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Journal of Lie Theory

Volume 36 (2026) 561–584

<https://doi.org/10.5802/jolt.1440>

Published online: October 1, 2026

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<https://www.centre-mersenne.org/>
e-ISSN: 2981-2917

Fourier multipliers on extensions of Heisenberg type groups and the contraction from K to $\bar{N}M$

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Communicated by Michael Pevzner

Abstract. If $G = KAN$ is the Iwasawa decomposition of a rank one semisimple Lie group, then the semi-direct product MN is a contraction of the compact group K . We use this fact to transfer harmonic analysis on K to harmonic analysis on MN , specifically transferring estimates for Fourier multipliers of L^p . The critical novelty is a relationship between the representation theory of the two groups: we show how to embed the irreducible representations of K into the Fock spaces for the irreducible representations of MN , and prove a limiting formula for the associated matrix entries. As an application we prove a version of de Leeuw’s theorem on transferring L^p -multiplier estimates from K to MN .

Introduction

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras with the same underlying vector space V , and Lie brackets $[\cdot, \cdot]_1, [\cdot, \cdot]_2$ respectively. We say that $\mathfrak{g}_1$ is a *contraction* of $\mathfrak{g}_2$ if there exists a family $\{\phi_\varepsilon : \varepsilon \in \mathbb{R}^+\}$ of invertible linear maps on V such that

$$\phi_\varepsilon^{-1}[\phi_\varepsilon X, \phi_\varepsilon Y]_2 \longrightarrow [X, Y]_1$$

as $\varepsilon \rightarrow 0$.

The Lie group G_1 is said to be a contraction of G_2 if they have the same dimension and there exists a family $\pi_\varepsilon : G_1 \rightarrow G_2$ of local diffeomorphisms whose differentials at the identity $\phi_\varepsilon = \pi_{\varepsilon*, e}$ define a contraction of the associated Lie algebras.

Contractions are used in physics (e.g. [BL68; Sal61]) to describe limiting behaviour of physical laws as certain physical constants go to infinity: for example, one recovers classical physics from relativistic physics by letting the speed of light tend to infinity. George Mackey [Mac75] suggested that there was a natural correspondence between the representations of a semisimple Lie group and those of its Cartan motion group. The authors, partly inspired by Mackey’s question, studied the contraction from a semisimple Lie group to its associated Cartan motion group, see [DG86; DR85a; DR85b] and references therein. Mackey’s suggestion was regarded as somewhat controversial at the time, see Afgoustidis’ interesting paper [Afg15] which gives details of the history and the work of Higson and others in C^* -algebras. Seen as a purely mathematical construct, contractions may be used to transfer harmonic analysis from one Lie group to another.

Another contraction which we have studied is the contraction from K to MN . Here, we start with a rank one semisimple Lie group G and write its Iwasawa decomposition as $G = KAN$. As usual, we take M to be the centraliser of A in K . Then M acts on N by conjugation, and we may form the semi-direct product MN . The

dilations coming from A then define a family of contraction maps from MN to K . We will define this more carefully below. Actually, some of our formulae work out more nicely if we consider the isomorphic situation where we consider the contraction of $\bar{N}M$ to K , so we will adopt this in the sequel. In the case where the rank one group is $SU(2, 1)$, the group K is the unitary group $U(2)$, N is the three-dimensional Heisenberg group, and M is the torus $\mathbb{T}$ which acts by rotation on the non-central component of N . This example was studied in [RR86]. It is also of interest in physics, where the Heisenberg group represents the canonical commutation relations and the unitary group the spin group. The authors studied several particular cases of this contraction: in [DG02] where we looked at $G = Sp(2p - 1)$.

In the present work, we will treat the general rank one case. As is well-known these are harmonic extensions of Heisenberg type (H -type) groups of Iwasawa type. We will use the formalism of H -type groups [Cow+91; Cow+98] to unify our treatment.

In the articles cited above, it is shown that the fact that G_1 is a contraction of G_2 allows us to deduce a limiting relationship between the representation theory of the two groups, more specifically, for each irreducible representation ρ of G_1 there exists a family of irreducible representations σ_ε of G_2 in the sense that $\sigma_\lambda \circ \pi_\varepsilon$ tends to ρ as $\varepsilon \rightarrow 0$. This limiting statement needs qualification and has been made more precise in each of the examples above. The strongest theorems, a generalisation of which we shall obtain below in the setting of K to $\bar{N}M$, look as follows.

Theorem 0.1. *In the context of an Iwasawa decomposition $G = KAN$ of a rank one semisimple Lie group, let $G_1 = K$, $G_2 = \bar{N}M$, and $\pi_\lambda: G_1 \rightarrow G_2$ be the contraction of G_2 to G_1 .*

Then for each irreducible representation ρ of G_1 , acting in $\mathcal{H}_\rho$, there exists a sequence σ_n^ρ of irreducible representations of G_2 , acting in Hilbert spaces $\mathcal{H}_n^\rho$, and for each n , an injection $\mathcal{R}_n: \mathcal{H}_n^\rho \rightarrow \mathcal{H}_\rho$ so that the images of $\mathcal{R}_n$ form an increasing family of subspaces of $\mathcal{H}_\rho$. A sequence λ_n of real numbers such that for all f in $\bigcup_{k=1}^\infty \text{Im } \mathcal{R}_k$,

$$\|\rho(g)f - \mathcal{R}_n \sigma_n^\rho \circ \pi_{\varepsilon_n}(g) \mathcal{R}_n^{-1} f\|_{\mathcal{H}_\rho} \quad (1)$$

tends to zero uniformly on compacta for $g \in G_1$ as $\varepsilon \rightarrow 0$.

A corollary of this theorem is that the union $\bigcup_{k=1}^\infty \text{Im } \mathcal{R}_k$ is dense in $\mathcal{H}_\rho$, as it is an invariant subspace for ρ . We can think of the operators $\mathcal{R}_n$ as “approximate intertwining operators” between the sequence σ_n^ρ and ρ induced by the contraction relationship. Indeed, the Kirillov orbit method [Doo83] suggests that such a relationship could exist in the setting of a much more general contraction.

In [DR85a], we proved that the “highest weight” element of the Fock space of $\bar{N}M$ could be approximated in this way by highest weight vectors in a sequence of representations of K , composed with contraction maps. We conjectured that all elements of the Fock space could be approximated by matrix coefficients of K . In this paper, we prove this conjecture, defining these injections and proving Theorem 0.1. Specifically, we construct the Lie algebra of K from the H -type data, and define explicit embeddings of the representations of K into the Fock spaces associated with the unitary dual of $\bar{N}M$, again using just the formalism of H -type groups.

The one-dimensional case is the very special contraction relationship between the real line $\mathbb{R}$ and the circle $\mathbb{T}$ induced by dilations of $\mathbb{R}$ composed with the exponential

map: all the representations are one-dimensional characters. The classical theorems of Karel de Leeuw [Lee65] give estimates for norms of L^p multipliers of $\mathbb{R}$ in terms of the circle L^p multiplier norms of their restrictions to the families $\varepsilon\mathbb{Z}$. As an illustration of the scope of our method, we prove an analogue of the de Leeuw theorems in the non-commutative setting of K and $\bar{N}M$. We believe that there is a substantial body of results linking harmonic analysis on $\mathbb{R}$ and $\mathbb{T}$ which can be similarly generalised.

The paper is organised as follows. Section 1 defines the structure of G , K and $\bar{N}M$, and the contraction mapping. In Section 2 we describe the irreducible representations of $\bar{N}$ generalised Fock space, induced from a specific character of the centre $\mu \in \mathfrak{g}_{-2\lambda}^*$. In Section 3, we combine this with the Mackey method to describe the representations of the semi-direct product $\bar{N}M$. Section 4 then describes the representation theory of the compact group K , the sequence of representations σ_n and the injections $\mathcal{R}_n$. Then in Section 5, we prove the main result. We give a statement and proof of our version of de Leeuw's theorem in Section 7.

The results of this paper are interesting in the context of the representation of the semisimple group G . As is well-known [Kna02], these can be realised in the “ K -picture”, i.e. as acting on $L^2(K)$ or in the “ N -picture”, i.e. acting on $L^2(N)$. The embeddings which we construct of the irreducible representations of K into the Fock spaces for N make a connection between these two realisations. An application of this theory, proving the Baum–Connes conjecture for rank one groups was discussed in [Doo11]. It would be interesting to know whether there is a version of the Mackey–Higson correspondence in the context of the contraction of K to $\bar{N}M$.

1. The contraction of K to $\bar{N}M$

Let G be a connected semisimple Lie group of real rank 1 and K a maximal compact subgroup of G . Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and choose $\mathfrak{a}$ to be a maximal abelian subalgebra of $\mathfrak{p}$. Let Σ (resp. Σ^+) denote the set of restricted roots (resp. positive restricted roots) of $\mathfrak{g}$ with respect to $\mathfrak{a}$ (with respect to a fixed positive cone $\mathfrak{a}^+$ in $\mathfrak{a}$). Then Σ^+ has either one or two elements; in the latter case, we take $\Sigma^+ = \{\lambda, 2\lambda\}$; in the former case, it will unify our presentation if we take $\Sigma^+ = \{\lambda, 2\lambda\}$ with $\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda} = \{0\}$.

We will denote by $\kappa(\cdot, \cdot)$ the Killing form of $\mathfrak{g}$. Choose $H \in \mathfrak{a}$ such that $\lambda(H) = 1$. Let $\langle X, Y \rangle = \frac{B(X, \theta Y)}{B(H, \theta H)}$ define an inner product on $\mathfrak{g}$. Note that $\|H\| = 1$ under this inner product.

The algebra $\bar{\mathfrak{n}} = \mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_{-2\lambda}$ is nilpotent of at most two steps: $\mathfrak{g}_{-2\lambda}$ is the centre. Also, $[\mathfrak{m}, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_\beta$ for $\beta \in \Sigma$, where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, so we have $[\mathfrak{m}, \bar{\mathfrak{n}}] \subseteq \bar{\mathfrak{n}}$, and $\mathfrak{m} \oplus \bar{\mathfrak{n}}$ is a Lie subalgebra of $\mathfrak{g}$. The Lie algebra of K may be decomposed as:

$$\mathfrak{k} = \mathfrak{m} \oplus \sum_{\beta \in \Sigma^+} \{X + \theta X : X \in \mathfrak{g}_{-\beta}\}. \quad (2)$$

Next we define a family $\{\pi_\varepsilon\}_{\varepsilon>0}$ of contraction mappings $\bar{N}M \rightarrow K$ as follows. Each element of $\bar{N}M$ can be expressed uniquely as $\exp(Y + X)m$, where $m \in M$, $Y \in \mathfrak{g}_{-\lambda}$, $X \in \mathfrak{g}_{-2\lambda}$, and we denote it by $(Y, X)m$. Then, for $\varepsilon > 0$, define

$$\pi_\varepsilon(\bar{n}m) = \exp(\sqrt{\varepsilon}(Y + \theta Y) + \varepsilon(X + \theta X))m. \quad (3)$$

In [DR85a] we showed that there is a relatively compact, open, M -invariant neighbourhood $\mathcal{O}_1$ of the identity in $\bar{N}$ such that π_1 is a diffeomorphism on $\mathcal{O} = \mathcal{O}_1 M$.

2. Representation theory of $\overline{N}$

If $\mathfrak{g}_{-\lambda} = \{0\}$, $\overline{N}$ is abelian and there is nothing to say. We assume therefore that $\mathfrak{g}_{-\lambda} \neq \{0\}$.

In this section, we define a canonical basis and complex structure for $\mathfrak{g}_{-\lambda}$, and realise the Fock space representation of $\overline{N}$, acting in a space of holomorphic functions thereon.

To each $\mu \in \mathfrak{g}_{-2\lambda}^* \setminus \{0\}$, with norm $|\mu|$, there corresponds a vector $X_\mu \in \mathfrak{g}_{-2\lambda}$ so that $\mu(X) = \langle X, X_\mu \rangle$ for all $X \in \mathfrak{g}_{-2\lambda}$. The vector $X_0 = \frac{X_\mu}{\sqrt{2}|\mu|}$ then satisfies $2\|X_0\|^2 = 1$.

Following the approach of [Cow+98], we define $J := J_{X_0} : \mathfrak{g} \rightarrow \mathfrak{g}$ by $JX = [X_0, \theta X]$. Then $J^2|_{\mathfrak{g}_{-\lambda}} = -I$, so that J induces a complex structure on $\mathfrak{g}_{-\lambda}$. In particular, $\mathfrak{g}_{-\lambda}$ is even-dimensional. We denote by $2p$ the dimension of $\mathfrak{g}_{-\lambda}$ and by q the dimension of $\mathfrak{g}_{-2\lambda}$.

Moreover, we have the identity

$$J_{\text{Ad}(m)X_0} = \text{Ad}(m) \circ J_{X_0} \circ \text{Ad}(m)^{-1} \quad (4)$$

for all X_0 as above and $m \in M$. By (4), the stabiliser M_{X_0} of X_0 preserves the complex structure of $(\mathfrak{g}_{-\lambda}, J)$.

Now $\langle Y, Y' \rangle_{\mathbb{C}} = \langle Y, Y' \rangle + i\langle Y, JY' \rangle$ is a J -hermitian inner product on $\mathfrak{g}_{-\lambda}$. Given a J -complex orthonormal basis $\{Y_1, \dots, Y_p\}$ of $\mathfrak{g}_{-\lambda}$, let $\mathcal{R} = \text{span}\{Y_1, \dots, Y_p\}$. Then $\mathfrak{g}_{-\lambda} = \mathcal{R} \oplus J\mathcal{R}$.

We let $\mathcal{F}_\mu$ be the corresponding Fock space with central character $e^{i\mu}$, specifically the set of complex-valued functions F on $\mathfrak{g}_{-\lambda}$ which are holomorphic for the complex structure associated to $J_{\frac{X_\mu}{\sqrt{2}|\mu|}}$ and satisfy

$$\|F\|_{\mathcal{F}_\mu}^2 = \left(\frac{|\mu|}{2\sqrt{2}\pi} \right)^{\frac{p}{2}} \int_{\mathfrak{g}_{-\lambda}} |F(Y)|^2 e^{-\frac{|\mu|}{2\sqrt{2}}|Y|^2} dY < \infty. \quad (5)$$

This space is a Hilbert space relative to the inner product:

$$\langle F, G \rangle = \left(\frac{|\mu|}{2\sqrt{2}\pi} \right)^{\frac{p}{2}} \int_{\mathfrak{g}_{-\lambda}} F(Y) \overline{G(Y)} e^{-\frac{|\mu|}{2\sqrt{2}}|Y|^2} dY. \quad (6)$$

Theorem 2.1. $\mathcal{F}_\mu$ carries an irreducible unitary representation σ_μ of $\overline{N}$ given by

$$(\sigma_\mu(\bar{n})F)(Y) = e^{i\mu(X')} e^{-\frac{|\mu|}{\sqrt{2}}(\frac{1}{4}|Y'|^2 + \frac{1}{2}\langle Y, Y' \rangle_{\mathbb{C}})} F(Y + Y'), \quad (7)$$

where $\bar{n} = (Y', X')$, $Y' \in \mathfrak{g}_{-\lambda}$, $X' \in \mathfrak{g}_{-2\lambda}$.

Up to unitary equivalence, σ_μ is the unique irreducible representation with central character $e^{i\mu}$, for each $\mu \neq 0$.

Furthermore $\{\sigma_\mu : \mu \in \mathfrak{g}_{-2\lambda}^* \setminus \{0\}\}$ is a set of representations of full Plancherel measure for $\overline{N}$.

Proof. Every irreducible unitary representation of $\overline{N}$ coincides with a character $e^{i\mu}$ on the centre $\mathfrak{g}_{-2\lambda}$. It is obvious, e.g., by Fourier transform on $\mathfrak{g}_{-2\lambda}$, that the Plancherel measure is concentrated on the representations for which $\mu \neq 0$. Such a representation factors to $\overline{N}/\exp(\ker \mu)$, isomorphic to a Heisenberg group. The conclusion is an easy

consequence of the Stone–von Neumann theorem, with formula (7) expressing the unique representation equal to $e^{i\mu}$ on $\mathfrak{g}_{-2\lambda}$ in Bargmann form [Fol16]. It is standard (see [Fol16]) that the Plancherel measure is supported on the set $\{\mathcal{F}_\mu : \mu \neq 0\}$. $\square$

We may realize σ_μ on a subspace of $L^2(\bar{N})$ as follows. For $F \in \mathcal{F}_\mu$, define $\widetilde{\Gamma}_\mu F$ by

$$\widetilde{\Gamma}_\mu F(Y, X) = G_\mu(Y, X)F(Y) \quad (8)$$

where $G_\mu(Y, X) = e^{-2i\pi\mu(X) + |\mu||Y|^2}$. Further define $\widetilde{\sigma}_\mu$ to be the restriction of the regular representation to $\{\widetilde{\Gamma}_\mu F : F \in \mathcal{F}_\mu\}$.

Proposition 2.2. *Suppose that $\mu(X_0) = |\mu|$ and $\mu(X') = 0$ for $X' \perp X_0$. Then $\widetilde{\Gamma}_\mu$ is an isometry from $\mathcal{F}_\mu$ into $L^2(\bar{N})$. Furthermore, $\widetilde{\Gamma}_\mu \circ \sigma_\mu = \widetilde{\sigma}_\mu \circ \widetilde{\Gamma}_\mu$, so that σ_μ is equivalent to the restriction of $\widetilde{\sigma}_\mu$ to the range of $\widetilde{\Gamma}_\mu$.*

Proof. Notice that

$$\begin{aligned} (\widetilde{\sigma}_\mu(Y', X')\widetilde{\Gamma}_\mu F)(Y, X) &= \widetilde{\Gamma}_\mu F((Y, X)(Y', X')) \\ &= \widetilde{\Gamma}_\mu F(Y + Y', X + X' + [Y, Y']) \\ &= F(Y + Y')e^{-\pi\mu(X + X' + [Y, Y']) + |\mu|(|Y|^2 + |X|^2)} \\ &= F(Y + Y')e^{-\pi\mu(X + X')}e^{-2\pi|\mu|((Y, Y')_{\mathbb{C}} + \frac{|Y|^2 + |Y'|^2}{2})} \end{aligned}$$

since $|Y + Y'|^2 + 2i\frac{\mu}{|\mu|}([Y, Y']) = |Y|^2 + |Y'|^2 + 2(Y, Y') + 2iJ(Y, Y') = |Y|^2 + |Y'|^2 + 2(Y, Y')_{\mathbb{C}}$.

Next,

$$\begin{aligned} \widetilde{\Gamma}_\mu(\sigma_\mu(Y', X')F)(Y, X) &= \sigma_\mu(Y', X')F(y)e^{-\pi(2i\mu(X) + |\mu||Y|^2)} \\ &= e^{-2i\pi\mu(X')}e^{-2\pi|\mu|(|Y'|^2 + |Y|^2 + (Y', Y)_{\mathbb{C}})}F(Y + Y')e^{-\pi(2i\mu(X) + |\mu||Y|)} \\ &= e^{-2i\mu(X' + X)}e^{-2\pi|\mu|(\frac{|Y'|^2 + |Y|^2}{2} + (Y', Y)_{\mathbb{C}})}F(Y + Y'). \end{aligned}$$

We have shown that $\widetilde{\Gamma}_\mu\sigma_\mu(Y', X')F = \widetilde{\sigma}_\mu(Y', X')\widetilde{\Gamma}_\mu F$. Hence $\widetilde{\sigma}_\mu$ is equivalent to σ_μ . $\square$

We now describe an orthonormal basis for $\mathcal{F}_\mu$, relative to our J -complex orthonormal basis $\{Y_1, \dots, Y_p\}$ of $\mathfrak{g}_{-\lambda}$. We can write each $Y \in \mathfrak{g}_{-\lambda}$ as $Y = \sum_{j=1}^p (\alpha_j Y_j + \beta_j JY_j)$, where $(Y, Y_j)_{\mathbb{C}} = (Y, Y_j) + i(Y, JY_j) = \alpha_j + i\beta_j$. Let us denote $(Y, Y_j)_{\mathbb{C}}$ by z_j . Then it is easy to see the following.

Lemma 2.3. *For each multi-index $\mathbf{n} = (n_1, \dots, n_p)$, the multinomial given by*

$$z^{\mathbf{n}} = z_1^{n_1} \dots z_p^{n_p} \quad (9)$$

belongs to $\mathcal{F}_\mu$. Furthermore, the multinomials $\{U_{\mu, \mathbf{n}} = \frac{z^{\mathbf{n}}}{\|z^{\mathbf{n}}\|}\}$ provide an orthonormal basis for $\mathcal{F}_\mu$.

See [Fol16] for a proof that this is an orthonormal basis.

3. Representations of $\bar{N}M$

One may use the Mackey method (see [Mac55, 3.12]) to describe the irreducible representations of $\bar{N}M$. In [DR85a], this was done via the Kirillov orbit method, but for the present purpose of finding an orthonormal basis, the Mackey method is somewhat cleaner. The Mackey method says that we should consider the action of M on the set of irreducible representations of $\bar{N}$. In the previous section, we parameterized a set of full Plancherel measure of irreducible representations of N by $\mu \in \mathfrak{g}_{-2\lambda}^* \setminus \{0\}$. Using formula (7), it is easy to check that $\sigma_\mu(m.(X, Z)) = \sigma_{m^{-1}.\mu}(X, Z)$. Now M acts isometrically on $\mathfrak{g}_{-\lambda} \times \mathfrak{g}_{-2\lambda}$, and by Kostant's double transitivity theorem (see [Cow+98]), the action is doubly transitive on the unit spheres of $\mathfrak{g}_{-\lambda}$ and $\mathfrak{g}_{-2\lambda}$. Hence in particular there is a unique orbit in the unit sphere of $\mathfrak{g}_{-2\lambda}^*$, and Mackey's theorem says that a set of full Plancherel measure of irreducible representations of $\bar{N}M$ is parameterized by pairs consisting of an M orbit: i.e. a sphere in $\mathfrak{g}_{-2\lambda}$ and an irreducible representation of the stabiliser group of an element of that sphere.

Fixing μ with $|\mu| = 1$, the set of all orbits is then parameterised by $\{s.\mu : s \in \mathbb{R}^+\}$. Hence the irreducible representations of $\bar{N}M$ are parameterized by a pair consisting of $s > 0$ and an irreducible representation of the stabiliser group of $s\mu$, which of course is just the stabiliser of μ , given by

$$M_\mu = \{m \in M : m.\mu = \mu\}.$$

If we take $\mathfrak{t}_1$ to be a Cartan subalgebra of M , the stabiliser has Lie algebra given by

$$\mathfrak{m}_\mu = \mathfrak{t}_\mu \oplus \bigoplus_{\alpha \in \Phi_0} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

where $\mathfrak{t}_\mu = \{H \in \mathfrak{t}_1 : \mu(H) = 0\}$ is the stabiliser of $X_\mu + \theta X_\mu$ in $\mathfrak{t}_1$, and Φ_0 is the set of roots which vanish on $X_\mu + \theta X_\mu$.

By a theorem of Kostant [Kos70], M_μ is connected. (This may also be seen by [DR85a, Proposition 4.5] or by an obvious case-by-case argument.) Hence, its irreducible representations are parameterized by the dominant weights of the Lie algebra.

Then $\mathfrak{t}_\mu$ is a Cartan subalgebra of $\mathfrak{m}_\mu$ and Φ_0 may be identified as the set of roots of $\mathfrak{m}_\mu$. Let Λ_0^+ be the set of dominant weights with respect to Φ_0^+ . To $\lambda_0 \in \Lambda_0^+$, associate the representation η_{λ_0} of M_μ of highest weight λ_0 , and form the induced representation

$$\rho_{\lambda_0, \mu} = \eta_{\lambda_0} \otimes \sigma_\mu \uparrow_{\bar{N}M_\mu}^{\bar{N}M}.$$

Mackey's theorem then assures us that $\rho_{\lambda_0, s\mu}$ is an irreducible representation and that almost every irreducible representation of $\bar{N}M$ can be obtained in this way as s runs over $\mathbb{R}^+$ and λ_0 over Λ_0^+ . (See also [Lip80].)

Now $\rho_{\lambda_0, s\mu}$ acts via the left regular representation in

$$\mathcal{F}_{\lambda_0, \mu} = \left\{ f: \bar{N}M \longrightarrow L^2(M_\mu) \otimes \tilde{\Gamma}_\mu \mathcal{F}_\mu \left| \begin{array}{l} f(\bar{n}m\bar{n}_1m_\mu) = \eta_{\lambda_0}(\bar{n}_1)\sigma_\mu(m_\mu)f(\bar{n}m), \\ f \text{ measurable and } \int_{M/M_\mu} \|f\|^2 dm < \infty \end{array} \right. \right\}$$

Now identifying $L^2(X, L^2(Y))$ with $L^2(X \times Y)$, we may identify $\mathcal{F}_{\lambda_0, \mu}$ with a certain space of functions on $\bar{N} \times M/M_\mu \times M_\mu \simeq \bar{N}M$. Writing $\bar{N}$ in (X, Z) coordinates,

these functions $f(X, Z, m)$ all have the form

$$f(X, Z, m) = e^{-\pi(s|X|^2 + 2is\mu(Z))} F(X, m),$$

where:

- (i) $F(\cdot, m)$ is analytic for the complex structure defined by J_{Z_μ} ;
- (ii) $\int_{\mathfrak{g}-\lambda} |F(X, m)|^2 e^{-2\pi|\mu|^2|X|^2} dX < \infty$ for all $m \in M$;
- (iii) $F(X, mt) = e^{2\pi i\lambda_0(\log t)} f(X, m)$ for all $t \in T_1$;
- (iv) $\frac{d}{dt} \Big|_{t=0} f(X, m \exp tX_\alpha) = 0$ for all $\alpha \in \Phi_0^+$.

In this realization of $\mathcal{F}_{\lambda_0, \mu}$, $\rho_{\lambda_0, s\mu}$ acts by

$$\begin{aligned} (\rho_{\lambda_0, s\mu}(\bar{n}_1 m_1) f)(\bar{n} m) &= f(\bar{n}_1 m_1 \cdot \bar{n}(m_1 m)) \\ &= f((X_1, Z_1)(m_1 X, Z), m_1 m) \\ &= f(X_1 + m_1 X, Z_1 + Z + [X_1, m_1 X], m_1 m) \\ &= e^{-2\pi i s\mu[X_1, m_1 X]} F(X_1 + m_1 X, m_1 m). \end{aligned}$$

An orthonormal basis for $\mathcal{F}_{\lambda_0, \mu}$ is found by identifying M with $M/M_\mu \times M_\mu$ and hence $L^2(M)$ with $L^2(M/M_\mu) \otimes L^2(M_\mu)$. Now choosing $\phi_{\lambda_0} \in L^2(M_\mu)$ to be the normalised highest weight vector for the representation η_{λ_0} , and $\{\psi_j : j \in \mathbb{N}\}$ to be any orthonormal basis for $L^2(M/M_\mu)$, we set

$$F_{\mathbf{n}, j, \mathbf{m}}(z, \alpha, \beta) = \frac{z^{\mathbf{n}}}{\|z^{\mathbf{n}}\|} \psi_j(\alpha) \frac{(X^{\mathbf{m}} \phi_{\lambda_0})(\beta)}{\|X^{\mathbf{m}} \phi_{\lambda_0}\|_{L^2(M_\mu)}}$$

where $\mathbf{n} \in \mathbb{N}^{\text{card } \Phi_1^+}$, $j \in \mathbb{N}$ and $\mathbf{m} \in F$, F being a finite subset of $\mathbb{N}^{\text{card } \Phi_0^+}$ chosen so that $\{X^{\mathbf{m}} \phi_{\lambda_0} : \mathbf{m} \in F\}$ is an orthonormal basis for $\mathcal{H}_{\lambda_0}$. Here, $X^{\mathbf{m}} = \prod_{\alpha \in \Phi_0^+} X_\alpha^{m_\alpha}$.

4. Representations of K

In order to prove our approximate intertwining theorem, we need to describe the irreducible representations of K in a way which is compatible with the representations of $\bar{N}M$ as described in Sections 2 and 3, choosing a particular Cartan subalgebra for $\mathfrak{k}$, and roots and weights which are aligned with the structure of $\mathfrak{n}$ which we used above.

We recall the standard representation theory of compact groups, given by the Borel–Weil theorem [Wal73]. Thus, let K be an arbitrary compact connected Lie group with maximal torus T , $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{t}^{\mathbb{C}}$ complexifications of the Lie algebra of K and T respectively.

If α is a root of $\mathfrak{g}^{\mathbb{C}}$ with respect to the CSA $\mathfrak{t}^{\mathbb{C}}$, we use the standard notation of a corresponding root vector X_α so that $[H, X_\alpha] = \alpha(H)X_\alpha$ for all $H \in \mathfrak{t}^{\mathbb{C}}$ and we choose $X_{-\alpha}$ so that $X_\alpha, X_{-\alpha}$, and $H_\alpha = [X_\alpha, X_{-\alpha}]$ span a three-dimensional simple subalgebra of $\mathfrak{g}^{\mathbb{C}}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Denote by Φ the roots of $\mathfrak{k}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$, and by Φ^+ a set of positive roots.

Let Λ^+ be the corresponding set of dominant weights, so that the irreducible representations of K are parameterised by Λ^+ .

Given $\lambda \in \Lambda^+$, let χ_λ be the corresponding character of T . Then the corresponding representation σ_λ acts by the right regular representation in $\mathcal{H}_\lambda = \{f \in L^2(K) : f(t^{-1}g) = \chi_\lambda(t)f(g) \text{ and } X_\alpha^L f = 0 \ \forall \alpha \in \Phi^+\}$. Here X^L denotes the left action of $\mathfrak{k}^{\mathbb{C}}$ on $L^2(K)$, given by $X^L f(g) = \frac{d}{dt} \Big|_{t=0} f(\exp(-tX)g)$.

Similarly we will denote the right action by $X^R f$, defined by

$$X^R f(g) = \frac{d}{dt} \bigg|_{t=0} f(g \exp(tX));$$

this is the action of $\mathfrak{k}^{\mathbb{C}}$ on $\mathcal{H}_\lambda$ given by the derivative of σ_λ . This action extends naturally to the action of the universal enveloping algebra $\mathcal{U}(\mathfrak{k}^{\mathbb{C}})$ on $\mathcal{H}_\lambda$.

Furthermore, there is a unique highest weight vector in $f_\lambda \in \mathcal{H}_\lambda$, defined by the equations $X^R f_\lambda = 0$ for all $X \in \mathfrak{k}_\alpha$, $\alpha \in \Phi^+$. Then $\mathcal{H}_\lambda = \mathcal{U}(\mathfrak{k}^{\mathbb{C}})f_\lambda$. It follows by the Poincaré–Birkhoff–Witt theorem, that $\mathcal{H}_\lambda$ is the span of functions of the form $X_1^{n_1} X_2^{n_2} \cdots X_k^{n_k} f_\lambda$, where $\{X_1, \dots, X_k\}$ is a basis for $\mathfrak{k}^{\mathbb{C}}$.

If we take the basis arising from the root space decomposition, use the fact that $\mathfrak{k}^{\mathbb{C}}$ acts on f_λ by scalars and the positive root directions annihilate it, we have the usual Verma module decomposition, that $\mathcal{H}_\lambda$ is the span of functions of the form $X_{-\alpha_1}^{m_1} X_{-\alpha_2}^{m_2} \cdots X_{-\alpha_k}^{m_k} f_\lambda$, where $\Phi^+ = \{\alpha_1, \dots, \alpha_k\}$ and the m_i are positive integers.

The previous two paragraphs summarised the situation for a general compact connected group. We now specialise to the situation where K is as in Section 1, i.e. it is the maximal compact group in a rank one semisimple group.

Recall from Helgason [Hel78] that $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^{\mathbb{C}}$, where the Cartan involution θ fixes $\mathfrak{k}$ and has eigenvalue -1 on $\mathfrak{p}$. The maximal abelian subalgebra $\mathfrak{a}^{\mathbb{C}}$ of $\mathfrak{p}^{\mathbb{C}}$ has dimension 1, spanned by H , and $\mathfrak{m} = \{X \in \mathfrak{k} : [X, H] = 0\}$ is the stabiliser of $\mathfrak{a}$ in $\mathfrak{k}$. With this notation, we have

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{m}^{\mathbb{C}} + \mathfrak{a}^{\mathbb{C}} + \sum_{\alpha \in P_+} (\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}),$$

where P_+ is the set of restricted roots, that is the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{a}^{\mathbb{C}}$. We have $\mathfrak{n} = \sum_{\alpha \in P_+} \mathfrak{g}^\alpha$.

Choose a Cartan subalgebra (CSA) $\mathfrak{t}_1^{\mathbb{C}}$ for the compact Lie algebra $\mathfrak{m}^{\mathbb{C}}$. Then it is easy to see from the rank one condition that $\mathfrak{t}_1^{\mathbb{C}} + \mathfrak{a}^{\mathbb{C}}$ is a CSA for $\mathfrak{g}^{\mathbb{C}}$, and we may choose positive roots Φ^+ for this CSA. Then Φ^+ can be written as a disjoint union $\Phi_0^+ \cup \Phi_1^+ \cup \Phi_2^+$, where $\Phi_i^+ = \{\alpha \in \Phi^+ : \alpha(H) = i\}$ for $i = 0, 1, 2$.

Then we have

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{t}_1^{\mathbb{C}} + \sum_{\alpha \in \Phi_0^+} (\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha})$$

and the restrictions of roots in Φ_1^+ and Φ_2^+ to $\mathfrak{a}^{\mathbb{C}}$ are exactly the restricted roots in P_+ of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{a}^{\mathbb{C}}$.

As noted in Helgason [Hel78], $\Phi_0^+ = \{\alpha \in \Phi^+ : \alpha^\theta = \alpha\}$ and $\Phi_1^+ \cup \Phi_2^+ = \{\alpha \in \Phi^+ : \alpha^\theta \neq \alpha\}$.

Thus the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ can be seen by writing

$$\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Phi_1^+ \cup \Phi_2^+} \mathbb{C}(X_\alpha - \theta X_\alpha)$$

and

$$\begin{aligned} \mathfrak{k}^{\mathbb{C}} &= \mathfrak{m} + \sum_{\alpha \in \Phi_1^+ \cup \Phi_2^+} \mathbb{C}(X_\alpha + \theta X_\alpha) \\ &= \mathfrak{t}_1^{\mathbb{C}} + \sum_{\alpha \in \Phi_0^+} \mathbb{C}X_\alpha + \sum_{\alpha \in \Phi_1^+ \cup \Phi_2^+} \mathbb{C}(X_\alpha + \theta X_\alpha). \end{aligned}$$

In this notation, $\mathfrak{k}$ is as above and $\mathfrak{n} = \sum_{\alpha \in P_+} \mathfrak{g}^\alpha = \sum_{\alpha \in \Phi_1^+} \mathfrak{g}^\alpha + \sum_{\alpha \in \Phi_2^+} \mathfrak{g}^\alpha$ are the Lie algebra of $\mathfrak{k}$ and $\mathfrak{n}$ as described in Section 1.

Now the abelian subalgebra $\mathfrak{t}_1^\mathbb{C}$ is not a Cartan subalgebra of $\mathfrak{k}^\mathbb{C}$, as it is not maximal: it is a CSA for $\mathfrak{m}^\mathbb{C}$. In the rank one case, however, the algebra $\mathfrak{t}^\mathbb{C} = \mathfrak{t}_1^\mathbb{C} + \mathbb{C}(X_0 + \theta X_0)$ is a Cartan subalgebra of $\mathfrak{k}^\mathbb{C}$ for any nonzero vector X_0 in $\mathfrak{g}_{-2\lambda}$. We now *choose and fix throughout this section an element X_0 of $\mathfrak{g}_{-2\lambda}$ of norm 1*. In fact, we can assume without loss of generality that X_0 is a $\mathfrak{g}$ root vector in $\mathfrak{g}_{-2\lambda}$, which in particular implies that it is orthogonal to the root vectors of the other roots of $\mathfrak{g}$ in Φ_2^+ . Our description of the roots and weights of the Lie algebra $\mathfrak{k}^\mathbb{C}$ (and hence the dual of the compact group K) will be carried out using Cartan subalgebra $\mathfrak{t}^\mathbb{C}$, which is defined via the vector X_0 . We denote by M_0 the stabiliser of $X_0 + \theta X_0$ in M (with Lie algebra $\mathfrak{m}_0$) and by $\mathfrak{t}_0$ the Lie algebra of the stabiliser of $X_0 + \theta X_0$ in $\mathfrak{t}_1$.

Remark 4.1. In Section 2 we chose an arbitrary nonzero vector $\mu \in \mathfrak{g}_{-2\lambda}^*$ and used it to define our representations of N . The entire description of the representations of K in the current section can be carried out without reference to $\mu \in \mathfrak{g}_{-2\lambda}^*$. A correspondence between X_0 and μ will need to be invoked when we define the relationship between representations of K and $\bar{N}M$ in Section 5. For $\mu \in \mathfrak{g}_{-2\lambda}^*$, dual to $X_\mu \in \mathfrak{g}_{-2\lambda}$, we will set $X_0 = \frac{X_\mu}{\sqrt{2}|\mu|}$ and $J = J_{X_0}$. With this choice of X_0 , we have $\mathfrak{m}_\mu = \mathfrak{m}_0$ and $\mathfrak{t}_\mu = \mathfrak{t}_0$.

We thus have multiple sets of roots.

- (i) The roots of $\mathfrak{g}$ with respect to the CSA $\mathfrak{t}_1 + \mathfrak{a}$, which we write as Φ (and Φ^+ for a choice of positive roots). If nonzero, their restrictions to $\mathfrak{a}$ are roots of $\mathfrak{g}$ with respect to the (non-maximal) abelian subalgebra $\mathfrak{a}$ and have the form $\pm\lambda$ or $\pm 2\lambda$. We are writing Φ_1^+ as the set of roots in Φ^+ whose restriction to $\mathfrak{a}$ is λ , and Φ_2^+ as the set of roots in Φ^+ whose restriction to $\mathfrak{a}$ is 2λ .
- (ii) The set Φ_0^+ of positive roots of $\mathfrak{g}$ whose restriction to $\mathfrak{a}$ is zero. The restriction of these roots to the CSA $\mathfrak{t}_1^\mathbb{C}$ of $\mathfrak{m}$ form a set of positive roots of $\mathfrak{m}^\mathbb{C}$ with respect to $\mathfrak{t}_1^\mathbb{C}$.
- (iii) The roots of the compact Lie algebra $\mathfrak{k}^\mathbb{C}$ with respect to the CSA $\mathfrak{t} = \mathfrak{t}_1^\mathbb{C} + \mathbb{C}(X_0 + \theta X_0)$. These contain the roots in Φ_0^+ , which can easily be seen to annihilate $X_0 + \theta X_0$. We will give a more detailed description of the other roots (and positive roots) in this category below.

It is clear that the choices of positivity can be made compatibly in the above descriptions.

In our description of item (i) above, we have glossed over the issue that either Φ_1^+ or Φ_2^+ may be empty. If one of them is empty, we assume (as at the beginning of Section 2) that it is Φ_1^+ , so that the algebra $\mathfrak{n} = \mathfrak{g}_{-2\lambda}$ is abelian of dimension q . If there exist nonzero roots in both Φ_1^+ and Φ_2^+ , then the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ implies that for all $\alpha_1, \alpha_2 \in \Phi_1^+$, $\alpha_1 + \alpha_2 \in \Phi_2^+$, and $\mathfrak{n}$ is a two-step nilpotent group with centre $\mathfrak{g}_{-2\lambda}$. Furthermore, if Φ_1^+ is nonzero, the fact that we may put a complex structure on $\mathfrak{g}_{-\lambda}$ implies that it is of even dimension.

Proposition 4.2. *Let X_0, J be as above. Then $\langle Y, Y' \rangle_\mathbb{C} = \langle Y, Y' \rangle + i\langle Y, JY' \rangle$ is a J -hermitian inner product on $\mathfrak{g}_{-\lambda}$. There exists a J -complex orthonormal basis $\{Y_1, \dots, Y_p\}$ of $\mathfrak{g}_{-\lambda}$ such that:*

- (i) for $T \in \mathfrak{t}_0$, $[T, Y_j] = \beta_j(T)JY_j$ and $[T, JY_j] = -\beta_j(T)Y_j$;
(ii) if $\mathcal{R} = \text{span}\{Y_1, \dots, Y_p\}$, then $\mathfrak{g}_{-\lambda} = \mathcal{R} \oplus J\mathcal{R}$.

Proof. Indeed, suppose that X_0 lies in a negative root space $\mathfrak{g}_{-\beta}$ of $(\mathfrak{g}, \mathfrak{t})$, $\beta \in \Phi_2^+$. Then for all $\alpha \in \Phi_1^+$ and for all $Y \in \mathfrak{g}_{-\alpha}$, $JY \in \mathfrak{g}_{-\beta+\alpha^\theta}$. Furthermore, if $Y \neq 0$, $JY \neq 0$. Thus $-\beta + \alpha^\theta \in \Phi_1^+$, and $\alpha \rightarrow -\beta + \alpha^\theta$ gives a pairing between the roots in Φ_1^+ . Letting $\Phi_1^r = \{\alpha_1, \dots, \alpha_p\}$ be a choice of roots containing exactly one from each such pair, and setting $\mathcal{R} = \bigoplus_{\alpha \in \Phi_1^r} \mathfrak{g}_{-\alpha}$, we see that $\mathfrak{g}_{-\lambda} = \mathcal{R} \oplus J_{X_0}\mathcal{R}$. The remainder of the claims follow from the basic properties of root spaces. $\square$

We have the following structure of roots of $\mathfrak{g}$.

Lemma 4.3. *Every root in Φ_2^+ is the sum of two roots in Φ_1^+ , i.e. we may choose a root vector Ξ_α for each $\alpha \in \Phi$ such that each $\Xi_{\alpha'}$, $\alpha' \in \Phi_2^+$ is the commutator $[\Xi_{\alpha_1}, \Xi_{\alpha_2}]$ for some $\alpha_1, \alpha_2 \in \Phi_1^+$.*

Proof. We will prove the lemma by explicitly constructing root vectors for roots in Φ_1 and Φ_2 . Set $J = J_{X_0}$. By [DR85a, Proposition 5.2], the root spaces of Φ_1 span $(I + \theta)\mathfrak{g}_{-\lambda}^\mathbb{C}$ and those of Φ_2 span $\mathfrak{m}_0^\perp \oplus (I + \theta)X_0^\perp$, where

$$X_0^\perp = \mathfrak{g}_{-2\lambda} \ominus \mathbb{R}X_0, \quad \mathfrak{m}_0^\perp = \mathfrak{m} \ominus \mathfrak{m}_0.$$

Here $\ominus$ denotes the orthogonal complement (for the Killing form). We start with Φ_1 . As pointed out in [DR85a, Proposition 5.2], the action of $\mathfrak{m}_0^\mathbb{C}$ on $\mathfrak{g}_{-\lambda}$ commutes with the action of J , which is given by $X_0 + \theta X_0$. Thus we may regard $(\mathfrak{g}_{-\lambda}, J)$ as a complex space, and choose a complex basis $Y_1, \dots, Y_n$ consisting of weight vectors for the action of $\mathfrak{t}_0^\mathbb{C}$. It follows that each of the vectors

$$Y_j - iJY_j, \quad Y_j + iJY_j$$

is a weight vector for the action of $\mathfrak{m}_0^\mathbb{C}$ on $\mathfrak{g}_{-\lambda}^\mathbb{C}$. Therefore,

$$\Xi_j^\pm = Y_j \mp iJY_j + \theta(Y_j \mp iJY_j)$$

is a root vector in $\mathfrak{k}^\mathbb{C}$. If β_j is the weight of Y_j on $\mathfrak{t}_0^\mathbb{C}$, then $(\beta_j, \pm i)$ is the root corresponding to $\Xi_j^\pm$.

The root vectors for roots in Φ_2 can be obtained in a similar way, extending the map $J = J_{X_0} : \cdot \mapsto [X_0, \theta \cdot]$ to $X_0^\perp \oplus \mathfrak{m}_0^\perp$.

Let $X_1, \dots, X_m$ be a basis for $X_0^\perp$, consisting of weight vectors for the action of $\mathfrak{t}_0^\mathbb{C}$. Then

$$Z_j = [X_0, \theta X_j] = JX_j$$

is a basis for $\mathfrak{m}_0^\perp$, also consisting of weight vectors for $\mathfrak{t}_0^\mathbb{C}$. (Since J intertwines the two actions, the weights are the same for both.) Since $\|X_0\|^2 = \frac{1}{2}$, [DR85a, Proposition 3.2] implies that $J^2 = -2I$.

Thus, setting

$$\Xi_j'^\pm = Z_j \pm i(X_j + \theta X_j),$$

we have

$$\begin{aligned} [X_0 + \theta X_0, \Xi_j^{\pm}] &= J^2 X_j + \theta J^2 X_j \pm i(JX_j + \theta X_j) \\ &= -2(X_j + \theta X_j) \pm 2iZ_j \\ &= \pm 2i\Xi_j^{\pm}. \end{aligned}$$

We claim that the commutators $[\Xi_1^-, \Xi_j^-]$ give all the root vectors for the roots in Φ_2^- .

Consider $Y_1 \in \mathfrak{g}_{-\lambda}$. By the J^2 property, for any $X, X' \in \mathfrak{g}_{-2\lambda}$, $X \perp X'$, there is X'' , orthogonal to both X and X' , such that

$$J_X J_{X'} Y_1 = J_{X''} Y_1.$$

This implies that the subspace $\{J_X Y_1 : X \in X_0^\perp\}$ is both J -invariant and $\mathfrak{t}_0$ -invariant. So we may assume that the elements $Y_2, \dots, Y_{m+1}$ of the basis of weight vectors selected above belong to this subspace. We then obtain a basis $X'_2, \dots, X'_{m+1}$ of $X_0^\perp$ such that, for $2 \leq j \leq m+1$,

$$\Xi_j^- = J_{X'_j} Y_1 + iJ J_{X'_j} Y_1 + \theta(J_{X'_j} Y_1 + iJ J_{X'_j} Y_1).$$

We isolate the $\mathfrak{g}_{-2\lambda}$ -component of $[\Xi_1^-, \Xi_j^-]$, equal to

$$[Y_1 + iJY_1, J_{X'_j} Y_1 + iJ J_{X'_j} Y_1].$$

By the J^2 property, this belongs to $X_0^{\perp \mathbb{C}}$. We identify its real and imaginary parts by pairing them with a generic element $X \in X_0^{\perp \mathbb{C}}$. We find that

$$\begin{aligned} \langle [Y_1, J_{X'_j} Y_1] - [JY_1, J J_{X'_j} Y_1], X \rangle &= \langle X'_j, X \rangle - \langle J_X JY_1, J J_{X'_j} Y_1 \rangle \\ &= \langle X'_j, X \rangle + \langle J_X Y_1, J_{X'_j} Y_1 \rangle \\ &= 2\langle X'_j, X \rangle, \end{aligned}$$

so that $[Y_1, J_{X'_j} Y_1] - [JY_1, J J_{X'_j} Y_1] = 2X'_j$.

In a similar fashion, the two imaginary terms give

$$\begin{aligned} \langle [Y_1, J J_{X'_j} Y_1] + [JY_1, J_{X'_j} Y_1], X \rangle &= \langle J_X Y_1, J J_{X'_j} Y_1 \rangle + \langle J_X JY_1, J_{X'_j} Y_1 \rangle \\ &= 2\langle J_X Y_1, J J_{X'_j} Y_1 \rangle \\ &= 2\langle X''_j, X \rangle, \end{aligned}$$

where X''_j is such that $J_{X''_j} Y_1 = J J_{X'_j} Y_1$.

This proves that $[\Xi_1^-, \Xi_j^-]$ is nonzero, so that it gives a root vector for a root in Φ_2^- . The fact that the X'_j form a basis for $X_0^\perp$ implies that all the roots in Φ_2^- are obtained in this way. $\square$

We now have a description of the root spaces of $\mathfrak{k}$ with respect to the Cartan subalgebra $\mathfrak{t}_1 + \mathbb{C}(X_0 + \theta X_0)$.

Corollary 4.4. *We have*

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{t}_1^{\mathbb{C}} + \mathbb{C}(X_0 + \theta X_0) + \sum_{\alpha \in \Phi_0} \mathbb{C}X_\alpha + \sum_{j=1}^p \mathbb{C}\Xi_j^+ + \mathbb{C}J\Xi_j^+ + \sum_{j,k=1}^p \mathbb{C}[\Xi_k^-, \Xi_j^-] + \mathbb{C}[\Xi_k^+, \Xi_j^+].$$

The positive roots may be chosen to be the elements $\alpha \in \Phi_0^+$, with root vectors X_α , the elements Φ_1^+ , with positive root vectors Ξ_j^+ for $j = 1, \dots, q$ and the elements of Φ_2^+ with positive root vectors $[\Xi_{\alpha_1}^+, \Xi_{\alpha_2}^+]$, for $\alpha_1, \alpha_2 \in \Phi_1^+$.

Furthermore, a set of fundamental roots is given by $\Phi_0^+ \cup \Phi_1^+$.

We have derived this picture of the structure of K using only the formalism of H -type groups. It is straightforward to check that this structure of K coincides with the Cartan–Helgason classification (see [Hel78; Kor09]). In particular, in the quaternionic case of $G = F_{4,-20}$, the roots Ξ_j correspond to the short roots and the roots $[\Xi_k^+, \Xi_j^+]$ correspond to the long roots for the group $K = G_2$.

We now describe the dominant integral weights of K . Let b be the smallest positive number such that $b(X_0 + \theta X_0)$ is in the integer lattice of $\mathfrak{t}$. Then every element in the integer lattice of $\mathfrak{t}$ is of the form $T + nb(X_0 + \theta X_0)$, where T belongs to the integer lattice in $\mathfrak{t}_0$ and $n \in \mathbb{Z}$. Define $\tilde{\gamma}$ to be the functional on $\mathfrak{t}$ such that $\tilde{\gamma}(\mathfrak{t}_0) = 0$ and $\tilde{\gamma}(b(X_0 + \theta X_0)) = 1$. Then the integral weights for K are all the functionals on $\mathfrak{t}$ of the form $(\psi, n\tilde{\gamma})$ where ψ is an integral weight for M_0 and $n \in \mathbb{Z}$. Such a weight is dominant if and only if ψ is dominant for M_0 and $n \geq n_\psi$, where n_ψ is an integer depending on ψ .

Associated to any such weight $(\psi, n\tilde{\gamma})$ we have the (finite dimensional) Hilbert spaces $\mathcal{H}_{(\psi, n)}(K)$ which consist of analytic functions on K such that:

- (i) $F(gh) = \chi_{(\psi, n)}(h^{-1})F(g)$ for $g \in K$ and $h \in M_\mu \exp(\mathbb{R}(X_\mu + \theta X_\mu))$ where $\chi_{(\psi, n)}$ is the character whose differential at the identity is $2\pi i(\psi, n\tilde{\gamma})$, and
- (ii) $XF = -2\pi i((\psi, n\tilde{\gamma}), X)F$ for $X \in \mathfrak{g}^\alpha$, $\alpha \in \Phi_1^+ \cup \Phi_2^+$.

By the Borel–Weil theorem, the right regular action of K on $\mathcal{H}_{(\psi, \ell)}$ gives an irreducible representation denoted σ_ℓ^ψ , and all irreducible representations occur from this construction.

Our next step is to find an explicit description of the representation σ_ℓ^ψ realised in $L^2(K)$. Consider the representation of K on the complexification $\mathfrak{p}^\mathbb{C}$ of $\mathfrak{p}$. This representation is of course closely related to the adjoint representation of K , a fact that we shall use in the sequel. With respect to the natural inner product, this action is unitary. In general this is not an irreducible representation as it may have two irreducible components. Using the structure of the roots of $\mathfrak{g}^\mathbb{C}$, we see that the action of $\mathfrak{t}^\mathbb{C} = \mathfrak{t}_0^\mathbb{C} \oplus \mathbb{C}(X_0 + \theta X_0)$ has weights belonging to one of three categories:

- (i) the weights of the form $(\alpha_j, 0)$ where α_j is a weight for the representation of $\mathfrak{m}_0$ on $X_0^{\perp \mathbb{C}}$, $j = 1, \dots, q-1$;
- (ii) the weights $(\beta_j, \pm i)$, for $j = 1, \dots, p$;
- (iii) the weights $(0, \pm 2i)$.

These three types of weights have corresponding weight spaces: $\{X - \theta X : X \in X_0^{\perp \mathbb{C}}\}$, $\{X - \theta X : X \in \mathfrak{g}_{-\lambda}^\mathbb{C}\}$ and $\mathfrak{a}^\mathbb{C} \oplus \mathbb{C}(X_0 - \theta X_0)$ respectively.

The weight space relative to $(0, 2i)$ is one-dimensional and is generated by the vector

$$B = H - i\lambda(H)(X_0 - \theta X_0).$$

Thus, the representation with this highest weight occurs with multiplicity one in $\mathfrak{p}^\mathbb{C}$. Also, $(0, 2i)$ is an integral multiple $n_0\tilde{\gamma}$ of $\tilde{\gamma}$, and since $(H, B) = 1 \neq 0$, the orthogonal

projection of H on the K -invariant subspace generated by B is nonzero and is an M -invariant vector. This proves that the representation $\sigma_{(0,n_0)}$ has a unique M -invariant subspace, i.e. is of M -class one.

It follows (see [DR85a, Proposition 6.2]) that if we set

$$\Gamma(k) = \kappa(H, \text{Ad}(k)B),$$

then $\Gamma \in \mathcal{H}_{(0,n_0)}$ is a left M -invariant, right M_0 -invariant function with $\|\Gamma\|_\infty = |\Gamma(e)| = 1$ so that the following holds.

Lemma 4.5. *For $j = 1, \dots, p$, let β_j and $\{Y_j\}$ be as in Proposition 4.2. Then:*

- (i) *for all $X \in \mathfrak{k}^\mathbb{C}$, $(X\Gamma)(k) = \kappa(H, \text{Ad}(k)[X, B])$;*
- (ii) *let $\Gamma_j(k) = 2(H, \text{Ad}(k)(Y_j - \theta Y_j))$, then $\Gamma_j = (Y_j + \theta Y_j)\Gamma$;*
- (iii) *$(X_j + \theta X_j)\Gamma = 0$ for $j = 1, \dots, q$;*
- (iv) *$[Y_j + \theta Y_j, B] = 2(Y_j - \theta Y_j)$ is a weight vector with weight $(-i\beta_j, i)$;*
- (v) *$[X_j + \theta X_j, B] = 2(X_j - \theta X_j)$ is a weight vector with weight $(-i\alpha_j, 0)$ for $j = 1, \dots, q - 1$.*

Proof. This follows from the basic properties of root vectors, together with Proposition 4.2 and Corollary 4.4. $\square$

Using Lemma 4.5 it follows that

$$\Xi_j \Gamma(k) = (H, \text{Ad}(k)(X_j - \theta X_j) + J(X_j - \theta X_j))$$

for $j = 1, \dots, p$. Hence the dimension of the space $\mathcal{H}_{(0,n_0)}$ is $p + 1$, and the space is spanned (as a complex vector space) by the vector Γ together with $\Gamma_j = \Xi_j \Gamma$, $j = 1, \dots, p$. Denote Γ by Γ_0 to see the following.

Corollary 4.6. *Γ is the highest weight vector of the representation $\sigma_{(0,n_0)}$, and Γ^ℓ is the highest weight vector of the representation $\mathcal{H}_{(0,\ell n_0)}$.*

To simplify notation, we write $\mathcal{H}_\ell = \mathcal{H}_{(0,\ell n_0)}$ and let σ_ℓ , acting in $\mathcal{H}_\ell$, denote the irreducible representation of K whose highest weight is Γ^ℓ .

Corollary 4.7. *The Hilbert space $\mathcal{H}_\ell$ has dimension $\ell(p + 1)$ and is spanned by the vectors $\Gamma_0^{i_0} \Gamma_1^{i_1}, \dots, \Gamma_p^{i_p}$ for $i_0, \dots, i_p \in \{0, \dots, \ell\}$. The representation σ_ℓ acts by the left regular representation.*

As explained at the beginning of this section, the Poincaré–Birkhoff–Witt theorem shows that $\mathcal{H}_\ell = \mathcal{U}(\sum_{\alpha \in \Phi - \mathfrak{k}_\alpha} \Gamma^\ell)$. As Γ is M invariant, the root vectors corresponding to roots in Φ_0 kill Γ^ℓ . As it is a highest weight, all the root vectors associated to positive roots kill Γ^ℓ . By Lemma 4.3, every root in Φ_2^- is a sum of two roots in Φ_1^- . Thus $\mathcal{H}_\ell = \mathcal{U}(\sum_{\alpha \in \Phi_1^-} \Gamma^\ell)$. Thus we obtain the following result.

Lemma 4.8. *Let $\{\Xi_{-\alpha_1}, \dots, \Xi_{-\alpha_p}\}$ be root vectors for the roots $\{-\alpha_1, \dots, -\alpha_p\}$ in Φ_1^- . Then $\mathcal{H}_\ell$ is the span of the vectors $\{\Xi_{-\alpha_{i_1}} \cdots \Xi_{-\alpha_{i_\nu}} \Gamma^\ell : \nu \in \mathbb{N}, 1 \leq i_1, \dots, i_\nu \leq p\}$.*

Now let ψ be a highest weight vector for M_0 associated to the highest weight $\lambda_0 \in \mathfrak{t}_0^{\mathbb{C}*}$. We extend the above discussion to analyse the structure of the representation space $\mathcal{H}_\ell^\psi$ for the representation σ_ℓ^ψ of highest weight $(\psi, \ell\tilde{\gamma})$ for $n = n_\psi + \ell n_0$. It follows from Lemma 4.5 that the following holds.

Theorem 4.9. *The representation σ_ℓ^ψ acts by the right regular representation in $\mathcal{H}_\ell^\psi$, which consists of the elements $f \in C^\infty(K)$ such that:*

- (i) *for all $k \in K$, for all $t \in T_0$, $f(t^{-1}k) = \chi_{\lambda_0}(t)f(k)$;*
- (ii) *for all $k \in K$, for all $r \in \mathbb{R}$, $f(\exp(r(X_0 + \theta X_0))k) = e^{in n_0 r} f(k)$;*
- (iii) *for all $\alpha \in \Phi_0^+$, $X_\alpha^L f = 0$;*
- (iv) *for all $\alpha \in \Phi_1^+$, $X_\alpha^L f = 0$.*

The Hilbert space structure is that of $L^2(K)$.

Note that $\psi\Gamma^\ell$ is the highest weight vector for $\mathcal{H}_\ell^\psi$. As before,

$$\mathcal{H}_\ell^\psi = \mathcal{U}(\sum_{\alpha \in \Phi_0^- \cup \Phi_1^-} \mathfrak{k}_\alpha) \psi \Gamma^\ell.$$

Now the root vectors corresponding to roots in Φ_0 kill Γ , so that

$$\mathcal{H}_\ell^\psi = (\mathcal{U}(\mathfrak{k}^\mathbb{C})\psi) \otimes \mathcal{U}(\sum_{\alpha \in \Phi_1^+} \mathfrak{k}_\alpha) \Gamma^\ell.$$

Now notice that $(\mathcal{U}(\mathfrak{k}^\mathbb{C})\psi)$ is exactly the representation space $\mathcal{H}_{\lambda_0}$ for the representation of K of highest weight λ_0 and that $\mathcal{U}(\sum_{\alpha \in \Phi_1^+} \mathfrak{k}_\alpha) \Gamma^\ell$ is exactly the space $\mathcal{H}_\ell$. Therefore, we get the following result.

Theorem 4.10. *Let $\lambda_0 \in \Lambda_0^+$, $\ell \in \mathbb{N}$ and $\mathcal{H}_\ell^{\lambda_0} = \mathcal{H}_{\lambda_0} \otimes \mathcal{H}_\ell$. Then the left regular representation of K on $\mathcal{H}_\ell^\psi$ is the irreducible representation σ_ℓ^ψ of K of highest weight $(\lambda_0, \ell n_0 \tilde{\gamma})$.*

Remark 4.11. We summarise the information and notation we have developed for the irreducible representations of K . We have $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{a}$ a maximal commutative subalgebra in $\mathfrak{p}$. Our assumption that $\mathfrak{g}$ is rank one means that $\mathfrak{a}^\mathbb{C}$ is one-dimensional: we take it to be the complex span of H , where $|H| = 1$. The group K acts by the adjoint representation on $\mathfrak{p}$, and every element of $\mathfrak{p}$ is conjugate in this action to an element of $\mathfrak{a}$ under the adjoint action. The Lie algebra $\mathfrak{m}$ is the stabiliser of $\mathfrak{a}$ in $\mathfrak{k}$, and the positive restricted roots of $\mathfrak{g}^\mathbb{C}$ for the abelian subalgebra $\mathfrak{a}^\mathbb{C}$ split into $\Phi_1^+ \cup \Phi_2^+$ of those which take value $\lambda(H)$ and $2\lambda(H)$ respectively on H . Inside $\mathfrak{m}^\mathbb{C}$ there is a Cartan subalgebra $\mathfrak{t}_1^\mathbb{C}$. We now choose a nonzero vector X_0 in $\mathfrak{g}_2^\mathbb{C}$ which we may (and do) assume is a positive root vector, and we define a Cartan subalgebra of $\mathfrak{k}$, by $\mathfrak{t}^\mathbb{C} = \mathfrak{t}_0^\mathbb{C} + \mathbb{C}(X_0 + \theta X_0)$, where $\mathfrak{t}_0$ is the set of all elements of $\mathfrak{t}_1^\mathbb{C}$ which commute with $X_0 + \theta X_0$. The choice of X_0 also defines a complex structure on $\mathfrak{g}_{-\lambda}$ via $JX = [X_0, \theta X]$, and hence $\mathfrak{g}_{-\lambda}$ is even dimensional. We showed in Lemma 4.3 that every root in Φ_2^+ is the sum of two roots in Φ_1^+ , and gave an explicit root space decomposition of $\mathfrak{k}^\mathbb{C}$ in Corollary 4.4.

By Cartan–Weyl theory, the set of all dominant integral weights of $\mathfrak{k}^\mathbb{C}$ with respect to $\mathfrak{t}^\mathbb{C}$ is in one-to-one correspondence with the set of irreducible representations of the compact group K . We describe these weights as pairs $(\psi, n\tilde{\gamma})$, where ψ is a dominant integral weight for M_0 , $\tilde{\gamma}$ is the dual vector of $b(X_0 + \theta X_0)$, and $n > n_\psi$. This enables

us to describe the irreducible representations of K as all having the form σ_ℓ^ψ described in Theorem 4.10.

5. The embeddings and the approximation of matrix coefficients

We now relate the representation theories of $\bar{N}M$ as described in Section 3 and of K as described in Section 4. In Section 3 we fixed $\mu \in \mathfrak{g}_{-2\lambda}^*$ with $|\mu| = 1$ and a dominant weight λ_0 for $\mathfrak{m}_\mu$ (corresponding to an irreducible representation of M_μ), and $s > 0$, and constructed the irreducible representation $\rho_{\lambda_0, s, \mu}$ of $\bar{N}M$. In Section 4 (see Remark 4.11) we chose $X_0 \in \mathfrak{g}_{-2\lambda}$ to define the Cartan subalgebra $\mathfrak{t}^C$ of $\mathfrak{k}$ and thence the representation theory of K . We now formally link the representation spaces by setting, for $\mu \in \mathfrak{g}_{-\lambda}^*$, dual to $X_\mu \in \mathfrak{g}_{-2\lambda}$, $X_0 = \frac{X_\mu}{\sqrt{2}|\mu|}$ and $J = J_{X_0}$. With this choice of X_0 , we have $\mathfrak{m}_\mu = \mathfrak{m}_0$ and $\mathfrak{t}_\mu = \mathfrak{t}_0$ (cf. Remark 4.1). Furthermore, we choose $\varphi \in L^2(K)$ to be a highest weight vector of norm 1 for the representation of highest weight λ_0 . In Lemma 4.3 we introduced the notation $\Xi_j (j = 1, \dots, q)$ for root vectors of $\mathfrak{k}$ which also arise from root vectors in $\mathfrak{g}_{-2\lambda}$ which are orthogonal to X_0 . We now let $\Xi_0 = X_0 + \theta X_0$. Recall that $\Xi_0 \Gamma = e^{\mu(\cdot)} \Gamma$.

With this additional notation, we see from Lemma 4.8 that, fixing an order $\Xi_1, \dots, \Xi_p$ for Φ_1^- , for fixed ℓ , $\mathcal{H}_\ell$ is spanned by the vectors $\Xi_0^{q_0} \Xi_1^{q_1} \dots \Xi_p^{q_p} \Gamma^\ell$ where $q_i \in \{0, \dots, \ell\}$ for all i . It follows that we may embed $\mathcal{H}_\ell$ into $\mathcal{H}_{\ell+1}$ by the map $\bar{\Gamma}(F)(k) = \Gamma(k)F(k)$, and it is clear from the definition of Γ that this map is an injection.

The functions in $\mathcal{H}_\ell^\psi$ likewise have a basis of the form $X\psi(m)F(k)$ where, for X a root vector in $\mathfrak{m}_0$, $X\psi$ is a function in $\mathcal{H}_\psi$ and $F \in \mathcal{H}_\ell$. Multiplication by Γ extends to a linear map $\tilde{\Gamma}: \mathcal{H}_\ell^\psi \rightarrow \mathcal{H}_{\ell+1}^\psi$ by $\mathcal{H}_\ell^\psi$, also an injection.

We set $\Psi_\ell(\Xi_1^{q_1} \Xi_2^{q_2} \dots \Xi_p^{q_p} \Gamma^\ell) = z_1^{q_1} z_2^{q_2} \dots z_p^{q_p}$, where $z_1^{q_1} z_2^{q_2} \dots z_p^{q_p}$ is the multinomial defined in Lemma 2.3. We extend the mapping Ψ_ℓ to a linear map from $\mathcal{H}_\ell$ to the Fock space $\mathcal{F}_\mu$.

Similarly, we can define mappings $\Psi_\ell^{\lambda_0}: \mathcal{H}_\ell^{\psi_0} \rightarrow \mathcal{F}_{\lambda_0, \mu}$ where for $\varphi \in \mathcal{H}_{\lambda_0}$, $f \in \mathcal{H}_\ell$, φf is mapped to $\varphi|_M \Psi_\ell(f)$.

Including also the $\mathfrak{g}_{-2\lambda}$ -vector Ξ_0 there is a natural linear mapping Ψ_ℓ of $\mathcal{H}_\ell$ into the Fock space $\mathcal{F}_\mu$ of Section 2, where the basis vectors correspond as follows:

$$\Psi_\ell: \Xi_0^{q_0} \Xi_1^{q_1} \dots \Xi_p^{q_p} \Gamma^\ell \longrightarrow e^{q_0 \mu(\cdot)} z_1^{q_1} z_2^{q_2} \dots z_\ell^{q_\ell}.$$

This map also extends to an injection Ψ_ℓ^ψ of $\mathcal{H}_\ell^\psi$ into the representation space $\mathcal{F}_{\lambda, s, \mu}$ defined in Section 3 by $X\psi(m)F(k) \mapsto X\psi(m)\Psi_\ell F(k)$.

The dominant integral weights λ of $\mathfrak{k}^C$ correspond to characters χ_λ of the maximal torus T (with Lie algebra $\mathfrak{t}^C$) via $\chi_\lambda(\exp(H)) = e^{i\lambda(H)}$. The irreducible representation σ_λ of K corresponding to λ acts by the left regular representation on the space $\mathcal{H}_\lambda$ of $f \in C^\infty(K)$ such that:

- $f(tg) = \chi_\lambda(t)f(g)$ for all $g \in K$, $t \in T$;
- $Xf = 0$ for $X \in \mathfrak{k}_\alpha$ for all $\alpha \in \Phi^+$.

Given $f \in C^\infty(K)$, satisfying the above two conditions, we consider the function $f \circ \pi_\varepsilon$ which we consider as the function $\bar{n}m \mapsto f(\pi_\varepsilon(\bar{n}m))$ on $\bar{N}M$.

Remark 5.1. Notice that if $\bar{n} = (Y, X) \in \bar{N}$,

$$\begin{aligned}
 & f \circ \pi_\varepsilon(\bar{n}m) \\
 &= f(m \exp_K(\sqrt{\varepsilon}(Y + \theta Y) + \varepsilon(X + \theta X))) \\
 &= f(m \exp_K(a\varepsilon(X_0 + \theta X_0) + \sqrt{\varepsilon}(Y + \theta Y) + \varepsilon(X_1 + \theta X_1))), \\
 &= f(\exp_K(\varepsilon a(X_0 + \theta X_0))(m \exp_K(\sqrt{\varepsilon}(Y + \theta Y) + \varepsilon(X_1 + \theta X_1) + O(\varepsilon^{3/2}))), \\
 &= \chi_\lambda(\exp_K(\varepsilon a(X_0 + \theta X_0)))f(m \exp_K(\sqrt{\varepsilon}(Y + \theta Y) + \varepsilon(X_1 + \theta X_1) + O(\varepsilon^{3/2}))) \\
 &= e^{ia\varepsilon\lambda(X_0 + \theta X_0)}f(m \exp_K(\sqrt{\varepsilon}(Y + \theta Y) + \varepsilon(X_1 + \theta X_1) + O(\varepsilon^{3/2})))
 \end{aligned}$$

Thus, if we have $\varepsilon\lambda(X_0 + \theta X_0) = \mu(X_0)$ the last expression converges to

$$e^{ia\mu(X_0)}f(m \exp_K(\sqrt{\varepsilon}(Y + \theta Y) + \varepsilon(X_1 + \theta X_1))),$$

as $\varepsilon \rightarrow 0$. Note that the third equality follows from the left invariance condition above.

Lemma 5.2. Let $f \in \mathcal{H}_\ell^\psi$. Then $g = f \circ \pi_\varepsilon$ satisfies:

- (i) $g(nmt) = \chi_{\lambda_0}(t)g(nm)$;
- (ii) for all $\alpha \in \Phi_0^+$, we have $X_\alpha g((Y, X)m) = 0$;
- (iii) for all $m \in M$, $Y \mapsto g((Y, 0)m)$ is analytic for the complex structure induced by J_{X_μ} .

Proof. (i) and (iii) follow from Theorem 4.9. For (ii), we use Theorem 4.9(iv) to see that for all $\nu \in \Phi_1^+$, we have $X^{(\nu)}g = 0$. By Theorem 4.9(iv), we have for all $\nu \in \Phi_2^+$, $X^{(\nu)}g = 0$ and by Theorem 4.9(ii), we see that $Z_{-\mu}(f \circ \pi_t)(X, 0, m) = 0$. $\square$

It follows from Lemma 5.2 that we can map the spaces $\mathcal{H}_\ell^\psi$ into the spaces $\mathcal{F}_{\lambda_0, \mu}$ as follows: we choose $\Psi_\ell^{\lambda_0} = \Psi_\ell: \mathcal{H}_\ell^\psi \rightarrow \mathcal{F}_{\lambda_0, \mu}$, where $\mu = \ell n_0 \gamma e^{-2t}$ by $\Psi_\ell(g) = g \circ \pi_t(X, 0, m)$. Lemma 5.2 guarantees that the image is in $\mathcal{F}_{\lambda_0, \mu}$. Item (ii) of the next lemma shows that Ψ_ℓ as defined above extends to all of $\mathcal{H}_\ell^\psi$.

Definition 5.3. Let $s \in \mathbb{R}^+$. For $\varepsilon > 0$, let $\ell(\varepsilon) = \lceil \frac{\pi s}{\varepsilon} \rceil$.

Lemma 5.4. Let $s > 0$, $Y \in \mathfrak{g}_{-\lambda}$ and $X \in \mathfrak{g}_{-2\lambda}$. Then:

- (i) $\lim_{\varepsilon \rightarrow 0} \varphi \Gamma^{\ell(\varepsilon)} \circ \pi_\varepsilon(Y, X, m) = \varphi(m) e^{-s\|Y\|^2/4} e^{is\langle X, X_\mu \rangle}$;
- (ii) we have:

$$\lim_{\varepsilon \rightarrow 0} \varphi \Xi_{i_1} \Xi_{i_2} \cdots \Xi_{i_\nu} \Gamma^{\ell(\varepsilon)} \circ \pi_\varepsilon(Y, X, m) = \varphi(m) e^{-s\|Y\|^2/4} X_{i_1} X_{i_2} \cdots X_{i_\nu} e^{is\langle X, X_\mu \rangle};$$

- (iii) the convergence is uniform on compact subsets of $\bar{N}M$.

Proof. (ii) follows from (i) by Remark 5.1, and the convergence in (iii) follows since all derivatives of Γ greater than one vanish.

To prove (i), note first that $\varphi \circ \pi_\varepsilon(Y, X, m) \rightarrow \varphi(m)$. Thus, taking $\varepsilon = \frac{\pi s}{\ell}$, we need to calculate the limit as $\ell \rightarrow \infty$ of $\Gamma^\ell(\pi_\varepsilon(Y, X, m))$. Substituting in the definitions

of π_ε and Γ , this expression becomes:

$$\begin{aligned}\Gamma^\ell(\exp(\sqrt{\varepsilon}(Y + \theta Y) + \varepsilon(X + \theta X))) &= (H, \text{Ad}(\exp(\sqrt{\varepsilon}(Y + \theta Y) + \varepsilon(X + \theta X)))B)^\ell \\ &= (H, e^{\text{ad}(\sqrt{\varepsilon}(Y + \theta Y) + \varepsilon(X + \theta X))}B)^\ell.\end{aligned}$$

Now the last expression is equal to:

$$\left(1 + \left(H, \sqrt{\varepsilon}[Y + \theta Y, B] + \varepsilon[X + \theta X, B] + \frac{1}{2}\varepsilon[Y + \theta Y, [Y + \theta Y, B]]\right) + O(\varepsilon^{\frac{3}{2}})\right)^\ell. \quad (10)$$

We analyse the terms in (10) of order $\sqrt{\varepsilon}$ and ε , using Lemma 5.2. Recall that $B = H - i\lambda(H)(X_0 - \theta X_0)$.

By Lemma 4.5(ii), for each $j = 1, \dots, p$, $[Y_j + \theta Y_j, B] = 2(Y_j - \theta Y_j)$ is in a weight space orthogonal to H . Thus the term of order $\sqrt{\varepsilon}$ vanishes, since each $Y \in \mathfrak{g}_{-\lambda}$ is a linear combination of such vectors.

Now by Lemma 4.5(iv), we similarly have $[X + \theta X, B] = 2(X - \theta X)$ for all X in the span of $\mathfrak{g}_{-2\lambda} \ominus (X_\mu)$. If, on the other hand, X is a multiple of X_μ , one calculates that $[X + \theta X, B] = X - \theta X - 2i\lambda(H)[\theta X, X_\mu]$ and thus $(H, [X + \theta X, B]) = -4i(X, X_0)$ for all $X \in \mathfrak{g}_{-2\lambda}$.

The third type of term to consider is a contribution of order ε from terms of the form $\text{ad}^2(Y + \theta Y)(H)$, where $Y \in \mathfrak{g}_{-\lambda}$. These terms contribute $[Y + \theta Y, [Y + \theta Y, B]] = 2[Y + \theta Y, Y - \theta Y] = 2[\theta Y, Y] = 2\|Y\|^2 H$.

Substituting these three expressions in (10), we see that we need to find the limit, as ℓ tends to infinity, of

$$\left(1 + \frac{s\pi}{\ell}(-4i(X, X_\mu) + 2\|Y\|^2) + O(\ell^{-\frac{3}{2}})\right)^\ell. \quad (11)$$

By elementary calculus, this expression tends to $e^{-s\|Y\|^2 - ir\langle X, X_\mu \rangle}$, as claimed. Note that the convergence is uniform on compacta. $\square$

We now state our main theorem for approximations of matrix coefficients. We fix $\rho_{\lambda_0, s\mu}$, an irreducible representation of $\bar{N}M$. We will approximate $\rho_{\lambda_0, s\mu}$ by the K -representations $\sigma_{\ell(\varepsilon)}^\psi$, composed with the contraction maps π_ε as $\varepsilon \rightarrow 0$. Let $\mathcal{B}_{\lambda_0, \ell} = \Psi_\ell^{\lambda_0}(\mathcal{H}_\ell^\psi)$ be the image of $\mathcal{H}_\ell^\psi$ in $\mathcal{F}_{\lambda_0, \mu}$.

Theorem 5.5. *For $f \in \mathcal{B}_{\lambda_0, \ell_1}$ and $g \in \mathcal{B}_{\lambda_0, \ell_2}$,*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-(r+p)} (\sigma_{\ell(\varepsilon)}^\psi (\pi_\varepsilon(\bar{n}m)) \Psi_{\ell(\varepsilon)}^{-1} f, \Psi_{\ell(\varepsilon)}^{-1} g) = (\rho_{\lambda_0, s\mu}(\bar{n}m) f, g).$$

Proof. The theorem follows from Lemmas 5.2 and 5.4, using the explicit form of the representation $\rho_{\lambda_0, s\mu}$ given in Section 3, and the remark that Lemma 5.4(ii) coincides exactly with the definition of the mapping Ψ_ℓ above. $\square$

Theorem 5.6. *The images $\mathcal{B}_{\lambda_0, \ell}$ in $\mathcal{F}_{\lambda_0, \mu}$ form an increasing family of finite dimensional subspaces whose union is dense. That is:*

(i) *if $\ell_1 \leq \ell_2$ then $\mathcal{B}_{\lambda_0, \ell_1} \subseteq \mathcal{B}_{\lambda_0, \ell_2}$;*

(ii) *$\Psi_\ell^{\lambda_0}$ is injective for all ℓ , and for $f \in \Psi_{\ell_1}^{\lambda_0}(\mathcal{H}_{\ell_1}^\psi)$, we have*

$$\Psi_{\ell_2}^{\lambda_0} \circ (\Psi_{\ell_1}^{\lambda_0})^{-1} f = \Gamma^{\ell_2 - \ell_1} f; \quad (12)$$

(iii) *$\bigcup_{\ell=0}^\infty \mathcal{B}_{\lambda_0, \ell}$ is dense in $\mathcal{F}_{\lambda_0, \mu}$.*

Proof. The operation of multiplication by Γ embeds $\mathcal{H}_\ell$ naturally into the space $\mathcal{H}_{\ell+1}$. In fact, for n fixed, successive multiplication by Γ gives a sequence of injections $\mathcal{H}_n \rightarrow \mathcal{H}_{n+1} \rightarrow \dots \rightarrow \mathcal{H}_{n+k} \rightarrow \dots$. Hence from the definition of Ψ_ℓ , we have $\mathcal{B}_{\lambda_0, \ell_1} \subseteq \mathcal{B}_{\lambda_0, \ell_2}$ for $\ell_1 \leq \ell_2$.

(ii) also follows from the definition of Ψ_ℓ and the above remark.

Finally, notice that by Theorem 5.5, the space $\bigcup_{\ell=\ell_0}^{\infty} \mathcal{B}_{\lambda_0, \ell}$ is an invariant space under the action of $\rho_{\lambda_0, s\mu}$; therefore its closure carries a subrepresentation of $\rho_{\lambda_0, s\mu}$. By irreducibility, this must be equivalent to $\rho_{\lambda_0, s\mu}$ itself, and hence the space $\bigcup_{\ell=\ell_0}^{\infty} \mathcal{B}_{\lambda_0, \ell}$ is dense in the Fock space $\mathcal{F}_{\lambda_0, \mu}$. $\square$

6. Jacobian

Before proving our L^p transference results in the next section, we need to calculate the Jacobian of the map π_ε .

First, we summarise some results on H -type groups from [Cow+98]. Let $X \in \mathfrak{g}_{-\lambda}$ and $Y \in \mathfrak{g}_{-2\lambda}$. The definition of J_X on H -type groups arising from rank one groups is

$$J_X Y = [X, \theta Y].$$

As in Section 1, choose $H \in \mathfrak{a}$ such that $\lambda(H) = 1$. Recall from [Cow+98, Proposition 5.9] the following results (actually, the third statement is [Cow+98, formula (2.6)]).

Lemma 6.1. *Let $Y \in \mathfrak{g}_{-\lambda}$, $X \in \mathfrak{g}_{-2\lambda}$. Then:*

- (i) $[Y, \theta Y] = -|Y|^2 H$;
- (ii) $[X, \theta X] = -2|X|^2 H$;
- (iii) $[Y, J_X Y] = |Y|^2 X$.

Lemma 6.2. *Let $Y \in \mathfrak{g}_{-\lambda}$, $X \in \mathfrak{g}_{-2\lambda}$. Then:*

- (i) $[J_X Y + \theta J_X Y, Y + \theta Y] = |Y|^2 (X + \theta X)$;
- (ii) $[J_X Y + \theta J_X Y, X + \theta X] = -|X|^2 (Y + \theta Y)$.

It follows from this lemma that $\text{ad}(J_X Y + \theta J_X Y)$ acts on the two-dimensional vector space $\{Y + \theta Y, X + \theta X\}$ as $\begin{pmatrix} 0 & -|X|^2 \\ |Y|^2 & 0 \end{pmatrix}$.

We now give the formula for the Jacobian $(\pi_\varepsilon)_{*,e}$ of the contraction map π_ε at the identity of $\bar{N}M$.

Proposition 6.3. *Let $W \in \mathfrak{m}$, $Y \in \mathfrak{g}_{-\lambda}$, $X \in \mathfrak{g}_{-2\lambda}$, and set $a = |Y|$ and $b = |X|$. Let $c = \sqrt{(\varepsilon a^2 + \varepsilon^2 b^2)}$.*

Then

$$(\pi_\varepsilon)_{*,e}(W, Y, X) = (\varepsilon)^{p+q} \prod_{\alpha \in \Phi_1^+} \left(\frac{\sin c/\sqrt{2}}{c/\sqrt{2}} \right)^2 \prod_{\alpha \in \Phi_2^+} \left(\frac{\sin \sqrt{2}c}{\sqrt{2}c} \right)^2.$$

Proof.

$$\pi_\varepsilon(\bar{n}m) = m \exp_K(\sqrt{\varepsilon}(Y + \theta Y) + \varepsilon(X + \theta X)).$$

Therefore we have

$$(\pi_\varepsilon)_{*,e}(W, Y, X) = W + (\exp_K)_{*,0}(\varepsilon^{1/2}(Y + \theta Y) + \varepsilon(X + \theta X)).$$

It follows from the elementary properties of Jacobians that $J\pi_\varepsilon(W, Y, X)$ is equal to $\varepsilon^{(p+q)}\tilde{J}(\varepsilon^{1/2}(Y + \theta Y) + \varepsilon(X + \theta X))$, where $\tilde{J}$ is the Jacobian for the exponential map of K . We have to work out a formula for $\tilde{J}$ restricted to the subalgebra of $\mathfrak{k}$ spanned by the $Y + \theta Y$'s and the $X + \theta X$'s.

Now for a compact group K , the Jacobian of the exponential map is well-known (see [Ada83]). It is the Ad-invariant function on $\mathfrak{k}$ whose restriction to the Cartan subalgebra $\mathfrak{t}$ is given by

$$\prod_{\alpha \in \Phi^+} \left(\frac{\sin \alpha(H)/2}{\alpha(H)/2} \right)^2. \quad (13)$$

A Cartan subalgebra for $\mathfrak{k}$ is given by the span of $\mathfrak{t}_1$, a Cartan subalgebra of $\mathfrak{m}$ and the vector $X_0 + \theta X_0$, where $X_0 \in \mathfrak{g}_{-2\lambda}$. Now M is a subgroup of K , so the values of $\tilde{J}$ are certainly constant on the orbits of Ad_M .

In fact, M acts by isometries on the unit ball of $\mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_{-2\lambda}$, and the Kostant double transitivity theorem (cf. [Cow+98, Theorem 6.2]) says that M acts doubly transitively, i.e. given any two pairs of vectors $Y, Y_1 \in B_1(\mathfrak{g}_{-\lambda})$ and $X, X_1 \in B_1(\mathfrak{g}_{-2\lambda})$ ($B_1(R)$ denotes the elements in R of norm 1), there exists $m \in M$ so that $mY = Y_1$ and $mX = X_1$. Since the action of M commutes with the involution θ , we also have $m(Y + \theta Y) = Y_1 + \theta Y_1$ and $m(X + \theta X) = X_1 + \theta X_1$.

Now, choose and fix two unit vectors $Y_0 \in \mathfrak{g}_{-\lambda}$ and $X_0 \in \mathfrak{g}_{-2\lambda}$. Choose an element of M which maps Y to aY_0 and X to bX_0 . Then

$$\tilde{J}(\varepsilon^{1/2}(Y + \theta Y) + \varepsilon(X + \theta X)) = \tilde{J}(\varepsilon^{1/2}a(Y_0 + \theta Y_0) + \varepsilon b(X_0 + \theta X_0)). \quad (14)$$

Now we want to find an element of K which maps (under Ad) the argument of the right-hand side of (14) to an element of $\mathfrak{t}$. We claim that for a suitable value of $t \in \mathbb{R}$, $\exp t(J_{X_0}Y_0 + \theta J_{X_0}Y_0)$ is such an element. Indeed, on the two-dimensional vector space spanned by $Y_0 + \theta Y_0$ and $X_0 + \theta X_0$, we have

$$\text{Ad}(\exp t(J_{X_0}Y_0 + \theta J_{X_0}Y_0)) = \exp t \begin{pmatrix} 0 & -|X_0|^2 \\ |Y_0|^2 & 0 \end{pmatrix}, \quad (15)$$

and hence we wish to choose t such that

$$\exp t \begin{pmatrix} 0 & -|X_0|^2 \\ |Y_0|^2 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon^{1/2}a \\ \varepsilon b \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix} \quad (16)$$

for some c .

Elementary calculations show that $c = (\varepsilon a^2 + \varepsilon^2 b^2)^{1/2}$ as $|X_0| = |Y_0| = 1$. We have shown that $\tilde{J}(\varepsilon^{1/2}(Y + \theta Y) + \varepsilon(X + \theta X)) = \tilde{J}(c(X_0 + \theta X_0))$. By (13), this is in turn equal to

$$\prod_{\alpha \in \Phi^+} \left(\frac{\sin c\alpha(X_0 + \theta X_0)/2}{c\alpha(X_0 + \theta X_0)/2} \right)^2.$$

Now recall that the roots are divided into two types: Φ_1^+ , which take the value 1 on $\frac{(X_0 + \theta X_0)}{\sqrt{2}}$, and Φ_2^+ , which take the value 2 on this element.

Thus, the above product is:

$$\prod_{\alpha \in \Phi_1^+} \left(\frac{\sin c/\sqrt{2}}{c/\sqrt{2}} \right)^2 \prod_{\alpha \in \Phi_2^+} \left(\frac{\sin \sqrt{2}c}{\sqrt{2}c} \right)^2.$$

This completes the proof. $\square$

7. The de Leeuw theorems

As an application of Theorem 5.5, we give a proof of an analogue of de Leeuw's theorem [Lee65] on multipliers of L^p , $1 < p < \infty$. In the context of $\mathbb{R}$ and $\mathbb{T}$, one defines a function ϕ on the Pontrjagin dual $\widehat{\mathbb{R}} \simeq \mathbb{R}$ to be a multiplier of $L^p(\mathbb{R})$ if whenever $f \in L^p(\mathbb{R})$, there is a function $g \in L^p(\mathbb{R})$ so that the Fourier transform

$$\widehat{g}(\xi) = \phi(\xi)\widehat{f}(\xi), \quad \text{for a.e. } \xi \in \mathbb{R}.$$

By an argument involving the closed graph theorem, the function $g \in L^p$ can be seen to be uniquely-defined, and the multiplier $M_\phi: f \mapsto g$ to be a bounded linear map on L^p with multiplier norm the operator norm $\|M_\phi\| = \sup_{f \in L^p(G)} \frac{\|M_\phi f\|_p}{\|f\|_p}$.

Similarly a function ψ on the Pontrjagin dual $\widehat{\mathbb{T}} \simeq \mathbb{Z}$ is a Fourier multiplier of $L^p(\mathbb{T})$ if for all $f \in L^p(\mathbb{T})$ there exists $g \in L^p(\mathbb{T})$ such that the Fourier series

$$\widehat{g}(n) = \psi(n)\widehat{f}(n), \quad \text{for all } n \in \mathbb{Z}.$$

For each $\varepsilon > 0$, the multiplier ϕ of $L^p(\mathbb{R})$ can be restricted to $\mathbb{T}$ by defining $\phi^{(\varepsilon)}(n) = \phi(\varepsilon n)$. De Leeuw's two theorems then say that:

- (i) if ϕ is a multiplier of $L^p(\mathbb{R})$, then each of the $\phi^{(\varepsilon)}$ is a multiplier of $L^p(\mathbb{T})$, with $\|\phi^{(\varepsilon)}\|_p \leq \|\phi\|_p$; and
- (ii) if every $\phi^{(\varepsilon)}$ is a multiplier of $L^p(\mathbb{T})$ with $\limsup_{\varepsilon \rightarrow 0} \|\phi^{(\varepsilon)}\|_p = C < \infty$, then ϕ is a multiplier of $L^p(\mathbb{R})$ with $\|\phi\|_p \leq C$.

Our aim is to establish similar theorems where $\mathbb{R}$ is replaced by $\overline{NM}$ and $\mathbb{T}$ is replaced by K . The first step is to define multipliers of L^p in this setting.

Definition 7.1. Let G be a locally compact group with left Haar measure dx , let $L^p(G)$ be defined as usual, and let Σ be the dual of G , a maximal set of pairwise inequivalent unitary irreducible representations of G . As usual, we define, for each $\sigma \in \Sigma$ and for $f \in L^1(G)$,

$$\widehat{f}(\sigma) = \int_G f(x)\sigma(x^{-1})dx$$

to be the Fourier transform of f at σ .

This expression is to be thought of as the Bochner integral, so that if σ acts in the Hilbert space $\mathcal{H}_\sigma$, then $\widehat{f}(\sigma)$ is an element of $\mathcal{B}(\mathcal{H}_\sigma)$, the set of bounded operators. It is easy to see that $\widehat{f * g}(\sigma) = \widehat{f}(\sigma)\widehat{g}(\sigma)$ and that other familiar properties of the Fourier transform hold.

We can think of $\widehat{f}$ as a kind of “section” of the “bundle” with total space $\bigcup \mathcal{B}(\mathcal{H}_\sigma)$ and “base space” Σ . By properties of the Bochner integral, we easily see that

$\|\widehat{f}(\sigma)\|_{\mathcal{B}(\mathcal{H}_\sigma)} \leq \|f\|_1$ for all $\sigma \in \Sigma$. As in [HR70], we denote the set of all sections Φ of our bundle satisfying $\|\Phi(\sigma)\|_{\text{op}} \leq C$ for some constant C by $\mathcal{L}^\infty(\Sigma)$.

Now given $\Phi \in \mathcal{L}^\infty(\Sigma)$, we say that Φ is a *Fourier multiplier of L^p* if for every $f \in L^p(G)$, there exists $g \in L^p(G)$ such that $\Phi(\sigma)\widehat{f}(\sigma) = \widehat{g}(\sigma)$. We denote g by Φf . If Φ is a multiplier of $L^p(G)$, its norm is $\|\Phi\|_p = \sup_{f \in L^p(G)} \frac{\|\Phi f\|_p}{\|f\|_p}$.

We are going to apply this definition to the duals of $\widehat{NM}$ and K as calculated in Sections 3 and 4 respectively.

Thus an element Φ of $\mathcal{L}^\infty(\widehat{NM})$ defines for each λ_0, μ , a bounded operator on the Hilbert space $\mathcal{F}_{\lambda_0, \mu}$. Similarly, an element Ψ of $\mathcal{L}^\infty(\widehat{K})$ defines, for each pair λ_0, ℓ , a bounded operator on $\mathcal{H}_\ell^{\lambda_0}$; in fact, as these are finite dimensional spaces, the bounded operators are simply finite matrices once a basis has been chosen for the representation space $\mathcal{H}_\ell^{\lambda_0}$.

Given an element Φ in $\mathcal{L}^\infty(\widehat{NM})$, we can use the contraction map, together with the embeddings of matrix entries, to define a family of restrictions $\Phi^{(\varepsilon)} \in \mathcal{L}^\infty(\widehat{K})$. Recall that $\ell(\varepsilon) = \lceil \frac{\pi s}{\varepsilon} \rceil$.

Definition 7.2. Let $\Phi \in \mathcal{L}^\infty(\widehat{NM})$ and let $\varepsilon > 0$. Define

$$\Phi^{(\varepsilon)}(\sigma_{\ell(\varepsilon)}^\psi) = \Psi_\varepsilon^{-1} \Phi(\rho_{\lambda_0, s\mu}) \Psi_\varepsilon.$$

Lemma 7.3. $\Phi^{(\varepsilon)}$ is an element of $\mathcal{L}^\infty(\widehat{K})$, and $\|\Phi^{(\varepsilon)}\|_\infty \leq \|\Phi\|_\infty$.

Proof. Recall that $\sigma_{\ell(\varepsilon)}^\psi$ is the irreducible representation of K of highest weight $\lambda_0 + \ell(\varepsilon)n_0\tilde{\gamma}$. So given an irreducible representation σ of K of highest weight $\lambda = \lambda_0 + \ell(n_0\tilde{\gamma})$, we recall that $\tilde{\gamma}(b(X_0 + \theta X_0)) = 1$, where $X_0 \in \mathfrak{g}_{-2\lambda}$ with $|X_0| = 1$. Define $\mu \in \mathfrak{g}_{-2\lambda}^*$ so that $\mu(X_0) = \frac{1}{\sqrt{2}}$ and $\mu(X) = 0$ if $X \perp X_0$. We now put $s = \frac{\varepsilon\ell}{\pi}$. Theorem 5.5 tells us that the projection of $\rho_{\lambda_0, s\mu}$ on $\mathcal{F}_{\lambda_0, s\mu}$ is the weak*-limit of the representations σ_ℓ^ψ , acting in the spaces $\mathcal{H}_\ell^\psi$, for $\ell = \ell(\varepsilon)$ as $\varepsilon \rightarrow 0$. Our assumption is that the operator norms of the operators $\Phi(\rho_{\lambda_0, s\mu})$ are uniformly bounded over all λ_0, s , and μ , and we can conclude that the operator norms of the operators $\Phi^{(\varepsilon)}(\sigma_\ell)$ are uniformly bounded by the same constant. $\square$

We can now state the main theorem of this section.

Theorem 7.4. Suppose $1 < p < \infty$. Suppose that for each $\varepsilon > 0$, $\Phi^{(\varepsilon)}$ is a Fourier multiplier of $L^p(K)$, and that $\liminf_{\varepsilon \rightarrow 0} \|\Phi^{(\varepsilon)}\|_p \leq C < \infty$.

Then Φ is a multiplier of $L^p(\widehat{NM})$ with $\|\Phi\|_p \leq C$.

Proof. The overall plan of the proof is to show that there exists $C > 0$ such that for $f \in L^p(\widehat{NM})$ and $g \in L^{p'}(\widehat{NM})$,

$$|\langle \Phi f, g \rangle| \leq C \|f\|_p \|g\|_{p'}. \quad (17)$$

(Note we are using the notation $\frac{1}{p} + \frac{1}{p'} = 1$, so that $L^{p'}$ is the dual of L^p .)

The aim is to change variables in the integral $\langle \Phi f, g \rangle$, using the contraction map, to convert it to an integral over K , involving $\Phi^{(\epsilon)}$, and then to use the K -boundedness of the maps $\Phi^{(\epsilon)}$.

However, before doing this, we wish to restrict f and g to belong to a particular dense subset of $L^p(\bar{N}M)$ and $L^{p'}(\bar{N}M)$ respectively.

Definition 7.5. Let $f \in C^\infty(\bar{N}M)$. We say that f is *band-limited* if the following conditions hold:

- (i) there exists $R_0 > 0$ such that $\widehat{f}(\rho_{\lambda_0, s\mu}) = 0$ whenever $|\lambda_0| \geq R_0$ and $s \geq R_0$;
- (ii) there exists $M_0 > 0$ such that for all $|\lambda_0| \leq R_0$, and for all $s \leq R_0$, $\widehat{f}(\rho_{\lambda_0, s\mu})$ belongs to $\mathcal{B}_{\lambda_0, M_0}$.

Lemma 7.6. Let $1 < p < \infty$. The set of band-limited functions is dense in $L^p(\bar{N}M)$.

Proof. The set of band limited functions is closed under addition and scalar multiplication. Furthermore it separates points of $\bar{N}M$. Therefore by the Stone–Weierstrass theorem, it is dense in $C_0(\bar{N}M)$ with the uniform metric. Now $C_0(\bar{N}M) \cap L^p(\bar{N}M)$ is dense in L^p . It follows that the band-limited functions are dense in $L^p(\bar{N}M)$. $\square$

Thus we may suppose that the functions f, g in (17) are band limited. Let A_0 be a compact M invariant subset of $\bar{N}M$ such that $\int_{A_0^c} |\Phi(f)|^p dx < \varepsilon$ and $\int_{A_0^c} |g|^{p'} dx < \varepsilon$: we may assume without loss of generality also that $A_0 = A_0^{-1}$.

If this is the case, then the expression on the left-hand side, $\langle \Phi f, g \rangle$, differs from $\int_{A_0} (\Phi(f)(x)\bar{g}(x^{-1})) dx$ by at most ε .

Now, by (3) we may choose ε small enough that π_ε is a diffeomorphism on A_0 . Making the change of variables $u = \pi_\varepsilon(x)$, we may use Proposition 6.3 to see that

$$\begin{aligned}
 & \left| \int_{A_0} \Phi(f)(x)\bar{g}(x^{-1}) dx \right| \\
 &= \varepsilon^{p+q} \left| \int_K \Phi(f) \circ \pi_\varepsilon^{-1}(u) \bar{g} \circ \pi_\varepsilon^{-1}(u^{-1}) \prod_{\alpha \in \Phi_1^+} \left(\frac{\sin c'/2}{c'/2} \right)^2 \prod_{\alpha \in \Phi_2^+} \left(\frac{\sin c'}{c'} \right)^2 du \right| \\
 &= \varepsilon^{p+q} \left| \int_K \Phi^\varepsilon(f \circ \pi_\varepsilon^{-1})(u) \bar{g} \circ \pi_\varepsilon^{-1}(u^{-1}) \prod_{\alpha \in \Phi_1^+} \left(\frac{\sin c'/2}{c'/2} \right)^2 \prod_{\alpha \in \Phi_2^+} \left(\frac{\sin c'}{c'} \right)^2 du \right| \\
 &\leq C \left(\int_K |(f \circ \pi_\varepsilon^{-1})(u)|^p \varepsilon^{p+q} du \right)^{\frac{1}{p}} \left(\int_K |(g \circ \pi_\varepsilon^{-1})(u)|^{p'} \varepsilon^{p+q} du \right)^{\frac{1}{p'}} \\
 &\leq C \left(\int_{A_0} |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{A_0} |g(x)|^{p'} dx \right)^{\frac{1}{p'}} \\
 &\leq C \|f\|_p \|g\|_{p'}.
 \end{aligned}$$

Here $c'(X, Z) = \sqrt{|X|^2 + \varepsilon|Z|^2}$. It follows that $\Phi(f)$ belongs to L^p . This completes the proof. $\square$

Acknowledgements

The authors would like to thank Fulvio Ricci for his support and encouragement throughout this work. We also owe a debt of gratitude to the referee, whose careful reading of our original submission has resulted in a considerably improved manuscript.

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Received October 8, 2024

Accepted December 4, 2025

Published online October 1, 2026

In final form October 2, 2026