_2)) \\
&\quad - [\beta(a_1), \varphi(x_2), \beta(a_3)]_{\widehat{\mathfrak{g}}} + [\beta(a_1), \varphi(x_2), \varphi(x_3)]_{\widehat{\mathfrak{g}}} - D_\rho(\alpha(x_3))(\beta(a_1), \varphi(x_2)) \\
&\quad - \rho(\alpha(x_2))(\beta(a_1), \beta(a_3)) + \rho(\alpha(x_2))(\beta(a_1), \varphi(x_3)) + \theta(\alpha(x_2), \alpha(x_3))\beta(a_1) \\
&\quad - [\varphi(x_1), \beta(a_2), \beta(a_3)]_{\widehat{\mathfrak{g}}} + [\varphi(x_1), \beta(a_2), \varphi(x_3)]_{\widehat{\mathfrak{g}}} - D_\rho(\alpha(x_3))(\varphi(x_1), \beta(a_2)) \\
&\quad + [\varphi(x_1), \varphi(x_2), \beta(a_3)]_{\widehat{\mathfrak{g}}} - [\varphi(x_1), \varphi(x_2), \varphi(x_3)]_{\widehat{\mathfrak{g}}} + D_\rho(\alpha(x_3))(\varphi(x_1), \varphi(x_2)) \\
&\quad + \rho(\alpha(x_2))(\varphi(x_1), \beta(a_3)) - \rho(\alpha(x_2))(\varphi(x_1), \varphi(x_3)) - \theta(\alpha(x_2), \alpha(x_3))\varphi(x_1) \\
&\quad + \rho(\alpha(x_1))(\beta(a_2), \beta(a_3)) - \rho(\alpha(x_1))(\beta(a_2), \varphi(x_3)) - \theta(\alpha(x_1), \alpha(x_3))\beta(a_2) \\
&\quad - \rho(\alpha(x_1))(\varphi(x_2), \beta(a_3)) + \rho(\alpha(x_1))(\varphi(x_2), \varphi(x_3)) + \theta(\alpha(x_1), \alpha(x_3))\varphi(x_2) \\
&\quad + D_\theta(\alpha(x_1), \alpha(x_2))\beta(a_3) - D_\theta(\alpha(x_1), \alpha(x_2))\varphi(x_3) + \omega(\alpha(x_1), \alpha(x_2), \alpha(x_3)) \\
&\quad + s[\alpha(x_1), \alpha(x_2), \alpha(x_3)]_{\mathfrak{g}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\gamma([\widehat{w}_1, \widehat{w}_2, \widehat{w}_3]_{\widehat{\mathfrak{g}}}) \\
&= \gamma([a_1, a_2, a_3]_{\widehat{\mathfrak{g}}} + [a_1, a_2, s(x_3)]_{\widehat{\mathfrak{g}}} + [a_1, s(x_2), a_3]_{\widehat{\mathfrak{g}}} + [a_1, s(x_2), s(x_3)]_{\widehat{\mathfrak{g}}} \\
&\quad + [s(x_1), a_2, s(x_3)]_{\widehat{\mathfrak{g}}} + [s(x_1), s(x_2), a_3]_{\widehat{\mathfrak{g}}} + [s(x_1), a_2, a_3]_{\widehat{\mathfrak{g}}} \\
&\quad + \omega(x_1, x_2, x_3) + s[x_1, x_2, x_3]_{\mathfrak{g}}) \\
&= \beta([a_1, a_2, a_3]_{\widehat{\mathfrak{g}}}) + \beta(D_\rho(x_3)(a_1, a_2) - \rho(x_2)(a_1, a_3) + \theta(x_2, x_3)a_1 - \theta(x_1, x_3)a_2 \\
&\quad + D_\theta(x_1, x_2)a_3 + \rho(x_1)(a_2, a_3)) + \beta\omega(x_1, x_2, x_3) - \varphi[x_1, x_2, x_3]_{\mathfrak{g}} + s\alpha[x_1, x_2, x_3]_{\mathfrak{g}}.
\end{aligned}
$$

Thanks to Eqs. (39)–(41), we have

$$\gamma[\widehat{w}_1, \widehat{w}_2, \widehat{w}_3]_{\widehat{\mathfrak{g}}} = [\gamma(\widehat{w}_1), \gamma(\widehat{w}_2), \gamma(\widehat{w}_3)]_{\widehat{\mathfrak{g}}},$$

which implies that γ is a homomorphism of Lie triple systems. Thus, $\gamma \in \mathrm{Aut}(\widehat{\mathfrak{g}})$. Finally, we show that $\gamma|_{\mathfrak{h}} = \beta$ and $p\gamma s = \alpha$. In fact,

$$\gamma(a) = \gamma(a + s(0)) = \beta(a), \quad \forall\, a \in \mathfrak{h}$$

and

$$(p\gamma s)(x)=p\gamma(0+s(x))=p(s\alpha(x)-\varphi(x))=ps\alpha(x)=\alpha(x),\quad \forall\, x\in\mathfrak{g}.$$

Therefore, $\gamma|_{\mathfrak{h}}=\beta$ and $p\gamma s=\alpha$. Thus, $(\alpha,\beta)\in \mathrm{Aut}(\mathfrak{g})\times\mathrm{Aut}(\mathfrak{h})$ is inducible. □

Let $0\to\mathfrak{h}\xrightarrow{i}\widehat{\mathfrak{g}}\xrightarrow{p}\mathfrak{g}\to 0$ be a non-abelian extension of the Lie triple system $\mathfrak{g}$ by $\mathfrak{h}$ with a section s of p and (ω,θ,ρ) be the corresponding non-abelian 3-cocycle induced by s.

For all $(\alpha,\beta)\in\mathrm{Aut}(\mathfrak{g})\times\mathrm{Aut}(\mathfrak{h})$. Define maps $\omega_{(\alpha,\beta)}:\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}\to\mathfrak{h},\ \theta_{(\alpha,\beta)}:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{gl}(\mathfrak{h}),\ \rho_{(\alpha,\beta)}:\mathfrak{g}\to\mathrm{Hom}(\mathfrak{h}\times\mathfrak{h},\mathfrak{h})$ respectively by

$$\omega_{(\alpha,\beta)}(x,y,z)=\beta\omega(\alpha^{-1}(x),\alpha^{-1}(y),\alpha^{-1}(z)),\tag{42}$$

$$\theta_{(\alpha,\beta)}(x,y)a=\beta\theta(\alpha^{-1}(x),\alpha^{-1}(y))\beta^{-1}(a),\tag{43}$$

$$\rho_{(\alpha,\beta)}(x)(a,b)=\beta\rho(\alpha^{-1}(x))(\beta^{-1}(a),\beta^{-1}(b)).\tag{44}$$

By direct computations, we get

$$\begin{aligned}&D_{\theta_{(\alpha,\beta)}}(x,y)=\beta D_\theta(\alpha^{-1}(x),\alpha^{-1}(y))\beta^{-1}(a),\\&D_{\rho_{(\alpha,\beta)}}(a,b)=\beta D_\rho(\alpha^{-1}(x))(\beta^{-1}(a),\beta^{-1}(b)).\end{aligned}\tag{45}$$

for all $x,y,z\in\mathfrak{g},a,b\in\mathfrak{h}$.

Proposition 5.3. *With the above notations, $(\omega_{(\alpha,\beta)},\theta_{(\alpha,\beta)},\rho_{(\alpha,\beta)})$ is a non-abelian 3-cocycle.*

Proof. Using Eqs. (12), (42), (43) and (45), we get for all $x_1,x_2,y_1,y_2,y_3\in\mathfrak{g}$,

$$\begin{aligned}
&D_{\theta_{(\alpha,\beta)}}(x_1,x_2)\omega_{(\alpha,\beta)}(y_1,y_2,y_3)+\omega_{(\alpha,\beta)}(x_1,x_2,[y_1,y_2,y_3]_{\mathfrak{g}})\\
&\quad=\beta\big(D_\theta(\alpha^{-1}(x_1),\alpha^{-1}(x_2))\beta^{-1}\big(\beta\omega_{(\alpha,\beta)}(\alpha^{-1}(y_1),\alpha^{-1}(y_2),\alpha^{-1}(y_3))\big)\big)\\
&\qquad-\beta\big(\omega_{(\alpha,\beta)}(\alpha^{-1}(x_1),\alpha^{-1}(x_2),\alpha^{-1}([y_1,y_2,y_3]_{\mathfrak{g}}))\big)\\
&\quad=\beta\omega([\alpha^{-1}(x_1),\alpha^{-1}(x_2),\alpha^{-1}(y_1)]_{\mathfrak{g}},\alpha^{-1}(y_2),\alpha^{-1}(y_3))\\
&\qquad+\beta\big(\theta(\alpha^{-1}(y_2),\alpha^{-1}(y_3))\omega(\alpha^{-1}(x_1),\alpha^{-1}(x_2),\alpha^{-1}(y_1))\big)\\
&\qquad+\beta\omega(\alpha^{-1}(y_1),[\alpha^{-1}(x_1),\alpha^{-1}(x_2),\alpha^{-1}(y_2)]_{\mathfrak{g}},\alpha^{-1}(y_3))\\
&\qquad-\beta\big(\theta(\alpha^{-1}(y_1),\alpha^{-1}(y_3))\omega(\alpha^{-1}(x_1),\alpha^{-1}(x_2),\alpha^{-1}(y_2))\big)\\
&\qquad+\beta\omega(\alpha^{-1}(y_1),\alpha^{-1}(y_2),[\alpha^{-1}(x_1),\alpha^{-1}(x_2),\alpha^{-1}(y_3)]_{\mathfrak{g}})\\
&\qquad+\beta\big(D_\theta(\alpha^{-1}(y_1),\alpha^{-1}(y_2))\omega(\alpha^{-1}(x_1),\alpha^{-1}(x_2),\alpha^{-1}(y_3))\big)\\
&\quad=\omega_{(\alpha,\beta)}([x_1,x_2,y_1]_{\mathfrak{g}},y_2,y_3)+\theta_{(\alpha,\beta)}(y_2,y_3)\omega_{(\alpha,\beta)}(x_1,x_2,y_1)\\
&\qquad+\omega_{(\alpha,\beta)}(y_1,[x_1,x_2,y_2]_{\mathfrak{g}},y_3)-\theta_{(\alpha,\beta)}(y_1,y_3)\omega_{(\alpha,\beta)}(x_1,x_2,y_2)\\
&\qquad+\omega_{(\alpha,\beta)}(y_1,y_2,[x_1,x_2,y_3]_{\mathfrak{g}})+D_{\theta_{(\alpha,\beta)}}(y_1,y_2)\omega_{(\alpha,\beta)}(x_1,x_2,y_3),
\end{aligned}$$

which implies that Eq. (12) holds for $\omega_{(\alpha,\beta)},\theta_{(\alpha,\beta)}$. Taking the same procedure, we can prove that Eqs. (11) and (13)–(22) hold for $\omega_{(\alpha,\beta)},\theta_{(\alpha,\beta)},D_{\theta_{(\alpha,\beta)}},\rho_{(\alpha,\beta)}$ and $D_{\rho_{(\alpha,\beta)}}$. The proof is finished. □

Theorem 5.4. *Let* $0 \to \mathfrak{h} \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0$ *be a non-abelian extension of* $\mathfrak{g}$ *by* $\mathfrak{h}$ *with a section* s *of* p *and* (ω, θ, ρ) *be the corresponding non-abelian 3-cocycle induced by* s. *A pair* $(\alpha, \beta) \in \mathrm{Aut}(\mathfrak{g}) \times \mathrm{Aut}(\mathfrak{h})$ *is inducible if and only if the non-abelian 3-cocycles* (ω, θ, ρ) *and* $(\omega_{(\alpha,\beta)}, \theta_{(\alpha,\beta)}, \rho_{(\alpha,\beta)})$ *are equivalent.*

Proof. Suppose $(\alpha, \beta) \in \mathrm{Aut}(\mathfrak{g}) \times \mathrm{Aut}(\mathfrak{h})$ is inducible, then by Theorem 5.2, there is a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ satisfying Eqs. (39)–(41). For all $x, y \in \mathfrak{g}, a \in \mathfrak{h}$, there exist $x_0, y_0 \in \mathfrak{g}, a_0 \in \mathfrak{h}$ such that $x = \alpha(x_0), y = \alpha(y_0), a = \beta(a_0)$. Therefore, using Eqs. (39) and (43),

$$\begin{aligned}
&\theta_{\alpha,\beta}(x,y)a - \theta(x,y)a \\
&\qquad = \beta(\theta(\alpha^{-1}(x), \alpha^{-1}(y))\beta^{-1}(a)) - \theta(x,y)a \\
&\qquad = \beta(\theta(x_0, y_0)a_0) - \theta(\alpha(x_0), \alpha(y_0))\beta(a_0) \\
&\qquad = [\beta(a_0), \varphi(x_0), \varphi(y_0)]_{\mathfrak{h}} - D_\rho(\alpha(y_0))(\beta(a_0), \varphi(x_0)) + \rho(x_0)(\beta(a_0), \varphi(y_0)) \\
&\qquad = [a, \varphi\alpha^{-1}(x), \varphi\alpha^{-1}(y)]_{\mathfrak{h}} - D_\rho(y)(a, \varphi\alpha^{-1}(x)) + \rho(x)(a, \varphi\alpha^{-1}(y)),
\end{aligned}$$

which yields that Eq. (25) holds. Analogously, Eqs. (24) and (26) hold. Hence, the non-abelian 3-cocycles $(\omega_{(\alpha,\beta)}, \theta_{(\alpha,\beta)}, \rho_{(\alpha,\beta)})$ and (ω, θ, ρ) are equivalent via the linear map $\varphi\alpha^{-1} : \mathfrak{g} \to \mathfrak{h}$.

The converse part can be checked similarly. □

6. Wells exact sequences for Lie triple systems

In this section, we are focus on the Wells map associated with non-abelian extensions of Lie triple systems.

Let $\mathcal{E} : 0 \to \mathfrak{h} \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0$ be a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$ with a section s of p. Suppose that (ω, θ, ρ) is the corresponding non-abelian 3-cocycle induced by s. Define a linear map $W : \mathrm{Aut}(\mathfrak{g}) \times \mathrm{Aut}(\mathfrak{h}) \to H^3_{nab}(\mathfrak{g}, \mathfrak{h})$ by

$$W(\alpha, \beta) = [(\omega_{(\alpha,\beta)}, \theta_{(\alpha,\beta)}, \rho_{(\alpha,\beta)}) - (\omega, \theta, \rho)]. \tag{46}$$

The map W is called the Wells map. In view of Lemma 3.9, the Wells map W does not depend on the choice of sections.

Theorem 6.1. *Assume that* $\mathcal{E} : 0 \to \mathfrak{h} \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0$ *is a non-abelian extension of* $\mathfrak{g}$ *by* $\mathfrak{h}$ *with a section* s *of* p. *Then there is an exact sequence:*

$$1 \longrightarrow \mathrm{Aut}^{\mathfrak{g}}_{\mathfrak{h}}(\widehat{\mathfrak{g}}) \xrightarrow{H} \mathrm{Aut}_{\mathfrak{h}}(\widehat{\mathfrak{g}}) \xrightarrow{\lambda} \mathrm{Aut}(\mathfrak{g}) \times \mathrm{Aut}(\mathfrak{h}) \xrightarrow{W} H^3_{nab}(\mathfrak{g}, \mathfrak{h}),$$

where $\mathrm{Aut}^{\mathfrak{g}}_{\mathfrak{h}}(\widehat{\mathfrak{g}}) = \{\gamma \in \mathrm{Aut}(\widehat{\mathfrak{g}}) \mid \lambda(\gamma) = (I_{\mathfrak{g}}, I_{\mathfrak{h}})\}$.

Proof. Obviously, $\mathrm{Ker}\,\lambda = \mathrm{Im}\,H$ and H is injective. By Theorem 5.4, one can easily check that $\mathrm{Ker}\,W = \mathrm{Im}\,\lambda$. This completes the proof. □

Let $\mathcal{E} : 0 \to \mathfrak{h} \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0$ be a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$ with a section s of p. Assume that (ω, θ, ρ) is the non-abelian 3-cocycle corresponding to $\mathcal{E}$ induced by

s. Denote

$$Z^1_{nab}(\mathfrak{g},\mathfrak{h})=\left\{\varphi:\mathfrak{g}\to\mathfrak{h}\;\middle|\;\begin{array}{l}[a,\varphi(x),b]_{\mathfrak{h}}=0,\\ \rho(x)(a,\varphi(y))-D_\rho(y)(a,\varphi(x))=0,\\ D_\rho(z)(\varphi(x),\varphi(y))-\rho(y)(\varphi(x),\varphi(z))\\ -\theta(y,z)\varphi(x)+\rho(x)(\varphi(y),\varphi(z))+\theta(x,z)\varphi(y)\\ =D_\theta(x,y)\varphi(z)-\varphi([x,y,z]_{\mathfrak{g}}),\quad\forall\, x,y,z\in\mathfrak{g},a,b\in\mathfrak{h}\end{array}\right\}. \tag{47}$$

It is easy to check that $Z^1_{nab}(\mathfrak{g},\mathfrak{h})$ is an abelian group, which is called a non-abelian 1-cocycle on $\mathfrak{g}$ with values in $\mathfrak{h}$.

Proposition 6.2. *With the above notations, we have*

(i) *The linear map $K:\operatorname{Ker}\lambda\to Z^1_{nab}(\mathfrak{g},\mathfrak{h})$ given by*

$$K(\gamma)(x)=\varphi_\gamma(x)=s(x)-\gamma s(x),\quad\forall\,\gamma\in\operatorname{Ker}\lambda,x\in\mathfrak{g} \tag{48}$$

is a homomorphism of groups.

(ii) *K is an isomorphism, that is, $\operatorname{Ker}\lambda\simeq Z^1_{nab}(\mathfrak{g},\mathfrak{h})$.*

Proof.

(i). Since $\gamma\in\operatorname{Ker}\lambda$, $\gamma|_{\mathfrak{h}}=I_{\mathfrak{h}}$. By Eqs. (31) and (48), we obtain for all $x,y,z\in\mathfrak{g}$,

$$\begin{aligned}[a,\varphi_\gamma(x),b]_{\mathfrak{h}}&=[a,s(x),b]_{\hat{\mathfrak{g}}}-[a,\gamma s(x),b]_{\hat{\mathfrak{g}}}\\&=-[s(x),a,b]_{\hat{\mathfrak{g}}}+[\gamma s(x),a,b]_{\hat{\mathfrak{g}}}\\&=-\rho_s(x)(a,b)+[\gamma s(x),\gamma(a),\gamma(b)]_{\hat{\mathfrak{g}}}\\&=-\rho_s(x)(a,b)+\gamma\rho_s(x)(a,b)\\&=-\rho_s(x)(a,b)+\rho_s(x)(a,b)\\&=0.\end{aligned}$$

Due to $[\varphi_\gamma(x),\varphi_\gamma(y),\varphi_\gamma(z)]_{\mathfrak{h}}=0$, using Eqs. (30)–(32) and (48), we have for all $x,y,z\in\mathfrak{g}$,

$$\begin{aligned}&\rho(y)(\varphi_\gamma(x),\varphi_\gamma(z))-D_\rho(z)(\varphi_\gamma(x),\varphi_\gamma(y))+\theta(y,z)\varphi_\gamma(x)\\&\qquad-\rho(x)(\varphi_\gamma(y),\varphi_\gamma(z))-\theta(x,z)\varphi_\gamma(y)+D_\theta(x,y)\varphi_\gamma(z)-\varphi_\gamma(\{x,y,z\}_{\mathfrak{g}})\\&=[\varphi_\gamma(x),\varphi_\gamma(y),\varphi_\gamma(z)]_{\mathfrak{h}}-D_\rho(z)(\varphi_\gamma(x),\varphi_\gamma(y))+\rho(y)(\varphi_\gamma(x),\varphi_\gamma(z))+\theta(y,z)\varphi_\gamma(x)\\&\qquad-\rho(x)(\varphi_\gamma(y),\varphi_\gamma(z))-\theta(x,z)\varphi_\gamma(y)+D_\theta(x,y)\varphi_\gamma(z)-\varphi_\gamma(\{x,y,z\}_{\mathfrak{g}})\\&=[s(x)-\gamma s(x),s(y)-\gamma s(y),s(z)-\gamma s(z)]_{\hat{\mathfrak{g}}}-[s(x)-\gamma s(x),s(y)-\gamma s(y),s(z)]_{\hat{\mathfrak{g}}}\\&\qquad+[s(y),s(x)-\gamma s(x),s(z)-\gamma s(z)]_{\hat{\mathfrak{g}}}+[s(x)-\gamma s(x),s(y),s(z)]_{\hat{\mathfrak{g}}}\\&\qquad-[s(x),s(y)-\gamma s(y),s(z)-\gamma s(z)]_{\hat{\mathfrak{g}}}-[s(y)-\gamma s(y),s(x),s(z)]_{\hat{\mathfrak{g}}}\\&\qquad+[s(x),s(y),s(z)-\gamma s(z)]_{\hat{\mathfrak{g}}}+\gamma s([x,y,z]_{\mathfrak{g}})-s([x,y,z]_{\mathfrak{g}})\\&=\gamma s([x,y,z]_{\mathfrak{g}})-[\gamma s(x),\gamma s(y),\gamma s(z)]_{\hat{\mathfrak{g}}}+[s(x),s(y),s(z)]_{\hat{\mathfrak{g}}}-s([x,y,z]_{\mathfrak{g}})\\&=\omega(x,y,z)-\gamma\omega(x,y,z)\\&=0.\end{aligned}$$

Analogously, we can check that φ_γ satisfies the other identities in $Z^1_{nab}(\mathfrak{g},\mathfrak{h})$. Thus, K is well-defined. For any $\gamma_1,\gamma_2\in\operatorname{Ker}\lambda$ and $x\in\mathfrak{g}$, suppose $K(\gamma_1)=\varphi_{\gamma_1}$ and $K(\gamma_2)=\varphi_{\gamma_2}$. By Eq. (48), we have

$$\begin{aligned}K(\gamma_1\gamma_2)(x)&=s(x)-\gamma_1\gamma_2 s(x)\\&=s(x)-\gamma_1(s(x)-\varphi_{\gamma_2}(x))\\&=s(x)-\gamma_1 s(x)+\gamma_1\varphi_{\gamma_2}(x)\\&=\varphi_{\gamma_1}(x)+\varphi_{\gamma_2}(x)\end{aligned}$$

which means that $K(\gamma_1\gamma_2)=K(\gamma_1)+K(\gamma_2)$ is a homomorphism of groups.

(ii). For all $\gamma\in\operatorname{Ker}\lambda$, we obtain that $\lambda(\gamma)=(p\gamma s,\gamma|_{\mathfrak{h}})=(I_{\mathfrak{g}},I_{\mathfrak{h}})$. If $K(\gamma)=\varphi_\gamma=0$, we can get $\varphi_\gamma(x)=s(x)-\gamma s(x)=0$, that is, $\gamma=I_{\hat{\mathfrak{g}}}$, which indicates that K is injective. Secondly, we prove that K is surjective. Since all $\widehat{x}\in\widehat{\mathfrak{g}}$ can be written as $a+s(x)$ for some $a\in\mathfrak{h},x\in\mathfrak{g}$, for any $\varphi\in Z^1_{nab}(\mathfrak{g},\mathfrak{h})$, we define a linear map $\gamma:\widehat{\mathfrak{g}}\to\widehat{\mathfrak{g}}$ by

$$\gamma(\widehat{x})=\gamma(a+s(x))=s(x)-\varphi(x)+a,\quad \forall\,\widehat{x}\in\widehat{\mathfrak{g}}. \tag{49}$$

It is obviously that $(p\gamma s,\gamma|_{\mathfrak{h}})=(I_{\mathfrak{g}},I_{\mathfrak{h}})$. We need to verify that γ is an automorphism of Lie triple system $\widehat{\mathfrak{g}}$. One can take the same procedure as the proof of the converse part of Theorem 5.2. It follows that $\gamma\in\operatorname{Ker}\lambda$. Thus, K is surjective. In all, K is bijective. So $\operatorname{Ker}\lambda\simeq Z^1_{nab}(\mathfrak{g},\mathfrak{h})$. □

Combining Theorem 6.1 and Proposition 6.2, we have

Theorem 6.3. *Let $\mathcal{E}:0\to\mathfrak{h}\xrightarrow{i}\widehat{\mathfrak{g}}\xrightarrow{p}\mathfrak{g}\to 0$ be a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$. There is an exact sequence:*

$$0\longrightarrow Z^1_{nab}(\mathfrak{g},\mathfrak{h})\xrightarrow{\ i\ }\operatorname{Aut}_{\mathfrak{h}}(\widehat{\mathfrak{g}})\xrightarrow{\ \lambda\ }\operatorname{Aut}(\mathfrak{g})\times\operatorname{Aut}(\mathfrak{h})\xrightarrow{\ W\ }H^3_{nab}(\mathfrak{g},\mathfrak{h}).$$

7. Particular case: abelian extensions of Lie triple systems

In this section, we interpret the results of previous section in particular case.

Let $\mathcal{E}:0\to\mathfrak{h}\xrightarrow{i}\widehat{\mathfrak{g}}\xrightarrow{p}\mathfrak{g}\to 0$ be an abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$ with a section s of p. Suupose that ω is a 3-cocycle corresponding to $\mathcal{E}$ induced by s. We have,

Theorem 7.1 ([Zha14])**.**

(i) *The triple $(\mathfrak{g}\oplus\mathfrak{h},[\ ,\ ,\ ]_{(\omega,\theta)})$ is a Lie triple system if and only if ω is a 3-cocycle of $\mathfrak{g}$ with coefficients in the representation $(\mathfrak{h},\theta)$.*

(ii) *Abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$ are classified by the cohomology group $H^3(\mathfrak{g},\mathfrak{h})$ of $\mathfrak{g}$ with coefficients in $(\mathfrak{h},\theta)$.*

The following statement can be got directly by Theorem 5.2.

Theorem 7.2. *Let $\mathcal{E}:0\to\mathfrak{h}\xrightarrow{i}\widehat{\mathfrak{g}}\xrightarrow{p}\mathfrak{g}\to 0$ be an abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$ with a section s of p. Assume that ω is a 3-cocycle and $(\mathfrak{h},\theta)$ is a representation of $\mathfrak{g}$ associated to $\mathcal{E}$. A pair $(\alpha,\beta)\in\operatorname{Aut}(\mathfrak{g})\times\operatorname{Aut}(\mathfrak{h})$ is inducible with respect to the abelian extension $\mathcal{E}$ if and only if there is a linear map $\varphi:\mathfrak{g}\to\mathfrak{h}$ satisfying the following* *conditions:*

$$\beta\omega(x,y,z)-\omega(\alpha(x),\alpha(y),\alpha(z))=\theta(\alpha(x),\alpha(z))\varphi(y)-\theta(\alpha(y),\alpha(z))\varphi(x)$$
$$-D_\theta(\alpha(x),\alpha(y))\varphi(z)+\varphi([x,y,z]_{\mathfrak{g}}), \quad (50)$$
$$\beta(\theta(x,y)a)=\theta(\alpha(x),\alpha(y))\beta(a). \quad (51)$$

According to Eqs. (7) and (51), we have

$$\beta D_\theta(x,y)a=D_\theta(\alpha(x),\alpha(y))\beta(a).$$

For all $(\alpha,\beta)\in \mathrm{Aut}(\mathfrak{g})\times\mathrm{Aut}(\mathfrak{h})$, $\omega_{(\alpha,\beta)}$ may not be a 3-cocycle. Indeed, $\omega_{(\alpha,\beta)}$ is a 3-cocycle if Eq. (51) holds. Thus, it is natural to introduce the space of compatible pairs of automorphisms:

$$C_\theta=\left\{(\alpha,\beta)\in \mathrm{Aut}(\mathfrak{g})\times\mathrm{Aut}(\mathfrak{h})\;\middle|\; \begin{array}{r}\beta(\theta(x,y)a)=\theta(\alpha(x),\alpha(y))\beta(a),\\ \forall\, x,y\in\mathfrak{g}, a\in\mathfrak{h}\end{array}\right\}.$$

Analogous to Theorem 5.4, we have

Theorem 7.3. *Let $\mathcal{E}:0\to\mathfrak{h}\xrightarrow{i}\hat{\mathfrak{g}}\xrightarrow{p}\mathfrak{g}\to 0$ be an abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$ with a section s of p and ω be a 3-cocycle associated to $\mathcal{E}$ induced by s. A pair $(\alpha,\beta)\in C_\theta$ is inducible with respect to the abelian extension $\mathcal{E}$ if and only if ω and $\omega_{(\alpha,\beta)}$ are in the same cohomological class.*

In the case of abelian extensions, $Z^1_{nab}(\mathfrak{g},\mathfrak{h})$ defined by (47) becomes to $H^1(\mathfrak{g},\mathfrak{h})$ given in Section 2. In the light of Theorem 6.3 and Theorem 7.3, we have the following exact sequence:

Theorem 7.4. *Let $\mathcal{E}:0\to\mathfrak{h}\xrightarrow{i}\hat{\mathfrak{g}}\xrightarrow{p}\mathfrak{g}\to 0$ be an abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$. There is an exact sequence:*

$$0\longrightarrow H^1(\mathfrak{g},\mathfrak{h})\xrightarrow{i}\mathrm{Aut}_{\mathfrak{h}}(\hat{\mathfrak{g}})\xrightarrow{\lambda}C_\theta\xrightarrow{W}H^3(\mathfrak{g},\mathfrak{h}).$$

Remark 7.5. After completing this paper, the first author and her coauthor generalized the results to Lie–Yamaguti algebras [SL24]. Although the results are analogous, they are specifically adapted to the distinct frameworks of Lie triple systems and Lie–Yamaguti algebras. This work remains mathematically meaningful.

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