Geometry of the Borel – de Siebenthal Discrete Series

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Abstract. Let G_0 be a connected, simply connected real simple Lie group. Suppose that G_0 has a compact Cartan subgroup T_0, so it has discrete series representations. Relative to T_0 there are several distinguished positive root systems Δ^+ for which there is a unique noncompact simple root ν, the "Borel – de Siebenthal system". There is a lot of fascinating geometry associated to the corresponding "Borel – de Siebenthal discrete series" representations of G_0. In this paper we explore some of those geometric aspects and we work out the K_0–spectra of the Borel – de Siebenthal discrete series representations. This has already been carried out in detail for the case where the associated symmetric space G_0/K_0 is of hermitian type, i.e. where ν has coefficient 1 in the maximal root μ, so we assume that the group G_0 is not of hermitian type, in other words that ν has coefficient 2 in μ.

Several authors have studied the case where G_0/K_0 is a quaternionic symmetric space and the inducing holomorphic vector bundle is a line bundle. That is the case where μ is orthogonal to the compact simple roots and the inducing representation is 1–dimensional.
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1. Introduction

One of Harish–Chandra's great achievements was the existence theorem for discrete series representations of a semisimple Lie group. He characterized the groups with discrete series representations by the equal rank condition, he found the explicit formulae on the regular elliptic set for the characters of the discrete series, and he showed that those formulae specify the characters. At the same time (and as a main motivation) he was able to explicitly construct a particularly simple series, the holomorphic discrete series, for those groups where the corresponding Riemannian symmetric space is a bounded symmetric domain in a complex Euclidean space.

For the other discrete series representations however, the actual construction has remained less explicit, although there are several beautiful realizations.

In this paper we initiate the study of a certain family, the so-called Borel – de Siebenthal discrete series, from a point of view as close as possible to that of Harish–Chandra for the holomorphic discrete series. This is motivated in part by the work of Gross and Wallach for the scalar case of the quaternionic discrete series [GW]. As in that case we obtain in particular the admissibility of the series for a small subgroup of the maximal compact subgroup. At the same time we discover a rather appealing geometry for the coadjoint orbit that one wants to attach to the discrete series in question. In particular we give a detailed classification of the possible structures of such orbits in terms of explicit prehomogeneous vector spaces with relative invariants. We feel these deserve attention in their own right; and while we do give the construction for the Borel – de Siebenthal discrete series here, including the explicit K_0-types and the (important) admissibility for a small subgroup K_1 of K_0, we defer further analysis of continuations of the series to a sequel to this paper. In particular, we shall then elucidate the role of the relative invariants in constructing rather singular representations in the continuation of the series. We mention that K_1–admissibility was first studied by Kobayashi in [Kob1] and [Kob2] (also see [Kob3]), and much of this has been carried out in the quaternion line bundle case in [GW]. We also mention the papers [Kn2] and [Kn3] treating such questions for the indefinite orthogonal and symplectic groups; here methods from [GW] are used, and the connection to the continuation of unitary modules in the sense of Vogan (with criteria for good and fair range of unitarity of cohomologically induced modules) is made clear. Our approach seeks to employ analytic methods and the geometry of the orbits, and in particular to use reproducing kernels, see e.g. [WaW].

Several questions concerning discrete series representations may be resolved by our methods, for example the question of finding admissible branching laws, where one obtains direct sum decompositions with finite multiplicities. By applying admissibility of K_1 such results may be obtained in complete analogy with what happens for holomorphic discrete series representations.

In this paper we give a complete description of the geometry of the elliptic coadjoint orbits corresponding to the Borel – de Siebenthal discrete series. They are open G_0–orbits in certain complex flag manifolds and we give precise results on their maximal compact subvarieties (which are compact hermitian symmetric spaces) and the holomorphic normal bundles to those subvarieties. We use this structural information to give a concrete geometric construction of the representations in this series, including the structure of the K_0–types. Our construction of the K_0–types provides an analogue of the K_0–type decomposition of holomorphic discrete series representations.

The quaternionic discrete series, studied by Gross and Wallach [GW] in the line bundle case, is the special case of the Borel – de Siebenthal discrete series, where the maximal root is compact and is orthogonal to all the other compact positive roots, or equivalently (see [W0]) where K_0 has a local direct factor isomorphic to $Sp(1)$. While each complex simple Lie group G has exactly one noncompact

real form G_0 that has quaternionic discrete series representations, every real simple Lie group G_0 with rank $G_0 = \operatorname{rank} K_0$ has either holomorphic discrete series representations or Borel – de Siebenthal discrete series representations. Thus the Borel – de Siebenthal discrete series is the natural extension of the holomorphic discrete series.

Our geometric approach allows us to extend several results from [GW]. While Gross and Wallach constructed quaternionic discrete series representations on Dolbeault cohomology spaces of holomorphic line bundles, we also allow vector bundles, and we treat a much larger class of discrete series. Our construction also provides good concrete examples of minimal cohomology degree realizations of discrete series representations in the sense of Kostant [Kos3].

Our basic tool is complex differential geometry and the associated cohomology groups. An important component of this is a collection of basic spectral sequence arguments, already implicit in the paper [S1]. See also [S2], [S3] and [W4]. Here we make use of some technical results of M. Eastwood and the second named author from [EW] for some crucial identifications of duals of finite dimensional representations of reductive Lie groups, in particular for keeping track of the action of the center in terms of the highest weights and the Dynkin diagrams.

Our results include a careful collection of the data attached to the orbits in question, and an explicit formula for the K_0–types in the Harish–Chandra module corresponding to the discrete series in question. As a by–product we find two natural sets of strongly orthogonal roots, one corresponding to the hermitian symmetric space K_0/L_0 and the other corresponding to the riemannian symmetric space G_0/K_0. They fit together to realize the orbit $G_0(z_0) = G_0/L_0$ as a kind of Siegel domain of Type II. This should provide useful coordinates for explicit calculations of the elements in the Harish–Chandra module.

In Section 2 we work out the general structure of the complex manifold $D = G_0(z_0) \cong G_0/L_0$. We describe the action of $\mathfrak{l}_0$ on the tangent and normal spaces to the maximal compact subvariety $Y = K_0(z_0) \cong K_0/L_0$, and on their duals. then we discuss a negativity condition that is crucial to the realization of our discrete series representations.

In Section 3 we list all instances of simple Lie algebras $\mathfrak{g}_0$ corresponding to Borel – de Siebenthal root orders. Setting aside the well–understood hermitian symmetric cases, we work out the precise structure of the algebras $\mathfrak{k}_0$ and $\mathfrak{l}_0$ and the parts of the complexified Lie algebra $\mathfrak{g}$ that correspond to the holomorphic tangent space of Y and the holomorphic normal space of Y in D, including the representations of $\mathfrak{l}_0$ on those two spaces. In each case this allows explicit parameterization of the Borel – de Siebenthal discrete series.

In Section 4 we consider the prehomogeneous space $(L, \mathfrak{u}_1)$ where $\mathfrak{u}_1 \subset \mathfrak{g}$ represents the holomorphic normal space to Y in D. There we describe the algebra of relative invariants, using our knowledge of the representation of L on the symmetric algebra $S(\mathfrak{u}_1)$. In most cases we can be explicit, but in some we must rely on general results of Sato and Kimura [SK]. These invariants are (in addition

to being interesting in themselves) relevant for the next step of understanding the analytic continuation of the discrete series; here the ring of regular functions on the zero set of an invariant will correspond to a module in this continuation. We intend to follow this idea in a sequel to this paper.

In Section 5 we assemble our preparations and work out the precise vanishing and negativity conditions to be used in describing the geometry of the Borel – de Siebenthal discrete series representations. Then in Section 6 we apply those results to give a precise description of the K_0–spectrum of the Borel – de Siebenthal discrete series representations. While an explicit general formula is not available here (just as it is not available for holomorphic discrete series representations), we have all cases listed in Section 3, and we give enough information so that in any particular case one can write out the K_0–spectrum with the aid of the LiE computer program [LiE]. Our main result here, which is the main result of the paper, is Theorem 6.8. Section 6 also contains examples where we compare our results and parameterizations with those in earlier work of Gross, Wallach and Knapp. Finally, in Section 6, we work out two examples in which we compare the negativity condition for discrete series with the corresponding condition for individual K_0–types, indicating a relatively accessible area in our planned analytic continuation of the Borel – de Siebenthal discrete series.

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2.   Notation and the Basic Fibration

In general we use capital Latin letters for Lie groups with subscript $_0$ for real groups and no subscript for complexifications. We use the corresponding small Gothic letters for Lie algebras, again with subscript $_0$ for real Lie algebras and no subscript for complexifications. Our basic objects are a connected simply connected simple real Lie group G_0, its Lie algebra $\mathfrak{g}_0$, the complexification G of G_0, and the Lie algebra $\mathfrak{g}$ of G. Here G is a connected simply connected complex Lie group and the inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}$ defines a homomorphism $G_0 \to G$ with discrete central kernel.

When we omit a subscript $_0$ where there had been one before, we mean complexification.

Fix a Cartan involution θ of G_0 and $\mathfrak{g}_0$. The fixed point set $K_0 = G_0^\theta$ is a maximal compactly embedded subgroup of G_0. As usual, we decompose $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ into (± 1)–eigenspaces of θ, where $\mathfrak{k}_0$ (resp. $\mathfrak{k}$) is the Lie algebra of K_0 (resp. K). We now make two assumptions:

(1) rank G_0 = rank K_0, and
(2) the symmetric space $S_0 := G_0/K_0$ is not of hermitian type.   (2.1)

In particular K_0 is a maximal compact subgroup of G_0. Both G_0 and K_0 are simply connected semisimple groups with finite center.

Fix a maximal torus $T_0 \subset K_0$. Then T_0 is a compact Cartan subgroup of G_0, and a celebrated theorem of Harish–Chandra says that G_0 has discrete series representations.

The construction of Borel and de Siebenthal [BoS] provides a positive root system $\Delta^+ = \Delta_G^+$ for $(\mathfrak{g}, \mathfrak{t})$ such that the associated simple root system $\Psi = \Psi_G$ contains just one noncompact root, and such that this root occurs with multiplicity 1 or 2 as a summand of the maximal root.

$$\begin{aligned} &\Psi = \{\psi_1, \ldots, \psi_\ell\} \text{ (Bourbaki root order) and} \\ &\nu \in \Psi \text{ is the noncompact simple root.} \end{aligned} \qquad (2.2)$$

Every root $\alpha \in \Delta^+$ has expression $\alpha = \sum n_i(\alpha)\psi_i$. Since we have excluded the hermitian case, the coefficient of ν in the maximal root μ is 2. Further, a root is compact just when the coefficient of ν in its expansion is 0 or ± 2, noncompact just when that coefficient is ± 1. Also, $(\Psi \setminus \{\nu\}) \cup \{-\mu\}$ is a simple root system for $(\mathfrak{k}, \mathfrak{t})$. Grading by the coefficient n_ν of ν we have a parabolic subalgebra of $\mathfrak{g}$ given by

$$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}_-, \text{ reductive part } \mathfrak{l} = \mathfrak{t} + \sum_{n_\nu = 0} \mathfrak{g}_\alpha \text{ and nilradical } \mathfrak{u}_- = \mathfrak{u}_{-2} + \mathfrak{u}_{-1} \quad (2.3)$$

where $\mathfrak{u}_i = \sum_{n_\nu = i} \mathfrak{g}_\alpha$. The opposite parabolic is $\mathfrak{q}^{opp} = \mathfrak{l} + \mathfrak{u}_+$ where $\mathfrak{u}_+ = \mathfrak{u}_1 + \mathfrak{u}_2$. Note that

$$\mathfrak{s} = \mathfrak{u}_{-1} + \mathfrak{u}_1, \text{ so } \mathfrak{q} \cap \mathfrak{s} = \mathfrak{u}_{-1}, \text{ and } \mathfrak{k} = \mathfrak{u}_{-2} + \mathfrak{l} + \mathfrak{u}_2. \qquad (2.4)$$

(The multiplicity condition on ν is automatic for the classical structures, but for F_4, E_6, E_7 and E_8 there are simple roots ψ of multiplicity 4, and for E_8 there is one of multiplicity 6, in the maximal root. Applying [BoS] in the cases where the maximal root is compact, even multiplicities $2r$ lead to noncompact simple Lie algebras $\mathfrak{g}$ such that the compact roots are those in which the coefficient of ν is even in the expansion as a linear combination of simple roots, and the noncompact roots are those in which the coefficient is odd. There if $r > 1$ the root order is not a Borel – de Siebenthal order. In the cases where the maximal root is noncompact and ν has multiplicity > 1 the situation is more complicated.)

On the group level, we have the parabolic subgroup $Q \subset G$ where Q has Lie algebra $\mathfrak{q}$. The group Q has Chevalley semidirect product decomposition LU_- where L is the reductive component and U_- is the unipotent radical. The inclusion $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}$ defines a map $p : G_0 \to G$. Note that $pG_0 \cap Q$ is a real form L_0 of L and that $p^{-1}L_0$ is the centralizer in K_0 of a circle subgroup of T_0. ¿From this point on, we will drop the notation p and p^{-1} to keep the notation from becoming too complicated. Thus the parabolic Q defines a complex flag manifold $Z = G/Q$, say with base point $z_0 = 1Q$, and and open orbit $D = G_0(z_0) \cong G_0/L_0$. The complex manifold D has maximal compact subvariety $Y = K_0(z_0) \cong K_0/L_0$, which is a smaller complex flag manifold $K/(K \cap Q)$.

Our choice of signs in (2.3) is such that

$$
\begin{aligned}
&\mathfrak{u}_+ \text{ is the holomorphic tangent space of } D \text{ at } z_0, \\
&\mathfrak{u}_2 \text{ is the holomorphic tangent space of } Y \text{ at } z_0, \text{ and} \\
&\mathfrak{u}_1 \text{ is the holomorphic normal space of } Y \text{ in } D \text{ at } z_0.
\end{aligned}
\tag{2.5}
$$

Since G_0/K_0 is irreducible but not hermitian we know that the action of $\mathfrak{k}_0$ on $\mathfrak{s}_0$ is absolutely irreducible. Thus the action of $\mathfrak{k}$ on $\mathfrak{s}$ is irreducible. From [W2, Theorem 8.13.3] we know that the action of $\mathfrak{l}$ on each $\mathfrak{u}_i$ is irreducible. It will be convenient to have the notation

$$
\tau_i : \text{ representation of } L \text{ on the vector space } \mathfrak{u}_i. \tag{2.6}
$$

The contragredient (dual) of τ_i is $\tau_i^* = \tau_{-i}$. Some obvious highest or lowest weight spaces of the τ_i are given by

$$
\begin{aligned}
&\tau_2 \text{ has highest weight space } \mathfrak{g}_\mu \text{ and } \tau_{-2} \text{ has lowest weight space } \mathfrak{g}_{-\mu}, \\
&\tau_1 \text{ has lowest weight space } \mathfrak{g}_\nu \text{ and } \tau_{-1} \text{ has highest weight space } \mathfrak{g}_{-\nu}.
\end{aligned}
\tag{2.7}
$$

Note that the degree $\deg \tau_i = \dim_{\mathbb{C}} \mathfrak{u}_i$. If $i \neq 0$ it is the number of roots α such that $n_\nu(\alpha) = i$.

The basic tool in this paper is the real analytic fibration

$$
D \to S_0 \text{ with fiber } Y, \text{ in other words } G_0/L_0 \to G_0/K_0 \text{ with fiber } K_0/L_0. \tag{2.8}
$$

The structure of the holomorphic tangent bundle and the holomorphic normal bundle to Y in D is given by (2.5), (2.6) and (2.7). In the next section we will make this explicit. The fibration (2.8) was first considered by W. Schmid in [S1] and [S2] for a related situation in which $L_0 = T_0$, and then somewhat later by R. O. Wells and one of us [WeW] without that restriction. A much more general setting, which drops the compactness assumption on L_0, is that of the double fibration transform (see [FHW] and the references there), where S_0 is replaced by a complexification $S \subset G/K$. Specialization of the double fibration transform to the Borel – de Siebenthal setting is carried out in [EW].

The simple root system $\Psi = \{\psi_1, \ldots, \psi_\ell\}$ of $\mathfrak{g}$ defines the system

$$
\Xi = \{\xi_1, \ldots, \xi_\ell\} \text{ where } \frac{2\langle \xi_i, \psi_j \rangle}{\langle \psi_j, \psi_j \rangle} = \delta_{i,j} \tag{2.9}
$$

of fundamental simple weights. Let γ be the highest weight of an irreducible representation of L_0. For our discussion of the Borel – de Siebenthal discrete series in Section 5 we will need to know exactly when $\langle \gamma + \rho_{\mathfrak{g}}, \alpha \rangle < 0$ for all positive complementary roots α (roots α that are not roots of $\mathfrak{l}$) where $\rho_{\mathfrak{g}}$ denotes half the sum of the positive roots of $\mathfrak{g}$. The condition is Theorem 2.12 below.

Define ν^* by $\langle \nu^*, \psi_j \rangle = 0$ for $\psi_j \neq \nu$ and $2\langle \nu^*, \nu \rangle = \langle \nu, \nu \rangle$ (i.e. the fundamental weight dual to ν). Then $\gamma \in i\mathfrak{t}_0^*$ decomposes as

$$
\gamma = \gamma_0 + t\nu^* \text{ where } \langle \gamma_0, \nu \rangle = 0 \text{ and } t \in \mathbb{R}. \tag{2.10}
$$

Define $\Delta_i = \{\alpha \in \Delta_G \mid n_\nu(\alpha) = i\}$, so $\mathfrak{u}_i = \sum_{\alpha \in \Delta_i} \mathfrak{g}_\alpha$ for $i \in \{\pm 1, \pm 2\}$. Thus the positive root system decomposes as $\Delta^+ = (\Delta_0 \cap \Delta^+) \cup \Delta_1 \cup \Delta_2$. The highest weight of τ_2, representation of $\mathfrak{l}$ on $\mathfrak{u}_2 = \sum_{\Delta_2} \mathfrak{g}_\alpha$ is μ. If we subtract a positive combination of roots of $\Psi \setminus \{\nu\}$ from μ we decrease the inner product with $\gamma + \rho_{\mathfrak{g}}$. Thus

$$\langle \gamma + \rho_{\mathfrak{g}}, \alpha \rangle < 0 \text{ for all } \alpha \in \Delta_2 \text{ if and only if } \langle \gamma + \rho_{\mathfrak{g}}, \mu \rangle < 0. \qquad (2.11a)$$

The highest weight of τ_{-1} is $-\nu$, so τ_1 has highest weight $w_{\mathfrak{l}}^0(\nu)$ where $w_{\mathfrak{l}}^0$ is the longest element of the Weyl group of $\mathfrak{l}$. Thus

$$\langle \gamma + \rho_{\mathfrak{g}}, \alpha \rangle < 0 \text{ for all } \alpha \in \Delta_1 \text{ if and only if } \langle \gamma + \rho_{\mathfrak{g}}, w_{\mathfrak{l}}^0(\nu) \rangle < 0. \qquad (2.11b)$$

As ν^* is orthogonal to the roots of $\mathfrak{l}$ it is fixed by the inverse of $w_{\mathfrak{l}}^0$, so $\langle \nu^*, w_{\mathfrak{l}}^0(\nu) \rangle = \langle \nu^*, \nu \rangle = 1$. Using the decomposition (2.10), and combining $(2.11a)$ and $(2.11b)$, we have

Theorem 2.12. The following conditions are equivalent.
    1. The inequality $\langle \gamma + \rho_{\mathfrak{g}}, \alpha \rangle < 0$ holds for every root $\alpha \in \Delta_1 \cup \Delta_2$ (i.e. every positive complementary root)
    2. Both $t < -\frac{1}{2}\langle \gamma_0 + \rho_{\mathfrak{g}}, \mu \rangle$ and $t < -\langle \gamma_0 + \rho_{\mathfrak{g}}, w_{\mathfrak{l}}^0(\nu) \rangle$.

In Theorem 2.12 it is automatic that $\langle \gamma + \rho_{\mathfrak{g}}, \beta \rangle > 0$ for every positive root of $\mathfrak{l}$, so the conditions of Theorem 2.12 ensure that $\langle \gamma + \rho_{\mathfrak{g}}, \alpha \rangle \neq 0$ for every root α. The relevance of Theorem 2.12 to this paper is described in [S2], [S3], and [W4]:

Proposition 2.13. The conditions of Theorem 2.12 ensure that $\gamma + \rho_{\mathfrak{g}}$ is the Harish–Chandra parameter of a discrete series representation of G_0.

Specifically, in our setting, the conditions of Theorem 2.12 will characterize the Borel – de Siebenthal discrete series.

3. Classification

In this section we give a complete list the simple Lie algebras $\mathfrak{g}_0$ for which the hypotheses (2.1) hold. We then specify the complex parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$ and the real subalgebras $\mathfrak{k}_0$ and $\mathfrak{l}_0$ of $\mathfrak{g}_0$. Next, we give precise descriptions of the representations τ_i and their representation spaces $\mathfrak{u}_i$. Much of this is done using the Dynkin diagrams to indicate highest weights of representations. There the special cases, where G_0/K_0 is a quaternionic symmetric space, are visible at a glance: they are the ones where $-\mu$ connects directly to ν in the extended Dynkin diagram of $\mathfrak{g}$.

We will denote highest weights of representations as follows. In Dynkin diagrams with two root lengths we denote short root nodes by black dots $\bullet$ and long roots by the usual circles $\circ$. Extended diagrams are those with the negative of the maximal root μ attached by the usual rules. Recall the system Ξ of fundamental

simple weights from (2.9). The (irreducible finite dimensional) representation of G and $\mathfrak{g}$ of highest weight $\sum n_i \xi_i$ is indicated by the Dynkin diagram of $\mathfrak{g}$ with n_i written next to the ith node, except that we omit writing zeroes. So for example the adjoint representations are indicated by

$A_\ell\,,\ell \geqq 1$	2 or 1 — ∘ ——∘ … ——∘ 1	$B_\ell\,,\ell \geqq 3$	1 ∘——∘ … ∘——•
$C_\ell\,,\ell \geqq 2$	2 •——• … •——∘	$D_\ell\,,\ell \geqq 4$	1 ∘——• … ∘⟨∘ ∘
G_2	1 •⇛∘	F_4	1 ∘——∘•——•
E_6	∘—∘—∘—∘—∘ with 1 below	E_7	1 ∘—∘—∘—∘—∘—∘ with node below
E_8	∘—∘—∘—∘—∘—∘ with 1 and node below		

We will use this notation for $\mathfrak{k}$ and $\mathfrak{l}$ as well. They can be identified from their simple root systems $\Psi_{\mathfrak{k}} = (\Psi \setminus \{\nu\}) \cup \{-\mu\}$ and $\Psi_{\mathfrak{l}} = (\Psi \setminus \{\nu\})$. The latter ignores the center of $\mathfrak{l}$, so we indicate it by putting in a $\times$ for the missing node corresponding to ν when ν is a long root, a $\boxtimes$ for that node when ν is a short root. Thus the diagram of $\mathfrak{l}$ consists of the diagram of $\mathfrak{g}$, except that the circle ∘ (resp. bullet •) at the node for the noncompact simple root ν is replaced by an $\times$ (resp. $\boxtimes$)[1]. In the diagram of $\mathfrak{l}$, a symbol $\times$ or $\boxtimes$ indicates the 1–dimensional center of L. The irreducible representation of L with highest weight $\sum n_i \xi_i$ now indicated by the Dynkin diagram of $\mathfrak{l}$ with n_i written next to the ith node, for $n_i \neq 0$, whether that node is ∘, •, $\times$ or $\boxtimes$.

If ν^* is the fundamental simple weight corresponding to the noncompact simple root ν, and $x \in \mathfrak{t}$ by $\alpha(x) = \langle \xi, \alpha \rangle$ for $\alpha \in \mathfrak{t}^*$, then $\mathfrak{l}_0$ has center $i\mathbb{R}\nu^*$.

Now we use the fact that K_0 is connected, simply connected and semisimple. The simple root system $\Psi_{\mathfrak{k}}$ decomposes into

$$\begin{aligned} \Psi_{\mathfrak{k}_1} &: \text{ connected component that contains } -\mu, \text{ and} \\ \Psi_{\mathfrak{k}_2} &: \text{ the complement of } \Psi_{\mathfrak{k}_1} \text{ in } \Psi_{\mathfrak{k}} \end{aligned} \qquad (3.1)$$

This results in decompositions $K_0 = K_1 \times K_2$ and $L_0 = L_1 \times L_2$, which we make explicit in each case.

Our primary object is the real group G_0. Corresponding to G_0 we can have several Borel – de Siebenthal root orders. Given one, specified by (Ψ, ν), we have its negative, specified by $(-\Psi, -\nu)$. Now we discuss uniqueness modulo that. If $\mathfrak{g}$ has an outer automorphism that preserves $\mathfrak{g}_0$, then modulo inner automorphisms of G_0 we realize it as a symmetry σ of the Dynkin diagram, and if $\sigma(\nu) \neq \nu$ then $(\Psi, \sigma(\nu))$ corresponds to the same root order but gives a different Dynkin

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[1] Of course one can also look at L_0 as a subgroup of K_0, and from that viewpoint the diagram of $\mathfrak{l}$ is obtained from that of $\mathfrak{k}$ on replacing the ∘ at the node for $-\mu$ with a $\times$. However it is more convenient to look at L_0 as a subgroup of G_0, and the diagram of $\mathfrak{l}$ from that viewpoint, when we consider the action of L on the subspaces $\mathfrak{u}_i$.

diagram description of the Borel – de Siebenthal root order for G_0. Finally, in some cases there are several real forms of G isomorphic to G_0, not related by outer automorphisms of $\mathfrak{g}$. They are derived from $\mathfrak{so}(m,n) \cong \mathfrak{so}(n,m)$ with m, n even, from $\mathfrak{sp}(m,n) \cong \mathfrak{sp}(n,m)$, from $\mathfrak{e}_{6,A_1A_5} \cong \mathfrak{e}_{6,A_5A_1}$, and from $\mathfrak{e}_{7,A_1D_6} \cong \mathfrak{e}_{7,D_6A_1}$. There the isomorphism comes from a symmetry of the extended Dynkin diagram that moves the negative of the maximal root, and it leads to another Borel – de Siebenthal root order for G_0. Conclusion: while the Borel – de Siebenthal pair (Ψ, ν) for G_0 is not unique for G_0, at least G_0 is unique for (Ψ, ν).

In the following we list the Dynkin diagrams, with the possibilities of the noncompact simple root ν among the simple roots ψ_i. Also in the picture one finds the extended Dynkin diagram node for $-\mu$ where μ is the maximal root. Diagrams of Type A do not occur because ν must have coefficient 2 in the expression of μ as a linear combination of simple roots. We now consider the cases where $\mathfrak{g}$ is of type B.

3.2. CASE $\mathfrak{g}_0 = \mathfrak{so}(4, 2\ell - 3)$. Here G_0 is the 4–sheeted universal cover of the group $SO(4, 2\ell - 3)$ and is 2–sheeted over the corresponding analytic subgroup of the complex simply connected group $Spin(2\ell + 1; \mathbb{C})$. Its extended Dynkin diagram is

$$\text{(type } B_\ell, \; \ell > 2) \tag{3.2a}$$

Thus $\mathfrak{k}$ is and $\mathfrak{l}$ is .

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus (\mathfrak{sp}(1) \oplus \mathfrak{so}(2\ell - 3)) \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus (\mathfrak{sp}(1) \times \mathfrak{so}(2\ell - 3)) \tag{3.2b}$$

where ν^* the fundamental simple weight corresponding to ν. The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_2$ so its diagram is .
Using (2.7), the representation

$$\tau_2 : \; \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is } \quad \text{} \tag{3.2c}$$

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is , so the dualizing diagram method of [EW] shows that the representation

$$\tau_1 : \; \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is } \quad \text{} \tag{3.2d}$$

Here $\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = (2\ell - 3)(2\ell - 4)$.

3.3. CASE $\mathfrak{g}_0 = \mathfrak{so}(2p, 2\ell - 2p + 1), 2 < p < \ell$. Here G_0 is the 4–sheeted universal cover of $SO(2p, 2\ell - 2p + 1)$, $2 < p < \ell$, and is 2–sheeted over the

corresponding analytic subgroup of $Spin(2\ell+1;\mathbb{C})$. Its extended Dynkin diagram is

$$
\begin{array}{c}
\underset{\psi_1\,\cdots\,\psi_2}{\circ\!-\!\circ} \;\cdots\; \overset{\nu}{\underset{\psi_p}{\circ}} \;\cdots\; \underset{\psi_{\ell-1}\;\;\psi_\ell}{\circ\!=\!\!\bullet} \\[4pt]
\underset{\mu}{\circ}
\end{array}
\qquad\qquad (\text{type } B_\ell,\; \ell>3)
$$

(3.3a)

Thus $\mathfrak{k}$ is $\underset{-\mu}{\overset{\psi_1\,\psi_2\,\cdots\,\psi_{p-1}\quad\psi_{p+1}\,\psi_{\ell-1}\psi_\ell}{\circ}}$ and $\mathfrak{l}$ is $\;\overset{\psi_1\,\psi_2\,\cdots\,\psi_{p-1}\,\times\,\psi_{p+1}\,\cdots\,\psi_{\ell-1}\psi_\ell}{}$

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$
\begin{aligned}
\mathfrak{k}_0 &= \mathfrak{so}(2p) \oplus \mathfrak{so}(2\ell - 2p + 1) \text{ and} \\
\mathfrak{l}_0 &= \mathfrak{u}(p) \oplus \mathfrak{so}(2\ell - 2p + 1) = i\mathbb{R}\nu^* \oplus \mathfrak{su}(p) \oplus \mathfrak{so}(2\ell - 2p + 1).
\end{aligned}
\qquad (3.3b)
$$

$\mathfrak{k}$ acts on $\mathfrak{s}$ with highest weight $-\nu = -\psi_p$: [Dynkin diagram]. Using (2.7), the representation

$$
\tau_2 : \;\mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is } \quad [\text{Dynkin diagram}] \,.
\qquad (3.3c)
$$

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is [Dynkin diagram] so the dualizing diagram method of [EW] shows that the representation

$$
\tau_1 : \;\mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is } \quad [\text{Dynkin diagram}] \,.
\qquad (3.3d)
$$

Here $\dim \mathfrak{u}_2 = p(p-1)/2$ and $\dim \mathfrak{u}_1 = p(2\ell - 2p + 1)$.

3.4.   CASE $\mathfrak{g}_0 = \mathfrak{so}(4,1)$. Here $G_0 = Spin(4,1)$ is the (universal) double cover of the group $SO(4,1)$. Its extended Dynkin diagram is

$$
\underset{\psi_1}{\circ}\!=\!\!\overset{\nu}{\underset{\psi_2}{\bullet}}\!\cdots\cdots\!\underset{-\mu}{\circ}
\qquad\qquad (\text{type } B_2)
$$

(3.4a)

Thus $\mathfrak{k}$ is $\;\underset{\psi_1}{\circ}\!-\!\underset{\mu}{\circ}$ and $\mathfrak{l}$ is $\;\overset{\psi_1}{\circ}\!\!-\!\!\boxtimes$. Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$
\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{sp}(1).
\qquad (3.4b)
$$

Here $\mathfrak{k}$ acts on $\mathfrak{s}$ with highest weight $-\nu = -\psi_2$: $\;\overset{\circ}{1}\;\overset{\circ}{1}$. Using (2.7), the representation

$$
\tau_2 : \;\mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is } \quad \underset{-2}{\circ\!=\!\!\boxtimes}.
\qquad (3.4c)
$$

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is $\;\underset{1\quad -2}{\circ\!=\!\!\boxtimes}$ so the dualizing diagram method of [EW] shows that the representation

$$
\tau_1 : \;\mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is } \quad \underset{1\quad -1}{\circ\!=\!\!\boxtimes}.
\qquad (3.4d)
$$

Here $\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = 2$.

3.5.   CASE $\mathfrak{g}_0 = \mathfrak{so}(2\ell, 1)$, $\ell > 1$. Here $G_0 = Spin(2\ell, 1)$ is the universal (two–sheeted) cover of the group $SO(2\ell, 1)$ with $\ell > 1$. Its extended Dynkin diagram is

$$\underset{\psi_1 \;\; \psi_2}{\circ\!\!-\!\!\circ} \cdots -\underset{\psi_{\ell-1}\;\;\psi_\ell}{\circ\!\!-\!\!\bullet}{}^{\nu} \qquad (\text{type } B_\ell,\ \ell > 2) \qquad\qquad\qquad (3.5a)$$
$$\underset{\mu}{\circ}$$

Thus $\mathfrak{k}$ is $\underset{\mu}{\overset{\psi_1\;\psi_2 \quad \cdots \quad \psi_{\ell-1}}{\circ}}$ and $\mathfrak{l}$ is $\underset{\psi_1\;\;\psi_2 \quad \cdots \quad \psi_{\ell-1}}{\circ\!\!-\!\!\circ \cdots -\!\boxtimes}$.

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$\mathfrak{k}_0 = \mathfrak{so}(2\ell) \text{ and } \mathfrak{l}_0 = \mathfrak{u}(\ell) = i\mathbb{R}\nu^* \oplus \mathfrak{su}(\ell). \qquad\qquad (3.5b)$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_\ell$: $\quad\overset{\nearrow}{\circ}\!\!-\!\!\cdots-\overset{1}{\circ}$. Using (2.7), the representation

$$\tau_2 : \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is } \quad \circ\!\!-\!\!\underset{1}{\circ}-\cdots-\circ\!\!=\!\!\boxtimes . \qquad\qquad (3.5c)$$

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is $\quad \circ\!\!-\!\!\circ-\cdots-\underset{1}{\circ}\!\!=\!\!\underset{-2}{\boxtimes}$ so the dualizing diagram method of [EW] shows that the representation

$$\tau_1 : \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is } \quad \underset{1}{\circ}\!\!-\!\!\circ-\cdots-\circ\!\!=\!\!\boxtimes . \qquad\qquad (3.5d)$$

Here $\dim \mathfrak{u}_2 = \ell(\ell-1)/2$ and $\dim \mathfrak{u}_1 = \ell$. This exhausts the cases where $\mathfrak{g}$ is of type B.

We go on to consider the cases where $\mathfrak{g}$ is of type C. There the isomorphisms $\mathfrak{sp}(m, n) \cong \mathfrak{sp}(n, m)$ are implemented by: $-\mu \longleftrightarrow \psi_1$ and $\psi_i \longleftrightarrow \psi_{\ell-i+1}$ for $1 < i < \ell$. That exchanges the cases $\nu = \psi_p$ and $\nu = \psi_{\ell-p}$.

3.6.   CASE $\mathfrak{g}_0 = \mathfrak{sp}(p, \ell-p), 1 < p < \ell$. Here $G_0 = Sp(p, \ell-p)$ and its extended Dynkin diagram is

$$\underset{-\mu\;\;\psi_1\;\;\psi_2}{\circ\!\!\cdots\!\!\bullet\!\!-\!\!\bullet}\cdots-\underset{\psi_p}{\bullet}-\cdots-\underset{\psi_{\ell-1}\;\psi_\ell}{\bullet\!\!=\!\!\circ} \qquad (\text{type } C_\ell,\ \ell > 1) \qquad (3.6a)$$

Thus $\mathfrak{k}$ is $\underset{-\mu\;\psi_1\;\psi_2}{\circ\!\!=\!\!\bullet\!\!-\!\!\bullet}\cdots-\underset{\psi_{p-1}}{\bullet}\;\underset{\psi_{p+1}}{\bullet}\cdots-\underset{\psi_\ell}{\bullet\!\!=\!\!\circ}$ with $\Psi_{\mathfrak{k}_1} = \{-\mu, \psi_1, \psi_2, \dots, \psi_{p-1}\}$ and $\Psi_{\mathfrak{k}_2} = \{\psi_{p+1}, \psi_{p+2}, \dots, \psi_\ell\}$, and $\mathfrak{l}$ is $\underset{\psi_1\;\;\psi_2}{\bullet\!\!-\!\!\bullet}\cdots-\boxtimes-\cdots-\underset{\psi_{\ell-1}\;\psi_\ell}{\bullet\!\!=\!\!\circ}$.

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$\mathfrak{k}_0 = \mathfrak{sp}(p) \oplus \mathfrak{sp}(\ell-p) \text{ and } \mathfrak{l}_0 = \mathfrak{u}(p) \oplus \mathfrak{sp}(\ell-p) = i\mathbb{R}\nu^* \oplus \mathfrak{su}(p) \oplus \mathfrak{sp}(\ell-p). \quad (3.6b)$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_p$:
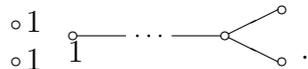
$$\circ\!\!-\!\!\bullet\!\!-\!\!\bullet-\cdots-\underset{1}{\bullet} \quad \underset{1}{\bullet}\!\!-\!\!\cdots-\bullet\!\!-\!\!\circ \quad .$$

Using (2.7), the representation

$$\tau_2 : \mathfrak{k} \text{ on } \mathfrak{u}_2 \text{ is } \quad \underset{2}{\bullet}\!\!\-\!\!-\!\cdots\!-\!\!\bullet\!-\!\!\boxtimes\!-\!\bullet\!-\cdots\!-\!\!\bullet\!\!\Rrightarrow\!\circ \tag{3.6c}$$

The action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is $\quad\bullet\!-\!\bullet\!-\cdots\!-\!\underset{1}{\bullet}\!-\!\underset{-2}{\boxtimes}\!-\!\underset{1}{\bullet}\!-\cdots\!-\!\!\bullet\!\Rrightarrow\!\circ$, so the dualizing diagram method of [EW] shows that the representation

$$\tau_1 : \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is } \quad \underset{1}{\circ}\!-\!\bullet\!-\cdots\!-\!\bullet\!-\!\underset{-1}{\boxtimes}\!-\!\underset{1}{\bullet}\!-\cdots\!-\!\!\bullet\!\Rrightarrow\!\circ \tag{3.6d}$$

Here $\dim \mathfrak{u}_2 = (p-1)(p+2)/2$ and $\dim \mathfrak{u}_1 = 2p(\ell - p)$.

3.7. CASE $\mathfrak{g}_0 = \mathfrak{sp}(1, \ell - 1)$. Here $G_0 = Sp(1, \ell - 1)$ and its extended Dynkin diagram is

$$\underset{-\mu}{\circ}\!\cdots\cdots\!\underset{\psi_1}{\bullet}\!-\!\underset{\psi_2}{\bullet}\!-\cdots-\!\underset{\psi_p}{\bullet}\!-\cdots-\!\underset{\psi_{\ell-1}}{\bullet}\!\!\Rrightarrow\!\underset{\psi_\ell}{\circ} \qquad (\text{type } C_\ell,\ \ell > 1) \tag{3.7a}$$

Thus $\mathfrak{k}$ is $\underset{-\mu}{\circ} \quad \underset{\psi_2}{\bullet}\!-\cdots-\!\underset{\psi_{\ell-1}}{\bullet}\!\Rrightarrow\!\underset{\psi_\ell}{\circ}$ and $\mathfrak{l}$ is $\quad \boxtimes\!-\!\underset{\psi_2}{\bullet}\!-\cdots-\!\underset{\psi_{\ell-1}}{\bullet}\!\Rrightarrow\!\underset{\psi_\ell}{\circ}$.

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(\ell - 1) \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{sp}(\ell - 1). \tag{3.7b}$$

$\mathfrak{k}$ acts on $\mathfrak{s}$ with highest weight $-\nu = -\psi_1$: $\quad \underset{1}{\circ} \qquad \underset{1}{\bullet}\!-\cdots-\!\bullet\!\Rrightarrow\!\circ$. Using (2.7), the representation

$$\tau_2 : \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is } \quad \underset{2}{\boxtimes}\!-\!\bullet\!-\cdots-\!\bullet\!\Rrightarrow\!\circ \tag{3.7c}$$

The action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is $\quad \underset{-2}{\boxtimes}\!-\!\underset{1}{\bullet}\!-\cdots-\!\bullet\!\Rrightarrow\!\circ$, so the dualizing diagram method of [EW] shows that the representation

$$\tau_1 : \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is } \quad \boxtimes\!-\!\underset{1}{\bullet}\!-\cdots-\!\bullet\!\Rrightarrow\!\circ . \tag{3.7d}$$

Here $\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = 2\ell$. This exhausts the cases where $\mathfrak{g}$ is of type C.

We go on to consider the cases where $\mathfrak{g}$ is of type D. When the rank $\ell > 4$ the only automorphisms of the extended Dynkin diagram that move $-\mu$ are given by (1) $\psi_i \mapsto \psi_i$ for $1 < i < \ell - 1$, $-\mu \longleftrightarrow \psi_1$ and $\psi_{\ell-1} \longleftrightarrow \psi_\ell$ and (2) $\psi_i \mapsto \psi_i$ for $1 < i \leq \ell$ and $-\mu \longleftrightarrow \psi_1$. Those exchange the cases $\nu = \psi_p$ and $\nu = \psi_{\ell-p}$. When $\ell = 4$ one has six possibilities: for $i = 1, 3, 4$ one has (1) $-\mu \longleftrightarrow \psi_i$ and $\psi_j \mapsto \psi_j$ for $j \neq i$ and (2) $-\mu \longleftrightarrow \psi_i$ and $\psi_j \longleftrightarrow \psi_k$ for $\{i, j, k\} = \{1, 3, 4\}$. None of these moves ν because $\nu = \psi_2$.

3.8. CASE $\mathfrak{g}_0 = \mathfrak{so}(4, 2\ell - 4), \ell > 4$. Here G_0 is the 4–sheeted universal cover of $SO(4, 2\ell - 4)$ and is 2–sheeted over the corresponding analytic subgroup of $Spin(2\ell; \mathbb{C})$. Its extended Dynkin diagram is
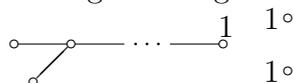
$$\underset{\psi_1}{\circ}\!\-\!\-\!\underset{\underset{\underset{\circ\,\mu}{\vdots}}{\psi_2}}{\overset{\nu}{\circ}}\!-\!\cdots\!-\!\underset{\psi_{\ell-2}}{\circ}\!\!\underset{\searrow}{\overset{\nearrow}{}}\!\!\begin{matrix}\circ\,\psi_{\ell-1}\\[4pt]\circ\,\psi_\ell\end{matrix} \qquad (\text{type } D_\ell,\ \ell > 4) \tag{3.8a}$$

Thus $\mathfrak{k}$ is
$$\psi_1 \;\; \psi_3 \quad \cdots \quad \psi_{\ell-2} \begin{smallmatrix} \circ\,\psi_{\ell-1} \\ \diagdown \\ \circ\,\psi_\ell \end{smallmatrix} \quad \underset{-\mu}{\overset{\psi_1}{\circ}}$$
and $\mathfrak{l}$ is
$$\psi_1 \overset{\times}{\quad} \psi_3 \quad \cdots \quad \psi_{\ell-2} \begin{smallmatrix} \circ\,\psi_{\ell-1} \\ \diagdown \\ \circ\,\psi_\ell \end{smallmatrix}$$

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus (\mathfrak{sp}(1) \oplus \mathfrak{so}(2\ell - 4)) \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus (\mathfrak{sp}(1) \oplus \mathfrak{so}(2\ell - 4)). \quad (3.8b)$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_2$ so its diagram is

$$\begin{smallmatrix} \circ\,1 \\ \circ\,1 \end{smallmatrix} \quad \underset{1}{\circ} \quad \cdots \quad \circ \begin{smallmatrix} \circ \\ \diagdown \\ \circ \end{smallmatrix} \; .$$

Using (2.7), the representation

$$\tau_2 : \; \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is } \quad \circ \overset{\times}{\underset{1}{\quad}} \circ \quad \cdots \quad \circ \begin{smallmatrix} \circ \\ \diagdown \\ \circ \end{smallmatrix} \qquad\qquad (3.8c)$$

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is $\underset{1}{\circ} \overset{\times}{\underset{-2}{\quad}} \underset{1}{\circ} \quad \cdots \quad \circ \begin{smallmatrix} \circ \\ \diagdown \\ \circ \end{smallmatrix}$, so the dualizing diagram method of [EW] shows that the representation

$$\tau_1 : \; \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is } \quad \underset{1}{\circ} \overset{\times}{\underset{-1}{\quad}} \underset{1}{\circ} \quad \cdots \quad \circ \begin{smallmatrix} \circ \\ \diagdown \\ \circ \end{smallmatrix} \; . \qquad\qquad (3.8d)$$

Here $\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = 4(\ell - 2)$.

3.9. CASE $\mathfrak{g}_0 = \mathfrak{so}(2\ell - 4, 4), \ell > 4$. As above, G_0 is the 4–sheeted universal cover of $SO(2\ell - 4, 4)$ and is 2–sheeted over the corresponding analytic subgroup of $Spin(2\ell; \mathbb{C})$. Its extended Dynkin diagram is

$$\underset{\psi_1}{\circ} \cdots \underset{\psi_2}{\circ} \quad \cdots \quad \underset{\psi_{\ell-2}}{\overset{\nu}{\circ}} \begin{smallmatrix} \circ\,\psi_{\ell-1} \\ \diagdown \\ \circ\,\psi_\ell \end{smallmatrix} \qquad \begin{matrix} \underset{-\mu}{\circ} \end{matrix} \qquad \text{(type } D_\ell,\; \ell > 4) \qquad (3.9a)$$

Thus $\mathfrak{k}$ is
$$\overset{\psi_1}{\diagup}\overset{\psi_2}{\circ} \quad \cdots \quad \underset{\psi_{\ell-3}}{\circ} \; \circ\,\psi_\ell \qquad \underset{-\mu}{\circ}$$
and $\mathfrak{l}$ is
$$\underset{\psi_1}{\circ} \;\; \underset{\psi_2}{\circ} \quad \cdots \quad \underset{\psi_{\ell-3}}{\circ} \overset{\times}{\diagup} \begin{smallmatrix} \circ\,\psi_{\ell-1} \\ \diagdown \\ \circ\,\psi_\ell \end{smallmatrix}$$
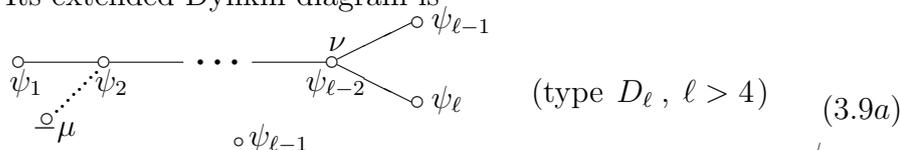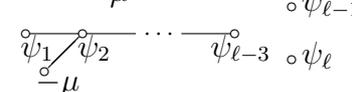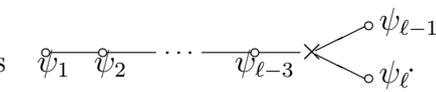
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are
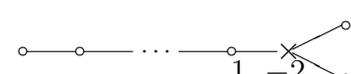
$$\begin{aligned} \mathfrak{k}_0 &= (\mathfrak{so}(2\ell - 4) \oplus \mathfrak{sp}(1)) \oplus \mathfrak{sp}(1) \text{ and} \\ \mathfrak{l}_0 &= (\mathfrak{su}(\ell - 2) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)) \oplus i\mathbb{R}\nu^*. \end{aligned} \qquad (3.9b)$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_{\ell-2}$ and diagram

$$\circ \overset{\diagup}{\underset{\diagdown}{\quad}} \circ \quad \cdots \quad \overset{1}{\underset{\circ}{\quad}} \begin{smallmatrix} 1\,\circ \\ \\ 1\,\circ \end{smallmatrix}$$

Using (2.7), the representation

$$\tau_2 : \; \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is } \quad \circ \underset{1}{\quad} \circ \quad \cdots \quad \circ \overset{\times}{\diagup}\begin{smallmatrix} \circ \\ \diagdown \\ \circ \end{smallmatrix} \; . \qquad\qquad (3.9c)$$

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is $\circ \underset{\quad}{\quad} \circ \quad \cdots \quad \underset{1}{\circ} \overset{\times}{\underset{-2}{\quad}} \begin{smallmatrix} \circ\,1 \\ \diagdown \\ \circ\,1 \end{smallmatrix}$, so the dualizing diagram method of [EW] shows that the representation

$$\tau_1 : \; \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is } \quad \underset{1}{\circ} \underset{\quad}{\quad} \circ \quad \cdots \quad \circ \overset{\times}{\underset{-1}{\quad}} \begin{smallmatrix} \circ\,1 \\ \diagdown \\ \circ\,1 \end{smallmatrix} \; . \qquad\qquad (3.9d)$$

Here $\dim \mathfrak{u}_2 = \frac{1}{2}(\ell - 2)(\ell - 3)$ and $\dim \mathfrak{u}_1 = 4(\ell - 2)$.

3.10.   Case $\mathfrak{g}_0 = \mathfrak{so}(4,4)$. Here G_0 is the 4–sheeted universal cover of the group $SO(4,4)$ and is 2–sheeted over the corresponding analytic subgroup of $Spin(8;\mathbb{C})$. Its extended Dynkin diagram is

$$
\text{(type } D_4 \text{)}
$$

$$(3.10a)$$

Thus $\mathfrak{k}$ is $\;\;$ and $\mathfrak{l}$ is $\;\;$. Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$
\begin{aligned}
\mathfrak{k}_0 &= \mathfrak{sp}(1) \oplus (\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)) \text{ and} \\
\mathfrak{l}_0 &= i\mathbb{R}\nu^* \oplus (\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)).
\end{aligned}
$$

$$(3.10b)$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_2$ so its diagram is

$\circ 1 \quad \circ 1$
$\circ 1 \quad \circ 1$. Using (2.7), the representation

$$
\tau_2 : \;\; \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is} \qquad\qquad .
$$

$$(3.10c)$$

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is $\;\;$ so the dualizing diagram method of [EW] shows that the representation

$$
\tau_1 : \;\; \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is} \qquad\qquad
$$

$$(3.10d)$$

Here $\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = 8$.

3.11.   Case $\mathfrak{g}_0 = \mathfrak{so}(2p, 2\ell - 2p), 2 < p < \ell - 2$. Here G_0 is the 4–sheeted universal cover of the group $SO(2p, 2\ell - 2p)$ and is 2–sheeted over the corresponding analytic subgroup of $Spin(2\ell;\mathbb{C})$, with $2 < p < \ell - 2$. Its extended Dynkin diagram is

$$
\text{(type } D_\ell \text{, } \ell > 5)
$$

$$(3.11a)$$

Thus $\mathfrak{k}$ is $\;\;$ and $\mathfrak{l}$ is $\;\;$. Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$
\begin{aligned}
\mathfrak{k}_0 &= \mathfrak{so}(2p) \oplus \mathfrak{so}(2\ell - 2p) \text{ and} \\
\mathfrak{l}_0 &= \mathfrak{u}(p) \oplus \mathfrak{so}(2\ell - 2p) = i\mathbb{R}\nu^* \oplus \mathfrak{su}(p) \oplus \mathfrak{so}(2\ell - 2p).
\end{aligned}
$$

$$(3.11b)$$

$\mathfrak{k}$ acts on $\mathfrak{s}$ with highest weight $-\nu = -\psi_p$: $\;\;$. Using (2.7), the representation

$$
\tau_2 : \;\; \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is} \qquad\qquad .
$$

$$(3.11c)$$

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is [diagram] so the dualizing diagram method of [EW] shows that the representation

$$\tau_1 : \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is } \quad [\text{diagram}] \qquad (3.11d)$$

Here $\dim \mathfrak{u}_2 = p(p-1)/2$ and $\dim \mathfrak{u}_1 = 2p(\ell - p)$. This exhausts the cases where $\mathfrak{g}$ is of type D, and thus exhausts the classical cases. We go on to the exceptional cases, where only $\mathfrak{e}_6$ has an outer automorphism.

3.12.   CASE $\mathfrak{g}_0 = \mathfrak{g}_{2,A_1A_1}$. Here G_0 is simply connected split real Lie group of type G_2. It has maximal compact subgroup $Spin(3) \times Spin(3)$. Its extended Dynkin diagram is

$$[\text{diagram}] \qquad (\text{type } G_2). \qquad (3.12a)$$

Thus $\mathfrak{k}$ is [diagram] and $\mathfrak{l}$ is [diagram]. Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{sp}(1). \qquad (3.12b)$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_2$: [diagram]. Using (2.7), the representation

$$\tau_2 : \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is } \quad [\text{diagram}] \qquad (3.12c)$$

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is [diagram] so the representation

$$\tau_1 : \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is } \quad [\text{diagram}] \qquad (3.12d)$$

Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 4, is self–dual, and has an antisymmetric bilinear invariant. Also, $\dim \mathfrak{u}_2 = 1$ and $\tau_2|_{[\mathfrak{l},\mathfrak{l}]}$ is trivial, so that bilinear invariant is given by the Lie algebra product $\mathfrak{u}_1 \times \mathfrak{u}_1 \to \mathfrak{u}_2$.

3.13.   CASE $\mathfrak{g}_0 = \mathfrak{f}_{4,A_1C_3}$. Here G_0 is the split simply connected real Lie group of type F_4. It has maximal compact subgroup $Sp(1) \times Sp(3)$. Its extended Dynkin diagram is

$$[\text{diagram}] \qquad (\text{Type } F_4) \qquad (3.13a)$$

Thus $\mathfrak{k}$ is [diagram] and $\mathfrak{l}$ is [diagram].
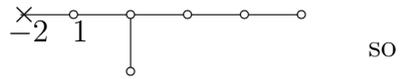
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(3) \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{sp}(3). \qquad (3.13b)$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_1$: $\underset{1}{\circ}$ $\underset{1}{\circ}\!=\!\!=\!\!\bullet\!-\!\bullet$.
Using (2.7), the representation

$$\tau_2 : \ \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is} \qquad \underset{1}{\times}\!-\!\!-\!\circ\!=\!\!=\!\bullet\!-\!\bullet \tag{3.13c}$$

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is $\quad \underset{-2}{\times}\ \underset{1}{\circ}\!=\!\!=\!\bullet\!-\!\bullet \quad$ so the representation

$$\tau_1 : \ \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is} \qquad \underset{-1}{\times}\!-\!\!-\!\underset{1}{\circ}\!=\!\!=\!\bullet\!-\!\bullet \ . \tag{3.13d}$$

Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 14, is self–dual, and has an antisymmetric bilinear invariant. Also, $\dim \mathfrak{u}_2 = 1$ and $\tau_2|_{[\mathfrak{l},\mathfrak{l}]}$ is trivial, so that bilinear invariant is given by the Lie algebra product $\mathfrak{u}_1 \times \mathfrak{u}_1 \rightarrow \mathfrak{u}_2$.

3.14.   CASE $\mathfrak{g}_0 = \mathfrak{f}_{4,B_4}$. Here G_0 is the simply connected real Lie group of type F_4 and real rank 1. It has maximal compact subgroup $Spin(9)$. Its extended Dynkin diagram is

$$\underset{-\mu}{\circ}\!\cdots\!\cdots\!\underset{\psi_1}{\circ}\!-\!\!-\!\underset{\psi_2}{\circ}\!=\!\!=\!\underset{\psi_3}{\bullet}\!-\!\overset{\nu}{\underset{\psi_4}{\bullet}} \qquad (\text{Type } F_4) \tag{3.14a}$$

Thus $\mathfrak{k}$ is $\underset{-\mu}{\circ}\!-\!\underset{\psi_1}{\circ}\!-\!\underset{\psi_2}{\circ}\!=\!\!=\!\underset{\psi_3}{\bullet}$ and $\mathfrak{l}$ is $\underset{\psi_1}{\circ}\!-\!\underset{\psi_2}{\circ}\!=\!\!=\!\underset{\psi_3}{\bullet}\!-\!\boxtimes$.

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$\mathfrak{k}_0 = \mathfrak{so}(9) \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{so}(7). \tag{3.14b}$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_4$: $\circ\!-\!\circ\!-\!\circ\!=\!\!=\!\overset{1}{\bullet}$.
Using (2.7), the representation

$$\tau_2 : \ \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is} \qquad \underset{1}{\circ}\!-\!\circ\!=\!\!=\!\bullet\!-\!\boxtimes \tag{3.14c}$$

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is $\quad \circ\!-\!\circ\!=\!\!=\!\overset{1}{\underset{\bullet}{}}\!-\!\overset{-2}{\boxtimes} \quad$ so the representation

$$\tau_1 : \ \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is} \qquad \circ\!-\!\circ\!=\!\!=\!\underset{1}{\bullet}\!-\!\underset{-1}{\boxtimes} \ . \tag{3.14d}$$

Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 8, is self–dual, and has a symmetric bilinear invariant. In effect τ_1 is the action of $Spin(7)$ on the Cayley numbers, and τ_2 is its action (factored through $SO(7)$) on the pure imaginary Cayley numbers. Thus $\dim \mathfrak{u}_2 = 7$ and $\dim \mathfrak{u}_1 = 8$.

3.15.   CASE $\mathfrak{g}_0 = \mathfrak{e}_{6,A_1A_5,1}$. Here G_0 is the group of type E_6 whose maximal compact subgroup is $SU(2) \times SU(6)$. The noncompact simple root $\nu = \psi_3$, so L
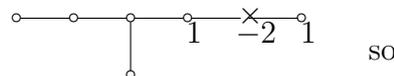
is of type $T_1 A_1 A_4$. We do not consider the case $\nu = \psi_5$ separately because the two differ only by an outer automorphism of E_6. Here the extended Dynkin diagram (Bourbaki root order) is

$$
\begin{array}{c}
\psi_1 \quad \psi_3 \quad \psi_4 \overset{\psi_5 \quad \psi_6}{\underset{\psi_2 \quad -\mu}{\diagup}} \\
\nu
\end{array}
\qquad (\text{Type } E_6)
\tag{3.15a}
$$

Thus $\mathfrak{k}$ is $\psi_1 \; \psi_4 \overset{\psi_5 \; \psi_6}{\underset{\psi_2 \; -\mu}{<}}$ and $\mathfrak{l}$ is $\psi_1 \times \psi_4 \overset{\psi_5 \; \psi_6}{\underset{\psi_2}{<}}$. Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$
\mathfrak{k}_0 = \mathfrak{su}(6) \oplus \mathfrak{su}(2) \quad \text{and} \quad \mathfrak{l}_0 = \big(\mathfrak{su}(5) \oplus i\mathbb{R}\nu^*\big) \oplus \mathfrak{su}(2).
\tag{3.15b}
$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$

has highest weight $-\nu = -\psi_3$: $\overset{1}{\underset{}{\circ}} \quad \overset{1}{\underset{}{\times}}{<}$. Using (2.7), the representation

$$
\tau_2 : \; \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is } \quad \circ \!\!-\!\!\times\!\!-\!\!\circ \overset{}{\underset{1}{<}}
\tag{3.15c}
$$

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is $\overset{1}{\circ}\!\!-\!\!\overset{-2}{\times}\!\!-\!\!\overset{1}{\circ}{<}$ so the representation

$$
\tau_1 : \; \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is } \quad \overset{1}{\circ}\!\!-\!\!\overset{-1}{\times}\!\!-\!\!\circ \overset{1}{\underset{}{<}}
\tag{3.15d}
$$

Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 20 and is not self–dual. We have $\dim \mathfrak{u}_2 = 5$ and $\dim \mathfrak{u}_1 = 20$.

3.16. CASE $\mathfrak{g}_0 = \mathfrak{e}_{6, A_1 A_5, 2}$. This case differs from Case $\mathfrak{g}_0 = \mathfrak{e}_{6, A_1 A_5, 1}$ by an order 3 symmetry of the extended Dynkin diagram. Again G_0 is the group of type E_6 with maximal compact subgroup $SU(2) \times SU(6)$. The noncompact simple root $\nu = \psi_2$, so L is of type $T_1 A_5$. The extended Dynkin diagram (Bourbaki root order) is

$$
\begin{array}{c}
\psi_1 \quad \psi_3 \quad \psi_4 \overset{\psi_5 \quad \psi_6}{\underset{\psi_2 \quad -\mu}{\diagup}} \\
\nu
\end{array}
\qquad (\text{Type } E_6)
\tag{3.16a}
$$

Thus $\mathfrak{k}$ is $\psi_1 \; \psi_3 \; \psi_4 \overset{\psi_5 \; \psi_6}{\underset{-\mu}{\diagup}}$ and $\mathfrak{l}$ is $\psi_1 \; \psi_3 \; \psi_4 \overset{\psi_5 \; \psi_6}{\underset{\times}{<}}$.

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$
\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{su}(6) \quad \text{and} \quad \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{su}(6).
\tag{3.16b}
$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_2$:

Using (2.7), the representation

$$\tau_2 : \ \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is} \qquad\qquad\qquad\qquad\qquad (3.16c)$$

Also, the action τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is   so the representation
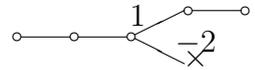
$$\tau_1 : \ \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is} \qquad\qquad\qquad\qquad\qquad (3.16d)$$

Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 20, is self–dual, and has an antisymmetric bilinear invariant. Also, $\dim \mathfrak{u}_2 = 1$ and $\tau_2|_{[\mathfrak{l},\mathfrak{l}]}$ is trivial, so that bilinear invariant is given by the Lie algebra product $\mathfrak{u}_1 \times \mathfrak{u}_1 \to \mathfrak{u}_2$. In brief, $\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = 20$.

3.17. CASE $\mathfrak{g}_0 = \mathfrak{e}_{7,A_1D_6,1}$. Here G_0 is the group of type E_7 with maximal compact subgroup $SU(2) \times Spin(12)$. The noncompact simple root $\nu = \psi_1$, so L is of type $T_1 D_6$ and the extended Dynkin diagram is

$$\text{(Type } E_7) \qquad\qquad\qquad\qquad\qquad (3.17a)$$

Thus $\mathfrak{k}$ is   and $\mathfrak{l}$ is   .
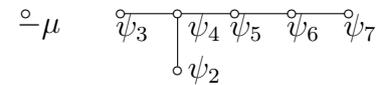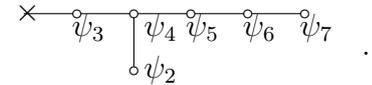
Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{so}(12) \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{so}(12). \qquad\qquad (3.17b)$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_1$:

Using (2.7), the representation

$$\tau_2 : \ \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is} \qquad\qquad\qquad\qquad\qquad (3.17c)$$

Also by (2.7), the representation τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is   so the representation

$$\tau_1 : \ \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is} \qquad\qquad\qquad\qquad\qquad (3.17d)$$

.

Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 32, is self–dual, and has an antisymmetric bilinear invariant. Also, $\dim \mathfrak{u}_2 = 1$ and $\tau_2|_{[\mathfrak{l},\mathfrak{l}]}$ is trivial, so that bilinear invariant is given by the Lie algebra product $\mathfrak{u}_1 \times \mathfrak{u}_1 \to \mathfrak{u}_2$. In brief, $\dim \mathfrak{u}_2 = 1$ and $\dim \mathfrak{u}_1 = 32$.

3.18. CASE $\mathfrak{g}_0 = \mathfrak{e}_{7,A_1 D_6,2}$. This case differs from Case $\mathfrak{g}_0 = \mathfrak{e}_{7,A_1 D_6,1}$ by an order 2 symmetry of the extended Dynkin diagram. Again G_0 is the group of type E_7 with maximal compact subgroup $SU(2) \times Spin(12)$, but now the noncompact simple root is $\nu = \psi_6$, so L is of type $T_1 A_1 D_5$ and the extended Dynkin diagram is

$$
-\mu \;\cdots\; \psi_1 \quad \psi_3 \quad \psi_4 \quad \psi_5 \quad \overset{\nu}{\psi_6} \quad \psi_7 \qquad \text{(Type } E_7)
$$
$$
\psi_2 \tag{3.18a}
$$

Thus $\mathfrak{k}$ is $\quad -\mu\ \psi_1\ \psi_3\ \psi_4\ \psi_5\ \psi_7\ ;\ \psi_2 \quad$ and $\mathfrak{l}$ is $\quad \psi_1\ \psi_3\ \psi_4\ \psi_5\ \times\ \psi_7\ ;\ \psi_2$.

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$
\mathfrak{k}_0 = \mathfrak{so}(12) \oplus \mathfrak{sp}(1) \text{ and}
$$
$$
\mathfrak{l}_0 = (\mathfrak{so}(2) \oplus \mathfrak{so}(10)) \oplus \mathfrak{sp}(1) = (i\mathbb{R}\nu^* \oplus \mathfrak{so}(10)) \oplus \mathfrak{sp}(1). \tag{3.18b}
$$

The representation of $\mathfrak{k}$ on $\mathfrak{s}$ has highest weight $-\nu = -\psi_6$: [Dynkin diagram with labels $1\ 1$]. Using (2.7), the representation
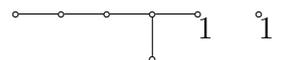
$$
\tau_2 : \ \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is} \qquad [\text{Dynkin diagram with label } 1,\ \times] \tag{3.18c}
$$

Also by (2.7), the representation τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is [Dynkin diagram with labels $1\ \overset{\times}{-2}\ 1$] so the representation

$$
\tau_1 : \ \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is} \qquad [\text{Dynkin diagram with labels } -1\ 1;\ 1] \tag{3.18d}
$$

Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 16 and is not self–dual. In brief, $\dim \mathfrak{u}_2 = 10$ and $\dim \mathfrak{u}_1 = 16$.

3.19. CASE $\mathfrak{g}_0 = \mathfrak{e}_{7,A_7}$. Here G_0 is the group of type E_7 with maximal compact subgroup $SU(8)$. The noncompact simple root $\nu = \psi_2$, so L is of type $T_1 E_6$ and the extended Dynkin diagram is

$$
-\mu \;\cdots\; \psi_1 \quad \psi_3 \quad \psi_4 \quad \psi_5 \quad \psi_6 \quad \psi_7
$$
$$
\overset{\nu}{\psi_2} \qquad \text{(Type } E_7) \tag{3.19a}
$$

Thus $\mathfrak{k}$ is $\quad -\mu\ \psi_1\ \psi_3\ \psi_4\ \psi_5\ \psi_6\ \psi_7 \quad$ and $\mathfrak{l}$ is $\quad \psi_1\ \psi_3\ \psi_4\ \psi_5\ \psi_6\ \psi_7\ ;\ \times$.

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$\mathfrak{k}_0 = \mathfrak{su}(8) \quad \text{and} \quad \mathfrak{l}_0 = (\mathfrak{u}(1) \oplus \mathfrak{u}(7)) \cap \mathfrak{su}(8) = i\mathbb{R}\nu^* \oplus \mathfrak{su}(7). \qquad (3.19b)$$

$\mathfrak{k}$ acts on $\mathfrak{s}$ with highest weight $-\nu = -\psi_2$: (Dynkin diagram with label 1) . Using (2.7), the representation

$$\tau_2 : \ \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is (Dynkin diagram with labels } 1, \times) \qquad (3.19c)$$

Also by (2.7), the representation τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is (Dynkin diagram with labels 1, $\times{-}2$) so the representation

$$\tau_1 : \ \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is (Dynkin diagram with labels } 1, \times{-}1) \qquad (3.19d)$$

Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 35 and is not self–dual. In brief, $\dim \mathfrak{u}_2 = 7$ and $\dim \mathfrak{u}_1 = 35$

3.20.   CASE $\mathfrak{g}_0 = \mathfrak{e}_{8,D_8}$. Here G_0 is the group of type E_8 with maximal compact subgroup $Spin(16)$. The noncompact simple root $\nu = \psi_1$, so L is of type $T_1 D_7$ and the extended Dynkin diagram is

(Dynkin diagram: $\nu \ \psi_1 \ \psi_3 \ \psi_4 \ \psi_5 \ \psi_6 \ \psi_7 \ \psi_8 \ -\mu$, with ψ_2 branch)   (Type E_8) $(3.20a)$

Thus $\mathfrak{k}$ is (Dynkin diagram: $\psi_3 \ \psi_4 \ \psi_5 \ \psi_6 \ \psi_7 \ \psi_8 \ -\mu$, with ψ_2 branch) and $\mathfrak{l}$ is (Dynkin diagram: $\times \ \psi_3 \ \psi_4 \ \psi_5 \ \psi_6 \ \psi_7 \ \psi_8$, with ψ_2 branch).

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are‘

$$\mathfrak{k}_0 = \mathfrak{so}(16) \quad \text{and} \quad \mathfrak{l}_0 = \mathfrak{so}(2) \oplus \mathfrak{so}(14) = i\mathbb{R}\nu^* \oplus \mathfrak{so}(14). \qquad (3.20b)$$

$\mathfrak{k}$ acts on $\mathfrak{s}$ with highest weight $-\nu = -\psi_1$: (Dynkin diagram with label 1) . Using (2.7), the representation

$$\tau_2 : \ \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is (Dynkin diagram with labels } \times, 1) \qquad (3.20c)$$

Also by (2.7), the representation τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is (Dynkin diagram with labels $\times{-}2$, 1) so the representation

$$\tau_1 : \ \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is (Dynkin diagram with labels } \times{-}2, 1) \qquad (3.20d)$$

Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$ has degree 64 and is not self–dual; $\dim \mathfrak{u}_2 = 14$ and $\dim \mathfrak{u}_1 = 64$.

3.21. CASE $\mathfrak{g}_0 = \mathfrak{e}_{8,A_1E_7}$. Here G_0 is the group of type E_8 with maximal compact subgroup $SU(2) \times E_7$ (where E_7 is simply connected). The noncompact simple root $\nu = \psi_8$, so L is of type $T_1 E_7$ and the extended Dynkin diagram is

$$\text{(Type } E_8 \text{)} \tag{3.21a}$$

Thus $\mathfrak{k}$ is ![diagram] $-\mu$ and $\mathfrak{l}$ is ![diagram] .

Now the decompositions $\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ and $\mathfrak{l}_0 = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ are

$$\mathfrak{k}_0 = \mathfrak{sp}(1) \oplus \mathfrak{e}_7 \text{ and } \mathfrak{l}_0 = i\mathbb{R}\nu^* \oplus \mathfrak{e}_7. \tag{3.21b}$$

$\mathfrak{k}$ acts on $\mathfrak{s}$ with highest weight $-\nu = -\psi_8$: ![diagram] $1\ 1$. Using (2.7), the representation

$$\tau_2 : \mathfrak{l} \text{ on } \mathfrak{u}_2 \text{ is} \qquad\qquad . \tag{3.21c}$$

Also by (2.7), the representation τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is ![diagram] $1\ -2$ so the representation

$$\tau_1 : \mathfrak{l} \text{ on } \mathfrak{u}_1 \text{ is} \qquad\qquad . \tag{3.21d}$$

Note that $\tau_1|_{[\mathfrak{l},\mathfrak{l}]}$, has degree 56, is self-dual, and has an antisymmetric bilinear invariant. Also, $\dim \mathfrak{u}_2 = 1$ and $\tau_2|_{[\mathfrak{l},\mathfrak{l}]}$ is trivial, so that bilinear invariant is given by the Lie algebra product $\mathfrak{u}_1 \times \mathfrak{u}_1 \to \mathfrak{u}_2$.

This completes our run through the exceptional cases.

4. Prehomogeneity and Relative Invariants for $(L, \mathfrak{u}_1)$

Consider a connected linear algebraic group with a rational representation on a complex vector space. We say that the triple consisting of the group, the representation and the vector space is *prehomogeneous* if there is a Zariski–dense orbit. When no confusion is possible we omit the representation. A general theorem of Vinberg on graded Lie algebras (see [Kn1, Theorem 10.19]) shows that $(L, \mathfrak{u}_1)$ is prehomogeneous. Or one can verify that fact by running through

the lists of Section 3 and the classification of [SK]. In fact we will do the latter in order to describe the algebra of relative–invariant polynomials on $\mathfrak{u}_1$ and the L–orbit structure of $\mathfrak{u}_1$ for each instance of $(L, \mathfrak{u}_1)$.

We recall some material on prehomogeneous spaces as it applies to $(L, \mathfrak{u}_1)$. There is no nonconstant L–invariant rational function $f : \mathfrak{u}_1 \to \mathbb{C}$ because the L–invariance would force it to be constant on the Zariski–dense L–orbit [SK, Proposition 3 in §2]. By *relative invariant* for $(L, \mathfrak{u}_1)$ we mean a nonconstant polynomial function $f : \mathfrak{u}_1 \to \mathbb{C}$ such that $f(\ell\xi) = \chi(\ell)f(\xi)$ for some rational character $\chi : L \to \mathbb{C}^\times$. The quotient of two relative invariants with the same character would be an L–invariant rational function of $\mathfrak{u}_1$, hence constant, so a relative invariant f is determined up to scalar multiple by its character χ [SK, Proposition 3 in §4]. In particular all the $f_c : \xi \mapsto f(c\xi)$ are proportional, so f is a *homogeneous* polynomial. It will be convenient to denote

$$\mathcal{A}(L, \mathfrak{u}_1) : \text{ the associative algebra of all relative invariants of } (L, \mathfrak{u}_1). \qquad (4.1)$$

The *regular set* for $(L, \mathfrak{u}_1)$ is the open L–orbit $\mathcal{O}_0 := \mathrm{Ad}(L)\xi_0 \subset \mathfrak{u}_1$ and the *singular set* is its complement $\mathfrak{u}_1 \setminus \mathcal{O}_0$. Let $\mathfrak{u}_{1,1}, \ldots, \mathfrak{u}_{1,e}$ be those components of the singular set that are of codimension 1 in $\mathfrak{u}_1$. For each i, $\mathfrak{u}_{1,i}$ is the zero set of an irreducible polynomial f_i. The algebra of relative invariants for $(L, \mathfrak{u}_1)$ is the polynomial algebra $\mathbb{C}[f_1, \ldots, f_e]$ [SK, Proposition 5 in §4]. In particular $(L, \mathfrak{u}_1)$ has a relative invariant if and only if the its singular set has a component of codimension 1. So far we haven't used irreducibility of L on $\mathfrak{u}_1$, but now we use it to see [SK, Proposition 12 in §4] that $e \leqq 1$, i.e. that either $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}$ (in other words $(L.\mathfrak{u}_1)$ has no relative invariant) or $\mathcal{A}(L, \mathfrak{u}_1)$ has form $\mathbb{C}[f]$.

4.2. Cases $SO(2p, r)$. We first consider the various cases where G_0 is the universal covering group of the indefinite orthogonal group $SO(2p, 2q)$ or $SO(2p, 2q+1)$. For convenience we write that as $SO(2p, r)$. Then L_0 consists of all $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ where a is in the image of the standard embedding $\iota : U(p) \hookrightarrow SO(2p)$ and where $b \in SO(r)$. Here $\mathfrak{s}_0 = \left\{ \left(\begin{smallmatrix} 0 & x \\ {}^t x & 0 \end{smallmatrix}\right) \big| x \in \mathbb{R}^{2p \times r} \right\} \cong \mathbb{R}^{2p \times r}$ and the (conjugation) action $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \in L_0$ on $\mathfrak{s}_0$ is given by $x \mapsto axb^{-1}$. Now $\mathfrak{u}_1 \cong \mathbb{C}^{p \times r}$ with the action of $L \simeq GL(p; \mathbb{C}) \times SO(r; \mathbb{C})$ given by $\ell = \left(\begin{smallmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{smallmatrix}\right) : z \mapsto \ell_1 z \ell_2^{-1}$. Then $f(z) := \det(z \cdot {}^t z)$ transforms by $f(\ell(z)) = \det(\ell)^2 f(z)$. However it is a relative invariant only when it is not identically zero, i.e. when $p \leqq r$.

On the other hand, if $p > r$ then the $(SL(p; \mathbb{C}) \times SO(r; \mathbb{C}))$–orbit of $\left(\begin{smallmatrix} I_r \\ 0_{p-r} \end{smallmatrix}\right)$ is open in $\mathbb{C}^{p \times r}$, so there is no nonconstant $(SL(p; \mathbb{C}) \times SO(r; \mathbb{C}))$–invariant. It follows that there is no relative invariant for L.

Summary: if $p \leqq r$ then $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}[f]$ where $f(z) := \det(z \cdot {}^t z)$, polynomial of degree $2p$. If $p > r$ then $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}$.

4.3. Cases $Sp(p, q)$. We next consider the cases where $G_0 = Sp(p, q)$. Then L_0 consists of all $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ where a is in the image of the standard embedding $\iota : U(p) \hookrightarrow Sp(p)$ and where $b \in Sp(q)$. Here $\mathfrak{s}_0 = \left\{ \left(\begin{smallmatrix} 0 & x \\ x^* & 0 \end{smallmatrix}\right) \big| x \in \mathbb{H}^{p \times q} \right\} \cong \mathbb{H}^{p \times q}$ and the (conjugation) action $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \in L_0$ on $\mathfrak{s}_0$ is given by $x \mapsto axb^{-1}$. Now $\mathfrak{u}_1 \cong \mathbb{C}^{p \times 2q}$

with the action of $L \simeq GL(p;\mathbb{C}) \times Sp(q;\mathbb{C})$ given by $\ell = \begin{pmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{pmatrix} : z \mapsto \ell_1 z \ell_2^{-1}$. Let $J = \begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix}$, so $Sp(q;\mathbb{C})$ is characterized by $b \cdot J \cdot {}^t b = J$. Let Pf denote the Pfaffian polynomial on the space of antisymmetric $2q \times 2q$ matrices, so $\mathrm{Pf}\,(m)^2 = \det(m)$. Then $f(z) := \mathrm{Pf}\,(z \cdot J \cdot {}^t z)$ transforms by $f(\ell z) = \det(\ell) f(z)$. And of course f is a relative invariant only when it is not identically zero. For that we must have the possibility that the $p \times p$ antisymmetric matrix $z \cdot J \cdot {}^t z$ is nonsingular, which is the case just when both that $p \leqq 2q$ and p is even.

As before, if $p > 2q$ then the $(SL(p;\mathbb{C}) \times Sp(q;\mathbb{C}))$–orbit of $\begin{pmatrix} I_{2q} \\ 0_{p-2q} \end{pmatrix}$ is open in $\mathbb{C}^{p \times 2q}$, so there is no nonconstant $(SL(p;\mathbb{C}) \times Sp(q;\mathbb{C}))$–invariant, and thus no relative invariant for L.

If $p \leqq 2q$ but p is odd we define $m \in \mathbb{C}^{p \times 2q}$ by $m_{i,i} = 1$ if $i \leqq p$ and i is odd, $m_{i,q+i} = 1$ if $i \leqq p$ and i is even, all other entries zero. The point is that the row space of m has dimension p and has nullity 1 relative to the bilinear form J that defines $Sp(q;\mathbb{C})$. Then $(SL(p;\mathbb{C}) \times Sp(q;\mathbb{C}))(m)$ consists of all elements of $\mathbb{C}^{p \times 2q}$ whose row space has dimension p and nullity 1 relative to J, and that is open in $\mathbb{C}^{p \times 2q}$. As above, it follows that there is no relative invariant for L.

Summary: if $p \leqq 2q$ and p is even then $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}[f]$ where $f(z) := \mathrm{Pf}\,(z \cdot J \cdot {}^t z)$, polynomial of degree p. If $p > 2q$ of if p is odd then $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}$.

We have completely described $\mathcal{A}(L\mathfrak{u}_1)$ when G is classical. Except for a few extreme cases, the representation $\tau_1|_{L'}$ of the derived group $L' = [L, L]$ failed to be self–dual because of a tensor factor $\underset{1}{\circ}\!\!-\!\!\circ\,\cdots\,\circ\!\!-\!\!\circ$. In many of the exceptional group cases, $\tau_1|_{L'}$ is self–dual, so it has a bilinear invariant, and when that bilinear invariant is symmetric it generates $\mathcal{A}(L, \mathfrak{u}_1)$. Also, if that bilinear invariant is antisymmetric, then $\tau_1(L')$ is contained in the symplectic group J of the bilinear invariant, and if σ is the representation of J on $\mathfrak{u}_1$ then $S^2(S^2(\sigma))$ contains the trivial representation with multiplicity 1, and that gives a quartic invariant that generates $\mathcal{A}(L, \mathfrak{u}_1)$. Before formalizing these statements we look back at Section 3 to see the first five columns of

Case	$\mathfrak{g}_0$	deg τ_1	self–dual?	bilinear invariant	rel. invariant
(3.12)	$\mathfrak{g}_{2,A_1 A_1}$	4	yes	antisymmetric	degree 4
(3.13)	$\mathfrak{f}_{4,A_1 C_3}$	14	yes	antisymmetric	degree 4
(3.14)	$\mathfrak{f}_{4,B_4}$	8	yes	symmetric	degree 2
(3.15)	$\mathfrak{e}_{6,A_1 A_5,1}$	20	no	none	none
(3.16)	$\mathfrak{e}_{6,A_1 A_5,2}$	20	yes	antisymmetric	degree 4
(3.17)	$\mathfrak{e}_{7,A_1 D_6,1}$	32	yes	antisymmetric	degree 4
(3.18)	$\mathfrak{e}_{7,A_1 D_6,2}$	16	no	none	none
(3.19)	$\mathfrak{e}_{7,A_7}$	35	no	none	degree 7
(3.20)	$\mathfrak{e}_{8,D_8}$	64	no	none	degree 8
(3.21)	$\mathfrak{e}_{8,A_1 E_7}$	56	yes	antisymmetric	degree 4

(4.4)

The information of the last column of Table 4.4 is contained in [SK], but we can give a short direct proof of the cases where there is a relative invariant, as follows.

Lemma 4.5.   If $\tau_1|_{L'}$ is self–dual there are two possibilities. Either it has a nonzero symmetric bilinear invariant b and $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}[b]$, or it has a nonzero antisymmetric bilinear invariant and $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}[f]$ where f has degree 4. In the non self–dual case (3.19) we have $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}[f]$ where f has degree 7, and in the non self–dual case (3.20) we have $\mathcal{A}(L, \mathfrak{u}_1) = \mathbb{C}[f]$ where f has degree 8.

Proof. If the bilinear invariant b is symmetric, then since it has degree 2 it must generate $\mathcal{A}(L, \mathfrak{u}_1)$. If b is antisymmetric, then in each of the five relevant cases of Table 4.4 we compute symmetric powers $S^2(\tau_1|_{L'})$, $S^3(\tau_1|_{L'})$ and $S^4(\tau_1|_{L'})$ to see that we first encounter a $\tau_1(L')$–invariant in degree 4. (This degree 4 semiinvariant can also be seen by a classification free argument [P, Proposition 1.4].)

Consider the two non self–dual cases of Table 4.4 for which we claim a $\tau_1|_{L'}$–invariant. In case (3.19) we compute the $S^r(\tau_1|_{L'})$ for $2 \leqq r \leqq 7$ to see that we first encounter a $\tau_1(L')$–invariant in degree 7, and and in case (3.20) we compute the $S^r(\tau_1|_{L'})$ for $2 \leqq r \leqq 8$ to see that we first encounter a $\tau_1(L')$–invariant in degree 8. $\square$

5.   Negativity and K_0–types

In this section we discuss negativity of a homogeneous holomorphic vector bundle over G_0/L_0 and the K_0–types of the resulting discrete series representations.

Recall some notation from Section 2. The flag domain $D = G_0(z_0) \cong G_0/L_0$ is an open G_0–orbit in the complex flag manifold $Z = G/Q$, where $z_0 = 1Q$ is the base point and $L_0 = G_0 \cap Q$. The parabolic subgroup Q of G has Lie algebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}_-$ and its nilradical $\mathfrak{u}_-$ is opposite to $\mathfrak{u}_+$, which in turn represents the holomorphic tangent space to D at z_0. Labeling by the multiplicity of the noncompact simple root, $\mathfrak{u}_+ = \mathfrak{u}_1 + \mathfrak{u}_2$. The maximal compact subvariety $Y = K_0(z_0) = K(z_0)$ has holomorphic tangent space at z_0 represented by $\mathfrak{u}_2$ and has holomorphic normal space represented by $\mathfrak{u}_1$. The group L acts irreducibly on both of them, and those representations were derived explicitly in Section 3. The variety Y is a complex flag manifold $K/(K \cap Q)$ in its own right, and is the fiber of the basic fibration (2.8) $D = G_0/L_0 \to G_0/K_0$.

Fix an irreducible representation τ_γ of L. Here γ is the highest weight, E_γ is the representation space, $\mathbb{E}_\gamma \to D$ is the associated homogeneous holomorphic vector bundle. and $\mathcal{O}(\mathbb{E}_\gamma) \to D$ is the sheaf of germs of holomorphic sections.

By $\mathcal{O}(\mathbb{E}_\gamma)|_Y \to D$ we mean the pull–back sheaf of $\mathcal{O}(\mathbb{E}_\gamma) \to D$ under $Y \hookrightarrow D$. It is a sheaf on D supported on Y. We filter it by order of vanishing:

$$\mathcal{F}^n(\mathbb{E}_\gamma) = \big\{ f \in \mathcal{O}(\mathbb{E}_\gamma)|_Y \ \big| \ f \text{ vanishes to order } \geqq n \text{ along } Y \big\}. \qquad (5.1)$$

The action of a vector field on such a section only depends on the vector field's

projection to the normal bundle. We also need the notation

$$
\begin{aligned}
&\mathbb{N}_Y \to Y : \text{ holomorphic normal bundle to } Y \text{ in } D, \\
&\mathbb{N}_Y^* \to Y : \text{ holomorphic conormal bundle to } Y \text{ in } D \text{ and} \\
&\mathcal{S}^n(\mathbb{N}_Y^*) = \mathcal{O}(S^n(\mathbb{N}_Y^*)) \text{ where } S^n(\mathbb{N}_Y^*) \to Y \text{ is the} \\
&\qquad\qquad n^{th} \text{ symmetric power of } \mathbb{N}_Y^* \to Y.
\end{aligned}
\tag{5.2}
$$

Then $\mathbb{N}_Y \to Y$ is the K–homogeneous holomorphic vector bundle over Y with fiber represented by $\mathfrak{u}_1$, its dual $\mathbb{N}_Y^* \to Y$ is the homogeneous holomorphic vector bundle with fiber $\mathfrak{u}_{-1}$, similarly for the third bundle with fibers $S^n(\mathfrak{u}_{-1})$, and we view $\mathcal{S}^n(\mathbb{N}_Y^*)$ as a sheaf on D supported on Y. Now we have short exact sequences

$$
0 \to \mathcal{F}^{n+1}(\mathbb{E}_\gamma) \to \mathcal{F}^n(\mathbb{E}_\gamma) \to \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*)) \to 0
\tag{5.3}
$$

of sheaves on D supported in Y. This leads to the long exact sequences

$$
\begin{aligned}
0 \to &H^0(D; \mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^0(D; \mathcal{F}^n(\mathbb{E}_\gamma)) \xrightarrow{b} H^0(D; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} \\
&H^1(D; \mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^1(D; \mathcal{F}^n(\mathbb{E}_\gamma)) \xrightarrow{b} H^1(D; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&H^{s-1}(D; \mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^{s-1}(D; \mathcal{F}^n(\mathbb{E}_\gamma)) \xrightarrow{b} H^{s-1}(D; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} \\
&H^s(D; \mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^s(D; \mathcal{F}^n(\mathbb{E}_\gamma)) \xrightarrow{b} H^s(D; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} 0.
\end{aligned}
\tag{5.4}
$$

where a and b are coefficient morphisms from (5.3), δ is the coboundary, and $s = \dim_{\mathbb{C}} Y$. If a sheaf on a locally compact space (such as D) is supported on a closed subspace (such as Y) then the inclusion induces a natural isomorphism of cohomologies [G, Corollary to Lemma 4.9.2]. So we can rewrite (5.4) as

$$
\begin{aligned}
0 \to &H^0(Y; \mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^0(Y; \mathcal{F}^n(\mathbb{E}_\gamma)) \xrightarrow{b} H^0(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} \\
&H^1(Y; \mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^1(Y; \mathcal{F}^n(\mathbb{E}_\gamma)) \xrightarrow{b} H^1(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&H^{s-1}(Y; \mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^{s-1}(Y; \mathcal{F}^n(\mathbb{E}_\gamma)) \xrightarrow{b} H^{s-1}(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} \\
&H^s(Y; \mathcal{F}^{n+1}(\mathbb{E}_\gamma)) \xrightarrow{a} H^s(Y; \mathcal{F}^n(\mathbb{E}_\gamma)) \xrightarrow{b} H^s(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) \xrightarrow{\delta} 0.
\end{aligned}
\tag{5.5}
$$

Note that (5.5) is an exact sequence of K–modules.

Lemma 5.6.   Let $n, j \geqq 0$. Then $H^j(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) = H^j(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y)) \otimes S^n(\mathfrak{u}_{-1})$ as K_2–module. If $\mathbb{E}_\gamma \to Y$ is a line bundle then K_2 acts trivially on the first factor $H^j(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y))$.

Proof. The group K_2 acts trivially on Y, so its action on $S^n(\mathbb{N}_Y^*)$ factors out of the cohomology. Recall that $\mathbb{N}_Y^* \to Y$ is the K_0–homogeneous vector bundle based on the L_0–module $\mathfrak{u}_{-1}$. If $\mathbb{E}_\gamma \to Y$ is a line bundle then K_2 acts trivially on each $H^j(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y))$ because it is semisimple. $\square$

Recall that the positive compact roots are those for which the coefficient of ν, as a linear combination from $\Psi = \Psi_G$, is 0 or 2. The ones of coefficient 0 are roots of $(\mathfrak{l}, \mathfrak{t})$. The others, forming the set Δ_2 of the discussion after (2.10), are the complementary compact positive roots. They give the holomorphic tangent space of Y. Let $\rho_\mathfrak{k}$ denote half the sum of the positive compact roots (positive roots of $\mathfrak{k}$). Then the proof of $(2.11a)$ gives us

$$\langle \gamma + \rho_\mathfrak{k}, \alpha \rangle < 0 \text{ for all } \alpha \in \Delta_2 \text{ if and only if } \langle \gamma + \rho_\mathfrak{k}, \mu \rangle < 0. \tag{5.7}$$

If $\alpha_1 \in \Delta_1$ and $\alpha_2 \in \Delta_2$ then $\alpha_1 + \alpha_2$ is not a root, because it would have coefficient 3 at ν. Thus $\langle \alpha_1, \alpha_2 \rangle \geqq 0$. That gives us

Lemma 5.8. If $\alpha_2 \in \Delta_2$ then $\langle \gamma + \rho_\mathfrak{k}, \alpha_2 \rangle \leqq \langle \gamma + \rho_\mathfrak{g}, \alpha_2 \rangle$. In particular if $\alpha_2 \in \Delta_2$ then $\langle \gamma + \rho_\mathfrak{g}, \alpha_2 \rangle < 0$ implies $\langle \gamma + \rho_\mathfrak{k}, \alpha_2 \rangle < 0$. Thus the G_0–negativity condition

$$\langle \gamma + \rho_\mathfrak{g}, \alpha \rangle < 0 \text{ for all } \alpha \in \Delta_1 \cup \Delta_2$$

implies the K_0–negativity condition

$$\langle \gamma + \rho_\mathfrak{k}, \alpha \rangle < 0 \text{ for all } \alpha \in \Delta_2.$$

We are going to need the following fact about tensor products of irreducible finite dimensional representations. It appears in [H] as Exercise 12 to Section 24, based on [Kos1].

Lemma 5.9. Let $E_{\gamma'}$ and $E_{\gamma''}$ be irreducible L_0–modules, where γ' is the highest weight of $E_{\gamma'}$ and γ'' is the highest weight of $E_{\gamma''}$. Then every irreducible summand of $E_{\gamma'} \otimes E_{\gamma''}$ has highest weight of the form $\gamma' + \varphi$ for some weight φ of $E_{\gamma''}$.

Now the K_0–negativity condition gives a vanishing result in (5.5), as follows, where we take (5.7) into account.

Theorem 5.10. Suppose $\langle \gamma + \rho_\mathfrak{k}, \mu \rangle < 0$. Then $H^j(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) = 0$ whenever $j \neq s$ and $n \geqq 0$.

Proof. Note that $\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*) \to Y$ is the K_0–homogeneous bundle based on the representation of L_0 on $E_\gamma \otimes S^n(\mathfrak{u}_{-1})$. In view of Lemma 5.9 that L_0–module is the sum of irreducibles with highest weights of the form $\gamma + \varphi$ where φ is a weight of $S^n(\mathfrak{u}_{-1})$. Thus, as a homogeneous holomorphic vector bundle, $\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*)$ has composition series with composition factors of the form $\mathbb{E}_{\gamma + \alpha_1 + \cdots + \alpha_n}$ where the $\alpha_i \in \Delta_{-1}$.

Let $\alpha \in \Delta_2$. Then (5.7) shows that $\langle \gamma + \rho_\mathfrak{k}, \alpha \rangle < 0$. The coefficient of ν in α is 2, so $\alpha - \alpha_i$ cannot be a root. This forces $\langle \alpha_i, \alpha \rangle \leqq 0$. Now $\langle \gamma + \rho_\mathfrak{k}, \alpha \rangle < 0$ forces $\langle \gamma + \alpha_1 + \cdots + \alpha_n + \rho_\mathfrak{k}, \alpha \rangle < 0$. The Bott–Borel–Weil Theorem now tells us that $H^j(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y \otimes S^n(\mathbb{N}_Y^*))) = 0$ for $j \neq s$. $\qquad \square$

Corollary 5.11. Suppose that $\langle \gamma + \rho_{\mathfrak{k}}, \mu \rangle < 0$. Then $H^j(Y; \mathcal{O}(\mathbb{E}_\gamma|_Y)) = 0$ whenever $j \neq s$.

As in the argument of Proposition 2.13, [S2], [S3] and [W4] now give us

Theorem 5.12. If $\langle \gamma + \rho_{\mathfrak{g}}, \alpha \rangle < 0$ whenever $\alpha \in \Delta_1 \cup \Delta_2$, then $H^j(D; \mathcal{O}(\mathbb{E}_\gamma)) = 0$ for $j \neq s$.

Recall the parameter t from the decomposition of (2.10):

$$\gamma = \gamma_t = \gamma_0 + t\nu^* \text{ where } \langle \gamma_0, \nu \rangle = 0 \text{ and } t \in \mathbb{R}. \tag{5.13}$$

In view of $(2.11a)$, $(2.11b)$ and Theorem 2.12, we reformulate Theorem 5.12 as follows.

Theorem 5.14. Let $\gamma = \gamma_t = \gamma_0 + t\nu^*$ as in (2.10) and (5.13). If $\langle \gamma + \rho_{\mathfrak{g}}, \mu \rangle < 0$ and $\langle \gamma + \rho_{\mathfrak{g}}, w_{\mathfrak{l}}^0(\nu) \rangle < 0$, in other words
$$\text{if } t < -\tfrac{1}{2}\langle \gamma_0 + \rho_{\mathfrak{g}}, \mu \rangle \text{ and } t < -\langle \gamma_0 + \rho_{\mathfrak{g}}, w_{\mathfrak{l}}^0(\nu) \rangle,$$
then $H^j(D; \mathcal{O}(\mathbb{E}_\gamma)) = 0$ for $j \neq s$.

Definition 5.15. To facilitate use of these vanishing theorems we will say that the bundle $\mathbb{E}_\gamma \to D$ is *sufficiently negative* if it satisfies $\langle \gamma + \rho_{\mathfrak{g}}, \alpha \rangle < 0$ whenever α is a complementary positive root, i.e. whenever $\alpha \in \Delta_1 \cup \Delta_2$. This means that $\mathbb{E}_\gamma \otimes \mathbb{K}_D^{1/2} \to D$ is negative in the sense of differential or algebraic geometry, where $\mathbb{K}_D \to D$ is the canonical line bundle. $\diamondsuit$

In the presence of sufficient negativity Theorem 5.14 trivializes the long exact sequences (5.4) and (5.5) as follows. Here recall $s = \dim_\mathbb{C} Y$.

Proposition 5.16. Suppose that $\mathbb{E}_{\gamma_t} \to D$ is sufficiently negative. Then

$$H^q(Y; \mathcal{F}^{n+1}(\mathbb{E}_{\gamma_t})) \cong H^q(Y; \mathcal{F}^n(\mathbb{E}_{\gamma_t})) \qquad \text{for } 0 \leqq q < s, \qquad \text{and}$$
$$H^s(Y; \mathcal{O}(\mathbb{E}_{\gamma_t}|_Y \otimes S^n(\mathbb{N}_Y^*))) \cong H^s(Y; \mathcal{F}^n(\mathbb{E}_{\gamma_t}))/H^s(Y; \mathcal{F}^{n+1}(\mathbb{E}_{\gamma_t})).$$

6. Analysis of the Borel – de Siebenthal Discrete Series

One can combine the results of Section 5 with the case-by-case analysis and diagrams of Section 3 to understand the Harish–Chandra module structure and geometric quantization construction of the family of discrete series representations of G_0 that we call the *Borel – de Siebenthal discrete series.* The latter amounts to geometric quantization of the elliptic coadjoint orbits of negative integral elements of the Lie algebra $\mathfrak{l}_0$.

The results above on the filtration are crucial for the construction of the cohomology groups carrying these representations. They are the analytic predecessors of the Vogan–Zuckerman derived functor modules, which are constructed purely algebraically. Here we construct the Borel – de Siebenthal discrete series

by direct analysis on orbits, and using the above results analyze the K_0–types explicitly. Then we end the paper with some remarks and immediate consequences.

We will treat the analytic continuation of the Borel – de Siebenthal discrete series in a later paper. But in this section we give some indications of part of the continuation for the groups $SO_0(2m, 2\ell - 2m)$ and E_{8,D_8}.

As is clear from the above discussion there are several approaches to the discrete series. It is the series of equivalence classes of unitary representations of G_0 that are (discrete) summands of the left regular representations. It is the series of equivalence classes of unitary representations π of G_0 for which the matrix coefficients $f_{u,v}(g) = \langle u, \pi(g)v \rangle$ of π belong to $L^2(G_0)$. It can be described as the action of G_0 on certain Dolbeault cohomology spaces $H^q(D; \mathbb{E})$ both as nuclear Fréchet spaces ([S2], [SW2]) and as Hilbert spaces [W4]. And of course the underlying Harish–Chandra modules can be described algebraically as Zuckerman derived functor modules. We use the Dolbeault cohomology constructions, both to define the Borel – de Siebenthal discrete series and to analyze it.

We first recall some structural facts about discrete series representations in general. We'll say that $\varphi \in i\mathfrak{t}_0^*$ is *integral* if $\exp(\varphi) : T_0 \to \mathbb{C}$ is well defined. If $\lambda \in i\mathfrak{t}^*$ is nonsingular and for some (hence every) positive root system $\lambda + \rho_{\mathfrak{g}}$ is integral we have the discrete series class $[\pi_\lambda] \in \widehat{G_0}$ of *Harish–Chandra parameter* λ. Discrete series representations π_λ and $\pi_{\lambda'}$ are equivalent exactly when λ and λ' differ only by an element of the Weyl group $W(\mathfrak{k}, \mathfrak{t})$. That is Harish–Chandra's parameterization of the discrete series. See [Kn1, Theorem 9.20].

In the cohomology construction [W4] one has a flag manifold $X = G/P$ and a flag domain $D = G_0(x) \subset X$ such that the isotropy subgroup U_0 of G_0 at x is compact. Then one arranges $T_0 \subset U_0 \subset K_0$ and works with a positive root system $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$ such that the nilradical of $\mathfrak{p}$ — which represents the antiholomorphic tangent space — is a sum of negative root spaces. If $\gamma \in i\mathfrak{t}_0^*$ is U_0–dominant integral and $\lambda := \gamma + \rho_{\mathfrak{g}}$ is nonsingular one defines

$$q(\lambda) = \#\{\alpha \in \Delta(\mathfrak{k}, \mathfrak{t}) \cap \Delta^+ \mid \langle \alpha, \lambda \rangle < 0\} + \#\{\alpha \in \Delta^+ \setminus \Delta(\mathfrak{k}, \mathfrak{t}) \mid \langle \alpha, \lambda \rangle > 0\}. \quad (6.1)$$

Now [S2], [S3], [S4] and [SW2] for the line bundle case, and [W4] for extension to the vector bundle case, say

Theorem 6.2. The L_2 cohomology $H_2^q(D; \mathcal{O}(\mathbb{E}_\gamma)) = 0$ for $q \neq q(\lambda)$, and the natural action of G_0 on $H_2^{q(\lambda)}(D; \mathcal{O}(\mathbb{E}_\gamma))$ is the discrete series representation π_λ. If further λ is negative in the sense of the equivalent conditions of Theorem 2.12, then

$$q(\lambda) = \#\{\alpha \in \Delta(\mathfrak{k}, \mathfrak{t}) \cap \Delta^+ \mid \langle \alpha, \lambda \rangle < 0\} = \#\Delta_2^+ = \dim_{\mathbb{C}} Y,$$

the Dolbeault cohomology $H^q(D; \mathcal{O}(\mathbb{E}_\gamma)) = 0$ for $q \neq q(\lambda)$, and $H^{q(\lambda)}(D; \mathcal{O}(\mathbb{E}_\gamma))$ is a nuclear Fréchet space on which natural action of G_0 is irreducible and infinitesimally equivalent to π_λ.

In order to describe the K_0–spectrum of a discrete series representation π_λ we use the *Harish–Chandra root order*

$$\Delta_\lambda^+ = \{\alpha \in \Delta \mid \langle \lambda, \alpha \rangle > 0\} \tag{6.3}$$

as described in [Kn1]. (Evidently Δ_λ^+ is different from our Borel – de Siebenthal root order Δ^+.) Conversely, to each Weyl chamber of $\mathfrak{g}$, modulo the action of $W(\mathfrak{k}, \mathfrak{t})$ we associate a family of discrete series representations. The chamber for Δ^+, where $q(\lambda)$ is minimized, is the one that corresponds to the Borel – de Siebenthal discrete series of G_0; see [Kos3, Theorem 6.5], which gives a formula for the degree in terms of the dimension of a corresponding abelian ideal in the Borel subalgebra defined by Δ^+.

Of particular interest is the lowest K_0–type contained in the (Harish-Chandra module for) π_λ given by its highest weight Λ (in the Harish–Chandra root order (6.3)),

$$\Lambda = \lambda + \rho_\mathfrak{g} - 2\rho_\mathfrak{k} \tag{6.4a}$$

in terms of the usual half sums of positive roots. This K_0–type has multiplicity one, and other K_0–types have highest weights of the form

$$\Lambda' = \Lambda + \sum_{\alpha \in \Delta^+} n_\alpha \alpha \tag{6.4b}$$

for integers $n_\alpha \geqq 0$. In the general theory of discrete series this statement about the K_0–types only amounts to an inclusion, whereas our results above analyzing the cohomology groups in terms of restriction and Taylor expansion in the normal direction ($\mathfrak{u}_1$) gives a concrete list of the K_0–types. We shall formulate this precisely as Theorem 6.8.

Example 6.5. We compare our parameters with those used in the line bundle case of the quaternionic discrete series [GW]. That is the setting in which $\dim \mathfrak{u}_2 = 1$, in fact Δ_2 consists only of the maximal root μ (denoted β in [GW]), and $\gamma_0 = 0$. The Harish–Chandra (and infinitesimal character) parameter of their representation π_λ^s is of the form $\lambda = -\frac{k}{2}\mu + \rho_\mathfrak{g}$, $k \in \mathbb{Z}$, where the integer $k \geqq 2d+1$ and $\dim_\mathbb{R} G_0/K_0 = 4d$. We consider the Harish–Chandra root order Δ_λ^+. Dividing as usual into compact and noncompact roots we have, for the Borel — de Siebenthal root order Δ^+, $\rho_\mathfrak{g} = \rho_\mathfrak{k} + \rho_{\mathfrak{g}/\mathfrak{k}}$, $\rho_\mathfrak{k} = \rho_\mathfrak{l} + \frac{\mu}{2}$ and $\rho_{\mathfrak{g}/\mathfrak{k}} = \frac{d}{2}\mu$, where $\rho_{\mathfrak{g}/\mathfrak{k}}$ is half the sum of the noncompact positive roots and $\rho_\mathfrak{l}$ is half sum of positive roots of $\mathfrak{l}$. Similarly for the Harish–Chandra root order we have $\rho_\mathfrak{k}' = \rho_\mathfrak{l} - \frac{\mu}{2}$ and $\rho_{\mathfrak{g}/\mathfrak{k}}' = -\frac{d}{2}\mu$, so the lowest K_0–type in the Harish–Chandra root order Δ_λ^+ is $\Lambda = -\frac{k}{2}\mu + \rho_\mathfrak{g} + \rho_\mathfrak{g}' - 2\rho_\mathfrak{k}'$. That simplifies to $\Lambda = \frac{-k+2}{2}\mu$. This is exactly the highest weight for the $(k-1)$–dimensional representation of the simple $SU(2)$ factor in K found as the lowest K_0–type by Gross and Wallach. In the following we shall find the analogous lowest K_0–type for the Borel – de Siebenthal discrete series, and at the same time realize it (and in fact all K_0–types) as cohomology groups on the compact Hermitian symmetric space Y. $\diamond$

Recall the noncompact simple root ν from Section 2. As before, ν^* denote the dual to ν in the system of fundamental simple weights (2.9). The parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ may also be defined by means of ν^*, and the centralizer of ν^* is $\mathfrak{l}$. Thus the coadjoint orbit $\mathrm{Ad}^*(G_0)(\nu^*)$ is our space G_0/L_0 and is fibered by the Hermitian symmetric space Y. Multiples of this ν^* will define the line bundles, and the line bundle representations in the Borel – de Siebenthal discrete series are then the cohomology groups in degree $s = \dim_{\mathbb{C}} Y$ with coefficients in those bundles. Or course most of the representations of the Borel – de Siebenthal discrete series are realized on s–cohomology of higher dimensional vector bundles

Recall the maximal compact subgroup $K_0 = K_1 \times K_2$ explicit in the classi-fication of Section 3, where the "small" factor K_1 corresponds to the component of the simple root system $\Psi_{\mathfrak{k}} = (\Psi \setminus \{\nu\}) \cup \{-\mu\}$ that contains $\{-\mu\}$. In the quater-nionic case $L_1 = Sp(1)$. Now $Y = K_0/L_0 = (K_1 \times K_2)/(L_1 \times K_2) = K_1/L_1$. Thus, as far as induced representations and cohomology, the action of the K_2 factor will be rather simple. This we will make explicit below. Also, it is important that the factor L_1 in L_0 contains the center of L_0, and that the action of that center on the holomorphic normal space $\mathfrak{u}_1$ is given explicitly in the case by case diagrams of Section 2.

Let $\mathbb{E}_{\gamma_t} \to D$ be the holomorphic vector bundle induced from the represen-tation of L_0 with highest weight $\gamma_t = \gamma_0 + t\nu^*$, $t \in \frac{1}{2}\mathbb{Z}$. As G is simply connected $\exp(\gamma_t)$ is the highest weight of a well defined representation of L_0. Denote

$$\pi_{\lambda_t} : \text{ representation of } G_0 \text{ on } H^s(D, \mathcal{O}(\mathbb{E}_{\gamma_t}))$$
$$\text{where } \lambda_t = \gamma_t + \rho_{\mathfrak{g}} = \gamma_0 + t\nu^* + \rho_{\mathfrak{g}}. \tag{6.6}$$

Using the filtration (5.1) and arguments analogous to those of [GW] we characterize the (vector bundle valued) Borel – de Siebenthal discrete series as in Theorem 6.8 below.

Example 6.7. In order to illustrate the conditions of negativity in Lemma 5.8 we consider the Case 3.11 of Section 3, where G_0 is the universal cover of $SO(2m, 2\ell - 2m)$ with $\ell > 2m$. For this we also refer to [Kn3] which contains precise results about the unitarity for the line bundle case (the case where $\gamma_0 = 0$) of the Borel – de Siebenthal discrete series of this group G_0 and its continuation (in a parameter which is essentially the negative of our parameter t). When we say t is *sufficiently negative* we mean the precise condition as of Definition 5.15. As noted in Proposition 2.13 and Theorem 6.2, it is the geometric condition for the cohomology representation in degree $q(\lambda)$ to belong to the discrete series. Let us be more specific about this. In terms of the usual basis for the Lie algebra of the compact torus we have

$$\begin{aligned}
\nu \quad &= e_m - e_{m+1} = (0, \ldots, 0, 1; -1, 0, \ldots, 0), \\
2\rho_{\mathfrak{k}} \quad &= (2m - 2, 2m - 4, \ldots, 0; 2\ell - 2m - 2, 2\ell - 2m - 4, \ldots, 0), \\
2\rho_{\mathfrak{g}/\mathfrak{k}} &= (2\ell - 2m, 2\ell - 2m, \ldots, 2\ell - 2m; 0, \ldots, 0), \\
2\rho_{\mathfrak{g}} \quad &= (2\ell - 2, 2\ell - 4, \ldots, 2\ell - 2m; 2\ell - 2m - 2, \ldots, 0), \text{ and} \\
\nu^* \quad &= (1, 1, \ldots, 1; 0, 0, \ldots, 0).
\end{aligned}$$

For any integer b denote
$$\chi_b = (-\ell + \tfrac{b}{2})\nu^*.$$
It is the differential of a well defined character $\exp(\chi_b)$ of L. This leads to the line bundle case of our construction. For $b > 2\ell - 2$ the cohomology group gives a discrete series representation with infinitesimal character $\chi_b + \rho_{\mathfrak{g}}$ and lowest K_0 - type $X_b = \chi_b + 2\rho_{\mathfrak{g}/\mathfrak{k}}$. (Here of course t and b have the opposite sign; in a moment we will check that $\tfrac{b}{2} = -t + m - \ell + 1$.) Note that the action of L_0 on its complement in the Lie algebra is the action of $U(m) \times SO(2\ell - 2m)$ on the complex $m \times (2\ell - 2m)$ matrices, and that our maximal compact complex subvariety $Y = O(2m)/U(m)$. Note also that the highest root is $\mu = e_1 + e_2$, and that the highest non-compact root is $e_1 + e_{m+1}$. All this fits with the Harish-Chandra condition for discrete series, as noted in Proposition 2.13. As b decreases, [Kn3] shows that the corresponding Zuckerman derived functor module has a unitary subrepresentation all the way down to $b = 0$. The particular case of $b = m + 1$ was proved unitary by D. Vogan; see [Kn3] for more details.

Now we specify the range of parameters for our *sufficiently negative* condition, in other words the discrete series condition. For this we observe that the longest Weyl group element in $W(\mathfrak{l}, \mathfrak{t})$ sends ν to $e_1 + e_{m+1}$, and that the inner product $\langle \rho_{\mathfrak{g}}, e_1 + e_{m+1} \rangle = 2l - m - 2$. Hence $t\nu^* + \rho_{\mathfrak{g}}$ is sufficiently negative just when $t < -2(l - m - 2)$. Computing $2\rho_{\mathfrak{k}/\mathfrak{l}} = (m - 1)\nu^*$ we see that the lowest K_0 - type in the cohomology group is $X_{-t} = (t + (m - 1))\nu^*$. Thus we have the identification of parameters

$$-t - m + 1 = \tfrac{b}{2} + l - 2m$$

and the conditions for negativity and discrete series both are

$$\tfrac{b}{2} - l > -1.$$

In this example we now consider the condition for vanishing of the cohomology groups over Y. The K_0 negativity condition in Lemma 5.8 is

$$\langle t\nu^* + \rho_{\mathfrak{k}}, \alpha \rangle < 0 \text{ for all } \alpha \in \Delta_2,$$

in other words for all positive roots α of $\mathfrak{k}$ that are not roots of $\mathfrak{l}$. Since $\langle \nu^*, \alpha \rangle = 2$ we get the condition $2t + m - 1 + m - 2 < 0$, i.e. $t < -m + \tfrac{3}{2}$. But given our parameter identifications this means

$$b \geqq 2(2m - l).$$

That is the overall condition in [Kn3] for consideration of these modules. In our picture this means the reasonable range where all the K_0–types of $(6.9a)$ continue to exist as cohomology groups, though of course they need not always fit together to form a discrete series Harish–Chandra module, or for that matter a $(\mathfrak{g}, \mathfrak{k})$–module. The starting point for geometric continuation in the parameter t of the Borel – de Siebenthal discrete series is the analysis of just when the remaining K_0–types

do fit together to form a $(\mathfrak{g}, \mathfrak{k})$–module. We indicate in Example 6.13 below, that such a range between the discrete series condition and the condition for existence of the K_0– types is to be expected in general. Comparing that example with the present one, one sees that the calculation depends on the specific structure of the group G_0, and will be different for other groups even though the general procedure is similar. This is why we just indicate the procedure and omit the specific case by case computations. $\diamond$

Observe that in the following theorem we deal with vector bundles $\mathbb{E}_\gamma$ that need not be line bundles. In other words we do not require $\gamma_0 = 0$.

Theorem 6.8. The Borel – de Siebenthal discrete series representations of G_0 are the cohomology representations π_{λ_t} of (6.6) for which γ_t is sufficiently negative in the sense of Proposition 2.13, in other words for which both $t < -\frac{1}{2}\langle \gamma_0 + \rho_\mathfrak{g}, \mu \rangle$ and $t < -\langle \gamma_0 + \rho_\mathfrak{g}, w_\mathfrak{l}^0(\nu) \rangle$. As a K–module, the underlying Harish–Chandra module of π_{λ_t} is

$$\sum_{m \geq 0} H^s(Y, \mathcal{O}(\mathbb{E}_{\gamma_t}|_Y \otimes S^m(\mathbb{N}^*))). \tag{6.9a}$$

It not only is K_0–admissible but is K_1–admissible. Further, the lowest K_0–type (corresponding to $m = 0$) is given by

$$W_{\lambda_t} = H^s(Y, \mathcal{O}(\mathbb{E}_{\gamma_t}|_Y)), \tag{6.9b}$$

and it has multiplicity 1 in π_{λ_t}.

Remark 6.10. Just as in the setting of the holomorphic discrete series, one cannot expect a general K_0–module decomposition formula for the cohomologies $H^s(Y, \mathcal{O}(\mathbb{E}_{\gamma_t}|_Y \otimes S^m(\mathbb{N}^*)))$ of (6.9a), because the pattern varies from case to case. However, in any particular case, one can use the information of Section 3 for the representation of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$, together with the computer program LiE (see [LiE] for instructions to obtain it), to explicitly decompose the fiber $E_{\gamma_t} \otimes S^m(\mathfrak{u}_{-1})$ into a sum of irreducible L_0–modules. This decomposes $\mathbb{E}_{\gamma_t}|_Y \otimes S^m(\mathbb{N}^*)$ as a direct sum of irreducible K–homogeneous holomorphic sub–bundles, and thus by the Bott–Borel–Weil Theorem explicitly decomposes $H^s(Y, \mathcal{O}(\mathbb{E}_{\gamma_t}|_Y \otimes S^m(\mathbb{N}^*)))$ as a direct sum of irreducible K_0–modules. $\diamond$

Remark 6.11. The L_0–modules $S^m(\mathfrak{u}_{-1})$ are not always multiplicity free, though they are multiplicity free in many cases. For example for the group of type D_9 and $m = 6$, calculation with the computer program LiE produces multiplicities, while there are none for F_4. Thus even in the scalar case, where $\mathbb{E}_{\gamma_n} \to D$ is a line bundle, i.e. when $\gamma_0 = 0$, π_{λ_t} need not be K_0–multiplicity free. This is of course in contrast the the K_0–multiplicity free property of the line bundle holomorphic discrete series. $\diamond$

Proof. We use the filtration (5.1) and the exact sequences (5.4) and (5.5), together with the fact that Y is a compact hermitian symmetric space for K_1. The action

of K_2 is part of the holomorphically induced representation, and the action of L_1 on $\mathfrak{u}_1$ and its dual $\mathfrak{u}_{-1}$ is given as above. Finally the admissibility can be read off from the K_0–types directly: each $H^s(Y, \mathcal{O}(\mathbb{E}_{\gamma_t} \otimes S^m(\mathfrak{u}_{-1})))$ is a sum of irreducible representations of K_1, disjoint for different m, and the $S^m(\mathfrak{u}_{-1})$ are finite dimensional representation spaces for K_2 and L_0.

Consider the parabolic subgroup $Q \cap K = LU_{-2}$ of K. Whenever M is a finite dimensional $(Q \cap K)$–module, the space of K–finite vectors in the induced representation $\operatorname{Ind}_{Q \cap K}^K(M)$ is $\sum_{\delta \in \widehat{K}} V_\delta \otimes (V_\delta^* \otimes M)^{Q \cap K}$. In particular the multiplicity of a K_0–type δ is equal to the number of times the highest weight vector of M occurs as a highest weight vector for L in V_δ. In our case the highest weights of M will grow with m in $S^m(\mathfrak{u}_{-1})$, and they are distinguished by the action of the center of L_0. Thus each K_1–type only occurs finitely many times, so π_{λ_t} is K_1–admissible. That, of course, implies admissibility for K_0. $\qquad\square$

Remark 6.12. We compare our parameter for the lowest K_0–type with the general description mentioned for the scalar quaternionic case, where $\gamma_t = t\nu^*$ and $\Delta_2 = \{\mu\}$. In that case the infinitesimal character of the representation π_{λ_t} is given by

$$\lambda_t = t\nu^* + \rho_\mathfrak{g}, \quad \rho_\mathfrak{k} = \rho_\mathfrak{l} + \tfrac{1}{2}\nu^* \quad \text{and} \quad \rho_{\mathfrak{g}/\mathfrak{k}} = c_1 \nu^*$$

for a positive constant c_1 depending only on the root system. Then for the Harish–Chandra root order $\Delta_{\lambda_t}^+$ we get

$$\rho_\mathfrak{k}' = \rho_\mathfrak{l} - \tfrac{1}{2}\nu^* \quad \text{and} \quad \rho_{\mathfrak{g}/\mathfrak{k}}' = -c_1 \nu^*.$$

Thus the lowest K_0–type has highest weight

$$\Lambda_t = t\nu^* + \rho_\mathfrak{g} + \rho_\mathfrak{g}' - 2\rho_\mathfrak{k}' = t\nu^* + 2\rho_{\mathfrak{k}/\mathfrak{l}}$$

where $\rho_{\mathfrak{k}/\mathfrak{l}} = \tfrac{1}{2}\nu^*$ is exactly the shift coming from the square root of the canonical bundle $\mathbb{K}_Y \to Y$. It corresponds to the lowest K_0–type above, viz. $W = H^s(Y, \mathcal{O}(\mathbb{E}_{\gamma_t}))$. $\qquad\diamondsuit$

It is an interesting problem to study the structure, including unitarity, of π_{λ_t} as t increases past the discrete series range, and to relate this to the projective varieties defined by the relative invariants. We will do this in a sequel to this paper. Here the ring of regular functions on L_0–orbits will be important, as in the paper [GW] of Gross and Wallach for the line bundle case of the quaternionic discrete series. Example 6.13 gives an indication of how this goes.

As an application of the K_1 admissibility above, the branching problems will be manageable in a way similar to the case of holomorphic discrete series. This will require that the embedding of the smaller group respects the relevant structure, i.e. that the orderings are compatible. For example if we want to branch to a symmetric subgroup H_0 of G_0, then the embedding will be compatible provided the symmetry fixes K_1. Then we use the admissibility of the action of K_1, so that admissibility for the branching law to H_0 will follow for the Borel – de Siebenthal discrete series. In this case a Borel – de Siebenthal discrete series will branch as a direct sum of Borel – de Siebenthal discrete series representations.

Example 6.13.   Example 6.7, where we considered the indefinite orthogonal groups $SO(2m, 2\ell - 2m)$, shows that continuation of the discrete series modules is closely connected with the geometry of the relative invariants for action of L on the holomorphic normal space $\mathfrak{u}_1$ to the maximal compact subvariety. Let us briefly indicate the case where G is of type E_8 and K of type D_8. There $\mathfrak{u}_1$ is of dimension 64 and admits a relative invariant of degree 8, and the maximal compact subvariety is the Grassmannian of 2–planes in $\mathbb{C}^{16}$.

Now we treat this case of $G_0 = E_{8,D_8}$ explicitly. The noncompact simple root $\nu = \psi_1$. As $\gamma_0 \perp \nu$ it has form

$$\gamma_0 = n_2\xi_2 + ... + n_8\xi_8.$$

We compute

$$\rho_{\mathfrak{k}} = 14\psi_1 + 28\psi_2 + 35\psi_3 + 55\psi_4 + 46\psi_5 + 36\psi_6 + 25\psi_7 + 13\psi_8\,,$$
$$\rho_{\mathfrak{g}} = 46\psi_1 + 68\psi_2 + 91\psi_3 + 135\psi_4 + 110\psi_5 + 84\psi_6 + 57\psi_7 + 29\psi_8\,, \text{ and}$$
$$\mu\ \ = 2\psi_1 + 3\psi_2 + 4\psi_3 + 6\psi_4 + 5\psi_5 + 4\psi_6 + 3\psi_7 + 2\psi_8\,.$$

It follows that the sufficient G_0–negativity condition of Theorems 2.12 and 5.14 is that both

$$t < -\tfrac{1}{2}(3n_2 + 4n_3 + 6n_4 + 5n_5 + 4n_6 + 3n_7 + 2n_8) - \tfrac{29}{2} \text{ and } t < -\langle\gamma_0 + \rho_{\mathfrak{g}}, w_{\mathfrak{l}}^0(\nu)\rangle.$$

On the other hand, the sufficient K_0–negativity condition of Corollary 5.11 is

$$t < -\tfrac{1}{2}(3n_2 + 4n_3 + 6n_4 + 5n_5 + 4n_6 + 3n_7 + 2n_8) + \tfrac{1}{2}.$$

Thus in the interval

$$M - \tfrac{29}{2} \leqq t < M + \tfrac{1}{2}\,,\ \ M = -\tfrac{1}{2}(3n_2 + 4n_3 + 6n_4 + 5n_5 + 4n_6 + 3n_7 + 2n_8),$$

we still have all the K_0–types as Dolbeault cohomology groups on Y, even though the Dolbeault cohomology group carrying the G_0–representation no longer is infinitesimally equivalent to a discrete series representation. Recall that we also saw this sort of phenomenon in Example 6.7. We shall study in more detail this phenomenon in a sequel to the present paper, making it the starting point of a geometric construction of unitary modules in the analytic continuation of the Borel – de Siebenthal discrete series. The continuation (in the parameter t) of the corresponding Harish–Chandra module will become reducible and a study of the reduction points will indicate just when the composition factor at the bottom of the (necessarily finite) composition series is the Harish–Chandra module of an irreducible unitary representation.

In order to see the K_0–types one must decompose the Q–modules $E_{\gamma_t} \otimes S^m(\mathfrak{u}_{-1})$ into indecomposables. The starting point is the decomposition of the $S^m(\mathfrak{u}_{-1})$ into irreducible L–modules. Let's look at this more closely for our E_{8,D_8}. We saw in Section 3.20 that the representation τ_{-1} of $\mathfrak{l}$ on $\mathfrak{u}_{-1}$ is
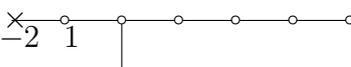
 . In order to apply LiE to symmetric powers $S^m(\tau_{-1})$ we note that the center takes care of itself and the $S^m(\tau_{-1}|_{[\mathfrak{l},\mathfrak{l}]})$ are given by

setdefault(E7) followed by sym_tensor(m,[0,0,0,0,0,0,1]). Here
$$[a,b,c,d,e,f,g]$$

indicates the representation of highest weight , so
$$[0,0,0,0,0,0,1]$$
indicates one of the half–spin representations, and kX[a,b,c,d,e,f,g] means that
it occurs with multiplicity k. LiE tells us
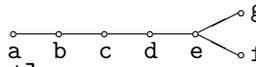
```
> sym_tensor(2,[0,0,0,0,0,0,1])
    1X[0,0,0,0,0,0,2] +1X[0,0,1,0,0,0,0]

> sym_tensor(3,[0,0,0,0,0,0,1])
    1X[0,0,0,0,0,0,3] +1X[0,0,1,0,0,0,1] +1X[1,0,0,0,0,0,1]

> sym_tensor(4,[0,0,0,0,0,0,1])
    1X[0,0,0,0,0,0,4] +1X[0,0,0,1,0,0,0] +1X[0,0,1,0,0,0,2] +
    1X[0,0,2,0,0,0,0] +1X[1,0,0,0,0,0,2] +1X[1,0,0,0,1,0,0] +
    1X[2,0,0,0,0,0,0]

> sym_tensor(5,[0,0,0,0,0,0,1])
    1X[0,0,0,0,0,0,5] +1X[0,0,0,1,0,0,1] +1X[0,0,1,0,0,0,3] +
    1X[0,0,1,0,0,1,0] +1X[0,0,2,0,0,0,1] +1X[1,0,0,0,0,0,3] +
    1X[1,0,0,0,0,1,0] +1X[1,0,0,0,1,0,1] +1X[1,0,1,0,0,0,1] +
    1X[2,0,0,0,0,0,1]

> sym_tensor(6,[0,0,0,0,0,0,1])
    1X[0,0,0,0,0,0,6] +1X[0,0,0,0,0,2,0] +1X[0,0,0,1,0,0,2] +
    1X[0,0,1,0,0,0,0] +1X[0,0,1,0,0,0,4] +1X[0,0,1,0,0,1,1] +
    1X[0,0,1,1,0,0,0] +1X[0,0,2,0,0,0,2] +1X[0,0,3,0,0,0,0] +
    1X[0,1,0,0,1,0,0] +1X[1,0,0,0,0,0,4] +1X[1,0,0,0,0,1,1] +
    1X[1,0,0,0,1,0,2] +1X[1,0,0,1,0,0,0] +1X[1,0,1,0,0,0,2] +
    1X[1,0,1,0,1,0,0] +2X[2,0,0,0,0,0,2] +1X[2,0,1,0,0,0,0]

> sym_tensor(7,[0,0,0,0,0,0,1])
    1X[0,0,0,0,0,0,7] +1X[0,0,0,0,0,1,0] +1X[0,0,0,0,0,2,1] +
    1X[0,0,0,1,0,0,3] +1X[0,0,0,1,0,1,0] +1X[0,0,1,0,0,0,1] +
    1X[0,0,1,0,0,0,5] +1X[0,0,1,0,0,1,2] +1X[0,0,1,1,0,0,1] +
    1X[0,0,2,0,0,0,3] +1X[0,0,2,0,0,1,0] +1X[0,0,3,0,0,0,1] +
    1X[0,1,0,0,0,1,0] +1X[0,1,0,0,1,0,1] +1X[0,1,0,1,0,1,0] +
    1X[1,0,0,0,0,0,5] +1X[1,0,0,0,0,1,2] +1X[1,0,0,0,1,0,3] +
    2X[1,0,0,1,0,0,1] +1X[1,0,1,0,0,0,3] +1X[1,0,1,0,0,1,0] +
    1X[1,0,1,0,1,0,1] +1X[1,0,2,0,0,0,1] +1X[1,1,0,0,0,0,1] +
    2X[2,0,0,0,0,0,3] +1X[2,0,0,0,1,0,1] +1X[2,0,1,0,0,0,1] +
    1X[3,0,0,0,0,0,1]

> sym_tensor(8,[0,0,0,0,0,0,1])
    1X[0,0,0,0,0,0,0] +1X[0,0,0,0,0,0,8] +1X[0,0,0,0,0,1,1] +
    1X[0,0,0,0,0,2,2] +1X[0,0,0,1,0,0,0] +1X[0,0,0,1,0,0,4] +
    1X[0,0,0,1,0,1,1] +2X[0,0,0,2,0,0,0] +1X[0,0,1,0,0,0,2] +
    1X[0,0,1,0,0,0,6] +1X[0,0,1,0,0,1,3] +1X[0,0,1,0,0,2,0] +
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1X[0,0,1,1,0,0,2] +1X[0,0,2,0,0,0,0] +1X[0,0,2,0,0,0,4] +
1X[0,0,2,0,0,1,1] +1X[0,0,2,1,0,0,0] +1X[0,0,3,0,0,0,2] +
1X[0,0,4,0,0,0,0] +2X[0,1,0,0,0,1,1] +1X[0,1,0,0,1,0,2] +
1X[0,1,0,1,0,0,0] +1X[0,1,0,1,0,1,1] +1X[0,1,1,0,1,0,0] +
1X[0,2,0,0,0,0,0] +1X[1,0,0,0,0,0,6] +1X[1,0,0,0,0,1,3] +
1X[1,0,0,0,1,0,0] +1X[1,0,0,0,1,0,4] +2X[1,0,0,1,0,0,2] +
1X[1,0,0,1,1,0,0] +1X[1,0,1,0,0,0,4] +2X[1,0,1,0,0,1,1] +
1X[1,0,1,0,1,0,2] +1X[1,0,1,1,0,0,0] +1X[1,0,2,0,0,0,2] +
1X[1,0,2,0,1,0,0] +1X[1,1,0,0,0,0,2] +1X[1,1,0,0,1,0,0] +
2X[2,0,0,0,0,0,4] +1X[2,0,0,0,0,1,1] +1X[2,0,0,0,1,0,2] +
1X[2,0,0,0,2,0,0] +1X[2,0,0,1,0,0,0] +2X[2,0,1,0,0,0,2] +
1X[2,0,2,0,0,0,0] +1X[3,0,0,0,0,0,2] +1X[3,0,0,0,1,0,0] +
1X[4,0,0,0,0,0,0]
```

etc. From this one notes a few small patterns. Of course [0,0,0,0,0,0,m] occurs just once in sym_tensor(m,[0,0,0,0,0,0,1]), but for $m > 1$ there is no [a,b,c,d,e,f,m-1]; and the only [a,b,c,d,e,f,m-2] are [0,0,1,0,0,0,m-2] and [1,0,0,0,0,0,m-2]. Further there is no [a,b,c,d,e,f,m-3]; and the only [a,b,c,d,e,f,m-4] are [0,0,0,1,0,0,m-4], [0,0,2,0,0,0,m-4], [1,0,0,0,1,0,m-4] and [2,0,0,0,0,0,m-4] for $m \geqq 4$, together with [1,0,1,0,0,0,m-4] for $m > 4$. Patterns of this sort continue up through the [a,b,c,d,e,f,m-7], but even they break down for the [a,b,c,d,e,f,m-k] with $m \geqq k \geqq 8$. Thus one cannot explect a clean formula for the decomposition of the $S^m(\mathfrak{u}_{-1})$ into irreducible L–modules. $\diamondsuit$

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