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Characterising semi-simple Lie algebras by their Borel nilpotent radical

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Abstract. We show that semi-simple Lie algebras can be characterized by their maximal nilpotent subalgebra, which is the same as the nilpotent radical of a Borel subalgebra.

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Key Words and Phrases: Lie algebras, semi-simple, nilpotent, Borel, characterising

1. Introduction

The classification of simple Lie algebras is well known and is made in terms of the combinatorics of root systems. While this is satisfactory for many purposes, one might wonder if they can also be classified by more algebraic means, or at least characterized that way. Indeed, such a theorem can perhaps be extended to cases where the theory of root system is less applicable, like the classification of finite groups.

The first option is to characterize the algebra from a Borel subalgebra, which is quite easy. A less obvious question is the question if one can do this from the nilpotent radical of a Borel subalgebra, which is also a maximal nilpotent subalgebra, see [DKK06], and we prove in this note:

Main Theorem. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two semi-simple Lie algebras, let $\mathfrak{b}(\mathfrak{g}_1), \mathfrak{b}(\mathfrak{g}_2)$ be Borel subalgebras, and let $\mathfrak{n}(\mathfrak{g}_i)$ be the nilpotent radical of $\mathfrak{b}(\mathfrak{g}_i)$. Assume that $\mathfrak{n}(\mathfrak{g}_1) \cong \mathfrak{n}(\mathfrak{g}_2)$. Then $\mathfrak{g}_1 \cong \mathfrak{g}_2$.*

In this theorem and throughout the paper we will always assume our Lie algebras are over $\mathbb{C}$ or any algebraically closed field of characteristic 0.

The note is made up of two parts, the simple case, and the semi-simple case. While the semi-simple case doesn't rely on the simple case, it is still included as an easy case to illustrate the techniques used also for the general case.

Idea of the proof. The nilpotent radical is a nilpotent algebra, distinguishing between such algebras is relatively simple, as by definition they have a lot of ideals and therefore invariants.

For the simple case, our main tool is the lower central sequence. It is defined by $\mathfrak{n}_1 = \mathfrak{n}$, $\mathfrak{n}_{i+1} = [\mathfrak{n}_i, \mathfrak{n}]$, and is obviously an invariant of the Lie algebra. Already $\dim(\mathfrak{n}_1/\mathfrak{n}_2)$ recovers the rank of the original simple algebra, and together with the dimension of the algebra this almost completes the proof. To deal with the remaining cases we consider the product $\mathfrak{n}_i/\mathfrak{n}_{i+1} \times \mathfrak{n}_j/\mathfrak{n}_{j+1} \rightarrow \mathfrak{n}_{i+j}/\mathfrak{n}_{i+j+1}$ induced by the Lie bracket, and show that in the remaining cases these are not always the same.

For the semi-simple case we have to add the upper central sequence $\mathfrak{u}_i$ which is defined in a similar way to $\mathfrak{n}_i$, and also the analogous product $\mathfrak{n}_i/\mathfrak{n}_{i+1} \times \mathfrak{u}_j/\mathfrak{u}_{j-1} \rightarrow \mathfrak{u}_{j-i}/\mathfrak{u}_{j-i-1}$.

2. The simple case

We follow the notations of [Kir08], everything is done over $\mathbb{C}$.

We have a (semi-)simple algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}(\mathfrak{g})$, its root system is denoted $R(\mathfrak{g})$, and its root spaces are $\mathfrak{g}_\alpha$, we will assume that some Borel subalgebra $\mathfrak{b}(\mathfrak{g})$ containing $\mathfrak{h}(\mathfrak{g})$ was chosen, and we denote by $\mathfrak{n}(\mathfrak{g}) = [\mathfrak{b}(\mathfrak{g}), \mathfrak{b}(\mathfrak{g})]$ its nilpotent radical. We have:

$$\mathfrak{n}(\mathfrak{g}) = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha$$

where R_+ is the set of positive roots with respect to our chosen Borel subalgebra. When there is no confusion about the simple Lie algebra, we will drop it from the notation.

Our goal is to prove that we can characterize a simple Lie algebra from the structure of $\mathfrak{n}$. Two invariants that we will explain now will suffice.

The lower central series. The lower central series of $\mathfrak{n}$ is defined by $\mathfrak{n}_1 = \mathfrak{n}$ and $\mathfrak{n}_{i+1} = [\mathfrak{n}_i, \mathfrak{n}]$.

Let us calculate the lower central series in our case. For a positive root $\alpha \in R_+$ we define its degree $\deg(\alpha)$ as the sum of coefficients when we write α as a linear combination of simple roots. Notice $\deg$ is additive i.e. $\deg(\alpha + \beta) = \deg(\alpha) + \deg(\beta)$ for roots α, β , and its values are integral.

Proposition 2.1. *For $i \geq 1$, $\mathfrak{n}_i$ is the subspace generated by $\mathfrak{g}_\alpha$ for α with $\deg(\alpha) \geq i$.*

Proof. For $i = 1$ this is obvious, we proceed by induction.

Let us denote the subspace generated by $\mathfrak{g}_\alpha$ for $\deg(\alpha) \geq i$ by $\mathfrak{n}'_i$, and assume we have proved $\mathfrak{n}_i = \mathfrak{n}'_i$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, and degree is additive, it is obvious that $\mathfrak{n}_{i+1} = [\mathfrak{n}_i, \mathfrak{n}] \subseteq \mathfrak{n}'_{i+1}$. On the other hand, let α be a root with $\deg(\alpha) \geq i + 1$, then we know that there is a simple root β s.t. $\alpha - \beta$ is also a root, in that case $[\mathfrak{g}_{\alpha-\beta}, \mathfrak{g}_\beta] = \mathfrak{g}_\alpha$, obviously $\deg(\alpha - \beta) = \deg(\alpha) - 1 \geq i$ since $\deg$ is additive, and so $[\mathfrak{g}_{\alpha-\beta}, \mathfrak{g}_\beta] \subseteq [\mathfrak{n}_i, \mathfrak{n}] = \mathfrak{n}_{i+1}$ and we are done. $\square$

The lower central series has an additional structure:

Definition 2.2. For $i \geq 1$ denote $\mathfrak{n}^i = \mathfrak{n}_i/\mathfrak{n}_{i+1}$.

Proposition 2.3. *For any $i, j \geq 1$ we have a bilinear map $\mathfrak{n}^i \times \mathfrak{n}^j \rightarrow \mathfrak{n}^{i+j}$.*

Proof. By Proposition 2.1 it's obvious that $[\mathfrak{n}_i, \mathfrak{n}_j] \subseteq \mathfrak{n}_{i+j}$, which therefore defines a product $\mathfrak{n}_i \times \mathfrak{n}_j \rightarrow \mathfrak{n}_{i+j}$. We can then quotient out $\mathfrak{n}_{i+j+1}$, and it only remains to see that $\mathfrak{n}_{i+1} \times \mathfrak{n}_j$ and $\mathfrak{n}_i \times \mathfrak{n}_{j+1}$ go to zero, but we already explained that their commutator is in $\mathfrak{n}_{i+j+1}$ as required. $\square$

Remark 2.4. While it may seem we used our description of $\mathfrak{n}_i$, we only used this for $[\mathfrak{n}_i, \mathfrak{n}_j] \subseteq \mathfrak{n}_{i+j}$ which is true in general by the Jacobi identity and induction.

Remark 2.5. Both of the claims can be described by saying that $\text{gr}(\mathfrak{n}) := \bigoplus_i \mathfrak{n}^i$ is a Lie algebra by itself, which in our case is in fact (non-canonically) isomorphic to $\mathfrak{n}$.

Rank. From the previous section we see that $\mathfrak{n}^i$ is isomorphic to the space generated by $\mathfrak{g}_\alpha$ with $\deg(\alpha) = i$, so the number of such roots is an invariant of $\mathfrak{n}$. In particular, the number of roots of degree 1, i.e. simple roots, is an invariant which is equal to the rank of the original Lie algebra. This means we only have to distinguish between A_n, B_n, C_n, D_n and the exceptionals for a constant n .

Now we can compare dimensions, since $\dim(\mathfrak{g}) = 2 \cdot \dim(\mathfrak{n}) + n$ we will compare the dimension of $\mathfrak{g}$ instead of the dimension of $\mathfrak{n}$, these numbers are well known:

Lie algebra	Rank	Dimension
A_n	n	$n(n+2)$
B_n	n	$n(2n+1)$
C_n	n	$n(2n+1)$
D_n	n	$n(2n-1)$
E_6	6	78
E_7	7	133
E_8	8	248
F_4	4	52
G_2	2	14

This table can be found for example at [MFM08, Table 2]. From the table, we easily see that B_n, C_n always have the same dimension, which is always different from that of D_n . A_n has the same dimension as B_n or C_n only for $n = 1$, and the same dimension as D_n only for $n = 3$, both of which correspond to the well-known fact that $A_1 = B_1 = C_1$ and $A_3 = D_3$.

Finally, the only exceptional that has the same dimension as a classic algebra of the same rank is E_6 with B_6 and C_6 .

In summary, we proved:

Proposition 2.6. *If $\mathfrak{g}_1, \mathfrak{g}_2$ are two simple Lie algebras, and we have $\mathfrak{n}(\mathfrak{g}_1) \cong \mathfrak{n}(\mathfrak{g}_2)$ then either $\mathfrak{g}_1 \cong \mathfrak{g}_2$, $\{\mathfrak{g}_1, \mathfrak{g}_2\} \cong \{B_n, C_n\}$ or one of the algebras is E_6 and the other one is B_6 or C_6 .*

The next natural step is to look at the number of roots of height i for a general i . It turns out that those numbers are the same for B_n and C_n , but do distinguish between E_6 and B_6, C_6 . Specifically E_6 has 5 roots of degree 4 while B_6, C_6 have four such roots. Both claims can be checked by writing down root systems. For B_6, C_6 we will do this later. One can also look at [Rin18, appendix] where root posets for those root systems are drawn, in particular, one can see this result.

B_n and C_n . As we mentioned, the number of roots of each degree for B_n and C_n is the same. Still, the combinatorics of the roots is different, and we claim that our bilinear product will distinguish between them.

First, we write down the root systems and our choice of simple roots. Both are written inside the vector space with basis $\{e_i\}_{1 \leq i \leq n}$ like in [Kir08, pp. 205–206]:

$$\begin{aligned} R(B_n) &= \{\pm e_i \pm e_j\}_{i \neq j} \cup \{\pm e_i\}, \\ \Pi(B_n) &= \{e_i - e_{i+1}\}_{1 \leq i \leq n-1} \cup \{e_n\}, \\ R(C_n) &= \{\pm e_i \pm e_j\}_{i \neq j} \cup \{\pm 2e_i\}, \\ \Pi(C_n) &= \{e_i - e_{i+1}\}_{1 \leq i \leq n-1} \cup \{2e_n\}. \end{aligned}$$

Now let us consider in both cases the bilinear product:

$$\mathfrak{n}^2 \times \mathfrak{n}^{2n-3} \longrightarrow \mathfrak{n}^{2n-1}.$$

Notice that in both cases $\mathfrak{n}_{2n-1}$ is 1 dimensional, while $\mathfrak{n}_{2n} = 0$. In B_n the root of highest degree is $e_1 + e_2$ with degree $2n - 1$, and in C_n it is $2e_1$ also with degree $2n - 1$.

Next we see that $\mathfrak{n}^{2n-3}$ is two dimensional in both cases, in the case of B_n the roots of degree $2n - 3$ are $e_1 + e_4, e_2 + e_3$ and for C_n there are $e_1 + e_3, 2e_2$.

Here we have assumed $n \geq 4$, since we had $e_1 + e_4$, this discussion however is also true for $n = 3$, by setting $e_4 = 0$ in this case.

For $n = 2$ or $n = 1$ the dimension calculation is wrong, but in those cases $B_2 = C_2$, $B_1 = C_1$ so there is nothing to prove.

Proposition 2.7. *In the case of C_n , $n \geq 3$, this bilinear product has non-trivial right kernel, specifically it contains $\mathfrak{g}_{2e_2}$.*

Proof. For any α of degree 2, $\alpha + 2e_2 \notin R$, if this was not the case, $\alpha + 2e_2$ would have to be $2e_1$, because its degree is $2n - 1$, but $2e_1 - 2e_2$ is not a root.

This means that $[\mathfrak{g}_\alpha, \mathfrak{g}_{2e_2}] = \mathfrak{g}_{\alpha+2e_2} = 0$ for any root of degree 2, but those $\mathfrak{g}_\alpha$ exactly generate $\mathfrak{n}^2$. $\square$

So now to finish we only have to prove:

Proposition 2.8. *In the case of B_n , $n \geq 3$, the bilinear product is non-degenerate on the right.*

Proof. Similar to before, notice that $(e_1 + e_4) + (e_2 - e_4) = (e_2 + e_3) + (e_1 - e_3) = e_1 + e_2$, which means that

$$\begin{aligned} [\mathfrak{g}_{e_2 - e_4}, \mathfrak{g}_{e_1 + e_4}] &= \mathfrak{g}_{e_1 + e_2}, \\ [\mathfrak{g}_{e_1 - e_3}, \mathfrak{g}_{e_1 + e_4}] &= 0, \\ [\mathfrak{g}_{e_2 - e_4}, \mathfrak{g}_{e_2 + e_3}] &= 0, \\ [\mathfrak{g}_{e_1 - e_3}, \mathfrak{g}_{e_2 + e_3}] &= \mathfrak{g}_{e_1 + e_2}. \end{aligned}$$

Which obviously means that the bilinear product is non-degenerate on the right.

For $n = 3$ the same calculation holds after erasing e_4 . $\square$

Now all that is to put things together.

Proof of the simple case. By Proposition 2.6 with addition to the remarks after it, it only remains to show that $\mathfrak{n}(B_n)$ is not isomorphic to $\mathfrak{n}(C_n)$ for $n \geq 3$ as for $n \leq 2$, $B_n \cong C_n$.

If they were isomorphic then we also would have had an isomorphism of the lower central series, and the bilinear product we constructed on it. But we just showed that for B_n one of those products is non-degenerate on the right, while for C_n the same one is, and therefore they are not isomorphic. $\square$

One notices that the calculation of the right kernel was completely combinatorial, this is true in general, and in the semi-simple case, we will not go into details about such calculations.

3. The semi-simple case

Any semi-simple Lie algebra can be written as a direct sum $\mathfrak{g} = \mathfrak{l}_1 \oplus \cdots \oplus \mathfrak{l}_k$ for simple algebras $\mathfrak{l}_i$, and our goal is to distinguish between different such sums. The invariants from the previous section can still be defined, and are additive over this sum, for example $\dim(\mathfrak{n}^1) = \text{rank}(\mathfrak{l}_1) + \cdots + \text{rank}(\mathfrak{l}_k)$. To use this we will need another set of invariants which in the previous case was redundant.

The upper central series. The upper central series is the dual notion to the lower central series.

Definition 3.1. $\mathfrak{u}_0 = 0$, and $\mathfrak{u}_{i+1}$ is defined as the unique subalgebra containing $\mathfrak{u}_i$ such that $\mathfrak{u}_{i+1}/\mathfrak{u}_i = Z(\mathfrak{n}/\mathfrak{u}_i)$ where Z denotes the center of a Lie algebra.

Like the lower central series, this is an invariant, we had no need of it in the case of a simple algebra, because of the following result.

Proposition 3.2. *If $\mathfrak{n}$ is the nilpotent radical of a Borel subalgebra of a simple Lie algebra, then the lower central series and the upper central series are the same up to a change of indexes.*

As we already have a description for the lower central series, we only have to show that this description holds for $\mathfrak{u}_i$. Let d be the degree of the highest root of $\mathfrak{g}$. Then, a more precise statement can be written.

Proposition 3.3. *If $\mathfrak{n}$ is the nilpotent radical of a Borel subalgebra of a simple Lie algebra and $i \geq 0$, then $\mathfrak{u}_i$ is the subspace generated by $\mathfrak{g}_\alpha$ for α with $\deg(\alpha) > d - i$.*

Proof. Again, we show this by induction, for $i = 0$ this is obvious. Assuming the result is true for i , we have to show it for $i + 1$.

If $\deg(\alpha) > d - i - 1$ and β is any root, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \subseteq \mathfrak{u}_i$. This is true because $\deg(\alpha + \beta) = \deg(\alpha) + \deg(\beta) > (d - i - 1) + 1 = d - i$ together with the induction hypothesis. This shows that $\mathfrak{g}_\alpha \subseteq \mathfrak{u}_{i+1}$.

On the other hand, assume $v \in \mathfrak{u}_{i+1}$, and that its projection to some $\mathfrak{g}_\alpha$ with $\deg(\alpha) \leq d - i - 1$ is not zero. Since α is not the highest root, we know that we can find some simple root β such that $\alpha + \beta$ is also a root. Then we see that $[\mathfrak{g}_\beta, v]$ has non-trivial projection to $\mathfrak{g}_{\alpha+\beta}$, but by the induction hypothesis $\mathfrak{g}_{\alpha+\beta}$ is not inside $\mathfrak{u}_i$, which means v is not in the center of $\mathfrak{n}/\mathfrak{u}_i$, and this is a contradiction. $\square$

While this claim might seem to imply that the upper central series holds no more information than the lower central series, this is untrue. The key point is that the reordering depends on the highest degree root, if $\mathfrak{g} = \mathfrak{l}_1 \oplus \cdots \oplus \mathfrak{l}_k$ and $\mathfrak{l}_i$ had a highest root of degree d_i then we would have

$$\mathfrak{u}_i(\mathfrak{g}) = \mathfrak{u}_i(\mathfrak{l}_1) \oplus \cdots \oplus \mathfrak{u}_i(\mathfrak{l}_k) = \mathfrak{n}_{d_1-i+1}(\mathfrak{l}_1) \oplus \cdots \oplus \mathfrak{n}_{d_k-i+1}(\mathfrak{l}_k)$$

which is not in general the same as $\mathfrak{n}_j$ for any j .

Similarly to before for a general semi-simple $\mathfrak{g}$ we denote:

Definition 3.4. For $i \geq 1$ denote $\mathfrak{u}^i = \mathfrak{u}_i/\mathfrak{u}_{i-1}$.

An example of the difference between the lower central series and upper central series can now be clearly seen, $\dim(\mathfrak{n}^1)$ is the total number of simple roots in all the $\mathfrak{l}_i$, while if we take the last non-zero $\mathfrak{u}^i$, it will give the total number of simple roots in only the $\mathfrak{l}_i$ whose highest degree root is maximal among the $\mathfrak{l}_i$.

Before concluding this section let us quickly mention that we have an analogue of the bilinear product:

Proposition 3.5. *The Lie bracket induces a product $\mathfrak{n}^i \times \mathfrak{u}^j \rightarrow \mathfrak{u}^{j-i}$. (When $j-i \leq 0$, $\mathfrak{u}^{j-i}$ is interpreted as zero.)*

Proof. Like the previous product we just need to show that the bracket takes $\mathfrak{n}_i \times \mathfrak{u}_j \rightarrow \mathfrak{u}_{j-i}$.

For $i = 1$ this is obvious, as $[\mathfrak{n}, \mathfrak{u}_j] \subseteq \mathfrak{u}_{j-1}$ by definition.

Using induction we get:

$$\begin{aligned} [\mathfrak{n}_{i+1}, \mathfrak{u}_j] &= [[\mathfrak{n}_i, \mathfrak{n}], \mathfrak{u}_j] \subseteq [\mathfrak{n}_i, [\mathfrak{n}, \mathfrak{u}_j]] + [\mathfrak{n}, [\mathfrak{n}_i, \mathfrak{u}_j]] \subseteq \\ &\subseteq [\mathfrak{n}_i, \mathfrak{u}_{j-1}] + [\mathfrak{n}, \mathfrak{u}_{j-i}] \subseteq \mathfrak{u}_{j-i-1}. \quad \square \end{aligned}$$

Now we wish to use the invariants we constructed to prove:

Main Theorem. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two semi-simple Lie algebras, let $\mathfrak{b}(\mathfrak{g}_1), \mathfrak{b}(\mathfrak{g}_2)$ be Borel subalgebras, and let $\mathfrak{n}(\mathfrak{g}_i)$ be the nilpotent radical of $\mathfrak{b}(\mathfrak{g}_i)$. Assume that $\mathfrak{n}(\mathfrak{g}_1) \cong \mathfrak{n}(\mathfrak{g}_2)$, then $\mathfrak{g}_1 \cong \mathfrak{g}_2$.*

First let us introduce a notation, any semi-simple Lie algebra $\mathfrak{g}$ is a sum of simple Lie algebras, we will denote the number of times a simple Lie algebra with root system R appears in $\mathfrak{g}$ by $\mathfrak{g}(R)$. For example $\mathfrak{g}(A_n)$ is the number of summands which are $\mathfrak{sl}_{n+1}$.

Our goal is to find $\mathfrak{g}(R)$ for all R using only $\mathfrak{n}(\mathfrak{g})$. As long as all our invariants are additive, i.e. are equal to a sum over the simple summand of $\mathfrak{g}$, then if we find $\mathfrak{g}(R)$, we can subtract its part, and calculate the invariants for the subalgebra of $\mathfrak{g}$ which is made up of all the summands not corresponding to R .

This suggests we should calculate all the $\mathfrak{g}(R)$ in steps, starting from “big” R and moving to small, we order the simple Lie algebras by the degree of their highest root. With that in mind, we list the degree of the highest root for all simple algebras.

Lie algebra	Maximal degree
A_n	n
B_n	$2n - 1$
C_n	$2n - 1$
D_n	$2n - 3$
E_6	11
E_7	17
E_8	29
F_4	11
G_2	5

For the proof we will denote n^i the dimension of $\mathfrak{n}^i$, u^i will be the dimension of $\mathfrak{u}^i$, and $n^{i,j}$ will be the codimension of the right kernel of $\mathfrak{n}^i \times \mathfrak{n}^j \rightarrow \mathfrak{n}^{i+j}$, which is n^j minus the dimension of the right kernel.

A few observations will be helpful:

Remark 3.6. $n^{i+j}, u^{i+j}, n^{i,j}$ are zero for simple algebras where the maximal degree is less than $i + j$.

Proof. For n^{i+j}, u^{i+j} this is obvious by our description.

For $n^{i,j}$ this is because the product takes values in $\mathfrak{n}^{i+j}$, which is zero. So the kernel is everything, and its codimension is zero. $\square$

Remark 3.7. For a simple Lie algebra there is always exactly one highest root, so if a simple algebra has highest root of degree d , then n^d is 1.

In that case n^{d-1} is the number of roots of degree $d-1$ (which is usually 1).

On the other hand, u^d is the number of simple roots, and u^{d-1} is the number of roots which are sums of two simple roots. Two simple roots sum to a root iff they are not orthogonal, i.e. if they are connected in the Dynkin diagram, so we always have $u^{d-1} = u^d - 1$.

Proof of Main Theorem. We will show that using our invariant one can find $\mathfrak{g}(R)$ for any simple algebra.

Our proof will be by induction, assuming we know these coefficients for all algebras with the degree of the highest root $\geq 2n$ we will show we can find those with degree of the highest root $2n-1, 2n-2$. (A base case is obvious, if $\mathfrak{n}_k = 0$ then there are no algebras with roots of degree $\geq k$.)

If $2n-1 \neq 1, 3, 5, 11, 17, 29$ then the only relevant coefficients are:

$$\mathfrak{g}(A_{2n-1}), \quad \mathfrak{g}(A_{2n-2}), \quad \mathfrak{g}(B_n), \quad \mathfrak{g}(C_n), \quad \mathfrak{g}(D_{n+1}).$$

The following table depicts five of our invariants, and their value on these simple algebras, and are easily seen to be zero on algebras with lower maximal degree:

	A_{2n-1}	A_{2n-2}	B_n	C_n	D_{n+1}
n^{2n-1}	1	0	1	1	1
n^{2n-2}	2	1	1	1	1
u^{2n-1}	$2n-1$	0	n	n	$n+1$
u^{2n-2}	$2n-2$	$2n-2$	$n-1$	$n-1$	n
$n^{2,2n-3}$	2	0	2	1	2

One can check by determinant that this matrix is non-degenerate, this can also easily be done by hand.

Ignoring the algebras with higher maximal degree (which we assumed we know their coefficients) we have:

$$u^{2n-1} - u^{2n-2} - n^{2n-1} = (2n-1)\mathfrak{g}(A_{2n-2}),$$

so we know $\mathfrak{g}(A_{2n-2})$, and

$$n^{2n-1} - n^{2n-2} - \mathfrak{g}(A_{2n-2}) = \mathfrak{g}(A_{2n-1}),$$

so we know this as well.

$u^{2n-1} - n \cdot n^{2n-1}$ is $\mathfrak{g}(D_{n+1})$ up to things we know, so now we know everything but $\mathfrak{g}(B_n), \mathfrak{g}(C_n)$.

But we know their sum from any of the first four equations, and with the last equation, we can find each of them.

Now we have to consider the case where there are also exceptional, we start with $2n - 1 = 29$, in this case, we need to also find $\mathfrak{g}(E_8)$, and we will add the invariant $n^{2,28}$.

	A_{29}	A_{28}	B_{15}	C_{15}	D_{16}	E_8
n^{29}	1	0	1	1	1	1
n^{28}	2	1	1	1	1	1
u^{29}	29	0	15	15	16	8
u^{28}	28	28	14	14	15	7
$n^{2,27}$	2	0	2	1	2	1
$n^{2,26}$	4	2	1	2	1	1

Again $u^{29} - u^{28} - n^{29}$ gives us $\mathfrak{g}(A_{28})$, and then $n^{28} - n^{29}$ gives us $\mathfrak{g}(A_{29})$, so we can ignore them.

This time $3n^{29} - n^{2,27} - n^{2,26}$ gives us $\mathfrak{g}(E_8)$ and from there we can continue as if this was the previous case.

Case $2n - 1 = 17$:

	A_{17}	A_{16}	B_9	C_9	D_{10}	E_7
n^{17}	1	0	1	1	1	1
n^{16}	2	1	1	1	1	1
u^{17}	17	0	9	9	10	7
u^{16}	16	16	8	8	9	6
$n^{2,15}$	2	0	2	1	2	1
$n^{2,14}$	4	2	1	2	1	1

This case is essentially identical to the last case.

Case $2n - 1 = 11$: this case has two exceptionals, so we need another invariant, we will add $n^{3,8}$.

	A_{11}	A_{10}	B_6	C_6	D_7	E_6	F_4
n^{11}	1	0	1	1	1	1	1
n^{10}	2	1	1	1	1	1	1
u^{11}	11	0	6	6	7	6	4
u^{10}	10	10	5	5	6	5	3
$n^{2,9}$	2	0	2	1	2	1	1
$n^{2,8}$	4	2	1	2	1	2	1
$n^{3,8}$	2	0	2	1	2	2	1

In the same way as before we get $\mathfrak{g}(A_{11}), \mathfrak{g}(A_{10})$. Then $n^{3,8} - n^{2,9}$ gives us $\mathfrak{g}(E_6)$. $3n^{11} - n^{2,9} - n^{2,8}$ gives us $\mathfrak{g}(F_4)$, and then we are left with only non-exceptionals so we can finish like in the original case.

We are left with algebras with a degree of the highest root ≤ 5 , sadly this case turns out to be the messiest.

It turns out that the invariants we used thus far are not enough, even if we take all of the $n^i, u^i, n^{i,j}$, and consider them as equations on the ten missing coefficients the solutions are not unique.

To fix this we consider the product $\mathfrak{n}^i \times \mathfrak{u}^j \rightarrow \mathfrak{u}^{j-i}$, which we have not used thus far, like with the other product we take its codimension of the right kernel, which we denote $nu^{i,j}$.

	A_5	A_4	B_3	C_3	D_4	G_2	A_3	B_2	A_2	A_1
n^5	1	0	1	1	1	1	0	0	0	0
n^4	2	1	1	1	1	1	0	0	0	0
n^3	3	2	2	2	3	1	1	1	0	0
n^2	4	3	2	2	3	1	2	1	1	0
n^1	5	4	3	3	4	2	3	2	2	1
u^5	5	0	3	3	4	2	0	0	0	0
u^4	4	4	2	2	3	1	0	0	0	0
u^3	3	3	2	2	3	1	3	2	0	0
$n^{2,3}$	2	0	2	1	3	1	0	0	0	0
$nu^{2,3}$	2	2	2	3	1	2	1	0	0	0

At this point showing by hand that the matrix is non-degenerate is unproductive.

By entering the matrix into any matrix calculator one can see that it is non-degenerate (its determinant is -4), and so we can recover the missing coefficients, and the proof is complete. $\square$

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