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Oscillating singular integral operators on graded Lie groups revisited

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Abstract. In this work, we extend the Euclidean theory of oscillating singular integrals due to Fefferman and Stein in [Fef70; FS72] to arbitrary graded Lie groups. Our approach reveals the strong compatibility between the geometric measure theory of a graded Lie group and the Fourier analysis associated with Rockland operators. Our criteria are presented in terms of the oscillating Fefferman condition of the kernel of the operator and its group Fourier transform. One of the novelties of this work is that we use the infinitesimal representation of a Rockland operator to measure the decay of the Fourier transform of the kernel.

Mathematics Subject Classification 2020: 35S30, 42B20, 42B37, 42B35

Key Words and Phrases: Calderón–Zygmund operator, Weak (1,1) inequality, Oscillating singular integrals

1. Introduction

1.1. Outline

This work is part of a series of papers [CR22a; CR22b; CR23] where the boundedness of oscillating integral operators on graded Lie groups and on compact Lie groups has been investigated. In [CR22b], we proved the H^1-L^1-boundedness of oscillating singular integrals on graded Lie groups, extending the analysis by Alexopoulos [Ale94] for the case of spectral multipliers of Hörmander sub-Laplacians. For instance, in [CR22b], our criteria are presented according to the modern theory of PDE in terms of Rockland operators. Having proved the H^1-L^1-boundedness estimate in [CR22b], here we consider the problem of extending the weak $(1,1)$ estimate by Fefferman [Fef70] to graded Lie groups. For the regularity properties of more general classes of spectral multipliers (of non-oscillating type) and of pseudo-differential operators on graded Lie groups, we refer the reader to [CDR21].

The results in [CR22b], together with the main result Theorem 1.1 extend the theory of oscillating singular integrals developed by Fefferman and Stein in [Fef70; FS72] to the setting of graded Lie groups. Our setting then covers a variety of groups of interest in analysis of PDE, namely, the Euclidean space, Heisenberg-type groups, stratified groups, etc. For our criteria, we will use the Fourier analysis associated with Rockland operators. These are hypoelliptic partial differential operators in view

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of the Helffer and Nourrigat solution of the Rockland conjecture (see [HN79]), and they could be of arbitrary order. In the particular case of stratified Lie groups, they encompass the family of Hörmander sub-Laplacians and their integer powers.

1.2. Historical aspects

We now turn to the historical aspects of the subject in the Euclidean setting that inspired our approach. As pointed out by Stein in [Ste98], there is probably no work in the last seventy years that has had as widespread an influence in analysis as the historic memoir [CZ52], "On the existence of certain singular integrals" by Calderón and Zygmund, published in 1952 in *Acta Mathematica.* Using the methods of real interpolation (e.g., the Marcinkiewicz interpolation theorem), one of the main problems of the Calderón–Zygmund theory is to identify the sharp conditions on a distribution K such that the convolution operator $f \mapsto f * K$ can be extended to a weak $(1,1)$-type operator, that is,

$$\|f * K\|_{L^{1,\infty}(\mathbb{R}^n)} \leq C\|f\|_{L^1(\mathbb{R}^n)}, \quad f \in C_0^\infty(\mathbb{R}^n).$$

For instance, singular integrals were important predecessors of so-called pseudo-differential operators, whose impact in analysis, partial differential equations, and differential geometry was recognized through the works of Kohn and Nirenberg, Hörmander, Stein and Fefferman, and Atiyah and Singer, among others (see Hörmander [Hör85, p. 178] for historical details). In terms of convolution operators, a remarkable kernel class H_∞ was introduced by Hörmander, consisting of all distributions satisfying the kernel estimate

$$[K]_{H_\infty} := \sup_{R>0} \left\| \int_{|x|\geq 2R} |K(x-y) - K(x)| \, \mathrm{d}x \right\|_{L^\infty(B(0,R),\, \mathrm{d}y)} < \infty. \tag{1}$$

Then, in [Hör60], Hörmander proved that a convolution operator $Tf = K * f$ with a kernel satisfying the "smoothness" condition in (1) and bounded on $L^2(\mathbb{R}^n)$ (that is, the Fourier transform of the kernel $\widehat{K} \in L^\infty(\mathbb{R}^n)$) is of weak $(1,1)$-type.

The theory of oscillating singular integrals developed by Fefferman and Stein in [Fef70; FS72] extends in some aspects the work of Calderón and Zygmund by enabling the analysis of oscillating multipliers. Indeed, generalizing the Hörmander condition, Fefferman [Fef70] and Fefferman and Stein [FS72] considered distributions with compact support satisfying the condition

$$[K]_{H_{\infty,\theta}} := \sup_{0<R<1} \left\| \int_{|x|\geq 2R^{1-\theta}} |K(x-y) - K(x)| \, \mathrm{d}x \right\|_{L^\infty(B(0,R),\, \mathrm{d}y)} < \infty. \tag{2}$$

Roughly speaking, if K satisfies (2) and its Fourier transform has the decay property

$$|\widehat{K}(\xi)| = O\left((1+|\xi|)^{-\frac{n\theta}{2}}\right), \quad 0<\theta<1, \tag{3}$$

then Fefferman's theorem establishes that T admits a bounded extension of weak $(1,1)$-type, provided the support of K is sufficiently small. While the small support condition does not explicitly appear in [Fef70, Theorem 2′], it is implicitly assumed in the proof given in [Fef70, p. 24]. This assumption is not restrictive, as an analysis similar to that in [Fef70, p. 23] enables the reduction from distributions K with arbitrary support size to those with small support.

Clearly, when $\theta = 0$, Fefferman's condition reduces to Hörmander's condition for distributions K with compact support. The boundedness of convolution operators T with kernels satisfying the Hörmander condition from the Hardy space $H^1(\mathbb{R}^n)$ to itself was established in [FS72].

From the perspective of harmonic analysis, our main Theorem 1.1 provides a critical estimate for oscillating singular integrals on L^p-spaces at the endpoint $p = 1$, specifically establishing their weak $(1, 1)$-boundedness. This result reveals profound connections between Fourier analysis on graded Lie groups and their geometric measure theory through the lens of representation theory. A key technical tool enabling this work is the existence of hypoelliptic left-invariant linear partial differential operators on (graded) Lie groups, which follows from Helffer and Nourrigat's solution to the Rockland conjecture [HN79].

A fundamental tool in proving our critical estimate is, as anticipated, the Calderón–Zygmund decomposition for spaces of homogeneous type developed by Coifman and Weiss in [CW71]. The key innovation of this work lies in combining three essential components:

- the Calderón–Zygmund decomposition of an integrable function $f = g + b = g + \sum_j b_j$ into its good part g and bad parts b_j,
- the geometric properties of the supports of b_j, and,
- the Fourier analysis associated with Rockland operators.

This synthesis is formally established in the proof of Lemma 3.2.

1.3. The main result

Let $T : X \to Y$ be a bounded operator between Banach spaces X and Y, with operator norm denoted by $\|T\|_{\mathrm{op}}$. To present our main result, we first introduce the necessary notation:

- $\widehat{G}$ denotes the unitary dual of G,
- $\mathcal{R} = \sum_{[\alpha]\leq\nu} a_\alpha X^\alpha$ is a positive Rockland operator on G, defined as a positive, left-invariant, hypoelliptic partial differential operator,
- $\pi(\mathcal{R}) = \widehat{k}_{\mathcal{R}}(\pi)$ represents the Fourier transform of its right-convolution kernel $k_{\mathcal{R}}$, characterized by the identity $\mathcal{R}f = f * k_{\mathcal{R}}$ for $f \in C_0^\infty(G)$.

The main theorem of this work is stated as follows:

Theorem 1.1. *Let G be a graded Lie group of homogeneous dimension Q, and let $T\colon C_0^\infty(G) \to \mathscr{D}'(G)$ be a left-invariant operator with right-convolution kernel $K \in L^1_{\mathrm{loc}}(G \setminus \{e\})$, that is T is defined by $Tf = f * K$. Assume that:*

(1) *K is a distribution with compact support and sufficiently small diameter, namely that,* $\operatorname{diam}(\operatorname{supp}(K)) < 1$.

(2) *For some $0 \leq \theta < 1$, K satisfies the group Fourier transform conditions:*

$$\sup_{\pi\in\widehat{G}} \left\| (1+\pi(\mathcal{R}))^{\frac{Q\theta}{2\nu}} \widehat{K}(\pi) \right\|_{\mathrm{op}} < \infty, \quad \sup_{\pi\in\widehat{G}} \left\| \widehat{K}(\pi)(1+\pi(\mathcal{R}))^{\frac{Q\theta}{2\nu}} \right\|_{\mathrm{op}} < \infty; \tag{4}$$

(3) *K satisfies the oscillating Hörmander condition:*

$$[K]_{H_{\infty,\theta}(G)} := \sup_{0<R<1} \sup_{|y|<R} \int_{|x|\geq 2R^{1-\theta}} \left| K(y^{-1}x) - K(x) \right| \mathrm{d}x < \infty. \tag{5}$$

Then T admits a bounded extension of weak $(1, 1)$*-type.*

Now, we briefly discuss our results.

Remark 1.2. For $G = \mathbb{R}^n$, we recover the classical weak $(1,1)$ boundedness result proved by Fefferman in [Fef70, Theorem 2′]. From the proof of Theorem 1.1 in Section 3, one can estimate the operator norm of $T : L^1 \to L^{1,\infty}$ by

$$\|T\|_{L^1\to L^{1,\infty}} \lesssim \|T\|_{L^2\to L^2} + [K]_{H_{\infty,\theta}} = \|\widehat{K}\|_{L^\infty} + [K]_{H_{\infty,\theta}},$$

in view of the Plancherel theorem.

Remark 1.3. The main contribution of Theorem 1.1 is when $0 < \theta < 1$. For $\theta = 0$, the statement in Theorem 1.1 follows from the general theory of Calderón–Zygmund operators developed by Coifman and Weiss in [CW71]. Hörmander–Mihlin criteria on graded groups were obtained by the second author and V. Fischer in [FR21]. For different classes of oscillating multipliers, we refer the reader to Bramati, Ciatti, Green, and Wright [Bra+21], and to Ciatti and Wright [CW18] for the case of stratified Lie groups.

Remark 1.4. The boundedness of an operator T satisfying the hypotheses in Theorem 1.1 from the Hardy space $H^1(G)$ into $L^1(G)$ has been proved by the authors in [CR22b], while the H^1-L^1 estimate for pseudo-differential operators on graded Lie groups was analyzed in [CDR21]. As for spectral multipliers, we refer the reader to the classical paper of Alexopoulos [Ale94].

Remark 1.5 (Historical note)**.** Oscillating singular integrals arose as generalisations of oscillating Fourier multipliers. These are multipliers associated to symbols of the form

$$\widehat{K}(\xi) = \psi(\xi)\frac{e^{i|\xi|^a}}{|\xi|^{\frac{n\alpha}{2}}}, \quad \psi \in C^\infty(\mathbb{R}^n), \quad 0 < a < 1,$$

where ψ vanishes near the origin and is equal to one for large $|\xi|$. It was proved by Wainger [Wai65] that $K(x)$ is essentially equal to $c_n|x|^{-n-\lambda}e^{ic_n'|x|^{a'}}$, where $\lambda = \frac{n(a-\alpha)}{2(1-a)}$ and $a' = \frac{a}{a-1}$. From this, one can deduce that $|\nabla K(x)| \lesssim |x|^{-n-\lambda-1+a'}$, from which it follows that for $a = \alpha = \theta$, the estimate (5) remains valid.

The L^p properties for convolution operators of this form were first studied by Hardy [Har13], Hirschman [Hir59], and Wainger [Wai65]. The sharp version of the L^∞-BMO boundedness for oscillating Fourier multipliers can be deduced from the classical work of Fefferman [Fef73]. Further works on the subject in the setting of manifolds and beyond can be found in Seeger [See88; See90; See91], Seeger and Sogge [SS89], and for the setting of Fourier integral operators, we refer the reader to Seeger, Sogge, and Stein [SSS91] and Tao [Tao04].

1.4. Open problems

In addition to our main result, in this paper we propose Conjecture 1.6 and Open Problem 1.7, both related to a modern problem in harmonic analysis: namely, the determination of optimal ranges of boundedness results in terms of the topological dimension n of the group, rather than its Hausdorff dimension. This concerns, in particular, the number or order of derivatives required of the symbol. For the relevant notions of graded Lie groups and Rockland operators, we refer the reader to Section 2.

Conjecture 1.6. *Let G be a graded Lie group of homogeneous dimension Q, and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu > 0$. Then, the weak $(1,1)$ boundedness of the oscillating spectral multiplier*

$$e^{i(1+\mathcal{R})^{a/\nu}}(1+\mathcal{R})^{-m/\nu}, \quad m = \frac{Qa}{2}, \quad 0 < a < 1,$$

can be deduced from the boundedness result in Theorem 1.1. Moreover, its H^1-L^1-boundedness can be shown by proving that it belongs to the Hörmander class $S^{-m}_{1-a,0}(G \times \widehat{G})$ on the phase space $G \times \widehat{G}$, where the boundedness properties for these classes have been established in [CDR21].

Remark 1.7 (Open Problem). The problem of improving the index Q (corresponding to the Hausdorff dimension of the group) in the weak $(1,1)$ estimate (4) of Theorem 1.1 by replacing it with the topological dimension n, or with another number $N > 0$ such that $n \leq N < Q$, remains open. Specifically, one may ask whether conditions of the form

$$\sup_{\pi \in \widehat{G}} \left\| (1+\pi(\mathcal{R}))^{\frac{N\theta}{2\nu}} \widehat{K}(\pi) \right\|_{\mathrm{op}} < \infty, \quad \sup_{\pi \in \widehat{G}} \left\| \widehat{K}(\pi)(1+\pi(\mathcal{R}))^{\frac{N\theta}{2\nu}} \right\|_{\mathrm{op}} < \infty, \qquad (6)$$

together with the oscillating Hörmander condition in (5), suffices to guarantee the weak $(1,1)$-boundedness (or the H^1-L^1-boundedness) of the operator T.

This problem remains open even in the setting where G is a general stratified Lie group and $\mathcal{R}$ is an arbitrary Rockland operator of any order. For the current state of the art when $\mathcal{R}$ is a second-order operator given by a Hörmander sub-Laplacian $\mathcal{L} = -\sum_i X_i^2$, we refer the reader to the pioneering work of Alexopoulos [Ale94], as well as to the more recent contributions by Martini, Müller, and Nicolussi Golo [MMN22], and the extensive references therein.

2. Fourier analysis on graded groups

The notation and terminology of this paper on the analysis of homogeneous Lie groups are mainly taken from Folland and Stein [FS82]. For the analysis of Rockland operators, we will follow [FR16, Chapter 4].

2.1. Homogeneous and graded Lie groups

Let G be a homogeneous Lie group. This means that G is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ is endowed with a family of dilations. We introduce it in the following definition.

A family of dilations $D_r^{\mathfrak{g}}$, $r > 0$, on the Lie algebra $\mathfrak{g}$ is a family of automorphisms on $\mathfrak{g}$ satisfying the following two conditions:

- For every $r > 0$, $D_r^{\mathfrak{g}}$ is a map of the form

$$D_r^{\mathfrak{g}} = \mathrm{Exp}(\ln(r)A)$$

for some diagonalisable linear operator $A \equiv \mathrm{diag}[\nu_1, \ldots, \nu_n]$ on $\mathfrak{g}$.

- For all $X, Y \in \mathfrak{g}$ and $r > 0$,

$$[D_r^{\mathfrak{g}} X, D_r^{\mathfrak{g}} Y] = D_r^{\mathfrak{g}}[X, Y].$$

Remark 2.1. We call the eigenvalues of A, $\nu_1, \nu_2, \ldots, \nu_n$, the dilation weights or weights of G.

In our analysis, the notion of the homogeneous dimension of the group is crucial. We introduce it as follows. The homogeneous dimension of a homogeneous Lie group G is given by

$$Q = \mathbf{Tr}(A) = \nu_1 + \cdots + \nu_n.$$

The dilations $D_r^{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$ induce a family of maps on G defined via

$$D_r := \exp_G \circ D_r^{\mathfrak{g}} \circ \exp_G^{-1}, \quad r > 0,$$

where $\exp_G : \mathfrak{g} \to G$ is the usual exponential map associated with the Lie group G. We refer to the family D_r, $r > 0$, as dilations on the group.

Remark 2.2. If we write $rx = D_r(x)$, $x \in G$, $r > 0$, then a relation on the homogeneous structure of G and the Haar measure $\mathrm{d}x$ on G is given by

$$\int_G (f \circ D_r)(x)\,\mathrm{d}x = r^{-Q} \int_G f(x)\,\mathrm{d}x.$$

Remark 2.3. A Lie group is graded if its Lie algebra $\mathfrak{g}$ may be decomposed as the sum of subspaces

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s,$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, and $\mathfrak{g}_{i+j} = \{0\}$ if $i + j > s$.

Examples of graded Lie groups are the Heisenberg group $\mathbb{H}^n$ and, more generally, any stratified groups where the Lie algebra $\mathfrak{g}$ is generated by $\mathfrak{g}_1$. Here, n is the topological dimension of G, $n = n_1 + \cdots + n_s$, where $n_k = \dim \mathfrak{g}_k$.

Remark 2.4. A Lie algebra admitting a family of dilations is nilpotent, and hence so is its associated connected, simply connected Lie group. The converse does not hold, i.e., not every nilpotent Lie group is homogeneous, although they exhaust a large class. See [FR16] for details. Indeed, the main class of Lie groups under our consideration is that of graded Lie groups.

Remark 2.5. A graded Lie group G is a homogeneous Lie group equipped with a family of weights ν_j, all of them positive rational numbers. Let us observe that if $\nu_i = \frac{a_i}{b_i}$ with a_i, b_i integers, and b is the least common multiple of the b_i's, the family of dilations

$$\mathbb{D}_r^{\mathfrak{g}} = \mathrm{Exp}(\ln(r^b)A) : \mathfrak{g} \longrightarrow \mathfrak{g},$$

has integer weights, $\nu_i = \frac{a_i b}{b_i}$. Thus, in this paper, we always assume that the weights ν_j, defining the family of dilations, are non-negative integers, which allows us to assume that the homogeneous dimension Q is a non-negative integer. This is a natural context for the study of Rockland operators (see [FR16, Remark 4.1.4]).

2.2. Fourier analysis on nilpotent Lie groups

Let G be a simply connected nilpotent Lie group. Then, the adjoint representation $\mathrm{ad}:\mathfrak{g}\to\mathrm{End}(\mathfrak{g})$ is nilpotent. Next, we define unitary and irreducible representations.

We say that π is a continuous, unitary, and irreducible representation of G if the following properties are satisfied:

- $\pi\in\mathrm{Hom}(G,\mathrm{U}(H_\pi))$ for some separable Hilbert space H_π; that is, $\pi(xy)=\pi(x)\pi(y)$ and, for the adjoint of $\pi(x)$, we have $\pi(x)^*=\pi(x^{-1})$ for every $x,y\in G$.
- The map $(x,v)\mapsto\pi(x)v$ from $G\times H_\pi$ into H_π is continuous.
- For every $x\in G$ and for every subspace $W_\pi\subset H_\pi$, if $\pi(x)W_\pi\subset W_\pi$, then $W_\pi=H_\pi$ or $W_\pi=\{0\}$.

Let $\mathrm{Rep}(G)$ be the set of unitary, continuous, and irreducible representations of G. The relation

$$\pi_1\sim\pi_2 \text{ if and only if there exists } A\in\mathscr{B}(H_{\pi_1},H_{\pi_2}) \text{ such that } A\pi_1(x)A^{-1}=\pi_2(x),$$

for every $x\in G$, is an equivalence relation, and the unitary dual of G, denoted by $\widehat{G}$, is defined via

$$\widehat{G}:=\mathrm{Rep}(G)/\sim.$$

Let us denote by $d\pi$ the Plancherel measure on $\widehat{G}$.

Next, we define the main object for our further analysis: the Fourier transform. Indeed, the Fourier transform of $f\in\mathscr{S}(G)$ (meaning that $f\circ\exp_G\in\mathscr{S}(\mathfrak{g})$, with $\mathfrak{g}\simeq\mathbb{R}^{\dim(G)}$) at $\pi\in\widehat{G}$ is defined by

$$\widehat{f}(\pi)=\int_G f(x)\pi(x)^*\,\mathrm{d}x: H_\pi\longrightarrow H_\pi,$$

and

$$\mathscr{F}_G:\mathscr{S}(G)\longrightarrow\mathscr{S}(\widehat{G}):=\mathscr{F}_G(\mathscr{S}(G)).$$

Remark 2.6 (Fourier inversion formula and Plancherel theorem)**.** Identifying each representation π with its equivalence class $[\pi]=\{\pi':\pi\sim\pi'\}$, for every $\pi\in\widehat{G}$, the Kirillov trace character Θ_π is defined by

$$(\Theta_\pi,f):=\mathbf{Tr}(\widehat{f}(\pi)),$$

where $\mathbf{Tr}$ denotes the trace of trace class operators on H_π. This defines Θ_π as a tempered distribution on $\mathscr{S}(G)$. In particular, the identity

$$f(e_G)=\int_{\widehat{G}}(\Theta_\pi,f)\,\mathrm{d}\pi$$

implies the Fourier inversion formula $f=\mathscr{F}_G^{-1}(\widehat{f})$, where

$$(\mathscr{F}_G^{-1}\sigma)(x):=\int_{\widehat{G}}\mathbf{Tr}(\pi(x)\sigma(\pi))\,\mathrm{d}\pi,\quad x\in G,$$

defines the inverse Fourier transform $\mathscr{F}_G^{-1}:\mathscr{S}(\widehat{G})\to\mathscr{S}(G)$. In this context, the Plancherel theorem asserts that

$$\|f\|_{L^2(G)}=\|\widehat{f}\|_{L^2(\widehat{G})},$$

where $L^2(\widehat{G})$ is the Hilbert space defined by

$$L^2(\widehat{G}) := \int_{\widehat{G}} H_\pi \otimes H_\pi^* \, \mathrm{d}\pi,$$

equipped with the norm

$$\|\sigma\|_{L^2(\widehat{G})} := \left(\int_{\widehat{G}} \|\sigma(\pi)\|_{\mathrm{HS}}^2 \, \mathrm{d}\pi \right)^{\frac{1}{2}},$$

and $\|\sigma(\pi)\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm of $\sigma(\pi)$.

2.3. Homogeneous linear operators and Rockland operators

There is a family of continuous linear operators that respect the action of the dilations of the group. These are called *homogeneous linear operators*. We introduce them in the following definition.

A linear operator $T : C^\infty(G) \to \mathscr{D}'(G)$ is said to be *homogeneous of degree* $\nu \in \mathbb{C}$ if, for every $r > 0$, the equality

$$T(f \circ D_r) = r^\nu (Tf) \circ D_r$$

holds for every $f \in \mathscr{D}(G)$.

Now, we introduce the main class of differential operators in the context of nilpotent Lie groups. The existence of these operators characterizes the family of graded Lie groups. We call them *Rockland operators*.

Let $\pi \in \widehat{G}$ be a unitary irreducible representation $\pi : G \to U(H_\pi)$. We denote by H_π^∞ the space of smooth vectors, that is, the set of all $v \in H_\pi$ such that the map $x \mapsto \pi(x)v$, $x \in G$, is smooth.

A *Rockland operator* is a left-invariant differential operator $\mathcal{R}$ which is homogeneous of positive degree $\nu = \nu_{\mathcal{R}} > 0$, and such that for every non-trivial unitary irreducible representation $\pi \in \widehat{G}$, the operator $\pi(\mathcal{R})$ is injective on H_π^∞.

The map $\sigma_{\mathcal{R}}(\pi) = \pi(\mathcal{R})$ is called the *symbol* associated to $\mathcal{R}$; it coincides with the infinitesimal representation of $\mathcal{R}$ viewed as an element of the universal enveloping algebra.

Remark 2.7. It can be shown that a Lie group G is graded if and only if there exists a differential Rockland operator on G.

Next, we record for our further analysis some aspects of the functional calculus for Rockland operators.

Remark 2.8 (Functional calculus for Rockland operators)**.** If the Rockland operator $\mathcal{R}$ is formally self-adjoint, then both $\mathcal{R}$ and $\pi(\mathcal{R})$ admit self-adjoint extensions on $L^2(G)$ and H_π, respectively. Preserving the same notation for their self-adjoint extensions, and denoting by E and E_π their corresponding spectral measures, we define the functional calculus by

$$f(\mathcal{R}) = \int_{-\infty}^{\infty} f(\lambda) \, \mathrm{d}E(\lambda), \quad \text{and} \quad \pi(f(\mathcal{R})) \equiv f(\pi(\mathcal{R})) = \int_{-\infty}^{\infty} f(\lambda) \, \mathrm{d}E_\pi(\lambda).$$

In general, we will reserve the notation $\{dE_A(\lambda)\}_{0 \leq \lambda < \infty}$ for the spectral measure associated with a positive and self-adjoint operator A on a Hilbert space H.

We now recall a lemma on dilations on the unitary dual $\widehat{G}$, which will be useful in our analysis of spectral multipliers. For the proof, see [FR16, Lemma 4.3].

Lemma 2.9. *For every $\pi \in \widehat{G}$ let us define*

$$D_r(\pi)(x) \equiv (r \cdot \pi)(x) := \pi(r \cdot x) \equiv \pi(D_r(x)), \tag{7}$$

for every $r > 0$ and all $x \in G$. Then, if $f \in L^\infty(\mathbb{R})$, we have

$$f(\pi^{(r)}(\mathcal{R})) = f(r^\nu \pi(\mathcal{R})).$$

Remark 2.10. For instance, for any $\alpha \in \mathbb{N}_0^n$ and for an arbitrary family $X_1, \dots, X_n$ of left-invariant vector fields, we will use the notation

$$[\alpha] := \sum_{j=1}^{n} \nu_j \alpha_j, \tag{8}$$

for the homogeneous degree of the operator $X^\alpha := X_1^{\alpha_1}, \dots, X_n^{\alpha_n}$, whose order is $|\alpha| := \sum_{j=1}^{n} \alpha_j$.

Remark 2.11. By considering the dilation $r \cdot x = D_r(x)$, $x \in G$, $r > 0$, the relation between the homogeneous structure of G and the Haar measure $\mathrm{d}x$ on G is given by (see [FR16, p. 100])

$$\int_G (f \circ D_r)(x)\, \mathrm{d}x = r^{-Q} \int_G f(x)\, \mathrm{d}x.$$

Note that if we define $f_r := r^{-Q} f(r^{-1} \cdot)$, then

$$\widehat{f_r}(\pi) = \int_G r^{-Q} f(r^{-1} \cdot x) \pi(x)^*\, \mathrm{d}x = \int_G f(y) \pi(r \cdot y)^*\, \mathrm{d}y = \widehat{f}(r \cdot \pi), \tag{9}$$

for any $\pi \in \widehat{G}$ and all $r > 0$, with $(r \cdot \pi)(y) = \pi(r \cdot y)$, $y \in G$, as in (7).

The following lemma present the action of the dilations of the group G into the kernels of bounded functions of a Rockland operator $\mathcal{R}$, see [FR16, p. 179].

Lemma 2.12. *Let $\varkappa \in L^\infty(\mathbb{R}_0^+)$ and let $r > 0$. Then, we have the following identity*

$$\varkappa(r^\nu \mathcal{R})\delta(x) = r^{-Q}[\varkappa(\mathcal{R})\delta](r^{-1} \cdot x), \tag{10}$$

for all $x \in G$.

3. Proof of the main theorem

3.1. Part 1: The Calderón–Zygmund decomposition

Let G be a graded Lie group with homogeneous dimension Q, and let $\mathcal{R}$ be a positive Rockland operator on G of homogeneous degree $\nu > 0$. In this section, we will prove that for a singular integral operator T satisfying conditions (4)–(5), there exists a constant $C > 0$ such that

$$\|Tf\|_{L^{1,\infty}(G)} := \sup_{\alpha > 0} \alpha \left| x \in G : |Tf(x)| > \alpha \right| \leq C \|f\|_{L^1(G)}, \tag{11}$$

with C independent of f.

We start by analysing the condition in (4).

Remark 3.1. Observe that, in view of the Plancherel theorem, the hypothesis (4) in Theorem 1.1 implies that the operator $(1+\mathcal{R})^{\frac{Q\theta}{2\nu}}T$ admits a bounded extension on $L^2(G)$. Indeed, the Plancherel theorem implies that

$$\left\|(1+\mathcal{R})^{\frac{Q\theta}{2\nu}}Tf\right\|^2_{L^2(G)} = \int_{\widehat{G}}\left\|(1+\pi(\mathcal{R}))^{\frac{Q\theta}{2\nu}}\widehat{K}(\pi)\widehat{f}(\pi)\right\|^2_{\mathrm{HS}}\,\mathrm{d}\pi$$
$$\leq \int_{\widehat{G}}\left\|(1+\pi(\mathcal{R}))^{\frac{Q\theta}{2\nu}}\widehat{K}(\pi)\right\|^2_{\mathrm{op}}\|\widehat{f}(\pi)\|^2_{\mathrm{HS}}\,\mathrm{d}\pi,$$

and in view of (4) we obtain

$$\left\|(1+\mathcal{R})^{\frac{Q\theta}{2\nu}}Tf\right\|^2_{L^2(G)} \leq C^2\int_{\widehat{G}}\|\widehat{f}(\pi)\|^2_{\mathrm{HS}}\,\mathrm{d}\pi = C^2\|f\|^2_{L^2(G)},$$

proving the boundedness of $(1+\mathcal{R})^{\frac{Q\theta}{2\nu}}T$ on $L^2(G)$. Since $(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}: L^2(G)\to L^2(G)$ is bounded, we deduce that

$$T=(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}(1+\mathcal{R})^{\frac{Q\theta}{2\nu}}T$$

is also bounded on $L^2(G)$.

Note also that the condition

$$\sup_{\pi\in\widehat{G}}\left\|\widehat{K}(\pi)(1+\pi(\mathcal{R}))^{\frac{Q\theta}{2\nu}}\right\|_{\mathrm{op}}<\infty,$$

provides the boundedness of the operator $T(1+\mathcal{R})^{\frac{Q\theta}{2\nu}}$ from $L^2(G)$ to itself.

For the proof of (11), fix $f\in L^1(G)$, and let us consider its Calderón–Zygmund decomposition, see Coifman and Weiss [CW71, pp. 73–74]. So, for any $\gamma,\alpha>0$, we have the decomposition

$$f=g+b=g+\sum_j b_j,$$

where the following properties are satisfied.

(1) $\|g\|_{L^\infty}\lesssim_G \gamma\alpha$ and $\|g\|_{L^1}\lesssim_G\|f\|_{L^1}$.

(2) The b_j's are supported in open balls $I_j=B(x_j,r_j)$ where they satisfy the cancellation property
$$\int_{I_j}b_j(x)\,\mathrm{d}x=0. \tag{12}$$

(3) Any component b_j satisfies the L^1-estimate
$$\|b_j\|_{L^1}\lesssim_G(\gamma\alpha)|I_j|. \tag{13}$$

(4) The sequence $\{|I_j|\}_j\in\ell^1$ and
$$\sum_j|I_j|\lesssim_G(\gamma\alpha)^{-1}\|f\|_{L^1}. \tag{14}$$

(5) $\|b\|_{L^1}\leq\sum_j\|b_j\|_{L^1}\lesssim_G\|f\|_{L^1}$.

(6) There exists $M_0\in\mathbb{N}$, such that any point $x\in G$ belongs at most to M_0 balls of the collection I_j.

So, by fixing $\alpha\gamma > 0$, note that in terms of g, b and of f one has the trivial estimate

$$|\{x : |Tf(x)| > \alpha\}| \leq |\{x : |Tg(x)| > \alpha/2\}| + |\{x : |Tb(x)| > \alpha/2\}|.$$

The estimates $\|g\|_{L^\infty} \lesssim \gamma\alpha$ and $\|g\|_{L^1} \lesssim \|f\|_{L^1}$, imply that

$$\|g\|_{L^2}^2 \leq \|g\|_{L^\infty}\|g\|_{L^1} \lesssim (\gamma\alpha)\|f\|_{L^1}.$$

So, by applying the Chebishev inequality and the L^2-boundedness of T, we have

$$\begin{aligned}|\{x : |Tg(x)| > \alpha/2\}| &\lesssim 2^2\alpha^{-2}\|Tg\|_{L^2}^2 \leq (2\|T\|_{\mathscr{B}(L^2)})^2\alpha^{-2}\|g\|_{L^2}^2 \\ &\leq (2\|T\|_{\mathscr{B}(L^2)})^2\alpha^{-2}(\gamma\alpha)\|f\|_{L^1} \lesssim \|T\|_{\mathscr{B}(L^2)}^2\gamma\alpha^{-1}\|f\|_{L^1} \\ &\lesssim_\gamma \alpha^{-1}\|f\|_{L^1}.\end{aligned}$$

In what follows, let us denote $I^* = \bigcup I_j^*$, where

$$I_j^* = B(x_j, 2r_j) = \left\{x \in G : |x_j^{-1}x| < 2R_j\right\},$$

and let us make use of the doubling condition in order to have the estimate

$$|I_j^*| \sim 2^Q|I_j|,$$

from which it follows that

$$|I^*| \lesssim \sum_j |I_j^*| \lesssim \sum_j |I_j| \lesssim \gamma^{-1}\alpha^{-1}\|f\|_{L^1}.$$

Consequently, we have the estimates

$$\begin{aligned}|\{x : |Tb(x)| > \alpha/2\}| &\leq |I^*| + |\{x \in G \setminus I^* : |Tb(x)| > \alpha/2\}| \\ &\lesssim \gamma^{-1}\alpha^{-1}\|f\|_{L^1} + |\{x \in G \setminus I^* : |Tb(x)| > \alpha/2\}|, \\ &\lesssim_\gamma \alpha^{-1}\|f\|_{L^1} + |\{x \in G \setminus I^* : |Tb(x)| > \alpha/2\}|.\end{aligned}$$

So, to conclude the inequality (11) we have to prove that

$$\sup_{\alpha>0} \alpha|\{x \in G \setminus I^* : |Tb(x)| > \alpha/2\}| \leq C\|f\|_{L^1}, \tag{15}$$

with C independent of f. So, the proof of Theorem 1.1 consists of estimating the term

$$|\{x \in G \setminus I^* : |Tb(x)| > \alpha/2\}|.$$

3.2. Part 2: Proof of the oscillating case

The relevant case in Theorem 1.1 is when $\theta \in (0, 1)$. We begin the proof of the main theorem by explaining this claim.

Proof of Theorem 1.1. We first consider the case $0 < \theta < 1$. Indeed, the statement for $\theta = 0$ in Theorem 1.1 follows from the fundamental theorem of singular integrals due to Coifman and Weiss; see [CW71, Theorem 2.4, p. 74]. In fact, a graded Lie group satisfies the doubling condition, and hence it is a homogeneous topological space in the sense of Coifman and Weiss [CW71].

For $0 < \theta < 1$, we require some geometric transformations on the support of K, which we present in the next subsection. To achieve this, we will use two modern techniques: the stability of the powers of Rockland operators within the Hörmander classes $S^m_{1,0}(G \times \widehat{G})$, and the Calderón–Vaillancourt theorem on graded Lie groups, as presented in [FR16, Section 5.7, Theorem 5.7.1].

Following our hypothesis, from now on, let us assume that the diameter of the support of K is small, for instance, that

$$\operatorname{diam}(\operatorname{supp}(K)) < c \leq 1,$$

where $0 < c < 1$ is suitable for our purposes. With this in mind, the next step is to construct a good replacement $\widetilde{b}$ for b, which allows us to exploit the hypothesis in (5).

3.3. Part 3: The function ϕ. New family of supports

Let us consider a function ϕ on G such that

$$\int_G \phi(x)\,\mathrm{d}x = 1, \quad \text{and} \quad \phi \in C_0^\infty(G,[0,\infty)) \cap \operatorname{Dom}\left[\mathcal{R}^{-\frac{Q\theta}{2\nu}}\right]. \tag{16}$$

For $\varepsilon > 0$, define

$$\phi(y,\varepsilon) := \varepsilon^{-Q}\phi(\varepsilon^{-1}\cdot x). \tag{17}$$

Now, for any j, define

$$\phi_j(y) := \phi\left(y, 2^{-\frac{1}{1-\theta}}\operatorname{diam}(I_j)^{\frac{1}{1-\theta}}\right), \tag{18}$$

$$\widetilde{b}_j := b_j(\,\cdot\,) * \phi_j, \tag{19}$$

and

$$\widetilde{b} := \sum_j \widetilde{b}_j. \tag{20}$$

Note that

$$Tb(x) = \sum_j Tb_j(x), \tag{21}$$

for a.e. $x \in G$. It is important to mention that in (21), the sums on the right-hand side only run over j such that $\operatorname{diam}(I_j) < c$. Indeed, for all $x \in G \setminus I^*$, the property of the support $\operatorname{diam}(\operatorname{supp}(K)) < c$ implies that for all j with $\operatorname{diam}(I_j) \geq c$, we have

$$Tb_j = b_j * K = 0.$$

So, we only need to analyze the case where $\operatorname{diam}(I_j) < c$. Indeed, for $x \in G \setminus I^*$ and j such that $\operatorname{diam}(I_j) \geq c$, we have

$$b_j * K(x) = \int_{I_j} K(y^{-1}x) b_j(y)\,\mathrm{d}y.$$

Since in the integral above $x \in G \setminus I^*$ and $y \in I_j$, we have

$$|y^{-1}x| = \operatorname{dist}(x,y) > \operatorname{diam}(I_j) > c,$$

we conclude that the element $y^{-1}x$ is not in the support of K, and thus the integral vanishes.

Now, going back to the analysis of (15), note that

$$\begin{aligned}
&\left|\left\{x \in G \setminus I^* : |Tb(x)| > \frac{\alpha}{2}\right\}\right| \\
&\quad \leq \left|\left\{x \in G \setminus I^* : |Tb(x) - T\widetilde{b}(x)| > \frac{\alpha}{4}\right\}\right| + \left|\left\{x \in G \setminus I^* : |T\widetilde{b}(x)| > \frac{\alpha}{4}\right\}\right| \\
&\quad \leq \frac{4}{\alpha}\left\|T(b-\widetilde{b})\right\|_{L^1(G\setminus I^*)} + \left|\left\{x \in G \setminus I^* : |T\widetilde{b}(x)| > \frac{\alpha}{4}\right\}\right|,
\end{aligned}$$

and let us take into account the estimate:

$$\|T(b-\widetilde{b})\|_{L^1(G\setminus I^*)}=\int_{G\setminus I^*}|Tb(x)-T\widetilde{b}(x)|\,\mathrm{d}x\quad\leq\sum_j\int_{G\setminus I^*}\left|Tb_j(x)-T\widetilde{b}_j(x)\right|\,\mathrm{d}x.$$

We are going to prove that $T\widetilde{b}$ and $T\widetilde{b}_j$ are good replacements for Tb and Tb_j, respectively, on the set $G\setminus I^*$. Observe that

$$\begin{aligned}
&\int_{G\setminus I^*}\left|Tb_j(x)-T\widetilde{b}_j(x)\right|\,\mathrm{d}x\\
&\qquad=\int_{G\setminus I^*}\left|b_j*K(x)-\widetilde{b}_j*K(x)\right|\,\mathrm{d}x\\
&\qquad=\int_{G\setminus I^*}|b_j*K(x)-[b_j*\phi_j*K](x)|\,\mathrm{d}x\\
&\qquad=\int_{G\setminus I^*}\left|\int_{I_j}K(y^{-1}x)b_j(y)\,\mathrm{d}y-\int_{I_j}(\phi_j*K)(y^{-1}x)b_j(y)\,\mathrm{d}y\right|\,\mathrm{d}x\\
&\qquad\leq\int_{I_j}\int_{G\setminus I^*}\left|K(y^{-1}x)-\phi_j*K(y^{-1}x)\right|\,\mathrm{d}x|b_j(y)|\,\mathrm{d}y\\
&\qquad\leq\int_{I_j}\int_{|z|>\mathrm{diam}(I_j)}|K(z)-\phi_j*K(z)|\,\mathrm{d}z|b_j(y)|\,\mathrm{d}y\\
&\qquad\leq\int_{I_j}\int_{|z|>\mathrm{diam}(I_j)}|K(z)-\phi_j*K(z)|\,\mathrm{d}z|b_j(y)|\,\mathrm{d}y\\
&\qquad=\int_{I_j}|b_j(y)|\,\mathrm{d}y\int_{|z|>\mathrm{diam}(I_j)}|K(z)-\phi_j*K(z)|\,\mathrm{d}z,
\end{aligned}$$

where, in the last line, we have used the change of variables $x\mapsto z=y^{-1}x$, and then we observe that $|z|>\mathrm{diam}(I_j)$ when $x\in G\setminus I^*$ and $y\in I_j$. Using that ϕ_j is supported in a ball of radius

$$R_j:=2^{-\frac{1}{1-\theta}}\,\mathrm{diam}(I_j)^{\frac{1}{1-\theta}},$$

and that $\|\phi_j\|_{L^1}=1$, we have that

$$\begin{aligned}
&\int_{|z|>\mathrm{diam}(I_j)}|K(z)-\phi_j*K(z)|\,\mathrm{d}z\\
&\qquad=\int_{|z|>\mathrm{diam}(I_j)}\Bigg|K(z)\int_{|y|<2^{-\frac{1}{1-\theta}}\,\mathrm{diam}(I_j)^{\frac{1}{1-\theta}}}\phi_j(y)\,\mathrm{d}y\\
&\qquad\qquad\qquad-\int_{|y|<2^{-\frac{1}{1-\theta}}\,\mathrm{diam}(I_j)^{\frac{1}{1-\theta}}}K(y^{-1}z)\phi_j(y)\,\mathrm{d}y\Bigg|\,\mathrm{d}z\\
&\qquad=\int_{|z|>\mathrm{diam}(I_j)}\left|\int_{|y|<2^{-\frac{1}{1-\theta}}\,\mathrm{diam}(I_j)^{\frac{1}{1-\theta}}}(K(z)-K(y^{-1}z))\phi_j(y)\,\mathrm{d}y\right|\,\mathrm{d}z\\
&\qquad\leq\int_{|y|<2^{-\frac{1}{1-\theta}}\,\mathrm{diam}(I_j)^{\frac{1}{1-\theta}}}\int_{|z|>\mathrm{diam}(I_j)}\left|K(y^{-1}z)-K(z)\right|\,\mathrm{d}z|\phi_j(y)|\,\mathrm{d}y\\
&\qquad\lesssim\frac{1}{R_j^Q}\int_{|y|<R_j}\int_{|z|>2R_j^{(1-\theta)}}\left|K(y^{-1}z)-K(z)\right|\,\mathrm{d}z\,\mathrm{d}y\ \leq\ [K]_{H_{\infty,\theta}(G)},
\end{aligned}$$

where we have used that $2R_j^{1-\theta}=\mathrm{diam}(I_j)$. Now, the inequalities above allow us to finish the estimate of $\|T(b-\widetilde{b})\|_{L^1(G\setminus I^*)}$. Indeed,

$$\begin{aligned}\|T(b-\widetilde{b})\|_{L^1(G\setminus I^*)} &\leq \sum_j \int_{I_j}\int_{G\setminus I^*} \left|K(y^{-1}x)-\phi_j * K(y^{-1}x)\right| \mathrm{d}x|b_j(y)|\,\mathrm{d}y \\ &\leq \sum_j \int_{I_j}\int_{|z|>\mathrm{diam}(I_j)} |K(z)-\phi_j * K(z)|\,\mathrm{d}z|b_j(y)|\,\mathrm{d}y \\ &\lesssim [K]_{H_{\infty,\theta}(G)} \sum_j \int_{I_j} |b_j(y)|\,\mathrm{d}y \leq [K]_{H_{\infty,\theta}(G)}\|b\|_{L^1} \\ &\lesssim [K]_{H_{\infty,\theta}(G)}\|f\|_{L^1}.\end{aligned}$$

Putting together the estimates above we deduce that

$$\begin{aligned}\left|\left\{x\in G\setminus I^* : |Tb(x)|>\frac{\alpha}{2}\right\}\right| &\leq \frac{4}{\alpha}\|Tb-T\widetilde{b}\|_{L^1(G\setminus I^*)} + \left|\left\{x\in G\setminus I^* : |T\widetilde{b}(x)|>\frac{\alpha}{4}\right\}\right| \\ &\lesssim \frac{4}{\alpha}[K]_{H_{\infty,\theta}(G)}\|f\|_{L^1} + \left|\left\{x\in G\setminus I^* : |T\widetilde{b}(x)|>\frac{\alpha}{4}\right\}\right|.\end{aligned}$$

Now, we will estimate the second term on the right hand side of this inequality. First, note that

$$\begin{aligned}\left|\left\{x\in G\setminus I^* : |T\widetilde{b}(x)|>\frac{\alpha}{4}\right\}\right| &\leq \left|\left\{x\in G\setminus I^* : |T\widetilde{b}(x)|^2>\frac{\alpha^2}{16}\right\}\right| \\ &\leq \frac{16}{\alpha^2}\|T\widetilde{b}\|_{L^2}^2.\end{aligned}$$

Now, using (4) we deduce that $T(1+\mathcal{R})^{\frac{Q\theta}{2\nu}}$ is bounded on L^2 (see Remark 3.1), and then

$$\|T\widetilde{b}\|_{L^2}^2 \leq \left\|T(1+\mathcal{R})^{\frac{Q\theta}{2\nu}}\right\|_{\mathscr{B}(L^2)}^2 \left\|(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}\widetilde{b}\right\|_{L^2}^2. \tag{22}$$

In the case of $G=\mathbb{R}^n$, and of the positive Laplace operator $\mathcal{R}=-\Delta$, it was proved by Fefferman that the function $F=(1+\Delta)^{-\frac{Q\theta}{2\nu}}\widetilde{b}$, admits a nice decomposition $F=F_1+F_2$, where $\|F_2\|_{L^2}^2\leq C\alpha\gamma\|f\|_{L^1}$, and $F_1=\sum_{j\,:\,\mathrm{diam}(I_j)<1}F_1^j$, where $\|F_1^j\|_{L^2}^2\leq A'\alpha^2|I_j|$. We will extend Fefferman's decomposition to an arbitrary Rockland operator $\mathcal{R}$ in Lemma 3.2 below.

3.4. Part 4: A Fefferman type decomposition for Rockland operators

In order to estimate the L^2-norm $\|(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}\widetilde{b}\|_{L^2}^2$, let us use the following lemma whose proof we postpone for a moment.

Lemma 3.2. *The function $F:=(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}\widetilde{b}$ can be decomposed in a sum $F=F_1+F_2$, where $\|F_2\|_{L^2}^2\leq C\alpha\gamma\|f\|_{L^1}$, and F_1 is also a sum of functions F_1^j with the following property:*

- *There exists $M_0\in\mathbb{N}$, and $A'>0$, such that $F_1=\sum_{j\,:\,\mathrm{diam}(I_j)<1}F_1^j$, $\|F_1^j\|_{L^2}^2\leq A'\alpha^2|I_j|$, and for any $x\in G$, there are at most M_0 values of j such that $F_1^j(x)\neq 0$.*

Let us continue with the proof of Theorem 1.1. Using Lemma 3.2 and the inequalities in (14) and (22) we have that

$$\begin{aligned}\|T\widetilde{b}\|_{L^2}^2 &\lesssim \|F_1\|_{L^2}^2 + \|F_2\|_{L^2}^2 \lesssim \alpha\gamma\|f\|_{L^1} + \sum_j \|F_1^j\|_{L^2}^2\\ &\lesssim \alpha\gamma\|f\|_{L^1} + \sum_j \alpha^2|I_j| \lesssim \alpha\gamma\|f\|_{L^1} + \alpha^2(\gamma\alpha)^{-1}\|f\|_{L^1}\\ &= (\gamma^{-1}+\gamma)\alpha\|f\|_{L^1}.\end{aligned}$$

Consequently

$$\left|\left\{x\in G\setminus I^* : |T\widetilde{b}(x)| > \frac{\alpha}{4}\right\}\right| \lesssim \frac{16}{\alpha^2}\|T\widetilde{b}\|_{L^2}^2 \lesssim (\gamma^{-1}+\gamma)\alpha^{-1}\|f\|_{L^1}.$$

Thus, we have proved that

$$|\{x\in\mathbb{R}^n : |Tf(x)|>\alpha\}| \leq C_{\gamma,[K]_{H_{\infty,\theta}(G)}}\alpha^{-1}\|f\|_{L^1}, \tag{23}$$

with $C := C_{\gamma,[K]_{H_{\infty,\theta}(G)}}$ independent of f.

Because γ is fixed, we have proved the weak $(1,1)$-type of T. Thus, the proof is complete once we prove the statement in Lemma 3.2. To do so, let us consider the right-convolution kernel $k_\theta := (1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}\delta$ of the operator $(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}$, and let us split the function $(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}\widetilde{b}(x)$ as follows:

$$\begin{aligned}(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}\widetilde{b}(x) &= \sum_j (1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}\widetilde{b}_j(x) = \sum_j \widetilde{b}_j * k_\theta(x)\\ &= \sum_{j:x\sim I_j}\widetilde{b}_j * k_\theta(x) + \sum_{j:x\nsim I_j}\widetilde{b}_j * k_\theta(x) =: G_1(x)+G_2(x),\end{aligned}$$

where $G_1(x) := \sum_{j:x\sim I_j}\widetilde{b}_j * k_\theta(x)$ and $G_2(x) := \sum_{j:x\nsim I_j}\widetilde{b}_j * k_\theta(x)$. We denote by $x\sim I_j$ if x belongs to I_j or to some $I_{j'}$ with a non-empty intersection with I_j. By the properties of these sets, there are at most M_0 sets $I_{j'}$ such that $I_j\cap I_{j'}\neq\emptyset$. Also, the notation $x\nsim I_j$ will be used to define the opposite of the previous property.

Let us prove the estimate

$$\|G_2\|_{L^2}^2 \leq C\alpha\gamma\|f\|_{L^1}. \tag{24}$$

Observe that

$$\begin{aligned}\|G_2\|_{L^1} &= \int_G\left|\sum_{j:x\nsim I_j} b_j*\phi_j*k_\theta(x)\right|\mathrm{d}x \leq \sum_{j:x\nsim I_j}\int_G |b_j*\phi_j*k_\theta(x)|\,\mathrm{d}x\\ &\leq \sum_j \|b_j*\phi_j*k_\theta\|_{L^1} \leq \sum_j \|b_j\|_{L^1}\|\phi_j*k_\theta\|_{L^1}.\end{aligned}$$

Note that $(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}$ is a pseudo-differential operator of order $-Q\theta/2$, and consequently, its kernel satisfies the estimate (see [FR16, Theorem 5.4.1])

$$|k_\theta(x)| \leq C_N|x|^{-\left(-\frac{Q\theta}{2}+Q\right)} \lesssim |x|^{-Q\left(1-\frac{\theta}{2}\right)},\quad x\in B(e,N)\setminus\{e\},$$

for any $N\in\mathbb{N}$, while outside of any ball $B(e,N)$, one has, for any $M>0$, that there exists $C_M>0$ such that (see [FR16, Theorem 5.4.1])

$$|k_\theta(x)| \leq C_M(1+|x|)^{-M}.$$

The condition $0<\theta<1$ implies that k_θ is an integrable distribution, and then

$$\|\phi_j * k_\theta\|_{L^1} \lesssim \|\phi_j\|_{L^1}\|k_\theta\|_{L^1} = \|k_\theta\|_{L^1} < \infty.$$

So, we have that

$$\|G_2\|_{L^1} \lesssim \sum_j \|b_j\|_{L^1} \lesssim \|f\|_{L^1}.$$

So, for the proof of (24), in view of the inequality $\|G_2\|_{L^2}^2 \leq \|G_2\|_{L^1}\|G_2\|_{L^\infty}$ is enough to show that

$$\|G_2\|_{L^\infty} \lesssim \alpha\gamma.$$

To do this, let us consider j such that $x \nsim I_j$. Since $\widetilde{b}_j * k_\theta = b_j * \phi_j * k_\theta$, one has that

$$\begin{aligned}|b_j * \phi_j * k_\theta(x)| &\leq \int_{I_j} \left|(\phi_j * k_\theta)(y^{-1}x)\right| |b_j(y)|\,\mathrm{d}y \leq \sup_{y\in I_j}\left|(\phi_j * k_\theta)(y^{-1}x)\right| \int_{I_j} |b_j(y)|\,\mathrm{d}y \\ &= \sup_{y\in I_j}\left|\phi_j * k_\theta(y^{-1}x)\right| |I_j| \times \frac{1}{|I_j|}\int_{I_j}|b_j(y)|\,\mathrm{d}y.\end{aligned}$$

To continue, we follow the observation of Fefferman in [Fef70, p. 26], which states that, in view of the property $x \nsim I_j$, we have that $\phi_j * |k_\theta|(y^{-1}x)$ is essentially constant over the ball $I_j = B(x_j, r_j)$, and we can estimate

$$\sup_{y\in I_j}\left|\phi_j * k_\theta(y^{-1}x)\right| |I_j| \leq \sup_{y\in I_j} \phi_j * |k_\theta\left|(y^{-1}x)\right| |I_j| \lesssim \int_{I_j} \phi_j * \left|k_\theta(y'^{-1}x)\right|\,\mathrm{d}y'. \tag{25}$$

On the other hand, using again the positivity of ϕ_j leads to

$$\begin{aligned}&\int_{I_j}\left|\phi_j * k_\theta(y'^{-1}x)\right|\,\mathrm{d}y' \frac{1}{|I_j|}\int_{I_j}|b_j(y)|\,\mathrm{d}y \\ &\qquad\leq \int_G (\phi_j * |k_\theta|)(y'^{-1}x)\left(\frac{1}{|I_j|}\int_{I_j}|b_j(y)|\,\mathrm{d}y\right)1_{I_j}(y')\,\mathrm{d}y' \\ &\qquad= \left(\frac{1}{|I_j|}\int_{I_j}|b_j(y)|\,\mathrm{d}y \times 1_{I_j}\right) * \phi_j * |k_\theta|(x).\end{aligned}$$

Consequently, we have that

$$\begin{aligned}|G_2(x)| &\leq \sum_{j:x\nsim I_j}\left|\widetilde{b}_j * k_\theta(x)\right| \lesssim \sum_{j:x\nsim I_j}\left(\frac{1}{|I_j|}\int_{I_j}|b_j(y)|\,\mathrm{d}y \times 1_{I_j}\right) * \phi_j * |k_\theta|(x) \\ &\lesssim \sum_{j:x\nsim I_j}\gamma\alpha \times 1_{I_j} * \phi_j * |k_\theta|(x) = \int_G \sum_{j:x\nsim I_j}\gamma\alpha\times 1_{I_j} * \phi_j(z)|k_\theta(z^{-1}x)|\,\mathrm{d}z \\ &\lesssim \gamma\alpha\|k_\theta\|_{L^1}\left\|\sum_{j:x\nsim I_j} 1_{I_j} * \phi_j\right\|_{L^\infty}.\end{aligned}$$

By observing that the supports of the functions $1_{I_j} * \phi_j$ have bounded overlaps we have that

$$\left\|\sum_{j:x\nsim I_j} 1_{I_j} * \phi_j\right\|_{L^\infty} < \infty,$$

and that $\|G_2\|_{L^\infty} \lesssim \gamma\alpha$. It remains only to prove that $\|G_1\|_{L^2}^2 \lesssim \alpha\gamma\|f\|_{L^1}$. Let us define

$$G^j(x) := \begin{cases} b_j * \phi_j * k_\theta(x), & x \in I_j, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Then $G_1 = \sum_j G^j$, and in view of the finite overlap of the balls I_j's, there exists $M_0 \in \mathbb{N}$ such that for any $x \in G$, $G^j(x) \neq 0$ for at most M_0 values of j. Therefore, we have that

$$\begin{aligned}\int_G |G_1(x)|^2 \,\mathrm{d}x &\leq M_0 \sum_j \int_G |G^j(x)|^2 \,\mathrm{d}x = \sum_j \int_{I_j} |b_j * \phi_j * k_\theta(x)|^2 \,\mathrm{d}x \\ &\leq \sum_j \|b_j\|_{L^1}^2 \|\phi_j * k_\theta\|_{L^2}^2 \leq \sum_j \alpha^2\gamma^2 |I_j|^2 \|\phi_j * k_\theta\|_{L^2}^2.\end{aligned}$$

To estimate the L^2-norm $\|\phi_j * k_\theta\|_{L^2}^2$, we will use the dilation properties of the spectral calculus of the Rockland operator $\mathcal{R}$.

3.5. Final Part: control of the Bessel operator associated to the Rockland operator

Now, let us apply the Plancherel theorem:

$$\|\phi_j * k_\theta\|_{L^2(G)}^2 = \left\|(1+\pi(\mathcal{R}))^{-\frac{Q\theta}{2\nu}} \widehat{\phi}_j\right\|_{L^2(\widehat{G})}^2.$$

Define for any j,

$$\psi_j(x) := R_j^Q \phi_j(R_j \cdot x),\, x \in G. \tag{27}$$

Observe that for any j,

$$\psi_j(x) := R_j^Q \phi_j(R_j \cdot x) = R_j^Q R_j^{-Q} \phi\left(R_j \cdot R_j^{-1} \cdot x\right) = \phi(x). \tag{28}$$

However, in order to use the dilation properties of the Fourier transform of distributions, let us keep in mind the identity for $\psi_j = \phi$ in (27). Also, let us remark that we have the following identities for the function ϕ_j:

$$\phi_j(x) = R_j^{-Q} \psi_j\left({R_j}^{-1} \cdot x\right), \quad x \in G.$$

as well as for its Fourier transform (in view of (9))

$$\widehat{\phi_j}(\pi) = \widehat{\psi_j}(R_j \cdot \pi), \quad \pi \in \widehat{G}.$$

Also, note that $\|\psi_j\|_{L^\infty} = 1$, and $\mathrm{supp}(\psi_j) \subset B(e,1)$. Using the Plancherel theorem, we have that

$$\begin{aligned}\|\phi_j * k_\theta\|_{L^2(G)}^2 &= \left\|(1+\pi(\mathcal{R}))^{-\frac{Q\theta}{2\nu}} \widehat{\phi}_j\right\|_{L^2(\widehat{G})}^2 \\ &= \left\|(1+\pi(\mathcal{R}))^{-\frac{Q\theta}{2\nu}} \widehat{\psi_j}\left(R_j \cdot \pi\right)\right\|_{L^2(\widehat{G})}^2 \\ &= \int_{\widehat{G}} \left\|(1+\pi(\mathcal{R}))^{-\frac{Q\theta}{2\nu}} \widehat{\psi_j}\left(R_j \cdot \pi\right)\right\|_{\mathrm{HS}}^2 \,\mathrm{d}\pi \\ &= {R_j}^{-Q} \int_{\widehat{G}} \left\|\left(1+({R_j}^{-1} \cdot \pi)(\mathcal{R})\right)^{-\frac{Q\theta}{2\nu}} \widehat{\psi_j}(\pi)\right\|_{\mathrm{HS}}^2 \,\mathrm{d}\pi,\end{aligned}$$

where we have used the rescaling property in Remark 2.11. Let us write:

$$\left(R_j{}^{-1}\cdot\pi\right)(\mathcal{R})=\widehat{\mathcal{R}\delta}\left(R_j{}^{-1}\cdot\pi\right).$$

Using the functional calculus for $\varkappa(t):=(1+t)^{-\frac{Q\theta}{2\nu}}$, we have that

$$\forall\, r>0,\ \ \forall\, x\in G,\quad \varkappa(r^{\nu}\mathcal{R})\delta(x)=r^{-Q}\varkappa(\mathcal{R})\delta(r^{-1}\cdot x),$$

that is equivalent to say that

$$\forall\, r>0,\ \ \forall\, x\in G,\quad (1+r^{\nu}\mathcal{R})^{-\frac{Q\theta}{2\nu}}\delta(x)=r^{-Q}(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}\delta(r^{-1}\cdot x).$$

Taking the Fourier transform in both sides of this equality, and using the functional calculus of Rockland operators, we obtain

$$\begin{aligned}\forall\, r>0,\ \ \forall\,\pi\in\widehat{G},\quad (1+r^{\nu}\pi(\mathcal{R}))^{-\frac{Q\theta}{2\nu}}&=\mathscr{F}\left[r^{-Q}(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}\delta(r^{-1}\cdot)\right](\pi)\\&=\mathscr{F}\left[(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}\delta(\cdot)\right](r\cdot\pi).\end{aligned}$$

Thus, we have

$$\forall\, r>0,\ \ \forall\,\pi\in\widehat{G},\quad (1+r^{\nu}\pi(\mathcal{R}))^{-\frac{Q\theta}{2\nu}}=\mathscr{F}\left[(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}\delta(\cdot)\right](r\cdot\pi). \tag{29}$$

By taking $r=R_j^{-1}$ in (29) we have that

$$\begin{aligned}\forall\, j,\ \ \forall\,\pi\in\widehat{G},\quad (1+R_j^{-\nu}\pi(\mathcal{R}))^{-\frac{Q\theta}{2\nu}}&=\mathscr{F}\left[(1+\mathcal{R})^{-\frac{Q\theta}{2\nu}}\delta(\cdot)\right]\left(R_j^{-1}\cdot\pi\right)\\&=\left(1+\left(R_j^{-1}\cdot\pi\right)(\mathcal{R})\right)^{-\frac{Q\theta}{2\nu}}.\end{aligned}$$

Note that

$$\begin{aligned}&R_j{}^{-Q}\int_{\widehat{G}}\left\|\left(1+(R_j{}^{-1}\cdot\pi)(\mathcal{R})\right)^{-\frac{Q\theta}{2\nu}}\widehat{\psi_j}(\pi)\right\|_{\mathrm{HS}}^2\mathrm{d}\pi\\&\qquad=R_j{}^{-Q}\int_{\widehat{G}}\left\|\left(1+R_j^{-\nu}\pi(\mathcal{R})\right)^{-\frac{Q\theta}{2\nu}}\widehat{\psi_j}(\pi)\right\|_{\mathrm{HS}}^2\mathrm{d}\pi\\&\qquad=R_j{}^{-Q}R_j^{Q\theta}\int_{\widehat{G}}\left\|\left(R_j^{\nu}+\pi(\mathcal{R})\right)^{-\frac{Q\theta}{2\nu}}\widehat{\psi_j}(\pi)\right\|_{\mathrm{HS}}^2\mathrm{d}\pi.\end{aligned}$$

Now, let us prove that for $a=R_j^{\nu}$ and $b=\frac{Q\theta}{2\nu}$, and $\tau(\pi):=\widehat{\psi_j}(\pi)$, we have the inequality:

$$\left\|\pi(a+\mathcal{R})^{-b}\tau(\pi)\right\|_{\mathrm{HS}}^2\leq\left\|\pi(\mathcal{R})^{-b}\tau(\pi)\right\|_{\mathrm{HS}}^2. \tag{30}$$

Indeed, let us denote by $\{dE_{\pi(\mathcal{R})}(\lambda)\}_{\lambda>0}$ the spectral measure associated to the operator $\pi(\mathcal{R})$. If $B_\pi=\{e_{\pi,k}\}_{k=1}^{\infty}$ is a basis of the representation space H_π, then,

$$\begin{aligned}\left\|\pi(a+\mathcal{R})^{-b}\tau(\pi)\right\|_{\mathrm{HS}}^2&=\sum_{k=1}^{\infty}\left\|\pi(a+\mathcal{R})^{-b}\tau(\pi)e_{\pi,k}\right\|_{H_\pi}^2\\&=\sum_{k=1}^{\infty}\left\|\int_0^{\infty}(a+\lambda)^{-b}\,\mathrm{d}E_{\pi(\mathcal{R})}(\lambda)\tau(\pi)e_{\pi,k}\right\|_{H_\pi}^2\\&=\sum_{k=1}^{\infty}\int_0^{\infty}(a+\lambda)^{-2b}\,\mathrm{d}\|E_{\pi(\mathcal{R})}(\lambda)\tau(\pi)e_{\pi,k}\|_{H_\pi}^2\end{aligned}$$

$$\begin{aligned}
&\le \sum_{k=1}^{\infty}\int_0^{\infty}\lambda^{-2b}\,\mathrm{d}\|E_{\pi(\mathcal{R})}(\lambda)\tau(\pi)e_{\pi,k}\|_{H_\pi}^2\\
&= \sum_{k=1}^{\infty}\left\|\int_0^{\infty}\lambda^{-b}\,\mathrm{d}E_{\pi(\mathcal{R})}(\lambda)\tau(\pi)e_{\pi,k}\right\|_{H_\pi}^2\\
&= \sum_{k=1}^{\infty}\left\|\pi(\mathcal{R})^{-b}\tau(\pi)e_{\pi,k}\right\|_{H_\pi}^2\\
&= \|\pi(\mathcal{R})^{-b}\tau(\pi)\|_{\mathrm{HS}}^2,
\end{aligned}$$

as desired. So, we can estimate

$$\begin{aligned}
&{R_j}^{-Q}\int_{\widehat{G}}\left\|\left(1+({R_j}^{-1}\cdot\pi)(\mathcal{R})\right)^{-\frac{Q\theta}{2\nu}}\widehat{\psi_j}(\pi)\right\|_{\mathrm{HS}}^2\,\mathrm{d}\pi\\
&\qquad\lesssim {R_j}^{-Q}R_j^{Q\theta}\int_{\widehat{G}}\left\|\pi(\mathcal{R})^{-\frac{Q\theta}{2\nu}}\widehat{\psi_j}(\pi)\right\|_{\mathrm{HS}}^2\,\mathrm{d}\pi\\
&\qquad= {R_j}^{-Q(1-\theta)}\int_{\widehat{G}}\left\|\pi(\mathcal{R})^{-\frac{Q\theta}{2\nu}}\widehat{\psi_j}(\pi)\right\|_{\mathrm{HS}}^2\,\mathrm{d}\pi\\
&\qquad= {R_j}^{-Q(1-\theta)}\left\|\mathcal{R}^{-\frac{Q\theta}{2\nu}}\psi_j\right\|_{L^2(G)}^2\\
&\qquad= {R_j}^{-Q(1-\theta)}\left\|\mathcal{R}^{-\frac{Q\theta}{2\nu}}\phi\right\|_{L^2(G)}^2\\
&\qquad\lesssim |I_j|^{-1},
\end{aligned}$$

where we have used (28) and the estimate $|I_j|\sim R_j^{Q(1-\theta)}$ because $2R_j^{1-\theta}=\mathrm{diam}(I_j)$. Thus, we deduce that

$$\begin{aligned}
\int_G|G_1(x)|^2\,\mathrm{d}x &\le \sum_j \alpha^2\gamma^2|I_j|^2\,\|\phi_j*k_\theta\|_{L^2}^2 \lesssim \sum_j\alpha^2\gamma^2|I_j|^2\,|I_j|^{-1}\\
&= \alpha^2\gamma^2\sum_j|I_j| \lesssim \alpha\gamma\|f\|_{L^1}\lesssim_\gamma \alpha\|f\|_{L^1},
\end{aligned}$$

in view of (14). Consequently, we can take $F_2:=G_2$, $F_1:=G_1$ and $F_1^j:=G_1^j$. Thus, the proof of Lemma 3.2 is complete as well as the proof of Theorem 1.1. ☐

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References

[Ale94] Georgios K. Alexopoulos, *Spectral multipliers on Lie groups of polynomial growth*, Proc. Am. Math. Soc. **120** (1994), no. 3, pp. 973–979.

[Bra+21] Roberto Bramati, Paolo Ciatti, John Green, and James Wright, *Oscillating spectral multipliers on groups of Heisenberg type*, 2021. arXiv: 2011.13987. to appear in *Rev. Mat. Iberoam.*

[CZ52] Alberto P. Calderón and Antoni Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), pp. 85–139.

[CDR21] Duván Cardona, Julio Delgado, and Michael Ruzhansky, *L^p-bounds for pseudo-differential operators on graded Lie groups*, J. Geom. Anal. **31** (2021), no. 12, pp. 11603–11647.

[CR22a] Duván Cardona and Michael Ruzhansky, *Björk-Sjölin condition for strongly singular convolution operators on graded Lie groups*, Math. Z. **302** (2022), no. 4, pp. 1957–1981.

[CR22b] Duván Cardona and Michael Ruzhansky, *Boundedness of oscillating singular integrals on Lie groups of polynomial growth*, 2022. arXiv: 2201.12883. submitted.

[CR23] Duván Cardona and Michael Ruzhansky, *Oscillating singular integral operators on compact Lie groups revisited*, Math. Z. **303** (2023), no. 2, article no. 26 (21 pages).

[CW18] Paolo Ciatti and James Wright, *Strongly singular integrals on stratified Lie groups*, in "Geometric aspects of Harmonic Analysis. Proceedings of the INdAM meeting, Cortona, Italy, June 25–29, 2018", Springer INdAM Series, no. 45, Springer, 2018, pp. 299–320.

[CW71] Ronald R. Coifman and Guido Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Mathematics, no. 242, Springer, 1971.

[Fef70] Charles L. Fefferman, *Inequalities for strongly singular integral operators*, Acta Math. **124** (1970), pp. 9–36.

[Fef73] Charles L. Fefferman, *L^p-bounds for pseudo-differential operators*, Isr. J. Math. **14** (1973), pp. 413–417.

[FS72] Charles L. Fefferman and Elias M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), pp. 137–193.

[FR16] Veronique Fischer and Michael Ruzhansky, *Quantization on Nilpotent Lie Groups*, Progress in Mathematics, no. 314, Birkhäuser, 2016.

[FR21] Veronique Fischer and Michael Ruzhansky, *Fourier multipliers on graded Lie groups*, Colloq. Math. **165** (2021), no. 1, pp. 1–30.

[FS82] Gerald B. Folland and Elias M. Stein, *Hardy Spaces on Homogeneous Groups*, Mathematical Notes (Princeton), no. 28, Princeton University Press, 1982.

[Har13] Godfrey H. Hardy, *A theorem concerning Taylor's series*, Quart. J. **44** (1913), pp. 147–160.

[HN79] Bernard Helffer and Jean-François Nourrigat, *Caractérisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe de Lie nilpotent gradué*, Commun. Partial Differ. Equations **4** (1979), pp. 899–958.

[Hir59] Isidore I. Hirschman, *On multiplier transformations*, Duke Math. J. **26** (1959), no. 2, pp. 222–242.

[Hör60] Lars Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), pp. 93–139.

[Hör85] Lars Hörmander, *The analysis of linear partial differential operators. III: Pseudo-differential operators*, Grundlehren der Mathematischen Wissenschaften, no. 274, Springer, 1985.

[MMN22] Alessio Martini, Detlef Müller, and Sebastiano Nicolussi Golo, *Spectral multipliers and wave equation for sub-Laplacians: lower regularity bounds of Euclidean type*, 2022. arXiv: 1812.02671. to appear in *J. Eur. Math. Soc.*

[See88] Andreas Seeger, *Some inequalities for singular convolution operators in L^p-spaces*, Trans. Am. Math. Soc. **308** (1988), no. 1, pp. 259–272.

[See90] Andreas Seeger, *Remarks on singular convolution operators*, Stud. Math. **97** (1990), no. 2, pp. 91–114.

[See91] Andreas Seeger, *Endpoint estimates for multiplier transformations on compact manifolds*, Indiana Univ. Math. J. **40** (1991), no. 2, pp. 471–533.

[SS89] Andreas Seeger and Christopher D. Sogge, *On the boundedness of functions of (pseudo-) differential operators on compact manifolds*, Duke Math. J. **59** (1989), no. 3, pp. 709–736.

[SSS91] Andreas Seeger, Christopher D. Sogge, and Elias M. Stein, *Regularity properties of Fourier integral operators*, Ann. Math. (2) **134** (1991), no. 2, pp. 231–251.

[Ste98] Elias M. Stein, *Singular Integrals: The roles of Calderón and Zygmund*, Notices Am. Math. Soc. **45** (1998), no. 9, pp. 1130–1140.

[Tao04] Terence Tao, *The weak-type* $(1,1)$ *of Fourier integral operators of order* $-(n-1)/2$, J. Aust. Math. Soc. **76** (2004), no. 1, pp. 1–21.

[Wai65] Stephen Wainger, *Special trigonometric series in* k*-dimensions*, Memoirs of the American Mathematical Society, no. 59, American Mathematical Society, 1965.

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