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Characterizations of Heisenberg-like Lie Algebras

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Abstract. We obtain characterizations of Heisenberg-like Lie algebras which are generalizations of results on Lie algebras of Heisenberg type, including a characterization for Heisenberg-like Lie algebras in terms of the curvature transformation. We also establish infinite families of examples of Lie algebras which are Heisenberg-like, but not Heisenberg type, including examples arising from representations of $\mathfrak{su}(2)$.
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1. Introduction

The purpose of this paper is to continue the investigation of the geometry of a two-step nilpotent Lie group endowed with a left-invariant metric, following the approach of P. Eberlein and others [E1, K1, K2, GM, LP, M, DeC, DeM]. We focus on Heisenberg-like Lie groups, introduced in [GM] as a natural generalization of Lie groups of Heisenberg type. Two-step nilpotent Lie groups have a structure that is both non-trivial and accessible. We study the geometry of a two-step nilpotent metric Lie group N by investigating its associated Lie algebra $\mathfrak{n}$. The main tool we use is the j-operator, first introduced by A. Kaplan [K1]; this operator determines a set of skew-symmetric linear transformations acting on the complement of the center of $\mathfrak{n}$ which completely determine the geometry of N endowed with a left-invariant metric.

The Heisenberg type Lie groups are considered model spaces in the class of simply connected, two-step nilpotent metric Lie groups; they play an important role in areas of research such as geometric analysis and mathematical physics. Heisenberg-like Lie groups were introduced by R. Gornet and M. Mast [GM] as a natural generalization, in a variety of ways, from the Heisenberg type condition. These generalizations include the formulation of the length spectrum of the resulting nilmanifold and the prevalence of periodic geodesics contained in three-dimensional totally geodesic submanifolds. We continue this investigation, establishing equivalent approaches to the Heisenberg-like condition and characterizing the Heisenberg-like Lie groups in terms of the curvature transformation. In

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[K2], Kaplan showed that there exist infinitely many Lie algebras of Heisenberg type with center of arbitrary dimension. We establish a similar result for the Heisenberg-like case. Finally, we construct new classes of examples of Heisenberg-like Lie groups, including a class of examples arising from representations of $\mathfrak{su}(2)$.

The outline of the paper is as follows. In section 2, we review basic information about two-step nilpotent Lie groups and introduce necessary notation. In Section 3 we strengthen a result of [E1] on Heisenberg type Lie groups and characterize Heisenberg-like Lie groups similarly. Additionally, we exploit a characteristic of Heisenberg-like Lie algebras to write the curvature transformation in terms of the eigenvalues of the transformations mentioned above. In the last three sections, we use representation theory and matrix algebras to construct infinite families of examples of Heisenberg-like Lie algebras.

2. Notation and Background

Let $\mathfrak{n}$ denote a finite-dimensional real Lie algebra with Lie bracket $[\ ,\ ]$. We say that $\mathfrak{n}$ is two-step nilpotent if $\mathfrak{n}$ is not abelian but $[X,[Y,W]] = 0$ for all $X, Y, W \in \mathfrak{n}$. Let N denote the unique, simply connected Lie group with Lie algebra $\mathfrak{n}$; N is said to be two-step nilpotent when $\mathfrak{n}$ is two-step nilpotent. By Raghunathan [R], the Lie group exponential map $\exp : \mathfrak{n} \to N$ is a diffeomorphism for two-step nilpotent $\mathfrak{n}$, with inverse $\log : N \to \mathfrak{n}$. Let $\mathfrak{z}$ denote the center of $\mathfrak{n}$.

Definition 2.1. [E1] A two-step nilpotent Lie algebra $\mathfrak{n}$ is *(strictly) nonsingular* if for every $Z \in \mathfrak{z}$ and for every $X \in \mathfrak{n} - \mathfrak{z}$, there exists $Y \in \mathfrak{n}$ such that $[X, Y] = Z$.

When $\mathfrak{n}$ has an inner product $\langle\, ,\rangle$, the notion of nonsingularity can be refined further; the key approach is to use a set of skew symmetric linear transformations which capture the geometry of N with a left invariant metric. Let $\{\mathfrak{n}, \langle\ ,\ \rangle\}$ have center $\mathfrak{z}$. We denote $\mathfrak{v} = \mathfrak{z}^{\perp}$ and write $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$.

Definition 2.2. For each nonzero $Z \in \mathfrak{z}$ define a skew symmetric linear transformation $j(Z) : \mathfrak{v} \to \mathfrak{v}$ by

$$\langle [X, Y], Z\rangle = \langle j(Z)X, Y\rangle \text{ for all } X, Y \in \mathfrak{v}.$$

Equivalently, let $\mathrm{ad}X(Y)$ be the Lie algebra adjoint defined by $\mathrm{ad}X(Y) = [X, Y]$ and let $(\mathrm{ad}X)^*$ denote the (metric) adjoint of $\mathrm{ad}X$. Then the map $j : \mathfrak{z} \to so(\mathfrak{v})$ defined by $j(Z)X = (\mathrm{ad}X)^*Z$ is a linear transformation such that $\langle [X, Y], Z\rangle = \langle j(Z)X, Y\rangle$. Observe that the skew symmetry of the Lie bracket leads to the skew symmetry of the $j(Z)$ maps for all $Z \in \mathfrak{z}$ by the following:

$$\langle j(Z)X, Y\rangle = \langle [X, Y], Z\rangle = -\langle [Y, X], Z\rangle = -\langle j(Z)Y, X\rangle.$$

The $j(Z)$ maps were first defined by Kaplan [K1] and were used extensively by Eberlein [E1] to investigate the geometry of metric two-step nilpotent Lie groups. These maps have proved to be valuable tools in understanding this geometry.

We use the $j(Z)$ maps to further refine the nonsingularity condition.

Definition 2.3. Let $\mathfrak{n}$ denote a two-step nilpotent metric Lie algebra.

1. If $j(Z)$ is nonsingular for every nonzero $Z \in \mathfrak{z}$, then $\mathfrak{n}$ is said to be *(strictly) nonsingular.*
2. If $j(Z)$ is nonsingular for every Z in an open dense subset of $\mathfrak{z}$, then $\mathfrak{n}$ is said to be *almost nonsingular.*
3. If $j(Z)$ is singular for all $Z \in \mathfrak{z}$, then $\mathfrak{n}$ is said to be *(strictly) singular.*

Proposition 2.4 ([GM], Lemma 1.16)**.** *Every two-step nilpotent Lie algebra is nonsingular, almost nonsingular, or singular.*

Remark 2.5. (See [E1], Proposition 3.5 and [GM] Definition 1.17)Let $\{\mathfrak{n}, \langle\ ,\ \rangle\}$ denote a two-step nilpotent Lie algebra and let $Z \in \mathfrak{z}$.

1. Denote the number of distinct eigenvalues of $j(Z)^2$ by $\mu(Z)$. We write $\mu = \mu(Z)$ when convenient.
2. Denote the μ distinct eigenvalues of $j(Z)^2$ by $\{-\vartheta_1(Z)^2, \ldots, -\vartheta_\mu(Z)^2\}$, where $0 \le \vartheta_1(Z)^2 < \vartheta_2(Z)^2 < \cdots < \vartheta_\mu(Z)^2$. Then the distinct eigenvalues of $j(Z)$ are $\{\pm i\vartheta_1(Z), \ldots, \pm i\vartheta_\mu(Z)\}$.
3. Let $W_m(Z)$ be the invariant subspace of $j(Z)$ associated to $\vartheta_m(Z)$, $m = 1, \ldots, \mu(Z)$. By skew-symmetry of $j(Z)$, $\mathfrak{v}$ is the direct sum of the invariant spaces $W_m(Z)$.

The distinct eigenvalues of $j(Z)^2$ give information about the geometry of N. In general, the number of distinct eigenvalues of $j(Z)^2$ can vary as Z varies. However, this number is constant on an open, dense subset of $\mathfrak{z}$, as the next result shows.

Theorem 2.6 ([GM], Prop. 1.19)**.** *Let $\mathfrak{U} = \{Z \in \mathfrak{z}$: there exists an open neighborhood $\mathfrak{O}$ of Z such that μ is constant on $\mathfrak{O}\}$. Then $\mathfrak{U}$ is an open, dense subset of $\mathfrak{z}$ and $\mu(Z)$, the number of distinct eigenvalues of $j(Z)^2$, is constant on $\mathfrak{U}$.*

3. Heisenberg type and Heisenberg-like Lie groups

3.1. Heisenberg type Lie groups.

Lie groups of Heisenberg type are a generalization of the well-studied Heisenberg group. Groups of Heisenberg type were introduced by Kaplan in [K1] and their geometry has been studied extensively, including in [E1], [E2] and [K2].

Definition 3.1. A two-step nilpotent metric Lie algebra $\mathfrak{n}$ is *of Heisenberg type* if

$$j(Z)^2 = -|Z|^2 Id|_{\mathfrak{v}} \tag{1}$$

for every choice of $Z \in \mathfrak{z}$. A simply connected two-step nilpotent metric Lie group N is *of Heisenberg type* if its Lie algebra $\mathfrak{n}$ is of Heisenberg type.

The next result shows the strong relationship between the behavior of the $j(Z)$-maps and the Heisenberg type condition. Eberlein [E1](1.7) established that the identities below hold when N is of Heisenberg type; we generalize the result to an if and only if statement.

Lemma 3.2. *Let N be a two-step nilpotent metric Lie group with Lie algebra $\mathfrak{n}$. The following are equivalent:*

a. $\mathfrak{n}$ *is of Heisenberg type.*

b. $j(Z)j(Z^*) + j(Z^*)j(Z) = -2\langle Z, Z^*\rangle Id|_{\mathfrak{v}}$ *for all* $Z, Z^* \in \mathfrak{z}$

c. $\langle j(Z)X, j(Z^*)X\rangle = \langle Z, Z^*\rangle |X|^2$ *for all* $Z, Z^* \in \mathfrak{z}, X \in \mathfrak{v}$

d. $\langle j(Z)X, j(Z)Y\rangle = |Z|^2\langle X, Y\rangle$ *for all* $Z \in \mathfrak{z}, X, Y \in \mathfrak{v}$

e. $|j(Z)X| = |Z||X|$ *for all* $Z \in \mathfrak{z}, X \in \mathfrak{v}$

f. $[X, j(Z)X] = |X|^2 Z$ *for all* $X \in \mathfrak{v}, Z \in \mathfrak{z}$.

Proof. [a. $\Longrightarrow$ b.] Apply the Heisenberg type condition to $j(Z + Z^*)^2 X$.
[b. $\Longrightarrow$ c.] From (b), $\langle (j(Z)j(Z^*)+j(Z^*)j(Z))X, X\rangle = -2\langle Z, Z^*\rangle |X|^2$. From the definition of $j(Z)$, $\langle j(Z^*)j(Z)X, X\rangle = -\langle j(Z^*)X, j(Z)X\rangle$. Then (c) follows.
[c. $\Longrightarrow$ d.] Apply (c) to $\langle j(Z)(X + Y), j(Z)(X + Y)\rangle$.
[d. $\Longrightarrow$ e.] Follows directly by letting $Y = X$.
[e. $\Longrightarrow$ f.] Apply (e) to $|j(Z+Z^*)X|$ to obtain $\langle j(Z)X, j(Z^*)X\rangle = \langle Z, Z^*\rangle |X|^2$. Therefore, if $Z \perp Z^*$, then $\langle [X, j(Z)X], Z^*\rangle = \langle j(Z)X, j(Z^*)X\rangle = 0$. It follows that when part (e) holds, $[X, j(Z)X]$ is a multiple of Z. Using part (e) again, $[X, j(Z)X] = |X|^2 Z$.
[f. $\Longrightarrow$ a.] Apply (f) to $X + Y$ to obtain $[X + Y, j(Z)(X + Y)] = |X + Y|^2 Z$. Therefore $|j(Z)(X+Y)|^2 = \langle [X+Y, j(Z)(X+Y)], Z\rangle = |X+Y|^2|Z|^2$. Expanding this and applying (f) again, we obtain $-\langle j(Z)^2X, Y\rangle = |Z|^2\langle X, Y\rangle$. Thus, if $X \perp Y$ then $\langle j(Z)^2X, Y\rangle = 0$. It follows that $j(Z)^2X$ is a multiple of X. Using (f), $\langle [X, j(Z)X], Z\rangle = |Z|^2|X|^2$. But $\langle [X, j(Z)X], Z\rangle = -\langle j(Z)^2X, X\rangle$. It follows that $j(Z)^2X = -|Z|^2X$, and so $\mathfrak{n}$ is of Heisenberg type. ∎

3.2. Heisenberg-like Lie groups.

Heisenberg-like Lie groups were introduced in [GM] as a generalization of Lie groups of Heisenberg type. The Lie groups that are Heisenberg-like reflect several of the properties of Heisenberg type Lie groups. By the following definitions and results, particularly Theorem 3.7, a Lie group of Heisenberg type is also a Heisenberg-like Lie group.

Definition 3.3. Let N denote a two-step nilpotent Lie group with a left-invariant metric. A simply connected subgroup H of N is said to be *totally geodesic* if every geodesic that starts in H remains in H. Equivalently, if $\mathfrak{h}$ denotes the Lie algebra of H, then H is *totally geodesic* if for all $X, Y \in \mathfrak{h}$, $\nabla_X Y \in \mathfrak{h}$.

If N is Heisenberg type, then for any $X \in \mathfrak{v}$ and $Z \in \mathfrak{z}$ the subalgebra given by $\mathrm{span}_{\mathbb{R}}\{X, j(Z)X, Z\}$ is totally geodesic [E2]. We obtain Heisenberg-like Lie groups by slightly generalizing this property.

Definition 3.4. A two-step nilpotent metric Lie algebra $\{\mathfrak{n}, \langle\ ,\ \rangle\}$ is *Heisenberg-like* if the subalgebra $\mathrm{span}_{\mathbb{R}}\{X_m, j(Z)X_m, Z\}$ is totally geodesic for every $Z \in \mathfrak{z}$ and every $X_m \in W_m(Z)$, $m = 1, \ldots, \mu$. A two-step nilpotent metric Lie group is *Heisenberg-like* if and only if its Lie algebra is.

Some results from [GM] about Heisenberg-like Lie algebras follow.

Lemma 3.5 ([GM], Lemma 3.3)**.** *A two-step nilpotent metric Lie algebra $\{\mathfrak{n}, \langle\ ,\ \rangle\}$ is Heisenberg-like if and only if $[j(Z)X_m, X_m] \in \mathrm{span}_{\mathbb{R}}\{Z\}$ for all $Z \in \mathfrak{z}$ and all $X_m \in W_m(Z)$, $m = 1, \ldots, \mu$.*

Lemma 3.5 generalizes the property given in equation (1) of definition 3.1. The next result eliminates the possibility that a Lie algebra could be Heisenberg-like and also almost nonsingular. Note that all Lie algebras of Heisenberg type are nonsingular.

Theorem 3.6 ([GM], Theorem 3.6)**.** *A Heisenberg-like metric Lie algebra is either strictly nonsingular or strictly singular.*

A significant property of Heisenberg-like groups is that the eigenvalues of the $j(Z)$ maps depend on the norm of Z, analogous to the Heisenberg type case.

Theorem 3.7 ([GM], Theorem 3.7)**.** *A two-step nilpotent metric Lie algebra $\{\mathfrak{n}, \langle\ ,\ \rangle\}$ is Heisenberg-like if and only if for every $i = 1, \ldots, \mu = \mu(\mathfrak{U})$ there is a constant $c_i \geq 0$ such that for every nonzero $Z \in \mathfrak{z}$, $\vartheta_i(Z) = c_i|Z|$.*

Note that both conditions in the theorem above imply that $\mathfrak{U} = \mathfrak{z} - \{0\}$. For clarity, we state this explicitly in the next result. Note also that when $\mathfrak{U} = \mathfrak{z} - \{0\}$ then we write μ instead of $\mu(\mathfrak{U})$.

In Lemma 3.2, we showed that certain properties related to the $j(Z)-$maps are equivalent to the Heisenberg type condition. This Lemma can be generalized to the Heisenberg-like case, with one exception which we address below.

Theorem 3.8. *Let N be a two-step nilpotent metric Lie group with Lie algebra $\mathfrak{n}$. Then $\mathfrak{n}$ is Heisenberg-like if and only if $\mathfrak{U} = \mathfrak{z} - \{0\}$ and any one of the following hold:*

a. For every $m \in \{1, \ldots, \mu = \mu(\mathfrak{U})\}$

$$[X_m, j(Z)X_m] = \left(\frac{\vartheta_m(Z)|X_m|}{|Z|}\right)^2 Z$$

for all nonzero $Z \in \mathfrak{z}$ and all $X_m \in W_m(Z)$.

b. *For every* $m \in \{1, \ldots, \mu = \mu(\mathfrak{U})\}$ *there exists a constant* $c_m \geq 0$ *such that*

$$\langle j(Z)X_m, j(Z^*)X_m\rangle = c_m^2\langle Z, Z^*\rangle |X_m|^2$$

for all $Z, Z^* \in \mathfrak{z}$ *with* $Z \neq 0$ *and all* $X_m \in W_m(Z)$.

c. *For every* $m \in \{1, \ldots, \mu = \mu(\mathfrak{U})\}$ *there exists a constant* $c_m \geq 0$ *such that*

$$\langle j(Z)X_m, j(Z)Y_m\rangle = c_m^2\langle X_m, Y_m\rangle |Z|^2$$

for all nonzero $Z \in \mathfrak{z}$ *and all* $X_m,\ Y_m \in W_m(Z)$.

d. *For every* $m \in \{1, \ldots, \mu = \mu(\mathfrak{U})\}$ *there exists a constant* $c_m \geq 0$ *such that*

$$|j(Z)X_m| = c_m|X_m||Z|$$

for all nonzero $Z \in \mathfrak{z}$ *and all* $X_m \in W_m(Z)$.

Proof. By Lemma 3.5, $\{\mathfrak{n}, \langle\ ,\ \rangle\}$ is Heisenberg-like if and only if $[j(Z)X_m, X_m] \in \mathrm{span}_{\mathbb{R}}\{Z\}$ for all $Z \in \mathfrak{z}$ and all $X_m \in W_m(Z)$, $m = 1, \ldots, \mu$. Since

$$\langle [j(Z)X_m, X_m], Z\rangle = \langle j(Z)^2X_m, X_m\rangle = -\langle \vartheta_m(Z)^2X_m, X_m\rangle,$$

part (a) follows. To show (b), let $\{\mathfrak{n}, \langle\ ,\ \rangle\}$ be Heisenberg-like and $c_1, \ldots, c_\mu$ be the constants given by Theorem 3.7. Thus $j(Z)^2|_{W_m(Z)} = -c_m^2|Z|Id|_{W_m(Z)}$ for each $Z \neq 0$. When Z^* is orthogonal to Z one has

$$\langle j(Z)X_m, j(Z^*)X_m\rangle = \langle Z^*, [X_m, j(Z)X_m]\rangle = 0$$

by Lemma 3.5. So (b) holds in this case. Next for a given $Z \neq 0$ and $Z^* \in \mathfrak{z}$ write $Z^* = aZ + Z'$ where $\langle Z, Z'\rangle = 0$ and $a = \langle Z, Z^*\rangle/|Z|^2$. Now

$$\begin{aligned}\langle j(Z)X_m, j(Z^*)X_m\rangle &= a\langle j(Z)X_m, j(Z)X_m\rangle + \langle j(Z)X_m, j(Z')X_m\rangle \\ &= -a\langle j(Z)^2X_m, X_m\rangle \text{ (by above and skew symmetry of } j(Z)) \\ &= c_m^2\langle Z, Z^*\rangle|X_m|^2\end{aligned}$$

as claimed. The remaining statements follow in a similar manner from Lemma 3.5 and Theorem 3.7. ∎

In Lemma 3.2 we showed that N is of Heisenberg-type if and only if

$$j(Z)j(Z^*) + j(Z^*)j(Z) = -2\langle Z, Z^*\rangle \mathrm{Id}|_{\mathfrak{v}} \text{ for all } Z, Z^* \in \mathfrak{z}.$$

The natural generalization of this statement in the Heisenberg-like case is that for all $Z, Z^* \in \mathfrak{z}$,

$$j(Z)j(Z^*) + j(Z^*)j(Z) = -2c_m^2\langle Z, Z^*\rangle \mathrm{Id} \text{ on } W_m(Z). \tag{2}$$

Unfortunately, this does not hold. Example 1.15 of [GM] is Heisenberg-like, but (2) will not hold for all $Z, Z^* \in \mathfrak{z}$. This example is described below in Example 5.2. If equation (2) does hold, the following Lemma shows that $\mathfrak{n}$ must either have one-dimensional center or be of Heisenberg type up to scaling.

Lemma 3.9. *Let $\{\mathfrak{n}, \langle\ ,\ \rangle\}$ be a two-step nilpotent metric Lie algebra. Suppose there exist non-negative constants c_m, $m = 1, \ldots, \mu$, such that for each $Z \in \mathfrak{z}$,*

$$j(Z)j(Z^*) + j(Z^*)j(Z) = -2c_m^2 \langle Z, Z^* \rangle \mathrm{Id} \text{ on } W_m(Z)$$

for all $Z^* \in \mathfrak{z}$. Then either $\dim \mathfrak{z} = 1$ and $\mathfrak{n}$ is Heisenberg-like or there is a constant $c > 0$ such that $j(Z)^2 = -c^2|Z|^2 Id|_{\mathfrak{v}}$ for all $Z \in \mathfrak{z}$.

Proof. If $\dim \mathfrak{z} = 1$ then $\mathfrak{n}$ must be Heisenberg-like by Theorem 3.7.

Suppose that $\dim \mathfrak{z} > 1$ and let $Z, Z^* \in \mathfrak{z}$ be non-zero. We may assume that Z and Z^* are not collinear and that they are not orthogonal. Furthermore, we may refine our choice of Z so that $Z \in \mathfrak{U}$ with $\mu(Z) > 1$; if this is not possible, then $\mathfrak{n}$ is Heisenberg type up to scaling and we are done.

Let $X \in W_k(Z)$, where $k > 1$. Since $\mathfrak{v} = \oplus_{m=1}^{\mu} W_m(Z^*)$, there exist $X_m^* \in W_m(Z^*), m = 1, ..., \mu$ such that

$$X = X_1^* + \cdots + X_\mu^*.$$

By the assumption in the statement of the lemma,

$$(j(Z)j(Z^*) + j(Z^*)j(Z))X = -2c_k^2 \langle Z, Z^* \rangle X.$$

Note that $c_k \neq 0$ by our assumption on k. Combining the previous two statements, we have

$$\begin{aligned} (j(Z)j(Z^*) + j(Z^*)j(Z))X &= (j(Z)j(Z^*) + j(Z^*)j(Z)) \sum_{m=1}^{\mu} X_m^* \\ &= -2 \sum_{m=1}^{\mu} c_m^2 \langle Z, Z^* \rangle X_m^*. \end{aligned}$$

Therefore,

$$\sum_{m=1}^{\mu} X_m^* = X = \sum_{m=1}^{\mu} \left(\frac{c_m^2}{c_k^2} \right) X_m^*. \tag{3}$$

We have two cases. If $c_1 \neq 0$ then $c_m^2 = c_k^2$ for all $m = 1, ..., \mu$. Hence $j(Z^*)^2$ is nonsingular and $j(Z^*)^2 = -c_k^2 |Z^*|^2 Id|_{\mathfrak{v}}$. Therefore, for all Z in $\mathfrak{z}$, $j(Z)^2 = -c_k^2|Z|^2 Id|_{\mathfrak{v}}$.

For the second case assume $c_1 = 0$. Then $c_m^2 = c_k^2$ for $m = 2, ..., \mu$. It follows that $j(Z^*)^2$ has two invariant subspaces, $W_1(Z^*) = \ker j(Z^*)$ and $W_2(Z^*) = \{X \in \mathfrak{v} : j(Z^*)^2 X = -c^2|Z^*|\}$ where $c^2 = c_2^2$. Using the symmetry between Z and Z^* in the assumption of this lemma, $j(Z)^2$ also has two invariant subspaces, $W_1(Z), W_2(Z)$. Furthermore, again using the decomposition of $X \in W_2(Z)$ and equation (3), we see that $W_2(Z)$ is orthogonal to $W_1(Z^*)$ and so $W_1(Z) = W_1(Z^*)$.

Now consider $Z^* \perp Z$. Since $\dim \mathfrak{z} > 1$ there exists nonzero $Z_0 \in \mathfrak{z}$ such that Z_0 is neither orthogonal nor colinear to either Z or Z^*. By the above argument, $W_i(Z) = W_i(Z_0) = W_i(Z^*)$, $i = 1, 2$.

We will show that $W_1(Z) = 0$ for all $Z \in \mathfrak{z}$. Let $X \in W_1(Z)$ and let $\{Z_1, \ldots Z_p\}$ be an orthonormal basis for $\mathfrak{z}$. Then for all $Y \in \mathfrak{v}$,

$$[X,Y] = \sum_{i=1}^{p} \langle [X,Y], Z_i \rangle Z_i = \sum_{i=1}^{p} \langle j(Z_i)X, Y \rangle = 0,$$

since $X \in \ker j(Z)$ for all Z. So $X \in W_1(Z)$ implies $X \in \mathfrak{z}$. Since $W_1(Z)$ and $\mathfrak{z}$ are orthogonal, $X = 0$.

Hence, $\mathfrak{n}$ is nonsingular and, as above, therefore, $j(Z)^2 = -c^2|Z|^2 Id|_{\mathfrak{v}}$ for all $Z \in \mathfrak{z}$. That is, $\mathfrak{n}$ is Heisenberg type up to scaling. ■

3.3. Curvature Transformations in the Heisenberg-like case. Let ξ_1, ξ_2, ξ_3 denote left-invariant vector fields on N. The curvature tensor is given by

$$R(\xi_1, \xi_2)\xi_3 = -\nabla_{[\xi_1,\xi_2]}\xi_3 + \nabla_{\xi_1}(\nabla_{\xi_2}\xi_3) - \nabla_{\xi_2}(\nabla_{\xi_1}\xi_3).$$

The curvature transformation R_ξ is defined by $R_\xi \nu = R(\nu, \xi)\xi$ for left invariant vector fields ξ, ν on N. Eberlein showed the following.

Lemma 3.10 ([E2], Lemma B). *Let $\{\mathfrak{n}, \langle\ ,\ \rangle\}$ be a nonsingular, two-step nilpotent metric Lie algebra. Then*

a. *$\{\mathfrak{n}, \langle\ ,\ \rangle\}$ is of Heisenberg type if and only if there exists a positive constant α such that*
$$R_Z|_{\mathfrak{v}} = \alpha |Z|^2 Id|_{\mathfrak{v}} \text{ for all } Z \in \mathfrak{z}.$$

b. *If $\{\mathfrak{n}, \langle\ ,\ \rangle\}$ is of Heisenberg type up to scaling with $j(Z)^2 = -\lambda|Z|^2 Id$ for some positive constant λ and all $Z \in \mathfrak{z}$, then $[X, j(Z)X] = \lambda|X|^2 Z$ for all $X \in \mathfrak{v}, Z \in \mathfrak{z}$. In particular,*
$$R_X|_{\mathfrak{z}} = \frac{1}{4}\lambda |X|^2 Id|_{\mathfrak{z}} \text{ for every } X \in \mathfrak{v}.$$

We generalize this result to the Heisenberg-like case:

Lemma 3.11. *A two-step nilpotent metric Lie algebra $\{\mathfrak{n}, \langle\ ,\ \rangle\}$ is Heisenberg-like if and only if there exist positive constants c_m such that $R_Z(X_m) = \frac{1}{4}c_m^2|Z|^2 X_m$, where $X_m \in W_m(Z)$, $m = 1, \ldots, \mu$.*

Proof. Using 1.11b in [E2], we calculate $R_Z(X_m) = R(X_m, Z)Z = -\frac{1}{4}j(Z)^2 X_m$ $= \frac{1}{4}\vartheta_m^2(Z)X_m$. By Theorem 3.7, $R_Z(X_m) = \frac{1}{4}c_m^2|Z|^2 X_m$ if and only if $\mathfrak{n}$ is Heisenberg-like. ■

3.4. Dimension of the center. Kaplan [K2] states that there are infinitely many Lie algebras of Heisenberg type with center $\mathfrak{z}$ of arbitrary dimension. An analogous result can be established for Heisenberg-like Lie algebras, using the families of examples in Example 6.3.

Proposition 3.12. *For each integer $n \geq 2$, there exist infinitely many Lie algebras which are Heisenberg-like, but not Heisenberg type, having center $\mathfrak{z}$ of dimension n.*

Proof. Fix an integer $n \geq 2$. Let k be any integer, $k \geq 1$. From Example 6.3 below, there exists a singular Heisenberg-like Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ such that $\dim \mathfrak{v} = k(n+1)$ and $\dim \mathfrak{z} = n$. Since $\mathfrak{n}$ is singular, it is not of Heisenberg type. As k varies the dimension of $\mathfrak{n}$ varies. The result follows. ■

4. A Class of Heisenberg-Like Lie Algebras from Representations of $\mathfrak{su}(2)$

A method for constructing a two-step nilpotent Lie algebra $\mathfrak{n}$ from an irreducible representation of a compact Lie group G was given in [EH] and also studied in [L]. The geodesic properties of these two-step nilpotent Lie algebras were studied in detail in the case of $\mathfrak{su}(2)$ in [DeM] and in the case of compact simple G in [DeC]. We give a brief description of the construction here and show that the algebras of the form $U_n \oplus \mathfrak{su}(2)$, for n even, are Heisenberg-like.

Let G denote a compact Lie group and $\rho : G \to \mathrm{Aut}(V)$ a finite dimensional real irreducible representation with discrete kernel. The derived representation $j = d\rho : \mathfrak{g} \to \mathrm{End}(V)$ is faithful. Since G is compact and ρ is irreducible, there is a $\rho(G)-$invariant inner product $\langle\, ,\rangle_V$ on V which is unique up to constant multiple. Let $\langle\, ,\rangle_{\mathfrak{g}}$ be any $Ad(G)$ invariant inner product on $\mathfrak{g}$. If $\mathfrak{g}$ is simple, then the only $Ad(G)$ invariant inner products on $\mathfrak{g}$ are of the form

$$\langle Z, Z^* \rangle_{\mathfrak{g}} = -c^2 \mathrm{trace}(j(Z)j(Z^*))$$

for all $Z, Z^* \in \mathfrak{g}$ where c is a nonzero constant.

From this representation, we construct a two-step nilpotent Lie algebra $\mathfrak{n}$ as follows. Let $\mathfrak{z} = \mathfrak{g}$ and $\mathfrak{v} = V$ and let $\mathfrak{n}$ denote the orthogonal direct sum $\mathfrak{n} = V \oplus \mathfrak{g}$ as a vector space with inner product. Define the bracket operation on $\mathfrak{n}$ by

$$[\mathfrak{n}, \mathfrak{g}] = 0, \qquad [X, Y] \in \mathfrak{g} \text{ for } X, Y \in V,$$

where $\langle [X, Y], Z \rangle_g = \langle j(Z)X, Y \rangle_V$ for $X, Y \in V, Z \in \mathfrak{z}$. Then $\mathfrak{n}$ is a metric two-step nilpotent Lie algebra.

Let U_n denote the irreducible real representation of $\mathfrak{su}(2)$ of dimension $n+1$. (See Fulton and Harris [FH] or [DeM] for full details.) In the case where n is even and the dimension of U_n is odd, the resulting two-step nilpotent Lie algebra is singular.

Let $\mathfrak{n} = U_n \oplus \mathfrak{su}(2)$ with metric as described above. The metric on $\mathfrak{z} = \mathfrak{su}(2)$ is $Ad(SU(2))$-invariant. By Lemma 2 of [DeM], the action of $Ad(SU(2))$ by conjugation on vectors of constant length in $\mathfrak{su}(2)$ is transitive.

Let $I(\mathfrak{n})$ denote the isometry group of $\mathfrak{n}$. The following two results were established by J. Lauret [L].

Proposition 4.1. *Let $\mathfrak{n}$ denote a two step nilpotent Lie algebra and write $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. Then*

$$Aut(\mathfrak{n}) \cap I(\mathfrak{n}) = \{(\rho, A) \in O(\mathfrak{v}) \times O(\mathfrak{z}) |\ \rho j(Z)\rho^{-1} = j(AZ) \ \textit{for all } Z \in \mathfrak{z}\}.$$

Theorem 4.2. *Let $\rho : G \to Aut(V)$ be an irreducible representation of a compact Lie group G with discrete kernel on a real, finite dimensional vector space V. Let $\mathfrak{n} = V \oplus \mathfrak{g}$ be a metric two-step nilpotent Lie algebra constructed as above, and let N be the simply connected two-step nilpotent Lie group with left invariant metric. Then $G \subseteq Aut(N) \cap I(N)$ and each $g \in G$ acts on $\mathfrak{n} = V \oplus \mathfrak{g}$ as an automorphism and an isometry by $(\rho(g), Ad(g))$.*

We now prove the main result of this section.

Theorem 4.3. *Let n be an even integer. Then the Lie algebras $\mathfrak{n} = U_n \oplus \mathfrak{su}(2)$ are Heisenberg-like.*

Proof. Let $\{Z_1, Z_2, Z_3\}$ be the standard basis for $\mathfrak{su}(2)$. In the metric given, this is an orthogonal basis with $|Z_1| = |Z_2| = |Z_3|$, but this basis is *not* orthonormal. Let $Z \in \{Z_1, Z_2, Z_3\}$. The $j(Z)$ maps may be explicitly calculated (see page 293 of [DeM]).

Let Y denote an eigenvector of $j(Z_1)$ with eigenvalue λ, and let $Z \in \mathfrak{g}$. Then there exist $g \in SU(2)$ and $\alpha \in \mathbb{R}$ so that $Ad(g)\alpha Z_1 = Z$; here $\alpha = \frac{|Z|}{|Z_1|}$ since $Ad(g)$ is an isometry on $\mathfrak{su}(2)$. Therefore, $j(Z)(\rho(g)Y) =$

$$j(Ad(g)\alpha Z_1)(\rho(g)Y) = \alpha\rho(g)j(Z_1)\rho(g)^{-1}(\rho(g)Y) = \alpha\rho(g)\lambda Y = \alpha\lambda(\rho(g)Y).$$

Hence $\rho(g)Y$ is an eigenvector of $j(Z)$ with eigenvalue $\alpha\lambda$. Therefore, for any $Z \in \mathfrak{z}$, the eigenvalues of $j(Z)$ depend only on the norm of Z. Hence $\mathfrak{n}$ is Heisenberg-like. ∎

Example 4.4. $n = 2$:
The real representations of $\mathfrak{su}(2)$ are given by vector spaces U_n where $\dim U_n = n + 1$. In this case $\mathfrak{n} = U_2 \oplus \mathfrak{su}(2)$ is a 6-dimensional Lie algebra with basis $\{v_1, v_2, v_3, Z_1, Z_2, Z_3\}$. Note that this basis arises from the representation and is an orthogonal basis, but is not orthonormal.

We obtain

$$j(Z_1) = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, j(Z_2) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, j(Z_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}.$$

If $Z = \alpha_1 Z_1 + \alpha_2 Z_2 + \alpha_3 Z_3$, then since the norms of the Z_i are all equal, $|Z| = |Z_1|\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$ and $j(Z)$ has eigenvalues

$$\left\{0, \pm 2i\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}\right\} = \left\{0, \pm 2i\frac{|Z|}{|Z_1|}\right\}.$$

Example 4.5. $n = 4$:
In this case $\mathfrak{n} = U_4 \oplus \mathfrak{su}(2)$ with basis $\{v_1, \ldots, v_5, Z_1, Z_2, Z_3\}$. The matrix repre-

sentations for $j(Z_1), j(Z_2), j(Z_3)$ with respect to $\{v_1, ..., v_5\}$ are as follows:

$$j(Z_1) = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad j(Z_2) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & 0 & -4 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{pmatrix},$$
$$j(Z_3) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -3 & 0 \end{pmatrix}.$$

Note that the elements $v_1, \ldots, v_5$ in the basis for U_4 do not all have the same length. This accounts for the lack of skew symmetry in the matrices.

For $Z = \alpha_1 Z_1 + \alpha_2 Z_2 + \alpha_3 Z_3$, $j(Z)$ has eigenvalues

$$\left\{0, \pm 2i\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}, \pm 4i\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}\right\} = \left\{0, \pm 2i\frac{|Z|}{|Z_1|}, \pm 4i\frac{|Z|}{|Z_1|}\right\}.$$

In general, for $\mathfrak{n} = U_n \oplus \mathfrak{su}(2)$ the map $j(Z) = j(\alpha_1 Z_1 + \alpha_2 Z_2 + \alpha_3 Z_3)$ has eigenvalues $\left\{0, \pm 2i\frac{|Z|}{|Z_1|}, \pm 4i\frac{|Z|}{|Z_1|}, \ldots, \pm ni\frac{|Z|}{|Z_1|}\right\}$. This follows from the proof of Theorem 4.3 above and the fact that $j(Z_1)$ has eigenvalues
$$\{0, \pm 2i, \pm 4i, \ldots, \pm(n-2)i, \pm ni\}.$$

5. Examples from Gornet-Mast [GM]

The examples of Heisenberg-like Lie algebras in section 6 below are based on examples first developed in [GM]. We use the examples presented here to generate families of examples in section 6.

Example 5.1. ([GM], example 3.9)
Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be a six-dimensional vector space with inner product. Let $\{X_1, X_2, X_3, X_4\}$ be an orthonormal basis for $\mathfrak{v}$ and $\{Z_1, Z_2\}$ be an orthonormal basis for $\mathfrak{z}$. Let $j(Z_1), j(Z_2)$ have the following matrix representations with respect to the basis given for $\mathfrak{v}$:

$$j(Z_1) = \begin{pmatrix} 0 & -a_1 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 \\ 0 & 0 & a_2 & 0 \end{pmatrix}, \; j(Z_2) = \begin{pmatrix} 0 & 0 & -b_1 & 0 \\ 0 & 0 & 0 & b_2 \\ b_1 & 0 & 0 & 0 \\ 0 & -b_2 & 0 & 0 \end{pmatrix}$$

for real numbers a_1, a_2, b_1, b_2.

Using the relationship $\langle j(Z)X, Y\rangle = \langle [X, Y], Z\rangle$, the $j(Z)$ maps define a Lie bracket on $\mathfrak{n}$ by

$$[X_1, X_2] = -[X_2, X_1] = a_1 Z_1, \quad [X_1, X_3] = -[X_3, X_1] = b_1 Z_2,$$
$$[X_2, X_4] = -[X_4, X_2] = -b_2 Z_2, \quad [X_3, X_4] = -[X_4, X_3] = a_2 Z_1,$$

with all other brackets zero.

Let $Z = \alpha_1 Z_1 + \alpha_2 Z_2$. The set of eigenvalues of $j(Z)$ is

$$\left\{ \pm \frac{i}{2} \left(\sqrt{(a_1+a_2)^2\alpha_1^2 + (b_1+b_2)^2\alpha_2^2} \pm \sqrt{(a_1-a_2)^2\alpha_1^2 + (b_1-b_2)^2\alpha_2^2} \right) \right\}.$$

Then $\mathfrak{n}$ is Heisenberg-like if and only if

$$a_1^2 + a_2^2 = b_1^2 + b_2^2 \text{ and } a_1 a_2 = b_1 b_2.$$

Note that $\mathfrak{n}$ is of Heisenberg type if $a_1 = a_2 = b_1 = b_2$.

Example 5.2. ([GM], example 1.15)
Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ denote a vector space with inner product; assume $\dim \mathfrak{v} = 3$ and $\dim \mathfrak{z} = 2$. Let $\{X_1, X_2, X_3\}$ denote an orthonormal basis for $\mathfrak{v}$ and $\{Z_1, Z_2\}$ denote an orthonormal basis for $\mathfrak{z}$. Define $j(Z_1), j(Z_2)$ as follows:

$$j(Z_1) = \begin{pmatrix} 0 & -a_1 & 0 \\ a_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; j(Z_2) = \begin{pmatrix} 0 & 0 & -a_2 \\ 0 & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix}$$

for nonzero constants a_1, a_2. Let $Z = \alpha_1 Z_1 + \alpha_2 Z_2$. Then $j(Z)$ has eigenvalues $\left\{ 0, \pm i \sqrt{\alpha_1^2 a_1^2 + \alpha_2^2 a_2^2} \right\}$. It follows that $\mathfrak{n}$ is Heisenberg-like if and only if $a_1^2 = a_2^2$. Note also that

$$\begin{aligned} W_1(Z) &= \operatorname{span}\{\alpha_2 X_2 - \alpha_1 X_3\} = \ker j(Z) \\ W_2(Z) &= \operatorname{span}\{X_1, \alpha_1 X_2 + \alpha_2 X_3\} \end{aligned}$$

Note that since $\mathfrak{n}$ is singular, it is not of Heisenberg type.

6. Generalizations of the Gornet-Mast examples

The above examples may be generalized to construct Heisenberg-like two-step nilpotent Lie groups with centers of arbitrary dimension. The main approach is to construct maps $j(Z_i)$ consisting of block matrices, where the blocks follow the patterns of the above examples. We then obtain conditions on the matrix entries which can be used to determine Heisenberg-like Lie algebras.

Recall that if $\mathfrak{n}$ is a two-step nilpotent metric Lie algebra with one-dimensional center then $\mathfrak{n}$ must be Heisenberg-like; thus we only consider $\mathfrak{z}$ with $\dim \mathfrak{z} \geq 2$.

In constructing the examples below, for $\dim \mathfrak{v} = m$ and $\dim \mathfrak{z} = r$, we let $\{X_1, ..., X_m\}$ denote an orthonormal basis for $\mathfrak{v}$ and $\{Z_1, ..., Z_r\}$ denote an orthonormal basis for $\mathfrak{z}$. Recall that we use the relationship $\langle j(Z)X, Y \rangle = \langle [X, Y], Z \rangle$ to determine a Lie bracket on $\mathfrak{n}$; thus the given definitions suffice to completely determine the Lie algebra structure, and thus the corresponding simply connected Lie group, in each case.

Example 6.1. Nonsingular $(4k+2)$-dimensional Heisenberg-like Lie groups. Let $\dim \mathfrak{v} = 4k$, $k \in \mathbb{Z}_+$, and $\dim \mathfrak{z} = 2$. We generalize the approach of example 5.1.

Define $j(Z_1), j(Z_2)$ by

$$j(Z_1) = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_k \end{pmatrix} \quad j(Z_2) = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_k \end{pmatrix}$$

where $A_l, B_l,\ 1 \le l \le k$, are 4×4 matrices of the form

$$A_l = \begin{pmatrix} 0 & -\gamma_l & 0 & 0 \\ \gamma_l & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_l \\ 0 & 0 & \delta_l & 0 \end{pmatrix} \quad B_l = \begin{pmatrix} 0 & 0 & -\rho_l & 0 \\ 0 & 0 & 0 & \eta_l \\ \rho_l & 0 & 0 & 0 \\ 0 & -\eta_l & 0 & 0 \end{pmatrix}.$$

For $Z = \alpha_1 Z_1 + \alpha_2 Z_2$, the set of eigenvalues of $j(Z)$ is

$$\left\{ \pm\frac{i}{2} \left(\sqrt{(\gamma_l + \delta_l)^2 \alpha_1^2 + (\rho_l + \eta_l)^2 \alpha_2^2} \pm \sqrt{(\gamma_l - \delta_l)^2 \alpha_1^2 + (\rho_l - \eta_l)^2 \alpha_2^2} \right) : 1 \le l \le k \right\}$$

Then $\mathfrak{n}$ is Heisenberg-like if and only if

$$\gamma_l^2 + \delta_l^2 = \rho_l^2 + \eta_l^2 \text{ and } \gamma_l \delta_l = \rho_l \eta_l \tag{4}$$

for $1 \le l \le k$. Furthermore, N is nonsingular provided $\gamma_l, \delta_l, \rho_l, \eta_l$ are all nonzero, for each l. With the Heisenberg-like assumption one of the following four relationships must hold: $\gamma_l = \rho_l$ and $\delta_l = \eta_l$, $\gamma_l = -\rho_l$ and $\delta_l = -\eta_l$, $\gamma_l = \eta_l$ and $\delta_l = \rho_l$, or $\gamma_l = -\eta_l$ and $\delta_l = -\rho_l$ for each l. Without loss of generality, assume the first. Then $W_1(Z) = \ker j(\alpha_1 Z + \alpha_2 Z) = \{0\}$ and the remaining invariant subspaces are as follows:

$$\begin{aligned} W_2(Z) &= \operatorname{span}\{X_1, \alpha_1 X_2 + \alpha_2 X_3\} \\ W_3(Z) &= \operatorname{span}\{X_4, \alpha_2 X_2 - \alpha_1 X_3\} \\ &\vdots \\ W_{2k}(Z) &= \operatorname{span}\{X_{4k-3}, \alpha_1 X_{4k-2} + \alpha_2 X_{4k-1}\} \\ W_{2k+1}(Z) &= \operatorname{span}\{X_{4k}, \alpha_2 X_{4k-2} - \alpha_1 X_{4k-1}\}. \end{aligned}$$

Example 6.2. Nonsingular $(4k+3)$-dimensional examples

The previous example may be generalized so that $\dim \mathfrak{z} = 3$. Define $j(Z_1)$ and $j(Z_2)$ as above and define

$$j(Z_3) = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & C_k \end{pmatrix} \text{ where } C_l = \begin{pmatrix} 0 & 0 & 0 & -\nu_l \\ 0 & 0 & -\omega_l & 0 \\ 0 & \omega_l & 0 & 0 \\ \nu_l & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $j(Z) = j(\alpha_1 Z_1 + \alpha_2 Z_2 + \alpha_3 Z_3)$ are

$$\left\{ \pm\frac{i}{2} \left(\sqrt{(\gamma_l + \delta_l)^2 \alpha_1^2 + (\rho_l + \eta_l)^2 \alpha_2^2 + (\nu_l + \omega_l)^2 \alpha_3^2} \right.\right.$$
$$\left.\left. \pm \sqrt{(\gamma_l - \delta_l)^2 \alpha_1^2 + (\rho_l - \eta_l)^2 \alpha_2^2 + (\nu_l - \omega_l)^2 \alpha_3^2} \right) : 1 \le l \le k \right\}.$$

The Lie algebra $\mathfrak{n}$ is Heisenberg-like if and only if the following hold for $1 \le l \le k$:

$$\gamma_l^2 + \delta_l^2 = \rho_l^2 + \eta_l^2 = \nu_l^2 + \omega_l^2 \text{ and } \gamma_l \delta_l = \rho_l \eta_l = \nu_l \omega_l. \tag{5}$$

Again, without loss of generality, assume $\gamma_l = \rho_l = \nu_l \neq 0$ and $\delta_l = \eta_l = \omega_l \neq 0$ for $1 \le l \le k$. As above $W_1 = \{0\}$, and the non-trivial invariant subspaces of $j(Z)$ are

$$\begin{aligned}
W_2(Z) &= \operatorname{span}\{X_1, \alpha_1 X_2 + \alpha_2 X_3 + \alpha_3 X_4\} \\
W_3(Z) &= \operatorname{span}\{\alpha_2 X_2 - \alpha_1 X_3, \alpha_3 X_2 - \alpha_1 X_4\} \\
&\vdots \\
W_{2k}(Z) &= \operatorname{span}\{X_{4k-3}, \alpha_1 X_{4k-2} + \alpha_2 X_{4k-1} + \alpha_3 X_{4k}\} \\
W_{2k+1}(Z) &= \operatorname{span}\{\alpha_2 X_{4k-2} - \alpha_1 X_{4k-1}, \alpha_3 X_{4k-2} - \alpha_1 X_{4k}\}
\end{aligned}$$

We can further alter this example by exchanging blocks between the matrices while maintaining the relationships between the constants given in equation (5). This exchange, for example swapping A_i with either B_i or C_i, leaves the eigenvalues unchanged.

Example 6.3. Singular $(k(n+1)+n)$-dimensional Heisenberg-like Lie algebras The first step is to construct a singular $(2n+1)$-dimensional Heisenberg-like Lie algebra. Let $\dim \mathfrak{v} = n+1$ and $\dim \mathfrak{z} = n$. Define $j(Z_l)$ to have (r,s) entry $j(Z_l)_{r,s}$, $1 \le r, s \le n+1$, where

$$j(Z_l)_{r,s} = \begin{cases} -a_l & r = 1,\ s = l+1 \\ a_l & r = l+1,\ s = 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

where a_l are real nonzero constants, $1 \le l \le n$. The resulting Lie bracket satisfies

$$[X_1, X_l] = -[X_l, X_1] = a_{l-1} Z_{l-1} \text{ for } 2 \le l \le n+1,$$

with all other brackets zero.

The eigenvalues of $j(Z_l)$ are $\{0, \pm i a_l\}$, $1 \le l \le n$, where the zero eigenvalue has multiplicity $n-1$. Thus there are only three distinct eigenvalues for each $j(Z)$ for nonzero $Z \in \mathfrak{z}$. Let $Z = \alpha_1 Z_1 + \alpha_2 Z_2 + \cdots + \alpha_n Z_n$. The eigenvalues of $j(Z)$ are

$$\left\{0, \pm i\sqrt{a_1^2\alpha_1^2 + a_2^2\alpha_2^2 + \cdots + a_n^2\alpha_n^2}\right\}.$$

Then $\mathfrak{n}$ is Heisenberg-like if and only if $a_1^2 = a_2^2 = \cdots = a_n^2$.

Assume that $\mathfrak{n}$ is Heisenberg-like. Without loss of generality, assume $a_i = a_1$, $2 \le i \le n$. In this case, the invariant subspaces of $j(Z)$ are

$$\begin{aligned}
W_1(Z) &= \operatorname{span}\{\alpha_2 X_2 - \alpha_1 X_3, \alpha_3 X_2 - \alpha_1 X_4, \ldots, \alpha_n X_2 - \alpha_1 X_{n+1}\} \\
W_2(Z) &= \operatorname{span}\{X_1, \alpha_1 X_2 + \alpha_2 X_3 + \cdots + \alpha_n X_{n+1}\}.
\end{aligned}$$

Now we generalize the above to construct a $(k(n+1)+n)$-dimensional Lie algebra. Let $\dim \mathfrak{v} = k(n+1)$ and $\dim \mathfrak{z} = n$, where $n \ge 2$. Construct the bracket on $\mathfrak{n}$ by letting each $j(Z_l)$ have matrix representation consisting of k blocks, where each $(n+1)\times(n+1)$ block is constructed as in equation (6). For example, if $n=3$ then

$$j(Z_1)=\begin{pmatrix} A_1^1 & 0 & \cdots & 0 \\ 0 & A_1^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_1^k \end{pmatrix},\quad j(Z_2)=\begin{pmatrix} A_2^1 & 0 & \cdots & 0 \\ 0 & A_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_2^k \end{pmatrix},$$

$$j(Z_3)=\begin{pmatrix} A_3^1 & 0 & \cdots & 0 \\ 0 & A_3^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_3^k \end{pmatrix}$$

where

$$A_1^l=\begin{pmatrix} 0 & -a_1^l & 0 & 0 \\ a_1^l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\quad A_2^l=\begin{pmatrix} 0 & 0 & -a_2^l & 0 \\ 0 & 0 & 0 & 0 \\ a_2^l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\quad A_3^l=\begin{pmatrix} 0 & 0 & 0 & -a_3^l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_3^l & 0 & 0 & 0 \end{pmatrix}$$

for $1\le l\le k$.

The eigenvalues of $j(Z_1)$ are $\{0,\pm ia_1^1,\pm ia_1^2,\ldots,\pm ia_1^k\}$, the eigenvalues of $j(Z_2)$ are $\{0,\pm ia_2^1,\pm ia_2^2,\ldots,\pm ia_2^k\}$, and so on. Then the eigenvalues of $j(Z)=j(\alpha_1Z_1+\alpha_2Z_2+\cdots+\alpha_nZ_n)$ are

$$\left\{0,\pm i\sqrt{(a_1^l)^2\alpha_1^2+(a_2^l)^2\alpha_2^2+\cdots+(a_n^l)^2\alpha_n^2}:1\le l\le k\right\}$$

where the zero eigenvalue has multiplicity $k(n+1)-2k$. Thus in order for $\mathfrak{n}$ to be Heisenberg-like, $(a_1^l)^2=(a_2^l)^2=\cdots=(a_n^l)^2$ for $1\le l\le k$, and then the eigenvalues of $j(Z)$ are

$$\left\{0,\pm ia_1^l\sqrt{\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2}:1\le l\le k\right\}.$$

Without loss of generality, assuming $a_i^l=a_1^l$ for $2\le i\le n$ and $1\le l\le k$, the invariant subspaces are

$$\begin{aligned}
W_1(Z) &= \operatorname{span}\{\alpha_2X_2-\alpha_1X_3,\alpha_3X_2-\alpha_1X_4,\ldots,\alpha_nX_2-\alpha_1X_{n+1},\\
&\qquad \alpha_2X_{n+3}-\alpha_1X_{n+4},\alpha_3X_{n+3}-\alpha_1X_{n+5},\ldots,\alpha_nX_{n+3}-\alpha_1X_{2(n+1)},\\
&\qquad \cdots\\
&\qquad \alpha_2X_{k(n+1)-n+1}-\alpha_1X_{k(n+1)-n+2},\alpha_3X_{k(n+1)-n+1}-\alpha_1X_{k(n+1)-n+3},\\
&\qquad \ldots,\alpha_nX_{k(n+1)-n+1}-\alpha_1X_{k(n+1)}\}\\
W_2(Z) &= \operatorname{span}\{X_1,\alpha_1X_2+\alpha_2X_3+\cdots+\alpha_nX_{n+1}\}\\
W_3(Z) &= \operatorname{span}\{X_{n+2},\alpha_1X_{n+3}+\alpha_2X_{n+4}+\cdots+\alpha_nX_{2(n+1)}\}\\
\vdots &= \vdots\\
W_{k+1}(Z) &= \operatorname{span}\{X_{k(n+1)-n},\alpha_1X_{k(n+1)-n+1}+\alpha_2X_{k(n+1)-n+2}+\cdots+\alpha_nX_{k(n+1)}\}.
\end{aligned}$$

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