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Spherical representation spaces of quivers

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Abstract. We consider representation spaces of quivers, together with their base change action, and classify the spherical varieties among them.

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Key Words and Phrases: spherical variety, quiver representation, representation space

1. Introduction

Spherical varieties, that is, varieties with a reductive group action for which a Borel subgroup acts with a dense orbit, form a particularly rewarding class of spaces, in particular since an extensive classification theory exists (see, for example, [Cup17; Kno98; KV06; Los09; Lun01; LV83]). It is thus desirable to identify the spherical ones among natural classes of reductive group actions on varieties.

A rather special, but particularly interesting, class of such actions is provided by the representation spaces of quivers with their base change action, for which the orbits naturally identify with isomorphism classes of quiver representations. The study of these actions can thus be enriched with representation-theoretic and homological techniques, leading to a fruitful theory of the geometry of orbits, invariants and quotients [Kac83; KR86].

In the present work we classify the spherical varieties among the representation spaces of quivers. The main result, Theorem 4.2, states that these correspond to the cases where the quiver with its dimension vector is given as a connected sum, along entries 1 of the dimension vector, of two special types of dimension vectors for quivers of Dynkin type A_3.

The proof of this classification is quiver-theoretic, mainly relying on the theorem of Kac on indecomposable representations [Kac80], and properties of infinite root systems. In principle, our main result could also be obtained from the general classification of spherical linear representations of Kac, Brion, Benson, Radcliff, Leahy [Kno98, Section 5]. However, the authors think that a concise quiver-theoretic derivation is preferable, and the methods developed for this should be useful for a general study of Borel orbits in quiver representation spaces.

The background on quiver representations, together with a key classical result of Brion and Vinberg on sphericity [Bri86; Vin86], are reviewed in Section 2. Using a standard fibre bundle construction (Lemma 3.1) [BR22], we identify in Section 3 Borel orbits in quiver representation spaces with certain orbits for the base change group in an extended quiver setting, for which finiteness can be characterized by a root-theoretic criterion (Proposition 3.3). This allows us to classify the spherical representation spaces in Section 4, by excluding certain quiver settings with infinitely many Borel orbits corresponding to imaginary roots of extended Dynkin diagrams.

We note that Shmelkin [Shm02] has studied the much finer problem when a single orbit in a quiver representation space is spherical, and has developed a general approach to this problem. However, there seems to be no direct way to apply the results of [Shm02] to the present problem. Namely, identifying dense orbits in representation varieties (with respect to the base change action), and describing the decomposition behaviour of the corresponding representations (which is a requirement for the criteria of [Shm02]) leads to the intricate recursive combinatorics of infinite root systems studied in [Sch92].

2. Prerequisites

We work over an algebraically closed base field K. Let Q be a finite quiver with set of vertices Q_0 and set of arrows Q_1, where arrows are written $\alpha : i \to j$ for $i, j \in Q_0$. Let $\mathbf{d} = (d_i)_i \in \mathbb{N}Q_0$ be a dimension vector for Q, and fix K-vector spaces V_i of dimension d_i for all $i \in Q_0$. The affine space

$$R_{\mathbf{d}}(Q) = \bigoplus_{\alpha:i\to j} \mathrm{Hom}_K(V_i, V_j)$$

encodes representations of Q on the vector spaces V_i. The connected reductive algebraic group

$$G_{\mathbf{d}} = \prod_{i\in Q_0} \mathrm{GL}(V_i)$$

acts on $R_{\mathbf{d}}(Q)$ via

$$(g_i)_i \cdot (f_\alpha)_\alpha = (g_j f_\alpha g_i^{-1})_{\alpha:i\to j},$$

and the orbits for this action correspond bijectively to the isomorphism classes of K-representations of Q of dimension vector $\mathbf{d}$. The pair $(Q, \mathbf{d})$ is called a quiver setting.

For the remainder of this section, let Q be acyclic. The Euler form of Q is the (non-symmetric) bilinear form $\langle \_, \_ \rangle_Q$ on $\mathbb{Z}Q_0$ given by

$$\langle \mathbf{d}, \mathbf{e} \rangle_Q = \sum_{i\in Q_0} d_i e_i - \sum_{\alpha:i\to j} d_i e_j.$$

We denote its symmetrization by $(\_, \_) = (\_, \_)_Q$. The Weyl group W_Q of Q is the subgroup of $\mathrm{Aut}(\mathbb{Z}Q_0)$ generated by the reflections

$$s_i(\mathbf{d}) = \mathbf{d} - (\mathbf{d}, \mathbf{i}) \cdot \mathbf{i}$$

for $i \in Q_0$, where $\mathbf{i} \in \mathbb{Z}Q_0$ denotes the i^{th} coordinate vector. The union Δ^{re} of W_Q-orbits of the $\mathbf{i}$ for $i \in Q_0$ is called the set of real roots. We denote by F_Q the set of all $0 \neq \mathbf{d} \in \mathbb{N}Q_0$ which have connected support and fulfill $(\mathbf{d}, \mathbf{i}) \leq 0$ for all $i \in Q_0$, called the fundamental domain of Q. Its W_Q-saturation $\Delta^{\mathrm{im}} = W_Q F_Q$ is called the set of imaginary roots. We denote by $\Delta^{\mathrm{re},+}$ (resp. $\Delta^{\mathrm{im},+}$) the intersection of Δ^{re} (resp. Δ^{im}) with $\mathbb{N}Q_0$, called the set of positive real (resp. imaginary) roots. For application in Section 4, we note that every imaginary root $\mathbf{d}$ admits a root $\mathbf{e} \in F_Q$ in the same Weyl group orbit such that $\mathbf{e} \leq \mathbf{d}$. The theorem of Kac relates the infinite root system to the indecomposable representations of Q:

Theorem 2.1 ([Kac80]). *There exists an indecomposable representation of Q of dimension vector $\mathbf{d}$ if and only if $\mathbf{d}$ is a positive real or imaginary root. There exist infinitely many isomorphism classes of such representations if and only if $\mathbf{d}$ is imaginary.*

This key result allows us to state a root-theoretic criterion for finiteness of orbits in representation spaces. We will not use the criterion itself, but derive a refinement (Proposition 3.3) relevant for our setup in the next section.

Proposition 2.2. *The group $G_{\mathbf{d}}$ acts on $R_{\mathbf{d}}(Q)$ with finitely many orbits if and only if there is no positive imaginary root $\mathbf{e} \leq \mathbf{d}$.*

Proof. Assume that $\mathbf{e} \leq \mathbf{d}$ is a positive imaginary root. Then, by the previous theorem, there exists an infinite family $(U_\alpha)_\alpha$ of pairwise non-isomorphic indecomposable representations of Q of dimension vector $\mathbf{e}$. Denoting by $S_{\mathbf{d}-\mathbf{e}}$ the representation of dimension vector $\mathbf{d} - \mathbf{e}$ in which all arrows are represented by zero maps, we thus find an infinite family $(U_\alpha \oplus S_{\mathbf{d}-\mathbf{e}})_\alpha$ of pairwise non-isomorphic representations of Q of dimension vector $\mathbf{d}$, which correspond to infinitely many $G_{\mathbf{d}}$-orbits in $R_{\mathbf{d}}(Q)$, a contradiction.

Conversely, assume that no such $\mathbf{e}$ exists. By the Krull–Remak–Schmidt theorem, every $V \in R_{\mathbf{d}}(Q)$ admits a decomposition

$$V = U_1 \oplus \cdots \oplus U_s,$$

into indecomposables U_k, and the dimension vectors of the U_k are real roots. Thus there are only finitely many isomorphism classes of such V. This proves finiteness of the number of $G_{\mathbf{d}}$-orbits in $R_{\mathbf{d}}(Q)$. □

Our approach to sphericity of $R_{\mathbf{d}}(Q)$ will be based on the following result:

Theorem 2.3 ([Bri86; Vin86]). *Let X be an irreducible algebraic variety with an action of a connected reductive algebraic group G, and let B be a Borel subgroup of G. Then X admits an open B-orbit if and only if B acts on X with finitely many orbits.*

3. Leg-extended quivers

We fix complete flags F_i^* in V_i for all $i \in Q_0$, and denote by $B_{\mathbf{d}} \subset G_{\mathbf{d}}$ the Borel subgroup fixing all F_i^*. We are interested in a description of the $B_{\mathbf{d}}$-orbits in $R_{\mathbf{d}}(Q)$.

We define the *leg-extended quiver* $Q_{\mathbf{d}}$ by:

- vertices (i, k) for $i \in Q_0$ and $k = 1, \ldots, d_i$,
- arrows $\beta_{i,k} : (i, k) \to (i, k + 1)$ for $i \in Q_0$ and $k = 1, \ldots, d_i - 1$,
- arrows $\alpha : (i, d_i) \to (j, d_j)$ for $\alpha : i \to j$ an arrow in Q.

We also define $\overline{Q}_{\mathbf{d}}$ by omitting the arrows α in $Q_{\mathbf{d}}$. We define a dimension vector $\widehat{\mathbf{d}}$ for $Q_{\mathbf{d}}$ by

$$\widehat{d}_{(i,k)} = k$$

for $i \in Q_0$ and $k = 1, \ldots, d_i$. We call a representation V of $Q_{\mathbf{d}}$ of dimension vector $\widehat{\mathbf{d}}$ *of flag type* if all $\beta_{i,k}$ are represented by injective maps. In this case, up to isomorphism, we can assume that $V_{(i,k)} = F_i^*$ for all i and k, and that $V_{\beta_{(i,k)}}$ equals the inclusion. We thus find a canonical bijection between the $G_{\widehat{\mathbf{d}}}$-orbits in the open subset $R_{\widehat{\mathbf{d}}}^{\mathrm{flag}}(Q_{\mathbf{d}})$ of $R_{\widehat{\mathbf{d}}}(Q_{\mathbf{d}})$ of representations of flag type and the $B_{\mathbf{d}}$-orbits in $R_{\mathbf{d}}(Q)$. More precisely, the restriction map

$$p : R_{\widehat{\mathbf{d}}}^{\mathrm{flag}}(Q_{\mathbf{d}}) \longrightarrow R_{\widehat{\mathbf{d}}}(\overline{Q}_{\mathbf{d}}),$$

is $G_{\widehat{\mathbf{d}}}$-equivariant onto a homogeneous space with stabilizer $B_{\mathbf{d}}$, with fibre isomorphic to $R_{\mathbf{d}}(Q)$. We conclude:

Lemma 3.1. *We have a $G_{\widehat{\mathbf{d}}}$-equivariant isomorphism*

$$R_{\widehat{\mathbf{d}}}^{\mathrm{flag}}(Q_{\mathbf{d}}) \simeq G_{\widehat{\mathbf{d}}} \times^{B_{\mathbf{d}}} R_{\mathbf{d}}(Q).$$

In particular, $B_{\mathbf{d}}$-orbits in $R_{\mathbf{d}}(Q)$ correspond bijectively to $G_{\widehat{\mathbf{d}}}$-orbits in $R_{\widehat{\mathbf{d}}}^{\mathrm{flag}}(Q_{\mathbf{d}})$.

We will now refine Proposition 2.2 to the case of representations of flag type for a leg-extended quiver. For this, we call a dimension vector $\mathbf{e} \leq \widehat{\mathbf{d}}$ for $Q_{\mathbf{d}}$ *of flag type* if

$$e_{(i,1)} \leq \cdots \leq e_{(i,d_i)}$$

for all $i \in Q_0$. We call $\mathbf{e}$ *gentle* if both $\mathbf{e}$ and $\widehat{\mathbf{d}} - \mathbf{e}$ are of flag type. Equivalently, $\mathbf{e}$ is gentle if and only if

$$e_{(i,k+1)} - e_{(i,k)} \in \{0, 1\}$$

for all $i \in Q_0$ and $k < d_i$.

Lemma 3.2. *If $\mathbf{e}$ is an imaginary root for $Q_{\mathbf{d}}$ and V is an indecomposable representation of dimension vector $\mathbf{e}$, then V is of flag type.*

Proof. Let $\mathbf{e}$ be an imaginary root for $Q_{\mathbf{d}}$, and let V be an indecomposable representation of dimension vector $\mathbf{e}$. If V is not of flag type, we can find $i \in Q_0$ such that $V_{\beta_{(i,k)}}$ is not injective for some $k < d_i$; let k be minimal with this property. Choose a non-zero vector v_k such that

$$V_{\beta_{(i,k)}}(v_k) = 0.$$

Choose $l \leq k$ minimal with the property that

$$v_k = V_{\beta_{(i,k-1)}} \ldots V_{\beta_{(i,l)}} v_l,$$

and define

$$v_p = V_{\beta_{(i,p-1)}} \ldots V_{\beta_{(i,l)}} v_l$$

for $l < p < k$. The v_p for $l \leq p \leq k$ clearly span a non-zero subrepresentation U of V. We claim that this is even a direct summand. Namely, we first define

$$W_{(i,p)} = V_{(i,p)}$$

for all $p < l$. For $l \leq p \leq k$, inductively, we choose a complement $W_{(i,p)}$ to v_p containing $V_{\beta_{(i,p-1)}} W_{(i,p-1)}$, which is possible by the minimality properties of k and l and by the choice of the v_p. We define $W_x = V_x$ for all other vertices x of $Q_{\mathbf{d}}$. Then W defines a subrepresentation of V complementing U, thus $V = U$ by indecomposability. But the dimension vector of U is clearly a real root, yielding a contradiction. □

The proof of the following follows closely the proof of Proposition 2.2.

Proposition 3.3. *For acyclic Q, the group $G_{\widehat{\mathbf{d}}}$ acts on $R^{\mathrm{flag}}_{\widehat{\mathbf{d}}}(Q)$ with finitely many orbits if and only if there is no gentle imaginary root $\mathbf{e} \leq \widehat{\mathbf{d}}$.*

Proof. Assume that $\mathbf{e} \leq \widehat{\mathbf{d}}$ is a gentle imaginary root. Then, there exists an infinite family $(U_\alpha)_\alpha$ of pairwise non-isomorphic indecomposable representations of Q of dimension vector $\mathbf{e}$, which are of flag type by the previous proposition. Since $\widehat{\mathbf{d}} - \mathbf{e}$ is again of flag type, we can choose a representation S of flag type of this dimension vector. We thus find an infinite family $(U_\alpha \oplus S)_\alpha$ of pairwise non-isomorphic representations of flag type of $Q_{\mathbf{d}}$ of dimension vector $\widehat{\mathbf{d}}$, which correspond to infinitely many $G_{\widehat{\mathbf{d}}}$-orbits in $R_{\widehat{\mathbf{d}}}(Q_{\mathbf{d}})$, a contradiction.

Conversely, assume that no such $\mathbf{e}$ exists, but that there are infinitely many isomorphism classes of representations of flag type. By the Krull–Remak–Schmidt theorem, every $V \in R^{\mathrm{flag}}_{\widehat{\mathbf{d}}}(Q_{\mathbf{d}})$ admits a decomposition

$$V = U_1 \oplus \cdots \oplus U_s,$$

into indecomposables U_k which are again of flag type. If the dimension vector $\mathbf{e}$ of one of them, say U_1, is an imaginary root, it is gentle, since both U_1 and $U_2 \oplus \cdots \oplus U_s$ are of flag type, a contradiction. Thus, the dimension vector of any such U_k is a real root, and there are only finitely many isomorphism classes of such V. This proves finiteness of the number of $G_{\widehat{\mathbf{d}}}$-orbits in $R^{\mathrm{flag}}_{\widehat{\mathbf{d}}}(Q)$. ☐

Corollary 3.4. *For $\mathbf{d} \in \mathbb{N}Q_0$, the representation space $R_{\mathbf{d}}(Q)$ is spherical if and only if there is no gentle imaginary root $\mathbf{e} \leq \widehat{\mathbf{d}}$ for $Q_{\mathbf{d}}$.*

Proof. Sphericity of $R_{\mathbf{d}}(Q)$ is equivalent to $B_{\mathbf{d}}$ acting with finitely many orbits on $R_{\mathbf{d}}$ by Theorem 2.3. This is equivalent to $G_{\widehat{\mathbf{d}}}$ acting with finitely many orbits on $R^{\mathrm{flag}}_{\widehat{\mathbf{d}}}(Q_{\mathbf{d}})$, and the previous Proposition 3.3 shows the claim. ☐

4. Classification

To approach the classification of spherical representation varieties, we first show that we can restrict to excluding minimal non-spherical situations.

Lemma 4.1. *If $\mathbf{d} \leq \mathbf{e}$ are dimension vectors for Q and $R_{\mathbf{d}}(Q)$ is not spherical, then $R_{\mathbf{e}}(Q)$ is not spherical.*

Proof. We have an embedding of quivers

$$\iota : Q_{\mathbf{d}} \longrightarrow Q_{\mathbf{e}}, \quad (i,k) \longmapsto (i, k + e_i - d_i)$$

for $i \in Q_0$ and $k \leq d_i$. It induces an embedding of Weyl groups

$$\iota : W_{Q_{\mathbf{d}}} \longrightarrow W_{Q_{\mathbf{e}}}$$

and a $W_{Q_{\mathbf{d}}}$-equivariant embedding

$$\iota : \mathbb{Z}(Q_{\mathbf{d}})_0 \longrightarrow \mathbb{Z}(Q_{\mathbf{e}})_0.$$

It is easily verified that ι maps $F_{Q_{\mathbf{d}}}$ to $F_{Q_{\mathbf{e}}}$, and thus imaginary roots for $Q_{\mathbf{d}}$ to imaginary roots for $Q_{\mathbf{e}}$. Obviously, ι also preserves gentleness. The claim then follows from the previous Corollary 3.4. $\square$

We say that a quiver Q is a *connected sum* of two full subquivers Q' and Q'' along a vertex $i \in Q_0$ if

$$(Q')_0 \cup (Q'')_0 = Q_0 \quad \text{and} \quad (Q')_0 \cap (Q'')_0 = \{i\} \quad \text{and} \quad (Q')_1 \cap (Q'')_1 = \emptyset.$$

In this case, we write

$$Q = Q' \#_i Q''.$$

Given a dimension vector $\mathbf{d}$ for Q, we write

$$(Q, \mathbf{d}) = (Q', \mathbf{d}|_{Q'}) \#_i (Q'', \mathbf{d}|_{Q''}).$$

We call such a connected sum of quiver settings *thin* if $d_i = 1$. In the following, the symbol $\leftrightarrow$ denotes an arrow of either orientation.

Theorem 4.2. *For a connected quiver setting $(Q, \boldsymbol{d})$, the representation space $R_{\boldsymbol{d}}(Q)$ is a spherical variety under the action of $B_{\boldsymbol{d}} \subset G_{\boldsymbol{d}}$ if and only if Q is of Dynkin type A_1 or A_2 or if $(Q, \boldsymbol{d})$ is a thin connected sum of quiver settings with Q of the form $1 \leftrightarrow 2 \leftrightarrow 3$, and $\boldsymbol{d} = (m, n, 1)$ or $\boldsymbol{d} = (m, 2, n)$.*

Proof. First assume $R_{\mathbf{d}}(Q)$ is spherical. We reduce to the claimed situation in several steps, always using previous Lemma 4.1.

(1) If Q contains a cycle, then $R_{\mathbf{d}}(Q)$ even admits infinitely many $G_{\mathbf{d}}$-orbits, since there are already infinitely many isomorphism classes of representations of dimension vector $(1, \ldots, 1)$ for any cycle quiver. Thus we can assume Q to be a tree.
(2) With the same argument applied to the Kronecker quiver, we can assume Q to have no parallel arrows between vertices.
(3) Assume that Q cannot be written as a proper thin connected sum.
(4) Then there is no vertex i incident with more than two other vertices: otherwise, we can assume $d_i \geq 2$ and i being connected to j_1, j_2, j_3, and the gentle imaginary root

$$(\mathbf{i}, \mathbf{1}) + 2(\mathbf{i}, \mathbf{2}) + (\mathbf{j_1}, \mathbf{1}) + (\mathbf{j_2}, \mathbf{1}) + (\mathbf{j_3}, \mathbf{1}),$$

for a $\widetilde{D}_4$-quiver contradicts sphericity. Thus we can assume Q to be of type A_n.
(5) If $n \geq 4$, we can find a subquiver $i_1 \leftrightarrow i_2 \leftrightarrow i_3 \leftrightarrow i_4$ such that $d_{i_2}, d_{i_3} \geq 2$, and the gentle imaginary root

$$(\mathbf{i_1}, \mathbf{1}) + (\mathbf{i_2}, \mathbf{1}) + 2(\mathbf{i_2}, \mathbf{2}) + (\mathbf{i_3}, \mathbf{1}) + 2(\mathbf{i_3}, \mathbf{2}) + (\mathbf{i_4}, \mathbf{1}),$$

for a $\widetilde{D}_5$ quiver contradicts sphericity. Thus Q is at most of type A_3.
(6) Say Q is of the form $i_1 \leftrightarrow i_2 \leftrightarrow i_3$. The gentle imaginary root $\widehat{(2, 3, 2)}$ for a $\widetilde{E}_6$-quiver shows that either $d_{i_2} \leq 2$, or $d_{i_1} \leq 1$, or $d_{i_3} \leq 1$.

This finishes the first part of the proof.

To prove the converse direction, we first show that the A_3-situations of the theorem are indeed spherical. We exemplify this for the case $\boldsymbol{d} = (m, n, 1)$ by using Corollary 3.4 and direct calculations.

$$\begin{tikzcd}
e_{(1,d_1)} \arrow[r, leftrightarrow] & e_{(2,d_2)} \arrow[r, leftrightarrow] & e_{(3,1)} \\
e_{(1,d_1-1)} \arrow[u] & e_{(2,d_2-1)} \arrow[u] & \\
\vdots \arrow[u] & \vdots \arrow[u] & \\
e_{(1,1)} \arrow[u] & e_{(2,1)} \arrow[u] &
\end{tikzcd}$$

FIGURE 1. Labelling of the vertices in the case $\boldsymbol{d} = (m, n, 1)$

So assume $\boldsymbol{e} \leq \widehat{\boldsymbol{d}}$ is an imaginary root for $Q_{\boldsymbol{d}}$. We can assume $e_{(3,1)} = 1$, otherwise the support of $\mathbf{e}$ is of type A, and $\boldsymbol{e}$ is a real root. We may also assume m and n to be minimal in the sense that $e_{(1,1)}$ and $e_{(2,1)}$ are non-zero, thus $e_{(1,1)} = e_{(2,1)} = 1$. Moreover, by applying reflections, we may assume that $\boldsymbol{e} \in F_{Q_{\mathbf{d}}}$. This allows us to use the conditions $(\boldsymbol{e}, (\boldsymbol{i}, \boldsymbol{k})) \leq 0$ for all $(\boldsymbol{i}, \boldsymbol{k}) \in Q_{\boldsymbol{d}}$. We first find

$$(\boldsymbol{e}, (\boldsymbol{i}, \mathbf{1})) = 2e_{(i,1)} - e_{(i,2)} \leq 0 \iff e_{(i,2)} \geq 2$$

for $i \in \{1, 2\}$. But we also have $e_{(i,2)} \leq d_{(i,2)} = 2$, thus $e_{(i,2)} = 2$. By a quick induction, we find $e_{(i,k)} = k$ for all $k \leq d_i$. Additionally, we find

$$(\boldsymbol{e}, (\mathbf{1}, \boldsymbol{d_1})) = 2e_{(1,d_1)} - e_{(1,d_1-1)} - e_{(2,d_2)} = 2d_1 - d_1 + 1 - d_2 \leq 0,$$

thus $d_1 + 1 \leq d_2$, and

$$(\boldsymbol{e}, (\mathbf{2}, \boldsymbol{d_2})) = 2e_{(2,d_2)} - e_{(2,d_2-1)} - e_{(1,d_1)} - e_{(3,1)} = 2d_2 - d_2 + 1 - d_1 - 1 \leq 0,$$

thus $d_2 \leq d_1$, a contradiction. The cases $\boldsymbol{d} = (m, n)$ and $\boldsymbol{d} = (m, 2, n)$ are proven similarly.

Now we prove that sphericity is preserved under thin connected sums. First note that a connected sum of trees is again a tree by disjointness of the respective sets of arrows. So let $(Q, \boldsymbol{f}) = (Q', \boldsymbol{c})\#_i(Q'', \boldsymbol{d})$ be the thin connected sum along a vertex $i \in Q_0$ such that $R_{\boldsymbol{c}}(Q')$ is spherical, Q'' is of type A_3, and $\boldsymbol{d} = (m, n, 1)$. Let $\boldsymbol{e} \leq \widehat{\boldsymbol{f}}$ be a positive root, which again we can assume to belong to $F_{Q_{\mathbf{f}}}$. We can then repeat the previous arguments word by word to arrive at the same contradiction, finishing the proof of Theorem 4.2. $\square$

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