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Abstract. Let $\mathfrak{g}$ be an exceptional complex simple Lie algebra and $\mathfrak{q}$ be a parabolic subalgebra. A generalized Verma module M is called a scalar generalized Verma module if it is induced from a one-dimensional representation of $\mathfrak{q}$. In this paper, we will determine the first diagonal-reducible point of scalar generalized Verma modules associated to minimal parabolic subalgebras of exceptional Lie algebras by computing explicitly the Gelfand–Kirillov (GK) dimension of the corresponding highest weight modules.

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Key Words and Phrases: Gelfand–Kirillov dimension, minimal parabolic subalgebra, exceptional type, PyCox

Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and $U(\mathfrak{g})$ be its universal enveloping algebra. Fix a Cartan subalgebra $\mathfrak{h}$ and denote by Δ the root system associated to $(\mathfrak{g}, \mathfrak{h})$. Choose a positive root system $\Delta^+ \subset \Delta$ and a simple system $\Pi \subset \Delta^+$. Denote ρ the half sum of positive roots. Let $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the triangular decomposition of $\mathfrak{g}$ with nilpotent radical $\mathfrak{n}$ and its opposite nilradical $\bar{\mathfrak{n}}$. Choose a subset $I \subset \Pi$, which generates a subsystem $\Delta_I \subset \Delta$. Let $\mathfrak{q}_I$ be the standard parabolic subalgebra corresponding to I with Levi decomposition $\mathfrak{q}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I$. So when $I = \emptyset$, we have $\mathfrak{q}_\emptyset = \mathfrak{h} \oplus \mathfrak{n} = \mathfrak{b}$.

Let $\mathfrak{q}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I$ and F_λ be a finite-dimensional irreducible $\mathfrak{l}_I$ -module with the highest weight $\lambda \in \mathfrak{h}^*$. It can also be viewed as a $\mathfrak{q}_I$ -module with trivial $\mathfrak{u}_I$ -action. The *generalized Verma module* $M_I(\lambda)$ is defined by

$$M_I(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{q}_I)} F_\lambda.$$

The irreducible quotient of $M_I(\lambda)$ is denoted by $L(\lambda)$, which is a highest weight module with highest weight λ . It is also the irreducible quotient of $M_\emptyset(\lambda)$. In the case when $\dim(F_\lambda) = 1$, we usually denote it by $\mathbb{C}_\lambda$ and $M_I(\lambda)$ is called a *scalar generalized Verma module*.

The theory of highest weight modules of simple complex finite-dimensional Lie algebras is built on the original work of Verma [Ver68]. In that work, Verma introduced and studied the generic family of highest weight modules known as Verma modules, which have become the foundation of the field. There have been several attempts to extend the theory of Verma modules, and one of the most natural ways is to generalize

the Verma modules themselves. This can be accomplished in various ways, such as in [Gyo93; Lep77; Roc80]. Generalized Verma modules (GVM) have been studied from various perspectives, and many of the properties of classical Verma modules have either been proved in GVMs or have been generalized. For a study of GVM, see Mazorchuk's work in [Maz00]. In this article, he provided a comprehensive survey of GVM for parabolic subalgebras of simple Lie algebras using parabolic induction and outlined the historical development of the subject.

The problem of reducibility of generalized Verma modules is of great significance in representation theory and is closely related to several other problems, as described in [BX21; EHW83; Mat06]. The essential tool for solving this problem is Jantzen's criterion [Jan77], although it can be quite complicated in practical applications. However, Kubo [Kub12] found some practical reducibility criteria for solving the reducibility of scalar generalized Verma modules associated with exceptional simple Lie algebras and certain maximal parabolic subalgebras. Using Kubo's results, He [He16] established the reducibility of all scalar generalized Verma modules of Hermitian symmetric pairs. Then He–Kubo–Zierau [HKZ19] extended it to all scalar generalized Verma modules associated with maximal parabolic subalgebras. Recently, Bai–Xiao [BX21] solved the reducibility problem for all generalized Verma modules of Hermitian symmetric pairs.

The GK dimension plays a fundamental role in describing algebraic structures with infinite dimensions. It has been used to measure the size of the representations of Lie algebras and Lie group since Joseph's work [Jos78]. Recently, a study led by Bai–Xiao [BX19a] proved that a scalar generalized Verma module is reducible if the GK dimension of its simple quotient is smaller than the dimension of $\mathfrak{u}$. (This approach does not rely on the reduction method outlined in [HKZ19].) For classical Lie algebras, by using this result and the algorithm of computing GK dimension as in [BXX23; BX19b], Bai–Jiang [BJ24] gave a new proof for the complete list of highest weights for which the scalar generalized Verma modules associated to maximal parabolic subalgebras are reducible. Jiang [Jia25] gave a proof for the complete list of highest weights for which the scalar generalized Verma modules associated to classical minimal parabolic subalgebras are reducible. For exceptional Lie algebras, we can use the same technique as in [Bai+25] to compute the GK dimension of the scalar generalized Verma modules associated to minimal parabolic subalgebras. The web page <http://test.slashblade.top:5000/lie/GKdim> in [Bai+25] is convenient to compute the GK dimension of $L(\lambda)$; the program takes λ as the input and returns $\text{GKdim } L(\lambda)$.

The paper is organized as follows. The necessary preliminaries for GK dimension, minimal parabolic subalgebra, Robinson–Schensted (RS) insertion algorithm, Lusztig's $\mathbf{a}$ -function and PyCox are given in Section 1. The reducibility of scalar generalized Verma modules of type G_2 and F_4 are given in Section 2. In Section 3, we give the reducibility of scalar generalized Verma modules of type E . In the appendix we tabulate the sets of reducible points of $M_I(z\xi_p)$ corresponding to each exceptional Lie algebra.

1. Preliminaries

In this section, we will give brief preliminaries for GK dimension, minimal parabolic subalgebra, Lusztig's $\mathbf{a}$ -function, RS algorithm, integral root system and PyCox. See [Bai+25; Jia25] for more details.

1.1. Minimal parabolic subalgebra

A parabolic subalgebra $\mathfrak{q}_I$ is said to be minimal if I is a minimal nonempty subset of Π . In other words, there is only one element in I .

Lemma 1.1 ([Jia25, Lemma 3.2]). *If $\mathfrak{q}_I$ is the minimal parabolic subalgebra, then $\dim(\mathfrak{u}) = |\Delta^+| - 1$.*

Proof. We know $\mathfrak{q}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I$ and $\dim(\Delta^+(\mathfrak{l})) = 1$, then the value of $\dim(\mathfrak{u})$ is obvious. $\square$

Thus we have the following table.

TABLE 1. $\dim(\mathfrak{u})$ of G with minimal parabolic.

G	$ \Delta^+ $	$\dim(\mathfrak{u})$
$\mathrm{SL}(n, \mathbb{C})$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2} - 1$
$\mathrm{SO}(2n+1, \mathbb{C})$	n^2	$n^2 - 1$
$\mathrm{Sp}(2n, \mathbb{C})$	n^2	$n^2 - 1$
$\mathrm{SO}(2n, \mathbb{C})$	$n^2 - n$	$n^2 - n - 1$
E_6	36	35
E_7	63	62
E_8	120	119
F_4	24	23
G_2	6	5

1.2. Lusztig's a-function

Let (W, S) be a Coxeter group with Coxeter matrix $(m_{st})_{s,t \in S}$ and length function ℓ . Following the seminal papers [KL79; Lus85], we define the Hecke algebra $\mathcal{H}$ of (W, S) as follows. Let $q^{\frac{1}{2}}$ be an indeterminate and $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ be the ring of Laurent polynomials in $q^{\frac{1}{2}}$. Let $\mathcal{H}$ be a free $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ -module with a formal basis $\{\tilde{T}_w\}_{w \in W}$. The multiplication on $\mathcal{H}$ is defined by

$$\tilde{T}_s \tilde{T}_w := \begin{cases} \tilde{T}_{sw}, & \text{if } \ell(sw) > \ell(w), \\ (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\tilde{T}_w + \tilde{T}_{sw}, & \text{if } \ell(sw) < \ell(w), \end{cases} \quad \forall s \in S, w \in W.$$

Then $\mathcal{H}$ is an associative $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ -algebra with unity $\tilde{T}_e$. Here $\tilde{T}_w = q^{-\frac{\ell(w)}{2}} T_w$ where T_w is the standard basis element in [KL79].

Besides the normalized standard basis $\{\tilde{T}_w \mid w \in W\}$, we have another basis $\{C_w \mid w \in W\}$ in [KL79] (now called the Kazhdan–Lusztig basis), such that

$$\begin{aligned} C_w &= \sum_{y \in W} (-1)^{\ell(w)+\ell(y)} q^{\frac{1}{2}(\ell(w)-\ell(y))} P_{y,w}(q^{-1}) \tilde{T}_y \\ &= \sum_{y \in W} (-1)^{\ell(w)+\ell(y)} q^{-\frac{1}{2}(\ell(w)-\ell(y))} P_{y,w}(q) \tilde{T}_y^{-1}, \end{aligned}$$

where $P_{y,w} \in \mathbb{Z}[q]$, $P_{y,w} = 0$ unless $y \leq w$, $P_{w,w} = 1$, and $\deg_q P_{y,w} \leq \frac{1}{2}(\ell(w)-\ell(y)-1)$ if $y < w$. Here $\leq$ is the Bruhat order on W , and $y < w$ indicates $y \leq w$ and $y \neq w$.

For any $x, y, w \in W$, let $h_{x,y,w} \in \mathbb{Z}[q^{\pm \frac{1}{2}}]$ be such that $C_x C_y = \sum_{w \in W} h_{x,y,w} C_w$. Following G. Lusztig [Lus85], for each $w \in W$ we define

$$\mathbf{a}(w) := \min\{i \in \mathbb{N} \mid q^{\frac{i}{2}} h_{x,y,w} \in \mathbb{Z}[q^{\frac{1}{2}}], \forall x, y \in W\}.$$

Then $\mathbf{a}(w)$ is a well defined natural number. We say that the function $\mathbf{a}$ is bounded if there is $N \in \mathbb{N}$ such that $\mathbf{a}(w) \leq N$ for any $w \in W$.

For $x, y \in W$, we say $y \leq_{\text{LR}} x$ if there exist $H_1, H_2 \in \mathcal{H}$, such that C_y has nonzero coefficient in the expression of $H_1 C_x H_2$ with respect to the basis $\{C_w\}_{w \in W}$. We say $x \sim_{\text{LR}} y$ if $x \leq_{\text{LR}} y$ and $y \leq_{\text{LR}} x$. It turns out that $\leq_{\text{LR}}$ is a pre-order on W , and $\sim_{\text{LR}}$ is an equivalence relation on W . The equivalence classes are called two-sided cells. The set of two-sided cells forms a partial order set with respect to $\leq_{\text{LR}}$.

We have the following standard properties of Kazhdan–Lusztig basis $\{C_w\}_{w \in W}$, two-sided cells and Lusztig’s $\mathbf{a}$ -function.

Proposition 1.2 ([KL79; Lus85; Lus03]). *Let $s \in S$ and $w, x, y \in W$.*

- (1) $P_{y,w}(0) = 1$ for any $y \leq w$.
- (2) $\mathbf{a}(w) = 0$ if and only if $w = e$. We have $\mathbf{a}(s) = 1$ for any $s \in S$.
- (3) $\mathbf{a}(w) = \mathbf{a}(w^{-1})$.
- (4) If $m_{rt} < \infty$ for some $r, t \in S$ with $r \neq t$, then $\mathbf{a}(w_{rt}) = m_{rt}$.
- (5) If $y \leq_{\text{LR}} x$, then $\mathbf{a}(y) \geq \mathbf{a}(x)$. If $y \sim_{\text{LR}} x$, then $\mathbf{a}(y) = \mathbf{a}(x)$.
- (6) Suppose $\mathbf{a}$ is bounded on W . If $\mathbf{a}(y) = \mathbf{a}(x)$ and $y \leq_{\text{LR}} x$, then $y \sim_{\text{LR}} x$.
- (7) If W is a direct product of Coxeter subgroups W_1 and W_2 , then

$$\mathbf{a}(w) = \mathbf{a}(w_1) + \mathbf{a}(w_2)$$

for $w = (w_1, w_2) \in W_1 \times W_2 = W$.

1.3. PyCox

In this paper, for a simple root $\alpha_i \in \Pi$ we simply write the simple reflection s_{α_i} by s_i . Here we adopt Geck’s notations in [Gec12], which can be used to compute the value of $\mathbf{a}(w)$ for $w \in W$. In type F_4 and E_n , we use $[i_1 - 1, i_2 - 1, \dots, i_k - 1]$ to represent $w = s_{i_1} s_{i_2} \cdots s_{i_k}$. In PyCox the function “klcellrepem” will give us the $\mathbf{a}$ value of $w \in W$ and the character of the left cell to which w belongs (note that the notation for characters in PyCox is different with Carter [Car85]).

Example 1.3. In the case of E_7 , we use $[3, 1, 3]$ to denote the element $w = s_4 s_2 s_4$. Then by PyCox we have:

```
>>> W = coxeter("E", 7)
>>> print(klcellrepem(W, [3, 1, 3]))
{'size': 77, 'character': [['56\_a', 1], ['21\_a', 1]], 'a': 3,
'special': '56\_a', 'index': 8, 'elms': False, 'distinv': False\}.
```

Then we have $\mathbf{a}(w) = 3$, the corresponding characters are 56_a and 21_a , and we can easily compute that the Gelfand–Kirillov dimension is 60 since $\text{GKdim } L(\lambda) = |\Delta^+| - \mathbf{a}(w)$ [BX19b].

1.4. RS algorithm

We recall the RS algorithm which will be used in our paper. Some details can be found in [BX19b].

For a totally ordered set Γ , we denote by $\text{Seq}_n(\Gamma)$ the set of sequences $x = (x_1, x_2, \dots, x_n)$ of length n with $x_i \in \Gamma$. In our paper, we usually take Γ to be $\mathbb{Z}$ or a coset of $\mathbb{Z}$ in $\mathbb{C}$. Then we have a Young tableau $P(x)$ obtained by applying the following RS algorithm to $x \in \text{Seq}_n(\Gamma)$.

Definition 1.4 (RS algorithm). For an element $x \in \text{Seq}_n(\Gamma)$, we write $x = (x_1, \dots, x_n)$. We associate to x a Young tableau $P(x)$ as follows: let P_0 be an empty Young tableau. Assume that we have constructed the Young tableau P_k associated to $(x_1, \dots, x_k)$, $0 \leq k < n$. Then P_{k+1} is obtained by adding x_{k+1} to P_k as follows. Firstly we add x_{k+1} to the first row of P_k by replacing the leftmost entry x_j in the first row which is *strictly* bigger than x_{k+1} . (If there is no such entry x_j , we just add a box with entry x_{k+1} to the right side of the first row, and end this process.) Then add this x_j to the next row as the same way of adding x_{k+1} to the first row. Finally we put $P(x) = P_n$.

Denote the shape of the Young tableau $P(x)$ by $\mathbf{p}(x) = \text{sh}(P(x)) = [p_1, p_2, \dots, p_k]$, where p_i is the number of boxes in the i -th row of $P(x)$. Then $\mathbf{p}(x)$ is a partition of $n = \sum 1 \leq i \leq k p_i$.

Example 1.5. Suppose $x = (1, -3, 2, -1, -3)$. Usually we write

$$x = (1, -3, 2, -1, -3')$$

and regard $-3 < -3'$. Then from the RS algorithm, we have

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \xrightarrow{-3} \begin{array}{|c|} \hline -3 \\ \hline 1 \\ \hline \end{array} \xrightarrow{2} \begin{array}{|c|c|} \hline -3 & 2 \\ \hline 1 & \\ \hline \end{array} \xrightarrow{-1} \begin{array}{|c|c|} \hline -3 & -1 \\ \hline 1 & 2 \\ \hline \end{array} \xrightarrow{-3} \begin{array}{|c|c|} \hline -3 & -3 \\ \hline -1 & 2 \\ \hline 1 & \\ \hline \end{array} = P(x).$$

Thus we have $\mathbf{p}(x) = [2, 2, 1]$, which is a partition of 5.

1.5. Gelfand–Kirillov dimension

Let M be a finite generated $U(\mathfrak{g})$ -module. Fix a finite-dimensional generating subspace M_0 of M . Let $U_n(\mathfrak{g})$ be the standard filtration of $U(\mathfrak{g})$. Set $M_n = U_n(\mathfrak{g}) \cdot M_0$

and $\text{gr}(M) = \bigoplus_{n=0}^{\infty} \text{gr}_n M$, where $\text{gr}_n M = M_n/M_{n-1}$. Thus $\text{gr}(M)$ is a graded module of $\text{gr}(U(\mathfrak{g})) \simeq S(\mathfrak{g})$.

Definition 1.6. The *Gelfand–Kirillov dimension* of M is defined by

$$\text{GKdim } M = \overline{\lim}_{n \rightarrow \infty} \frac{\log \dim(U_n(\mathfrak{g})M_0)}{\log n}.$$

It is easy to see that the above definition is independent of the choice of M_0 . Then we have the following lemma.

Lemma 1.7 ([BX19a, Lemma 4.4]). *For any $z \in \mathbb{C}$, we have*

$$\text{GKdim}(L((z+1)\xi)) \leq \text{GKdim}(L(z\xi)),$$

where ξ is the fundamental weight of a simple root α . In particular, if $M_I(z\xi)$ is reducible, then $M_I((z+1)\xi)$ is also reducible.

The following lemma is very useful in our proof of the main results.

Lemma 1.8 ([BX19a, Theorem 1.1]). *A scalar generalized Verma module $M_I(\lambda)$ is irreducible if and only if $\text{GKdim } L(\lambda) = \dim(\mathfrak{u})$.*

There is a formula connecting the Lusztig’s $\mathbf{a}$ -function and the GK dimension

$$\text{GKdim } L(\lambda) = |\Delta^+| - \mathbf{a}(w_\lambda) \quad \text{for some } w_\lambda \in W$$

in [BX19b], it also holds for exceptional types. In [Bai+25] the authors have found an efficient algorithm to compute the Gelfand–Kirillov dimension of highest weight modules of exceptional Lie algebras and we recall their notations and algorithms here.

Let $(-, -): \mathfrak{h} \times \mathfrak{h}^* \rightarrow \mathbb{C}$ be the canonical pairing. For $\mu \in \mathfrak{h}^*$, define

$$\Delta_{[\mu]} := \{\alpha \in \Delta \mid (\mu, \alpha^\vee) \in \mathbb{Z}\},$$

where $\alpha^\vee$ is the coroot of the root $\alpha \in \Delta$. Denote

$$W_{[\mu]} := \{w \in W \mid w\mu - \mu \in \mathbb{Z}\Delta\}.$$

Then $\Delta_{[\mu]}$ is a root system and let $\Pi_{[\mu]}$ be the simple system of $\Delta_{[\mu]}$. Set $S := \{\alpha \in \Pi_{[\mu]} \mid (\mu, \alpha^\vee) = 0\}$. Let ℓ be the length function on $W_{[\mu]}$ and

$$W_{[\mu]}^S := \{w \in W_{[\mu]} \mid \ell(ws_\alpha) = \ell(w) + 1 \text{ for all } \alpha \in S\}.$$

A weight μ is *anti-dominant* if $(\mu, \alpha^\vee) \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Delta^+$. For any $\lambda \in \mathfrak{h}^*$, there exists a unique anti-dominant weight $\mu \in \mathfrak{h}^*$ and a unique $w_\lambda \in W_{[\mu]}^S$.

For integral highest weight modules, we can easily find the required w_λ by using the positive index reduction algorithm [Bai+25, Lemma 3.2], then by using PyCox we can compute the GK dimension. For non-integral highest weight modules, we need more notations.

For a root system Δ , denote $\mathfrak{h}_\Delta$ the corresponding Cartan subalgebra of Lie algebra $\mathfrak{g}_\Delta$ associated to Δ . Denote $\Delta_{[\lambda]}$ the integral root system for $\lambda \in \mathfrak{h}_\Delta^*$ and $\Pi_{[\lambda]}$ the simple system of $\Delta_{[\lambda]} \cap \Delta^+$ in $\Delta_{[\lambda]}$. It is provided in [Bai+25] a way to find $\Pi_{[\lambda]}$ as follows.

Proposition 1.9 ([Bai+25, Lemma 3.5]). *For $\lambda \in \mathfrak{h}_\Delta^*$, decompose $\Delta_{[\lambda]}$ into several orthogonal components:*

$$\Delta_{[\lambda]} = \Delta_{[\lambda]_1} \cup \Delta_{[\lambda]_2} \cup \cdots \cup \Delta_{[\lambda]_k} \simeq \Delta_1 \times \Delta_2 \times \cdots \times \Delta_k$$

such that $\Delta_{[\lambda]_i} \simeq \Delta_i$ for $1 \leq i \leq k$. Then we have the following way to determine the simple system $\Pi_{[\lambda]_i}$ of $\Delta_{[\lambda]_i}$.

- (1) *If $|\Delta_{[\lambda]_i}^+| = \frac{n(n+1)}{2}$, then $\Delta_{[\lambda]_i} \simeq A_n$. We compute $\rho_i = \frac{1}{2} \sum_{\alpha \in \Delta_{[\lambda]_i}^+} \alpha$. Then we find out all the positive roots $\alpha \in \Phi_{[\lambda]_i}^+$ such that $(\rho_i, \alpha^\vee) = 1$ and denote them by $I_n = \{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_n}\}$. Then we have $\Pi_{[\lambda]_i} = I_n$.*
- (2) *If $|\Delta_{[\lambda]_i}^+| = n^2$ and the number of short roots is n ($n \geq 2$), then $\Delta_{[\lambda]_i} \simeq B_n$. We compute $\rho_i = \frac{1}{2} \sum_{\alpha \in \Delta_{[\lambda]_i}^+} \alpha$. Then we find out all the positive roots $\alpha \in \Delta_{[\lambda]_i}^+$ such that $(\rho_i, \alpha^\vee) = 1$ and denote them by $I_n = \{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_n}\}$. Then we have $\Pi_{[\lambda]_i} = I_n$.*
- (3) *If $|\Delta_{[\lambda]_i}^+| = n^2$ and the number of long roots is n ($n \geq 3$), then $\Delta_{[\lambda]_i} \simeq C_n$. Then similar to the case in (2), we have $\Pi_{[\lambda]_i} = I_n$.*
- (4) *If $|\Delta_{[\lambda]_i}^+| = n^2 - n$ ($n \geq 4$), then $\Delta_{[\lambda]_i} \simeq D_n$. Then similar to the case in (1), we have $\Pi_{[\lambda]_i} = I_n$.*
- (5) *If $|\Delta_{[\lambda]_i}^+| = 36$, then $\Delta_{[\lambda]_i} \simeq E_6$. Then similar to the case in (1), we have $\Pi_{[\lambda]_i} = I_6$.*
- (6) *If $|\Delta_{[\lambda]_i}^+| = 63$, then $\Delta_{[\lambda]_i} \simeq E_7$. Then similar to the case in (1), we have $\Pi_{[\lambda]_i} = I_7$.*

Denote the label “ $\sim$ ” attached to the connected component corresponding to short roots. In other cases it is similar. Let $\phi: \Delta_{[\lambda]} \rightarrow \phi(\Delta_{[\lambda]})$ be an isomorphism, then we have $\mathbf{a}(w) = \mathbf{a}(\phi(w))$. We will determine the value of $\mathbf{a}(w_\lambda)$ by Proposition 1.9 in the non-integral case. Then we have the following proposition.

Proposition 1.10 ([Bai+25, Theorems 4.1, 5.1, 6.1, 6.4 and 6.7]). *Let $L(\lambda)$ be a simple integral highest weight module of type E , F and G , then the following holds:*

- (1) $\mathbf{a}(w_\lambda) = |\Delta^+|$ if and only if $\lambda + \rho$ is dominant integral;
- (2) $\mathbf{a}(w_\lambda) = 0$ if and only if $\lambda + \rho$ is anti-dominant integral.

If $\lambda \in \mathfrak{h}^*$ is non-integral, we can get an integral weight $\phi(\lambda)$ of type $\phi(\Delta_{[\lambda]})$ by Proposition 1.9, in this case we have $\mathbf{a}(w_\lambda) = \mathbf{a}(w_{\phi(\lambda)})$ and then we can compute the value of $\mathbf{a}(w_{\phi(\lambda)})$ by RS algorithm and Proposition 1.2 if $\phi(\Delta_{[\lambda]})$ is of classical type and by PyCox and Proposition 1.2 if $\phi(\Delta_{[\lambda]})$ is of exceptional type. From now on, we will use $(k_1, k_2, \dots, k_t)$ to denote $\lambda = k_1 e_1 + k_2 e_2 + \cdots + k_t e_t$, where $\{e_1, e_2, \dots, e_t\}$ is a basis for $\mathfrak{h}^*$.

Example 1.11. Let $\mathfrak{g} = F_4$ and $L(\lambda)$ be the highest weight module of $\mathfrak{g}$ with $\lambda + \rho = (1, 2, \frac{3}{2}, \frac{7}{2})$, then $\Delta_{[\lambda]} \simeq C_4$ by Proposition 1.9. Suppose the simple system of C_4 is $\Pi = \{\beta_1 = e_1 - e_2, \beta_2 = e_2 - e_3, \beta_3 = e_3 - e_4, \beta_4 = 2e_4\}$. Define a map $\phi: \Delta_{[\lambda]} \rightarrow C_4$ by

$$\phi(e_2) = \beta_1, \phi(\alpha_1) = \beta_2, \phi(\alpha_2) = \beta_3, \phi(\alpha_3) = \beta_4.$$

Then we have

$$\lambda + \rho = -2\xi_1 + 7\xi_2 - 6\xi_3 + 4\xi_4,$$

where $\{\xi_i \mid 1 \leq i \leq 4\}$ are the fundamental weights for $\Pi_{[\lambda]}$, and

$$\phi(\lambda + \rho) = -2\phi(\xi_1) + 7\phi(\xi_2) - 6\phi(\xi_3) + 4\phi(\xi_4) = (3, -1, 5, -2),$$

which is an integral weight of type C_4 , we can get that $\mathbf{a}(w_\lambda) = 6$ by the RS algorithm. Thus $\text{GKdim } L(\lambda) = |\Delta^+| - \mathbf{a}(w_\lambda) = 24 - 6 = 18$ by [BX19b].

2. Reducibility of scalar generalized Verma modules for type G_2 and F_4

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra and let $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ be a fixed Borel subalgebra of $\mathfrak{g}$. For a minimal parabolic subalgebra $\mathfrak{q}$, we recall that $\mathfrak{q}$ corresponds to the subsets $\Pi \setminus \{\alpha_i\}_{i \neq p}$. It is easy to get that $\lambda = z\eta$ for some $z \in \mathbb{R}$ and $\eta = \sum_{i \neq p} k_i \xi_i$ by Weyl dimension formula, where $k_i \in \mathbb{R}$ and ξ_i is the fundamental weight associated with the simple root α_i . In this paper, we suppose that $\eta = \sum_{i \neq p} \xi_i = \widehat{\xi}_p$ and we call the corresponding reducible point a *diagonal-reducible point*.

From Lemma 1.7, the set of diagonal-reducible points of a scalar generalized Verma module $M_I(z\widehat{\xi}_p)$ is given in the following diagram:

$$\text{-----} \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \cdots$$

$a \qquad \quad b \qquad \quad \quad \quad$

where the diagonal-reducible points starting from $z = a \in \mathbb{R}$ are equally spaced at an interval of length b and can be written in the form $a + b\mathbb{Z}_{\geq 0}$. The point a will be called the *first diagonal-reducible point* of $M_I(z\widehat{\xi}_p)$.

From Lemma 1.7, we only need to find the first diagonal-reducible point of the scalar generalized Verma module $M_I(z\widehat{\xi}_p)$. The following lemma plays an important role in the process of determining the first diagonal-reducible point.

Lemma 2.1. *Let ξ_i be the fundamental weight of simple root α_i and ρ be half the sum of the members in Δ^+ , then $\rho = \sum_{i \geq 1} \xi_i$.*

Proof. It can be found in [Hum78, Section 13.3, Lemma A]. Of course, we can also draw the conclusion case by case in [Kna02]. For convenience, from now on we will use Knapp conventions to denote the simple root in type E , F and G , see [Kna02, Appendices C.2] for more details. $\square$

In this section we give the reducibility of scalar generalized Verma modules associated to minimal parabolic subalgebra for type G_2 and F_4 . It's worth noting that in G_2 , the case of minimal parabolic type is contrary to maximal parabolic type since there are only two simple roots. In other words, in the minimal parabolic type, the case of $p = 1$ is the same as $p = 2$ in the maximal parabolic type.

Theorem 2.2. *Let $\mathfrak{g}$ be of type G_2 , then $M_I(z\widehat{\xi}_p)$ is reducible if and only if*

- (1) $p = 1$, $z \in \{-\frac{1}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}\} \cup \{-\frac{2}{3} + \frac{1}{3}\mathbb{Z}_{\geq 0}\}$; or
- (2) $p = 2$, $z \in \{-\frac{3}{2} + \mathbb{Z}_{\geq 0}\} \cup \{\mathbb{Z}_{\geq 0}\}$.

Proof.

Proof of (1). Let $\Delta^+(\mathfrak{l}) = \{\alpha_1\} = \{(1, -2, 0)\}$. When $M_I(\lambda)$ is of scalar type, we know that $\lambda = z\xi_2$ for some $z \in \mathbb{R}$, where ξ_2 is the fundamental weight of simple root α_2 and $\xi_2 = (-1, -1, 2)$.

In [Kna02] we know that $\rho = (-1, -2, 3)$ and $\lambda + \rho = (-z - 1, -z - 2, 2z + 3)$.

- (1) If λ is integral, then we have the following.
 - (a) When $z = 0$, then $\lambda + \rho = (-1, -2, 3)$, which is dominant integral, by Proposition 1.10 we can get that $\text{GKdim } L(\lambda) = 0 < \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = 0$ is a diagonal-reducible point.
 - (b) When $z = -1$, then $\lambda + \rho = (0, -1, 1)$, one can easily check that $\lambda + \rho$ is not dominant or anti-dominant, by Proposition 1.10 we can get that $\text{GKdim } L(\lambda) = 5 = \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -1$ is an irreducible point.
- (2) If λ is half-integral, one can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2 \simeq A_1 \times \widetilde{A}_1$. The simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (-1, -1, 2)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (1, -1, 0)\}$. Suppose the simple system of A_1 is $\{\alpha_1 = (1, -1, 0)\}$, and the simple system of $\widetilde{A}_1$ is $\{\alpha'_1 = (1, -1, 0)\}$. We define a map $\phi: \Delta_{[\lambda]} \rightarrow A_1 \times \widetilde{A}_1$ such that $\phi(\beta_1) = \alpha_1$, $\phi(\gamma_1) = \alpha'_1$, and we can have $\phi(\lambda + \rho)|_{A_1} = (\frac{1}{4}(2z + 3), -\frac{1}{4}(2z + 3))$, $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (\frac{1}{2}, -\frac{1}{2})$. We will have the following.
 - (a) When $z = -\frac{1}{2}$, then $\lambda + \rho = (-\frac{1}{2}, -\frac{3}{2}, 2)$, $\phi(\lambda + \rho)|_{A_1} = (\frac{1}{2}, -\frac{1}{2})$, and $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (\frac{1}{2}, -\frac{1}{2})$. By using RS algorithm, we have $\mathbf{a}(w_\lambda) = 2$ and $\text{GKdim } L(\lambda) = 4 < \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -\frac{1}{2}$ is a diagonal-reducible point.
 - (b) When $z = -\frac{3}{2}$, then $\lambda + \rho = (\frac{1}{2}, -\frac{1}{2}, 0)$, $\phi(\lambda + \rho)|_{A_1} = (0, 0)$, and $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (\frac{1}{2}, -\frac{1}{2})$. By using RS algorithm, we have $\mathbf{a}(w_\lambda) = 1$ and $\text{GKdim } L(\lambda) = 5 = \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -\frac{3}{2}$ is an irreducible point.
- (3) If λ is one-third-integral, one can easily check that $\Delta_{[\lambda]} \simeq A_2$. The simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (1, -1, 0), \beta_2 = (-1, 0, 1)\}$. Suppose the simple system of A_2 is $\{\alpha_1 = (1, -1, 0), \alpha_2 = (0, 1, -1)\}$. We define a map $\phi: \Delta_{[\lambda]} \rightarrow A_2$ such that $\phi(\beta_1) = \alpha_1$, $\phi(\beta_2) = \alpha_2$, and we can have $\phi(\lambda + \rho) = (z + 2, z + 1, -2z - 3)$. We will have the following.
 - (a) When $z = -\frac{2}{3}$, then $\lambda + \rho = (-\frac{1}{3}, -\frac{4}{3}, \frac{5}{3})$ and $\phi(\lambda + \rho) = (\frac{4}{3}, \frac{1}{3}, -\frac{5}{3})$. By using RS algorithm, we have $\mathbf{a}(w_\lambda) = 3$ and $\text{GKdim } L(\lambda) = 3 < \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -\frac{2}{3}$ is a diagonal-reducible point.
 - (b) When $z = -\frac{5}{3}$, then $\lambda + \rho = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and $\phi(\lambda + \rho) = (\frac{1}{3}, -\frac{2}{3}, \frac{1}{3})$. By using RS algorithm, we have $\mathbf{a}(w_\lambda) = 1$ and $\text{GKdim } L(\lambda) = 5 = \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -\frac{5}{3}$ is an irreducible point.
 - (c) When $z = -\frac{1}{3}$, then $\lambda + \rho = (-\frac{2}{3}, -\frac{5}{3}, \frac{7}{3})$ and $\phi(\lambda + \rho) = (\frac{5}{3}, \frac{2}{3}, -\frac{7}{3})$. By using RS algorithm, we have $\mathbf{a}(w_\lambda) = 3$ and $\text{GKdim } L(\lambda) = 3 < \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -\frac{1}{3}$ is a diagonal-reducible point.
 - (d) When $z = -\frac{4}{3}$, then $\lambda + \rho = (\frac{1}{3}, -\frac{2}{3}, \frac{1}{3})$ and $\phi(\lambda + \rho) = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$. By using RS algorithm, we have $\mathbf{a}(w_\lambda) = 1$ and $\text{GKdim } L(\lambda) = 5 = \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -\frac{4}{3}$ is an irreducible point.

- (4) If λ is non-integral, non-half-integral and non-one-third-integral, one can easily check that $\Delta_{[\lambda]} \simeq A_1$. The simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = e_1 - e_2\}$. We can get $\mathbf{a}(w_\lambda) = 1$ by using RS algorithm, thus $\text{GKdim } L(\lambda) = 5 = \dim(\mathbf{u})$. In this case $M_I(z\widehat{\xi}_1)$ is irreducible.

All in all, when $p = 1$, $M_I(z\widehat{\xi}_1)$ is reducible if and only if

$$\begin{aligned} z &\in \left\{-\frac{1}{2} + \mathbb{Z}_{\geq 0}\right\} \cup \left\{\mathbb{Z}_{\geq 0}\right\} \cup \left\{-\frac{2}{3} + \mathbb{Z}_{\geq 0}\right\} \cup \left\{-\frac{1}{3} + \mathbb{Z}_{\geq 0}\right\} \\ &= \left\{-\frac{1}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}\right\} \cup \left\{-\frac{2}{3} + \frac{1}{3}\mathbb{Z}_{\geq 0}\right\}. \end{aligned}$$

Proof of (2). Next we consider the case when $p = 2$. Let $\Delta^+(\mathbf{l}) = \{\alpha_2\} = \{(-2, 1, 1)\}$. When $M_I(z\widehat{\xi}_p)$ is of scalar type, we know that $\lambda = z\xi_1$ for some $z \in \mathbb{R}$, where ξ_1 is the fundamental weight of simple root α_1 and $\xi_1 = (0, -1, 1)$. In [Kna02] we know that $\rho = (-1, -2, 3)$ and $\lambda + \rho = (-1, -z - 2, z + 3)$.

- (1) If λ is integral, then we have the following.
- (a) When $z = 0$, then $\lambda + \rho = (-1, -2, 3)$, which is dominant integral, by Proposition 1.10 we can get that $\text{GKdim } L(\lambda) = 0 < \dim(\mathbf{u})$. By Lemma 1.8 we can obtain that $z = 0$ is a diagonal-reducible point.
 - (b) When $z = -1$, then $\lambda + \rho = (-1, -1, 2)$, one can easily check that $\lambda + \rho$ is not dominant or anti-dominant, by Proposition 1.10 we can get that $\text{GKdim } L(\lambda) = 5 = \dim(\mathbf{u})$. By Lemma 1.8 we can obtain that $z = -1$ is an irreducible point.
- (2) If λ is half-integral, one can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2 \simeq A_1 \times \widetilde{A_1}$. The simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (-2, 1, 1)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (0, -1, 1)\}$. Suppose the simple system of A_1 is $\{\alpha_1 = (1, -1, 0)\}$, and the simple system of $\widetilde{A_1}$ is $\{\alpha'_1 = (1, -1, 0)\}$. We define a map $\phi: \Delta_{[\lambda]} \rightarrow A_1 \times \widetilde{A_1}$ such that $\phi(\beta_1) = \alpha_1$, $\phi(\gamma_1) = \alpha'_1$, and we can have $\phi(\lambda + \rho)|_{A_1} = (\frac{1}{2}, -\frac{1}{2})$, $\phi(\lambda + \rho)|_{\widetilde{A_1}} = (z + \frac{5}{2}, -z - \frac{5}{2})$. Then we have the following.
- (a) When $z = -\frac{3}{2}$, then $\lambda + \rho = (-1, -\frac{1}{2}, \frac{3}{2})$, $\phi(\lambda + \rho)|_{A_1} = (\frac{1}{2}, -\frac{1}{2})$, and $\phi(\lambda + \rho)|_{\widetilde{A_1}} = (1, -1)$. By using RS algorithm, we have $\mathbf{a}(w_\lambda) = 2$ and $\text{GKdim } L(\lambda) = 4 < \dim(\mathbf{u})$. By Lemma 1.8 we can obtain that $z = -\frac{3}{2}$ is a diagonal-reducible point.
 - (b) When $z = -\frac{5}{2}$, then $\lambda + \rho = (-1, \frac{1}{2}, \frac{1}{2})$, $\phi(\lambda + \rho)|_{A_1} = (\frac{1}{2}, -\frac{1}{2})$, and $\phi(\lambda + \rho)|_{\widetilde{A_1}} = (0, 0)$. By using RS algorithm, we have $\mathbf{a}(w_\lambda) = 1$ and $\text{GKdim } L(\lambda) = 5 = \dim(\mathbf{u})$. By Lemma 1.8 we can obtain that $z = -\frac{5}{2}$ is an irreducible point.
- (3) If λ is non-integral and non-half-integral, one can easily check that $\Delta_{[\lambda]} \simeq A_1$. The simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (-2, 1, 1)\}$. We can get $\mathbf{a}(w_\lambda) = 1$ by using RS algorithm, thus $\text{GKdim } L(\lambda) = 5 = \dim(\mathbf{u})$. In this case $M_I(z\widehat{\xi}_2)$ is irreducible.

All in all, when $p = 2$, $M_I(z\widehat{\xi}_2)$ is reducible if and only if

$$z \in \left\{-\frac{3}{2} + \mathbb{Z}_{\geq 0}\right\} \cup \left\{\mathbb{Z}_{\geq 0}\right\}.$$

So far, we have completed the proof of all cases of type G_2 . □

Theorem 2.3. *Let $\mathfrak{g}$ be of type F_4 , then $M_I(z\hat{\xi}_p)$ is reducible if and only if*

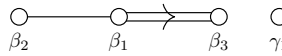
- (1) $p = 1, z \in \bigcup_{i=5}^9 \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (2) $p = 2, z \in \bigcup_{i=5}^8 \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (3) $p = 3, z \in \bigcup_{i=4}^5 \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=6}^7 \{-\frac{i+3}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (4) $p = 4, z \in \bigcup_{i=5}^6 \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=7}^9 \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$.

Proof.

Proof of (1). Let $\Delta^+(\mathfrak{l}) = \{\alpha_1\} = \{(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$. When $M_I(\lambda)$ is of scalar type, we know that $\lambda = z(\xi_2 + \xi_3 + \xi_4) = z(\rho - \xi_1)$ for some $z \in \mathbb{R}$, where ξ_1 is the fundamental weight of simple root α_1 and $\xi_1 = (1, 0, 0, 0)$.

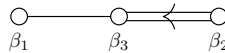
In [Kna02] we know that $\rho = (\frac{11}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$ and $\lambda + \rho = (\frac{11}{2} + \frac{9}{2}z, \frac{5}{2} + \frac{5}{2}z, \frac{3}{2} + \frac{3}{2}z, \frac{1}{2} + \frac{1}{2}z)$.

- (1) If λ is integral, then we have the following.
 - (a) When $z = -1$, then $\lambda + \rho = (1, 0, 0, 0)$, we can get $w_\lambda = [0, 1, 2, 1, 3, 2, 1, 0, 1, 2, 1, 3, 2, 1, 0]$. By using PyCox we can get that $\text{GKdim } L(\lambda) = 21 < \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -1$ is a diagonal-reducible point.
 - (b) When $z = -2$, then $\lambda + \rho = (-\frac{7}{2}, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2})$, we can get $w_\lambda = [0]$. By using PyCox we can get that $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -2$ is an irreducible point.
- (2) If λ is half-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 0, 1, 1), \beta_2 = (0, 1, 0, -1), \beta_3 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow B_3 \times \widetilde{A}_1$. Then $\phi(\lambda + \rho)|_{B_3} = (4z + \frac{9}{2}, 2z + \frac{5}{2}, \frac{1}{2})$, $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (3z + \frac{7}{2}, -3z - \frac{7}{2})$. By using RS algorithm, we can easily get that:

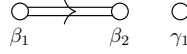
- (a) when $z = -\frac{1}{2}$, $\mathbf{a}(w_\lambda) = 10$, and $\text{GKdim } L(\lambda) = 14 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that this point is a reducible point;
- (b) when $z = -\frac{3}{2}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that this point is an irreducible point.
- (3) If λ is one-third-integral, we can easily check that the simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (0, 0, 1, 0), \beta_2 = (0, 1, 0, 1), \beta_3 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow C_3$. Then $\phi(\lambda + \rho)|_{C_3} = (6z + 7, 3z + 4, 3z + 3)$. By using RS algorithm, we can easily get that:

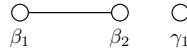
- (a) when $z \in \{-\frac{2}{3}, -\frac{1}{3}\}$, $\mathbf{a}(w_\lambda) = 9$, and $\text{GKdim } L(\lambda) = 15 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
- (b) when $z \in \{-\frac{5}{3}, -\frac{4}{3}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.

- (4) If λ is one-fourth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 1, 1, 0), \beta_2 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow B_2 \times \widetilde{A}_1$. Then $\phi(\lambda + \rho)|_{B_2} = (4z + \frac{9}{2}, \frac{1}{2})$, $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (2z + \frac{5}{2}, -2z - \frac{5}{2})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{3}{4}, -\frac{1}{4}\}$, $\mathbf{a}(w_\lambda) = 5$, and $\text{GKdim } L(\lambda) = 19 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{7}{4}, -\frac{5}{4}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (5) If λ is one-fifth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 1, 0, 0), \beta_2 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (1, 0, 0, 1)\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow \widetilde{A}_2 \times A_1$. Then $\phi(\lambda + \rho)|_{\widetilde{A}_2} = (\frac{10}{3}z + \frac{11}{3}, -\frac{5}{3}z - \frac{4}{3}, -\frac{5}{3}z - \frac{7}{3})$, $\phi(\lambda + \rho)|_{A_1} = (\frac{5}{2}z + 3, -\frac{5}{2}z - 3)$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{4}{5}, -\frac{3}{5}, -\frac{2}{5}, -\frac{1}{5}\}$, $\mathbf{a}(w_\lambda) = 4$, and $\text{GKdim } L(\lambda) = 20 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{9}{5}, -\frac{8}{5}, -\frac{7}{5}, -\frac{6}{5}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (6) If λ is one-sixth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2 \cup \Delta_3$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (1, 0, 1, 0)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$, and the simple system of Δ_3 is $\Pi_3 = \{\delta_1 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_1 \times \widetilde{A}_1 \times \widetilde{A}_1$. Then $\phi(\lambda + \rho)|_{A_1} = (3z + \frac{7}{2}, -3z - \frac{7}{2})$, $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (3z + \frac{7}{2}, -3z - \frac{7}{2})$, $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (\frac{1}{2}, -\frac{1}{2})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{5}{6}, -\frac{1}{6}\}$, $\mathbf{a}(w_\lambda) = 3$, and $\text{GKdim } L(\lambda) = 21 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{11}{6}, -\frac{7}{6}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (7) If λ is one-seventh-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (1, 1, 0, 0)\}$, and the simple system of Δ_2 is

$\Pi_2 = \{\gamma_1 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_1 \times \widetilde{A_1}$. Then $\phi(\lambda + \rho)|_{A_1} = (\frac{7}{2}z + 4, -\frac{7}{2}z - 4)$, $\phi(\lambda + \rho)|_{\widetilde{A_1}} = (\frac{1}{2}, -\frac{1}{2})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{6}{7}, -\frac{5}{7}, -\frac{4}{7}, -\frac{3}{7}, -\frac{2}{7}, -\frac{1}{7}\}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 22 < \dim(\mathbf{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{13}{7}, -\frac{12}{7}, -\frac{11}{7}, -\frac{10}{7}, -\frac{9}{7}, -\frac{8}{7}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathbf{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (8) If λ is one-eighth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow \widetilde{A_1} \times \widetilde{A_1}$. Then $\phi(\lambda + \rho)|_{\widetilde{A_1}} = (4z + \frac{9}{2}, -4z - \frac{9}{2})$, $\phi(\lambda + \rho)|_{\widetilde{A_1}} = (\frac{1}{2}, -\frac{1}{2})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{7}{8}, -\frac{5}{8}, -\frac{3}{8}, -\frac{1}{8}\}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 22 < \dim(\mathbf{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{15}{8}, -\frac{13}{8}, -\frac{11}{8}, -\frac{9}{8}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathbf{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (9) If λ is one-ninth-integral, we can easily check that the simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \beta_2 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow \widetilde{A_2}$. Then $\phi(\lambda + \rho)|_{\widetilde{A_2}} = (6z + 7, -3z - 3, -3z - 4)$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{8}{9}, -\frac{7}{9}, -\frac{5}{9}, -\frac{4}{9}, -\frac{2}{9}, -\frac{1}{9}\}$, $\mathbf{a}(w_\lambda) = 3$, and $\text{GKdim } L(\lambda) = 21 < \dim(\mathbf{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{17}{9}, -\frac{16}{9}, -\frac{14}{9}, -\frac{13}{9}, -\frac{11}{9}, -\frac{10}{9}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathbf{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (10) If λ does not belong to any of the types mentioned above, one can easily check that $\Delta_{[\lambda]} \simeq A_1$. The simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$. We can get $\mathbf{a}(w_\lambda) = 1$ by using RS algorithm, thus $\text{GKdim } L(\lambda) = 23 = \dim(\mathbf{u})$. In this case $M_I(z\xi_1)$ is irreducible.

All in all, when $p = 1$, $M_I(z\hat{\xi}_1)$ is reducible if and only if

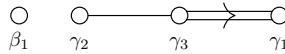
$$\begin{aligned}
 z \in & \{-1 + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{1}{2} + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{2}{3} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{3} + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{3}{4} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{4} + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{4}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{2}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{5} + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{5}{6} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{6} + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{6}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{5}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{4}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{2}{7} + \mathbb{Z}_{\geq 0}\} \\
 & \quad \cup \{-\frac{1}{7} + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{7}{8} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{5}{8} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{8} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{8} + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{8}{9} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{7}{9} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{5}{9} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{4}{9} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{2}{9} + \mathbb{Z}_{\geq 0}\} \\
 & \quad \cup \{-\frac{1}{9} + \mathbb{Z}_{\geq 0}\} \\
 = & \bigcup_{i=5}^9 \left\{ -1 + \frac{1}{i} \mathbb{Z}_{\geq 0} \right\}.
 \end{aligned}$$

Proof of (2). Let $\Delta^+(1) = \{(0, 0, 0, 1)\} = \{\alpha_2\}$. When $M_I(\lambda)$ is of scalar type, we know that $\lambda = z(\xi_1 + \xi_3 + \xi_4) = z(\rho - \xi_2)$ for some $z \in \mathbb{R}$, where ξ_2 is the fundamental weight of simple root α_2 and $\xi_2 = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. In [Kna02] we know that $\rho = (\frac{11}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$ and $\lambda + \rho = (\frac{11}{2} + 4z, \frac{5}{2} + 2z, \frac{3}{2} + z, \frac{1}{2})$.

(1) If λ is integral, then we have the following.

- (a) When $z = -1$, then $\lambda + \rho = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, we can get $w_\lambda = [1, 2, 1, 0, 1, 3, 2, 1, 2, 0, 1, 3, 2, 1, 2, 0, 1, 3, 2, 1]$. By using PyCox we can get that $\text{GKdim } L(\lambda) = 20 < \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -1$ is a diagonal-reducible point.
- (b) When $z = -2$, then $\lambda + \rho = (-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2})$, we can get $w_\lambda = [1]$. By using PyCox we can get that $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -2$ is an irreducible point.

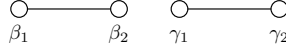
(2) If λ is half-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 0, 1, 0)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (0, 0, 0, 1), \gamma_2 = (1, -1, 0, 0), \gamma_3 = (0, 1, 0, -1)\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow \widehat{A_1} \times B_3$. Then $\phi(\lambda + \rho)|_{\widehat{A_1}} = (z + \frac{3}{2}, -z - \frac{3}{2})$, $\phi(\lambda + \rho)|_{B_3} = (4z + \frac{11}{2}, 2z + \frac{5}{2}, \frac{1}{2})$. By using RS algorithm, we can easily get that:

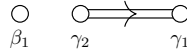
- (a) when $z = -\frac{1}{2}$, $\mathbf{a}(w_\lambda) = 10$, and $\text{GKdim } L(\lambda) = 14 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that this point is a reducible point;
- (b) when $z = -\frac{3}{2}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that this point is an irreducible point.

- (3) If λ is one-third-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 0, 0, 1), \beta_2 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (0, 1, 1, 0), \gamma_2 = (1, 0, -1, 0)\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



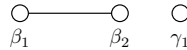
and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow \widetilde{A}_2 \times A_2$. Then $\phi(\lambda + \rho)|_{\widetilde{A}_2} = (z + 2, z + 1, -2z - 3)$, $\phi(\lambda + \rho)|_{A_2} = (3z + 4, 0, -3z - 4)$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{2}{3}, -\frac{1}{3}\}$, $\mathbf{a}(w_\lambda) = 6$, and $\text{GKdim } L(\lambda) = 18 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{5}{3}, -\frac{4}{3}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (4) If λ is one-fourth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 1, 0, 0)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (0, 0, 0, 1), \gamma_2 = (1, 0, 0, -1)\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow \widetilde{A}_1 \times B_2$. Then $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (2z + \frac{5}{2}, -2z - \frac{5}{2})$, $\phi(\lambda + \rho)|_{B_2} = (4z + \frac{11}{2}, \frac{1}{2})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{3}{4}, -\frac{1}{4}\}$, $\mathbf{a}(w_\lambda) = 5$, and $\text{GKdim } L(\lambda) = 19 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{7}{4}, -\frac{5}{4}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (5) If λ is one-fifth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 0, 0, 1), \beta_2 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (1, 0, 1, 0)\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow \widetilde{A}_2 \times A_1$. Then $\phi(\lambda + \rho)|_{\widetilde{A}_2} = (\frac{5}{3}z + \frac{8}{3}, \frac{5}{3}z + \frac{5}{3}, -\frac{10}{3}z - \frac{13}{3})$, $\phi(\lambda + \rho)|_{A_1} = (\frac{5}{2}z + \frac{7}{2}, -\frac{5}{2}z - \frac{7}{2})$. By using RS algorithm, we can easily get that:

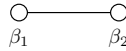
- (a) when $z \in \{-\frac{4}{5}, -\frac{3}{5}, -\frac{2}{5}\}$, $\mathbf{a}(w_\lambda) = 4$, and $\text{GKdim } L(\lambda) = 20 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z = -\frac{6}{5}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 22 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that this point is a reducible point;
 - (c) when $z \in \{-\frac{9}{5}, -\frac{8}{5}, -\frac{7}{5}, -\frac{11}{5}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (6) If λ is one-sixth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 0, 0, 1)\}$, and the simple system of Δ_2 is

$\Pi_2 = \{\gamma_1 = (1, 1, 0, 0)\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow \widetilde{A}_1 \times A_1$. Then $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (\frac{1}{2}, -\frac{1}{2})$, $\phi(\lambda + \rho)|_{A_1} = (3z + 4, -3z - 4)$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{5}{6}, -\frac{7}{6}\}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 22 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{11}{6}, -\frac{13}{6}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (7) If λ is one-seventh-integral, we can easily check that the simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (0, 0, 0, 1), \beta_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow \widetilde{A}_2$. Then $\phi(\lambda + \rho)|_{\widetilde{A}_2} = (\frac{7}{3}z + \frac{11}{3}, \frac{7}{3}z + \frac{8}{3}, -\frac{14}{3}z - \frac{19}{3})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{6}{7}, -\frac{5}{7}, -\frac{4}{7}, -\frac{3}{7}, -\frac{2}{7}, -\frac{8}{7}\}$, $\mathbf{a}(w_\lambda) = 3$, and $\text{GKdim } L(\lambda) = 21 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{13}{7}, -\frac{12}{7}, -\frac{11}{7}, -\frac{10}{7}, -\frac{9}{7}, -\frac{15}{7}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (8) If λ is one-eighth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (1, 0, 0, 0)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (0, 0, 0, 1)\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow \widetilde{A}_1 \times \widetilde{A}_1$. Then $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (4z + \frac{11}{2}, -4z - \frac{11}{2})$, $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (\frac{1}{2}, -\frac{1}{2})$. By using RS algorithm, we can easily get that:

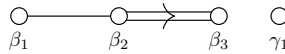
- (a) when $z \in \{-\frac{7}{8}, -\frac{5}{8}, -\frac{3}{8}, -\frac{9}{8}\}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 22 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{15}{8}, -\frac{13}{8}, -\frac{11}{8}, -\frac{17}{8}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (9) If λ does not belong to any of the types mentioned above, one can easily check that $\Delta_{[\lambda]} \simeq A_1$. The simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (0, 0, 0, 1)\}$. We can get $\mathbf{a}(w_\lambda) = 1$ by using RS algorithm, thus $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$. In this case $M_I(z\xi_2)$ is irreducible.

All in all, when $p = 2$, $M_I(z\widehat{\xi}_2)$ is reducible if and only if

$$\begin{aligned}
 z \in & \{-1 + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{1}{2} + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{2}{3} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{3} + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{3}{4} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{4} + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{4}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{2}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{6}{5} + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{5}{6} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{7}{6} + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{6}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{5}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{4}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{2}{7} + \mathbb{Z}_{\geq 0}\} \\
 & \quad \cup \{-\frac{8}{7} + \mathbb{Z}_{\geq 0}\} \\
 & \cup \{-\frac{7}{8} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{5}{8} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{8} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{9}{8} + \mathbb{Z}_{\geq 0}\} \\
 = & \bigcup_{i=5}^8 \left\{ -\frac{i+1}{i} + \frac{1}{i} \mathbb{Z}_{\geq 0} \right\}.
 \end{aligned}$$

Proof of (3). Let $\Delta^+(\mathbf{t}) = \{(0, 0, 1, -1) = \{\alpha_3\}\}$. When $M_I(\lambda)$ is of scalar type, we know that $\lambda = z(\xi_1 + \xi_2 + \xi_4) = z(\rho - \xi_3)$ for some $z \in \mathbb{R}$, where ξ_3 is the fundamental weight of simple root α_3 and $\xi_3 = (2, 1, 1, 0)$. In [Kna02] we know that $\rho = (\frac{11}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$ and $\lambda + \rho = (\frac{11}{2} + \frac{7}{2}z, \frac{5}{2} + \frac{3}{2}z, \frac{3}{2} + \frac{1}{2}z, \frac{1}{2} + \frac{1}{2}z)$.

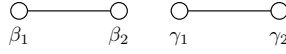
- (1) If λ is integral, then we have the following.
 - (a) When $z = -1$, then $\lambda + \rho = (2, 1, 1, 0)$, we can get $w_\lambda = [2, 3, 1, 2, 3, 1, 2, 0, 1, 2, 1, 0, 3, 2, 1, 2, 3, 0, 1, 2]$. By using PyCox we can get that $\text{GKdim } L(\lambda) = 20 < \dim(\mathbf{u})$. By Lemma 1.8 we can obtain that $z = -1$ is a diagonal-reducible point.
 - (b) When $z = -2$, then $\lambda + \rho = (-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, we can get $w_\lambda = [1, 2]$. By using PyCox we can get that $\text{GKdim } L(\lambda) = 23 = \dim(\mathbf{u})$. By Lemma 1.8 we can obtain that $z = -2$ is an irreducible point.
- (2) If λ is half-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 1, 0, 1), \beta_2 = (0, 0, 1, -1), \beta_3 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow B_3 \times \widetilde{A}_1$. Then $\phi(\lambda + \rho)|_{B_3} = (3z + 5, z + 2, z + 1)$, $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (2z + 3, -2z - 3)$. By using RS algorithm, we can easily get that:

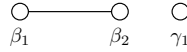
- (a) when $z = -\frac{3}{2}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 22 < \dim(\mathbf{u})$; by Lemma 1.8, we can obtain that this point is a reducible point;
- (b) when $z = -\frac{5}{2}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathbf{u})$; by Lemma 1.8, we can obtain that this point is an irreducible point.
- (3) If λ is one-third-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 1, 0, 0), \beta_2 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (1, 0, -1, 0), \gamma_2 = (0, 0, 1, -1)\}$. The Dynkin

diagram of $\Delta_{[\lambda]}$ is:



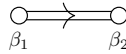
and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow \widetilde{A_2} \times A_2$. Then $\phi(\lambda + \rho)|_{\widetilde{A_2}} = (3z + 5, 0, -3z - 5)$, $\phi(\lambda + \rho)|_{A_2} = (2z + 3, -z - 1, -z - 2)$. By using RS algorithm, we can easily get that:

- (a) when $z = -\frac{2}{3}$, $\mathbf{a}(w_\lambda) = 6$, and $\text{GKdim } L(\lambda) = 18 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that this point is a reducible point;
 - (b) when $z = -\frac{4}{3}$, $\mathbf{a}(w_\lambda) = 4$, and $\text{GKdim } L(\lambda) = 20 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that this point is a reducible point;
 - (c) when $z \in \{-\frac{5}{3}, -\frac{7}{3}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (4) If λ is one-fourth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (1, 0, 0, 1), \beta_2 = (0, 0, 1, -1)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_2 \times \widetilde{A_1}$. Then $\phi(\lambda + \rho)|_{A_2} = (\frac{8}{3}z + \frac{13}{3}, -\frac{4}{3}z - \frac{5}{3}, -\frac{4}{3}z - \frac{8}{3})$, $\phi(\lambda + \rho)|_{\widetilde{A_1}} = (2z + 3, -2z - 3)$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{3}{4}, -\frac{5}{4}\}$, $\mathbf{a}(w_\lambda) = 4$, and $\text{GKdim } L(\lambda) = 20 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{7}{4}, -\frac{9}{4}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (5) If λ is one-fifth-integral, we can easily check that the simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (0, 0, 1, -1), \beta_2 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow B_2$. Then $\phi(\lambda + \rho)|_{B_2} = (\frac{5}{2}z + \frac{9}{2}, \frac{5}{2}z + \frac{7}{2})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{4}{5}, -\frac{3}{5}, -\frac{2}{5}, -\frac{6}{5}\}$, $\mathbf{a}(w_\lambda) = 4$, and $\text{GKdim } L(\lambda) = 20 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{9}{5}, -\frac{8}{5}, -\frac{7}{5}, -\frac{11}{5}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (6) If λ is one-sixth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 0, 1, -1)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_1 \times \widetilde{A}_1$. Then $\phi(\lambda + \rho)|_{A_1} = (\frac{1}{2}, -\frac{1}{2})$, $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (3z + 5, -3z - 5)$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{5}{6}, -\frac{7}{6}\}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 22 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{11}{6}, -\frac{13}{6}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (7) If λ is one-seventh-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (1, 0, 0, 0)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (0, 0, 1, -1)\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:

$$\begin{array}{cc} \circ & \circ \\ \beta_1 & \gamma_1 \end{array}$$

and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow \widetilde{A}_1 \times A_1$. Then $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (\frac{7}{2}z + \frac{11}{2}, -\frac{7}{2}z - \frac{11}{2})$, $\phi(\lambda + \rho)|_{A_1} = (\frac{1}{2}, -\frac{1}{2})$. By using RS algorithm, we can easily get that:

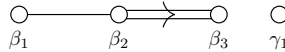
- (a) when $z \in \{-\frac{6}{7}, -\frac{5}{7}, -\frac{4}{7}, -\frac{10}{7}, -\frac{9}{7}, -\frac{8}{7}\}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 22 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{13}{7}, -\frac{12}{7}, -\frac{11}{7}, -\frac{17}{7}, -\frac{16}{7}, -\frac{15}{7}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (8) If λ does not belong to any of the types mentioned above, one can easily check that $\Delta_{[\lambda]} \simeq A_1$. The simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (0, 0, 1, -1)\}$. We can get $\mathbf{a}(w_\lambda) = 1$ by using RS algorithm, thus $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$. In this case $M_I(z\widehat{\xi}_3)$ is irreducible.

All in all, when $p = 3$, $M_I(z\widehat{\xi}_3)$ is reducible if and only if

$$\begin{aligned} & z \in \{-1 + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{3}{2} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{2}{3} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{4}{3} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{3}{4} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{5}{4} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{4}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{2}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{6}{5} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{5}{6} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{7}{6} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{6}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{5}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{4}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{10}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{9}{7} + \mathbb{Z}_{\geq 0}\} \\ & \quad \cup \{-\frac{8}{7} + \mathbb{Z}_{\geq 0}\} \\ & = \bigcup_{i=4}^5 \left\{ -\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0} \right\} \cup \bigcup_{i=6}^7 \left\{ -\frac{i+3}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0} \right\}. \end{aligned}$$

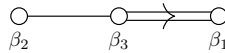
Proof of (4). Let $\Delta^+(\Gamma) = \{(0, 1, -1, 0)\} = \{\alpha_4\}$. When $M_I(\lambda)$ is of scalar type, we know that $\lambda = z(\xi_1 + \xi_2 + \xi_3) = z(\rho - \xi_4)$ for some $z \in \mathbb{R}$, where ξ_4 is the fundamental weight of simple root α_4 and $\xi_4 = (1, 1, 0, 0)$. In [Kna02] we know that $\rho = (\frac{11}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$ and $\lambda + \rho = (\frac{11}{2} + \frac{9}{2}z, \frac{5}{2} + \frac{3}{2}z, \frac{3}{2} + \frac{3}{2}z, \frac{1}{2} + \frac{1}{2}z)$.

- (1) If λ is integral, then we have the following.
- (a) When $z = -1$, then $\lambda + \rho = (1, 1, 0, 0)$, we can get $w_\lambda = [3, 2, 1, 2, 3, 0, 1, 2, 3, 1, 2, 0, 1, 2, 3]$. By using PyCox we can get that $\text{GKdim } L(\lambda) = 21 < \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -1$ is a diagonal-reducible point.
 - (b) When $z = -2$, then $\lambda + \rho = (-\frac{7}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2})$, we can get $w_\lambda = [3]$. By using PyCox we can get that $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -2$ is an irreducible point.
- (2) If λ is half-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 0, 1, 1), \beta_2 = (0, 1, -1, 0), \beta_3 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



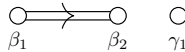
and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow B_3 \times \widetilde{A}_1$. Then $\phi(\lambda + \rho)|_{B_3} = (4z + 5, 2z + 3, 2z + 2)$, $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (z + 1, -z - 1)$. By using RS algorithm, we can easily get that:

- (a) when $z = -\frac{1}{2}$, $\mathbf{a}(w_\lambda) = 10$, and $\text{GKdim } L(\lambda) = 14 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that this point is a reducible point;
 - (b) when $z = -\frac{3}{2}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that this point is an irreducible point.
- (3) If λ is one-third-integral, we can easily check that the simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (0, 0, 1, 0), \beta_2 = (1, -1, 0, 0), \beta_3 = (0, 1, -1, 0)\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow B_3$. Then $\phi(\lambda + \rho)|_{B_3} = (\frac{9}{2}z + \frac{11}{2}, \frac{3}{2}z + \frac{5}{2}, \frac{3}{2}z + \frac{3}{2})$. By using RS algorithm, we can easily get that:

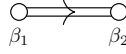
- (a) when $z \in \{-\frac{2}{3}, -\frac{1}{3}\}$, $\mathbf{a}(w_\lambda) = 9$, and $\text{GKdim } L(\lambda) = 15 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{5}{3}, -\frac{4}{3}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (4) If λ is one-fourth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 1, -1, 0), \beta_2 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow B_2 \times A_1$. Then $\phi(\lambda + \rho)|_{B_2} = (2z + 3, 2z + 2)$, $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (4z + 5, -4z - 5)$. By using RS algorithm, we can easily get that:

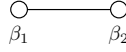
- (a) when $z \in \{-\frac{3}{4}, -\frac{1}{4}\}$, $\mathbf{a}(w_\lambda) = 5$, and $\text{GKdim } L(\lambda) = 19 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
- (b) when $z \in \{-\frac{7}{4}, -\frac{5}{4}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.

- (5) If λ is one-fifth-integral, we can easily check that the simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (0, 1, -1, 0), \beta_2 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow B_2$. Then $\phi(\lambda + \rho)|_{B_2} = (\frac{5}{2}z + \frac{7}{2}, \frac{5}{2}z + \frac{5}{2})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{4}{5}, -\frac{3}{5}, -\frac{2}{5}, -\frac{1}{5}\}$, $\mathbf{a}(w_\lambda) = 4$, and $\text{GKdim } L(\lambda) = 20 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{9}{5}, -\frac{8}{5}, -\frac{7}{5}, -\frac{6}{5}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (6) If λ is one-sixth-integral, we can easily check that the simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (1, 0, 1, 0), \beta_2 = (0, 1, -1, 0)\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_2$. Then $\phi(\lambda + \rho)|_{A_2} = (4z + 5, -2z - 2, -2z - 3)$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{5}{6}, -\frac{1}{6}\}$, $\mathbf{a}(w_\lambda) = 3$, and $\text{GKdim } L(\lambda) = 21 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{11}{6}, -\frac{7}{6}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (7) If λ is one-seventh-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 1, -1, 0)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_1 \times \widetilde{A}_1$. Then $\phi(\lambda + \rho)|_{A_1} = (\frac{1}{2}, -\frac{1}{2})$, $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (\frac{7}{2}z + \frac{9}{2}, -\frac{7}{2}z - \frac{9}{2})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{6}{7}, -\frac{5}{7}, -\frac{4}{7}, -\frac{3}{7}, -\frac{2}{7}, -\frac{8}{7}\}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 22 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{13}{7}, -\frac{12}{7}, -\frac{11}{7}, -\frac{10}{7}, -\frac{9}{7}, -\frac{15}{7}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (8) If λ is one-eighth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 1, -1, 0)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_1 \times \widetilde{A}_1$. Then $\phi(\lambda + \rho)|_{A_1} = (\frac{1}{2}, -\frac{1}{2})$, $\phi(\lambda + \rho)|_{\widetilde{A}_1} = (4z + 5, -4z - 5)$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{7}{8}, -\frac{5}{8}, -\frac{3}{8}, -\frac{9}{8}\}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 22 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
- (b) when $z \in \{-\frac{15}{8}, -\frac{13}{8}, -\frac{11}{8}, -\frac{17}{8}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (9) If λ is one-ninth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (1, 0, 0, 0)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (0, 1, -1, 0)\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:

$$\begin{array}{cc} \bigcirc & \bigcirc \\ \beta_1 & \gamma_1 \end{array}$$

and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow \widetilde{A_1} \times A_1$. Then $\phi(\lambda + \rho)|_{\widetilde{A_1}} = (\frac{9}{2}z + \frac{11}{2}, -\frac{9}{2}z - \frac{11}{2})$, $\phi(\lambda + \rho)|_{A_1} = (\frac{1}{2}, -\frac{1}{2})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{8}{9}, -\frac{7}{9}, -\frac{5}{9}, -\frac{4}{9}, -\frac{2}{9}, -\frac{10}{9}\}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 22 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
- (b) when $z \in \{-\frac{17}{9}, -\frac{16}{9}, -\frac{14}{9}, -\frac{13}{9}, -\frac{11}{9}, -\frac{19}{9}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (10) If λ does not belong to any of the types mentioned above, one can easily check that $\Delta_{[\lambda]} \simeq A_1$. The simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (0, 1, -1, 0)\}$. We can get $\mathbf{a}(w_\lambda) = 1$ by using RS algorithm, thus $\text{GKdim } L(\lambda) = 23 = \dim(\mathfrak{u})$. In this case $M_I(z\widehat{\xi}_4)$ is irreducible.

All in all, when $p = 4$, $M_I(z\widehat{\xi}_4)$ is reducible if and only if

$$\begin{aligned} & z \in \{-1 + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{1}{2} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{2}{3} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{3} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{3}{4} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{4} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{4}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{2}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{5} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{5}{6} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{6} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{6}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{5}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{4}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{2}{7} + \mathbb{Z}_{\geq 0}\} \\ & \quad \cup \{-\frac{8}{7} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{7}{8} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{5}{8} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{8} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{9}{8} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{8}{9} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{7}{9} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{5}{9} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{4}{9} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{2}{9} + \mathbb{Z}_{\geq 0}\} \\ & \quad \cup \{-\frac{10}{9} + \mathbb{Z}_{\geq 0}\} \\ & = \bigcup_{i=5}^6 \left\{ -1 + \frac{1}{i} \mathbb{Z}_{\geq 0} \right\} \cup \bigcup_{i=7}^9 \left\{ -\frac{i+1}{i} + \frac{1}{i} \mathbb{Z}_{\geq 0} \right\}. \end{aligned}$$

So far, we have completed the proof of all cases of type F_4 . □

3. Reducibility of scalar generalized Verma modules for type E

Next we give the reducibility of scalar generalized Verma modules associated to minimal parabolic subalgebras for type E_6 , E_7 and E_8 .

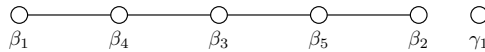
Theorem 3.1. *Let $\mathfrak{g}$ be of type E_6 , then $M_I(z\hat{\xi}_p)$ is reducible if and only if*

- (1) $p = 1$, $z \in \bigcup_{i=6}^{10} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (2) $p = 2$, $z \in \bigcup_{i=5}^9 \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (3) $p = 3$, $z \in \bigcup_{i=5}^7 \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=8}^9 \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (4) $p = 4$, $z \in \bigcup_{i=4}^8 \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (5) $p = 5$, $z \in \bigcup_{i=5}^7 \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=8}^9 \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (6) $p = 6$, $z \in \bigcup_{i=6}^{10} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$.

Proof.

Proof of (1). Let $\mathfrak{g}$ be of type E_6 and $\Delta^+(\mathfrak{l}) = \{\alpha_p\}$. Let $\Delta^+(\mathfrak{l}) = \{(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\} = \{\alpha_1\}$. When $M_I(\lambda)$ is of scalar type, we know that $\lambda = z(\xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6) = z(\rho - \xi_1)$ for some $z \in \mathbb{R}$, where ξ_1 is the fundamental weight of simple root α_1 and $\xi_1 = (0, 0, 0, 0, 0, -\frac{2}{3}, -\frac{2}{3}, \frac{2}{3})$. In [Kna02] we know that $\rho = (0, 1, 2, 3, 4, -4, -4, 4)$ and $\lambda + \rho = (0, 1 + z, 2 + 2z, 3 + 3z, 4 + 4z, -4 - \frac{10}{3}z, -4 - \frac{10}{3}z, 4 + \frac{10}{3}z)$.

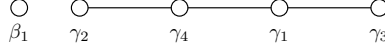
- (1) If λ is integral, then we have the following.
 - (a) When $z = -1$, then $\lambda + \rho = (0, 0, 0, 0, 0, -\frac{2}{3}, -\frac{2}{3}, \frac{2}{3})$, we can get $w_\lambda = [5, 4, 3, 2, 0, 1, 3, 2, 4, 3, 1, 5, 4, 3, 2, 0]$. By using PyCox we can get that $\text{GKdim } L(\lambda) = 34 < \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -1$ is a diagonal-reducible point.
 - (b) When $z = -2$, then $\lambda + \rho = (0, -1, -2, -3, -4, \frac{8}{3}, \frac{8}{3}, -\frac{8}{3})$, we can get $w_\lambda = [0]$. By using PyCox we can get that $\text{GKdim } L(\lambda) = 35 = \dim(\mathfrak{u})$. By Lemma 1.8 we can obtain that $z = -2$ is an irreducible point.
- (2) If λ is half-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (1, 0, 1, 0, 0, 0, 0, 0), \beta_2 = (0, 1, 0, 1, 0, 0, 0, 0), \beta_3 = (-1, 0, 1, 0, 0, 0, 0, 0), \beta_4 = (0, 0, -1, 0, 1, 0, 0, 0), \beta_5 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (0, -1, 0, 1, 0, 0, 0, 0)\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_5 \times A_1$. Then $\phi(\lambda + \rho)|_{A_5} = (\frac{14}{3}z + 5, \frac{8}{3}z + 3, \frac{2}{3}z + 1, -\frac{4}{3}z - 1, -\frac{4}{3}z - 2, -\frac{16}{3}z - 6)$, $\phi(\lambda + \rho)|_{A_1} = (z + 1, -z - 1)$. By using RS algorithm, we can easily get that:

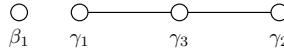
- (a) when $z = -\frac{1}{2}$, $\mathbf{a}(w_\lambda) = 16$, and $\text{GKdim } L(\lambda) = 20 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that this point is a reducible point;
- (b) when $z = -\frac{3}{2}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 35 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that this point is an irreducible point.

- (3) If λ is one-third-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (1, 0, 0, 1, 0, 0, 0, 0)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (0, 1, 1, 0, 0, 0, 0, 0), \gamma_2 = (-1, 0, 0, 1, 0, 0, 0, 0), \gamma_3 = (0, -1, 0, 0, 1, 0, 0, 0), \gamma_4 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



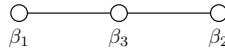
and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_1 \times A_4$. Then $\phi(\lambda + \rho)|_{A_1} = (\frac{3}{2}z + \frac{3}{2}, -\frac{3}{2}z - \frac{3}{2})$, $\phi(\lambda + \rho)|_{A_4} = (\frac{21}{5}z + \frac{24}{5}, \frac{6}{5}z + \frac{9}{5}, \frac{6}{5}z + \frac{4}{5}, -\frac{9}{5}z - \frac{11}{5}, -\frac{24}{5}z - \frac{26}{5})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{2}{3}, -\frac{1}{3}\}$, $\mathbf{a}(w_\lambda) = 11$, and $\text{GKdim } L(\lambda) = 25 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{5}{3}, -\frac{4}{3}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 35 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (4) If λ is one-fourth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (1, 0, 0, 0, 1, 0, 0, 0)\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (0, 1, 0, 1, 0, 0, 0, 0), \gamma_2 = (-1, 0, 0, 0, 1, 0, 0, 0), \gamma_3 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_1 \times A_3$. Then $\phi(\lambda + \rho)|_{A_1} = (2z + 2, -2z - 2)$, $\phi(\lambda + \rho)|_{A_3} = (4z + \frac{9}{2}, \frac{1}{2}, -\frac{1}{2}, -4z - \frac{9}{2})$. By using RS algorithm, we can easily get that:

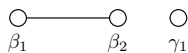
- (a) when $z \in \{-\frac{3}{4}, -\frac{1}{4}\}$, $\mathbf{a}(w_\lambda) = 7$, and $\text{GKdim } L(\lambda) = 29 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{7}{4}, -\frac{5}{4}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 35 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (5) If λ is one-fifth-integral, we can easily check that the simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (0, 1, 0, 0, 1, 0, 0, 0), \beta_2 = (0, 0, 1, 1, 0, 0, 0, 0), \beta_3 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_3$. Then $\phi(\lambda + \rho)|_{A_3} = (5z + \frac{11}{2}, \frac{1}{2}, -\frac{1}{2}, -5z - \frac{11}{2})$. By using RS algorithm, we can easily get that:

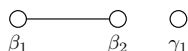
- (a) when $z \in \{-\frac{4}{5}, -\frac{3}{5}, -\frac{2}{5}, -\frac{1}{5}\}$, $\mathbf{a}(w_\lambda) = 6$, and $\text{GKdim } L(\lambda) = 30 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{9}{5}, -\frac{8}{5}, -\frac{7}{5}, -\frac{6}{5}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 35 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (6) If λ is one-sixth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 0, 1, 0, 1, 0, 0, 0), \beta_2 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$.

The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_2 \times A_1$. Then $\phi(\lambda + \rho)|_{A_2} = (4z + \frac{13}{3}, -2z - \frac{5}{3}, -2z - \frac{8}{3})$, $\phi(\lambda + \rho)|_{A_1} = (3z + \frac{7}{2}, -3z - \frac{7}{2})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{5}{6}, -\frac{1}{6}\}$, $\mathbf{a}(w_\lambda) = 4$, and $\text{GKdim } L(\lambda) = 32 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{11}{6}, -\frac{7}{6}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 35 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (7) If λ is one-seventh-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (0, 0, 0, 1, 1, 0, 0, 0), \beta_2 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_2 \times A_1$. Then $\phi(\lambda + \rho)|_{A_2} = (\frac{14}{3}z + 5, -\frac{7}{3}z - 2, -\frac{7}{3}z - 3)$, $\phi(\lambda + \rho)|_{A_1} = (\frac{7}{2}z + 4, -\frac{7}{2}z - 4)$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{6}{7}, -\frac{5}{7}, -\frac{4}{7}, -\frac{3}{7}, -\frac{2}{7}, -\frac{1}{7}\}$, $\mathbf{a}(w_\lambda) = 4$, and $\text{GKdim } L(\lambda) = 32 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{13}{7}, -\frac{12}{7}, -\frac{11}{7}, -\frac{10}{7}, -\frac{9}{7}, -\frac{8}{7}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 35 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (8) If λ is one-eighth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_1 \times A_1$. Then $\phi(\lambda + \rho)|_{A_1} = (\frac{1}{2}, -\frac{1}{2})$, $\phi(\lambda + \rho)|_{A_1} = (4z + \frac{9}{2}, -4z - \frac{9}{2})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{7}{8}, -\frac{5}{8}, -\frac{3}{8}, -\frac{1}{8}\}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 34 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
 - (b) when $z \in \{-\frac{15}{8}, -\frac{13}{8}, -\frac{11}{8}, -\frac{9}{8}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 35 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.
- (9) If λ is one-ninth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:



and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_1 \times A_1$. Then $\phi(\lambda + \rho)|_{A_1} = (\frac{1}{2}, -\frac{1}{2})$, $\phi(\lambda + \rho)|_{A_1} = (\frac{9}{2}z + 5, -\frac{9}{2}z - 5)$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{8}{9}, -\frac{7}{9}, -\frac{5}{9}, -\frac{4}{9}, -\frac{2}{9}, -\frac{1}{9}\}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 34 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
- (b) when $z \in \{-\frac{17}{9}, -\frac{16}{9}, -\frac{14}{9}, -\frac{13}{9}, -\frac{11}{9}, -\frac{10}{9}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 35 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.

- (10) If λ is one-tenth-integral, we can easily check that $\Delta_{[\lambda]} = \Delta_1 \cup \Delta_2$, where the simple system of Δ_1 is $\Pi_1 = \{\beta_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$, and the simple system of Δ_2 is $\Pi_2 = \{\gamma_1 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$. The Dynkin diagram of $\Delta_{[\lambda]}$ is:

$$\begin{array}{cc} \bigcirc & \bigcirc \\ \beta_1 & \gamma_1 \end{array}$$

and with the comparison with Dynkin diagrams of classical Lie algebra, we can easily get the isomorphism $\phi: \Delta_{[\lambda]} \rightarrow A_1 \times A_1$. Then $\phi(\lambda + \rho)|_{A_1} = (5z + \frac{11}{2}, -5z - \frac{11}{2})$, $\phi(\lambda + \rho)|_{A_1} = (\frac{1}{2}, -\frac{1}{2})$. By using RS algorithm, we can easily get that:

- (a) when $z \in \{-\frac{9}{10}, -\frac{7}{10}, -\frac{3}{10}, -\frac{1}{10}\}$, $\mathbf{a}(w_\lambda) = 2$, and $\text{GKdim } L(\lambda) = 34 < \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are reducible points;
- (b) when $z \in \{-\frac{19}{10}, -\frac{17}{10}, -\frac{13}{10}, -\frac{11}{10}\}$, $\mathbf{a}(w_\lambda) = 1$, and $\text{GKdim } L(\lambda) = 35 = \dim(\mathfrak{u})$; by Lemma 1.8, we can obtain that these points are irreducible points.

- (11) If λ does not belong to any of the types mentioned above, one can easily check that $\Delta_{[\lambda]} \simeq A_1$. The simple system of $\Delta_{[\lambda]}$ is $\Pi = \{\beta_1 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\}$. We can get $\mathbf{a}(w_\lambda) = 1$ by using RS algorithm, thus $\text{GKdim } L(\lambda) = 35 = \dim(\mathfrak{u})$. In this case, $M_I(z\hat{\xi}_1)$ is irreducible.

All in all, when $p = 1$, $M_I(z\hat{\xi}_1)$ is reducible if and only if

$$\begin{aligned} & z \in \{-1 + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{2} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{2}{3} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{3} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{3}{4} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{4} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{4}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{2}{5} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{5} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{5}{6} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{6} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{6}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{5}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{4}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{7} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{2}{7} + \mathbb{Z}_{\geq 0}\} \\ & \quad \cup \{-\frac{1}{7} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{7}{8} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{5}{8} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{8} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{8} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{8}{9} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{7}{9} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{5}{9} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{4}{9} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{2}{9} + \mathbb{Z}_{\geq 0}\} \\ & \quad \cup \{-\frac{1}{9} + \mathbb{Z}_{\geq 0}\} \\ & \cup \{-\frac{9}{10} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{7}{10} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{3}{10} + \mathbb{Z}_{\geq 0}\} \cup \{-\frac{1}{10} + \mathbb{Z}_{\geq 0}\} \\ & = \bigcup_{i=6}^{10} \left\{ -1 + \frac{1}{i} \mathbb{Z}_{\geq 0} \right\}. \end{aligned}$$

The computations for E_6 are tedious, so we only present the detailed procedure for the first case of E_6 , and omit subsequent derivations for brevity. $\square$

Theorem 3.2. *Let $\mathfrak{g}$ be of type E_7 , then $M_I(z\hat{\xi}_p)$ is reducible if and only if*

- (1) $p = 1$, then $z \in \bigcup_{i=8}^{15} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (2) $p = 2$, then $z \in \bigcup_{i=8}^{12} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=13}^{15} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (3) $p = 3$, then $z \in \bigcup_{i=8}^{13} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \{-\frac{8}{7} + \frac{1}{14}\mathbb{Z}_{\geq 0}\}$; or
- (4) $p = 4$, then $z \in \bigcup_{i=7}^8 \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=9}^{11} \{-\frac{i+2}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=12}^{13} \{-\frac{i+3}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (5) $p = 5$, then $z \in \bigcup_{i=8}^{11} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=12}^{14} \{-\frac{i+2}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (6) $p = 6$, then $z \in \bigcup_{i=8}^{10} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=11}^{15} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (7) $p = 7$, then $z \in \bigcup_{i=8}^{16} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$.

Proof. The computations for E_7 are tedious and the same methodology has been employed in the previous cases, so we omit detailed procedures here. $\square$

Theorem 3.3. *Let $\mathfrak{g}$ be of type E_8 , then $M_I(z\hat{\xi}_p)$ is reducible if and only if*

- (1) $p = 1$, then $z \in \bigcup_{i=14}^{20} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=21}^{27} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (2) $p = 2$, then $z \in \bigcup_{i=14}^{21} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=22}^{26} \{-\frac{i+2}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (3) $p = 3$, then $z \in \{-\frac{14}{13} + \frac{1}{13}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=14}^{20} \{-\frac{i+2}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=21}^{25} \{-\frac{i+3}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (4) $p = 4$, then $z \in \{-\frac{7}{6} + \frac{1}{12}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=13}^{15} \{-\frac{i+3}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=16}^{20} \{-\frac{i+4}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=21}^{23} \{-\frac{i+5}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (5) $p = 5$, then $z \in \bigcup_{i=13}^{16} \{-\frac{i+2}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=17}^{21} \{-\frac{i+3}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=22}^{24} \{-\frac{i+4}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (6) $p = 6$, then $z \in \bigcup_{i=13}^{16} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=17}^{23} \{-\frac{i+2}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\} \cup \bigcup_{i=24}^{25} \{-\frac{i+3}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (7) $p = 7$, then $z \in \bigcup_{i=13}^{26} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$; or
- (8) $p = 8$, then $z \in \bigcup_{i=14}^{27} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$.

Proof. The computations for E_8 are tedious and the same methodology has been employed in the previous cases. To avoid lengthy articles, we omit detailed procedures here. $\square$

Appendix

In the following tables, p means the case of $\lambda = z\hat{\xi}_p$, and we give the sets of corresponding reducible points in the second column of the tables.

TABLE 2. Diagonal-reducible points of G_2 .

p	diagonal-reducible points of $M_I(z\hat{\xi}_p)$
1	$-\frac{1}{2} + \frac{1}{2}\mathbb{Z}_{\geq 0}, -\frac{2}{3} + \frac{1}{3}\mathbb{Z}_{\geq 0}$
2	$-\frac{3}{2} + \mathbb{Z}_{\geq 0}, \mathbb{Z}_{\geq 0}$

TABLE 3. Diagonal-reducible points of F_4 .

p	diagonal-reducible points of $M_I(z\hat{\xi}_p)$
1	$\bigcup_{i=5}^9 \left\{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}$
2	$\bigcup_{i=5}^8 \left\{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}$
3	$\bigcup_{i=4}^5 \left\{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}, \bigcup_{i=6}^7 \left\{-\frac{i+3}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}$
4	$\bigcup_{i=5}^6 \left\{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}, \bigcup_{i=7}^9 \left\{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}$

TABLE 4. Diagonal-reducible points of E_6 .

p	diagonal-reducible points of $M_I(z\hat{\xi}_p)$
1	$\bigcup_{i=6}^{10} \left\{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}$
2	$\bigcup_{i=5}^9 \left\{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}$
3	$\bigcup_{i=5}^7 \left\{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}, \bigcup_{i=8}^9 \left\{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}$
4	$\bigcup_{i=4}^8 \left\{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}$
5	$\bigcup_{i=5}^7 \left\{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}, \bigcup_{i=8}^9 \left\{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}$
6	$\bigcup_{i=6}^{10} \left\{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\right\}$

TABLE 5. Diagonal-reducible points of E_7 .

p	diagonal-reducible points of $M_I(z\widehat{\xi}_p)$
1	$\bigcup_{i=8}^{15} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$
2	$\bigcup_{i=8}^{12} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=13}^{15} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$
3	$\bigcup_{i=8}^{13} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, -\frac{8}{7} + \frac{1}{14}\mathbb{Z}_{\geq 0}$
4	$\bigcup_{i=7}^8 \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=9}^{11} \{-\frac{i+2}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=12}^{13} \{-\frac{i+3}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$
5	$\bigcup_{i=8}^{11} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=12}^{14} \{-\frac{i+2}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$
6	$\bigcup_{i=8}^{10} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=11}^{15} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$
7	$\bigcup_{i=8}^{16} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$

TABLE 6. Diagonal-reducible points of E_8 .

p	diagonal-reducible points of $M_I(z\widehat{\xi}_p)$
1	$\bigcup_{i=14}^{20} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=21}^{27} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$
2	$\bigcup_{i=14}^{21} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=22}^{26} \{-\frac{i+2}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$
3	$-\frac{14}{13} + \frac{1}{13}\mathbb{Z}_{\geq 0}, \bigcup_{i=14}^{20} \{-\frac{i+2}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=21}^{25} \{-\frac{i+3}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$
4	$-\frac{7}{6} + \frac{1}{12}\mathbb{Z}_{\geq 0}, \bigcup_{i=13}^{15} \{-\frac{i+3}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=16}^{20} \{-\frac{i+4}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=21}^{23} \{-\frac{i+5}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$
5	$\bigcup_{i=13}^{16} \{-\frac{i+2}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=17}^{21} \{-\frac{i+3}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=22}^{24} \{-\frac{i+4}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$
6	$\bigcup_{i=13}^{16} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=17}^{23} \{-\frac{i+2}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}, \bigcup_{i=24}^{25} \{-\frac{i+3}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$
7	$\bigcup_{i=13}^{26} \{-\frac{i+1}{i} + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$
8	$\bigcup_{i=14}^{27} \{-1 + \frac{1}{i}\mathbb{Z}_{\geq 0}\}$

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