Whittaker Functions on Orthogonal Groups of Odd Degree

Taku Ishii

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Abstract. We give explicit formulas for Whittaker functions for the class one principal series representations of the orthogonal groups $\mathrm{SO}_{2n+1}(\mathbf{R})$ of odd degree. Our formulas are similar to the recursive formulas for Whittaker functions on $\mathrm{SL}_n(\mathbf{R})$ given by Stade and the author.
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Introduction

Special functions on a semisimple Lie group G have been studied by many authors, however, most of examples given in the literature are limited to the cases of rank one, and they are reduced to classical special functions such as hypergeometric functions, Whittaker functions, Bessel functions and so on. As an effective example for higher rank case, we here give explicit formulas of certain special functions on the orthogonal group $SO_{2n+1}(\mathbf{R}) = SO_{n+1,n}(\mathbf{R})$ of odd degree.

Let us briefly recall the spherical functions of Harish-Chandra [4]. We fix a maximal compact subgroup K of G and denote by $\mathbf{D}(G/K)$ the algebra of invariant differential operators on G/K. Let $\chi_\nu : \mathbf{D}(G/K) \to \mathbf{C}$ be a homomorphism. A smooth function f on G is called a spherical function if $f(e) = 1$ and

(1) f is bi-K invariant,

(2) $Df = \chi_\nu(D)f$, for all $D \in \mathbf{D}(G/K)$.

Harish-Chandra gave the following integral representation for the spherical function:

$$\psi_\nu(g) = \int_K a(kg)^{\nu+\rho} dk$$

(see §1 for the notation), and obtained the expansion formula

$$\psi_\nu = \sum_{w \in \mathcal{W}} c(w\nu) \Psi_{w\nu},$$

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where $\mathcal{W}$ is the (small) Weyl group and $\{\Psi_{w\nu} \mid w \in \mathcal{W}\}$ is a basis of the solution of the system (2) and $c(\nu)$ denotes Harish-Chandra's c-function. The c-functions play many roles in harmonic analysis on G, for example they determine Plancherel measure for the spherical transform on G.

Our target in this paper is Whittaker function which is also a smooth function on G satisfying the system (2). Instead of the condition (1), we impose on

$$f(ngk) = \eta(n)f(g), \text{ for all } (n,g,k) \in N \times G \times K,$$

where N is a maximal unipotent subgroup of G and η a (nondegenerate) unitary character of N.

As is well known Whittaker functions appear in Fourier expansions of automorphic forms ([12]), and therefore they are indispensable in the various constructions of automorphic L-functions. Jacquet [10] introduced an integral representation of Whittaker function of the form

$$J_{\nu,\eta}(g) = \int_N \eta^{-1}(n)\, a(w_0^{-1}ng)^{\nu+\rho} dn$$

where w_0 is the longest element in $\mathcal{W}$. We refer to this Jacquet's Whittaker function as *class one Whittaker function*. Inspired by the work of Harish-Chandra, Hashizume [3] proved an expansion formula for the class one Whittaker function:

$$J_{\nu,\eta} = \sum_{w \in \mathcal{W}} c'(w\nu) M_{w\nu,\eta}.$$

Here $c'(\nu)$ is a product of c-functions and certain ratio of gamma functions and $M_{\nu,\eta}$ is a power series solution (we call it *fundamental Whittaker function*) of the system (2) around the regular singularity. We will recall the results of Hashizume in section 1.

Despite the development of the study of the expansion formulas, explicit formulas of the spherical functions or Whittaker functions themselves seem to be still missing in most cases. It is necessary to understand deeper the both sides of the expansion formula above for serious applications to automorphic forms.

As for the fundamental Whittaker functions, recurrence relations characterizing the coefficients of them are easily given since they are essentially controlled by the Casimir operators. At the present these coefficients for $SL_3(\mathbf{R})$, $SO_5(\mathbf{R})$ and $SL_4(\mathbf{R})$ are known to be expressed in terms of (terminating) generalized hypergeometric series $_2F_1(1)$ (=ratio of gamma functions by Gauss' formula), $_3F_2(1)$ and $_4F_3(1)$, respectively (see [1], [7], [14]). But such a direction, that is, unit arguments of generalized hypergeometric series do not seem be appropriate for higher rank cases. Recently Stade and the author [9] reached a very satisfactory expression, which is a recursive relation between $SL_{n-1}(\mathbf{R})$ and $SL_n(\mathbf{R})$. In section 2, a similar formula for $SO_{2n+1}(\mathbf{R})$ will be given. See also [8] for other groups.

Jacquet integrals are actually integral representations of the class one Whittaker functions. But it does not seem to be suitable form for applications to automorphic forms, such as computations of archimedean L-factors, or giving sharp estimates for Whittaker functions. Then we need to modify Jacquet integrals to

more desirable expressions such as Mellin-Barnes type. In the case of $SL_2(\mathbf{R})$ the class one Whittaker function is essentially the modified K-Bessel function and it has the integral representations

$$K_\nu(z) = 2^{-1} \int_0^\infty \exp\left\{-\frac{z}{2}\left(t+\frac{1}{t}\right)\right\}t^\nu \frac{dt}{t}$$
$$= \frac{2^{-2}}{2\pi\sqrt{-1}} \int_{-\sqrt{-1}\infty}^{\sqrt{-1}\infty} \Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)\left(\frac{z}{2}\right)^{-s} ds.$$

Extension to $SL_n(\mathbf{R})$ was done by Stade [13], [15]. Starting from the Jacquet integral, he reached recursive formulas between the class one Whittaker functions on $SL_n(\mathbf{R})$ and $SL_{n-2}(\mathbf{R})$ of both types above. The formulas of Stade are very useful for an application. Actually he computed certain archimedean zeta integrals in [15] and [16]. Based on his formulas, Stade and the author [9] find another recursive formula between $SL_n(\mathbf{R})$ and $SL_{n-1}(\mathbf{R})$ corresponding to that for fundamental Whittaker functions.

Our approach here is different from [9], because we do not use the Jacquet integral. Recall that the result of [9] highly relies on evaluation of the Jacquet integral which needs very complicated computation as shown in [13]. Therefore we firstly try to guess a right formula by taking notice of an analogy between recursive formula for the fundamental Whittaker functions and the Mellin-Barnes integral representations of the class one Whittaker functions, roughly speaking, the residue of the integrand of the Mellin-Barnes integral representation gives the coefficient of the fundamental Whittaker functions. Then we can arrive a conjectural form in view of the above integral representations of K-Bessel functions (Theorem 3.1). This argument is of course heuristic and does not give a proof. In our proof in section 3, the expansion formulas of Hashizume and the recursive formula for the fundamental Whittaker functions obtained in section 2 play central roles. We remark that our idea of proof discussed here can be also applicable to the case of $SL_n(\mathbf{R})$.

In section 4 we will observe our recursive relation for $SO_{2n+1}(\mathbf{R})$ and $SO_{2n-1}(\mathbf{R})$ is similar to that for $SL_{2n}(\mathbf{R})$ and $SL_{2n-2}(\mathbf{R})$ in [13]. Following the argument of [15], we will compute the Mellin transforms of the class one Whittaker functions.

1. Preliminaries

In this section we recall basic facts about Whittaker functions for the class one principal series representations of $SO_{2n+1}(\mathbf{R})$. Our main reference is [3], which discusses the class one Whittaker functions on arbitrary semisimple Lie groups.

1.1. Group structures.

Let $G = SO_{2n+1}(\mathbf{R})$ be the special orthogonal group of degree n with respect to an anti-diagonal matrix $\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ of size $2n+1$. Let $\mathfrak{g} = \mathrm{Lie}(G) =$

$\mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition where $\mathfrak{k}$ and $\mathfrak{p}$ are $+1$ and -1 eigenspaces, respectively with respect to a Cartan involution $\theta(X) = -{}^t X$ $(X \in \mathfrak{g})$. We take a maximal compact subgroup K of G by $K = \exp \mathfrak{k} \cong SO(n) \times SO(n+1)$.

Take a maximal abelian subalgebra $\mathfrak{a} = \{\mathrm{diag}(t_1, \ldots, t_n, 0, -t_n, \ldots, -t_1) \mid t_i \in \mathbf{R}\}$ of $\mathfrak{p}$. The restricted root system

$$\Delta = \Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm e_i \pm e_j, \pm e_k \mid 1 \le i < j \le n, \ 1 \le k \le n\}$$

is of type B_n, where e_i is a linear form on $\mathfrak{a}$ such that

$$e_i(\mathrm{diag}(t_1, \ldots, t_n, 0, -t_n, \ldots, -t_1)) = t_i.$$

We take a positive system Δ_+ and the simple system Π by

$$\Delta_+ = \{e_i \pm e_j \mid 1 \le i < j \le n\} \cup \{e_k \mid 1 \le k \le n\}$$

and

$$\Pi = \{\alpha_i = e_i - e_{i+1} \mid 1 \le i \le n-1\} \cup \{\alpha_n = e_n\}.$$

Denote by $\mathfrak{g}_\alpha$ the root space for $\alpha \in \Delta$ and put $\mathfrak{n} = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$. Then we have the Iwasawa decompositions $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ and $G = NAK$ with

$$N = \exp \mathfrak{n} = \{\text{upper triangular unipotent matrices in } G\}, \quad A = \exp \mathfrak{a}.$$

Let $\langle \, , \, \rangle$ be an inner product on the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ induced from the Killing form on $\mathfrak{g}$, and we extend it to the complex dual $\mathfrak{a}_{\mathbf{C}}^*$. Fix an element ν of $\mathfrak{a}_{\mathbf{C}}^*$. Since $\mathfrak{a}_{\mathbf{C}}^* \cong \mathbf{C}^n$, we can identify ν with $(\nu_1, \ldots, \nu_n) \in \mathbf{C}^n$ via $\langle e_i, \nu \rangle / \langle e_i, e_i \rangle = 2\nu_i$. Let $\rho_n = (1/2) \sum_{\alpha \in \Delta_+} \alpha$ be the half sum of positive roots. Then

$$a^{\nu + \rho_n} = \exp(\nu + \rho_n)(\log a) = \prod_{i=1}^n a_i^{2\nu_i + n - i + 1/2}$$

for $a = \mathrm{diag}(a_1, \ldots, a_n, 1, a_n^{-1}, \ldots, a_1^{-1}) \in A$. We introduce a coordinate $y = (y_1, \ldots, y_n)$ on A by

$$y_i = \frac{a_i}{a_{i+1}} \ (1 \le i \le n-1), \quad y_n = a_n.$$

Then

$$a^{\nu + \rho_n} = y^{\nu + \rho_n} = \prod_{i=1}^n y_i^{2(\nu_1 + \cdots + \nu_i) + i(n - i/2)}.$$

We denote by $\mathcal{W}_n = \langle w_i \mid 1 \le i \le n \rangle \cong \mathfrak{S}_n \times (\mathbf{Z}/2\mathbf{Z})^n$ the Weyl group of Δ. Here w_i is the simple reflection with respect to the simple root α_i. We note the action of w_i on $\nu = (\nu_1, \ldots, \nu_n) \in \mathfrak{a}_{\mathbf{C}}^*$:

$$w_i \nu = (\nu_1, \ldots, \nu_{i-1}, \nu_{i+1}, \nu_i, \nu_{i+2}, \ldots, \nu_n) \quad (1 \le i \le n-1);$$
$$w_n \nu = (\nu_1, \ldots, \nu_{n-1}, -\nu_n).$$

A nondegenerate unitary character η of N is of the form

$$\eta\big((u_{i,j})\big) = \exp\Big(2\pi\sqrt{-1}\sum_{i=1}^{n}\eta_i u_{i,i+1}\Big), \quad (u_{i,j}) \in N$$

with nonzero real numbers η_i $(1 \le i \le n)$. We remark that $|\eta_{\alpha_i}| = \sqrt{2}\pi\eta_i/\sqrt{2n-1}$ in the notation of [3].

Let $U(\mathfrak{g}_{\mathbf{C}})$ and $U(\mathfrak{a}_{\mathbf{C}})$ be the universal enveloping algebras of $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{a}_{\mathbf{C}}$, the complexifications of $\mathfrak{g}$ and $\mathfrak{a}$, respectively. Set

$$U(\mathfrak{g}_{\mathbf{C}})^K = \{X \in U(\mathfrak{g}_{\mathbf{C}}) \mid \mathrm{Ad}(k)X = X \text{ for all } k \in K\}.$$

Let p be the projection $U(\mathfrak{g}_{\mathbf{C}}) \to U(\mathfrak{a}_{\mathbf{C}})$ along the decomposition

$$U(\mathfrak{g}_{\mathbf{C}}) = U(\mathfrak{a}_{\mathbf{C}}) \oplus (\mathfrak{n}U(\mathfrak{g}_{\mathbf{C}}) + U(\mathfrak{g}_{\mathbf{C}})\mathfrak{k}).$$

Define an automorphism γ of $U(\mathfrak{a}_{\mathbf{C}})$ by $\gamma(H) = H + \rho_n(H)$ for $H \in \mathfrak{a}_{\mathbf{C}}$. For the linear form ν above we define an algebra homomorphism $\chi_\nu : U(\mathfrak{g}_{\mathbf{C}})^K \to \mathbf{C}$ by

$$\chi_\nu(z) = \nu(\gamma \circ p(z)), \quad z \in U(\mathfrak{g}_{\mathbf{C}})^K.$$

Note that χ_ν is trivial on $U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}$.

Definition 1.1. Under the notation above, we denote by $\mathrm{Wh}(\nu, \eta)$ the space of smooth functions $f : G \to \mathbf{C}$ satisfying

- $f(ngk) = \eta(n)f(g)$ for all $n \in N$, $g \in G$ and $k \in K$,

- $Zf = \chi_\nu(Z)f$ for all $Z \in U(\mathfrak{g}_{\mathbf{C}})^K$.

The Iwasawa decomposition $G = NAK$ implies that any function f in the space $\mathrm{Wh}(\nu, \eta)$ is determined by its restriction $f|_A$ to A, which we call the *radial part* of f.

We mention a relation with the principal series representation of G. Let M be the centralizer of K in A and $P_{\min} = MAN$ a minimal parabolic subgroup of G. The induced representation

$$\pi_\nu = \mathrm{Ind}_{P_{\min}}^{G}(1_M \otimes \exp(\nu + \rho_n) \otimes 1_N)$$

is called the *class one principal series representation* of G. Consider an intertwining space $\mathrm{Hom}_G(\pi_\nu, \mathrm{Ind}_N^G(\eta))$, where

$$\mathrm{Ind}_N^G(\eta) = \{f \in C^\infty(G, \mathbf{C}) \mid f(ng) = \eta(n)f(g) \text{ for all } (n, g) \in N \times G\}.$$

Then our target space $\mathrm{Wh}(\nu, \eta)$ can be thought as a realization of π_ν in the induced module $\mathrm{Ind}_N^G(\eta)$, that is, for a nonzero intertwiner $\Phi \in \mathrm{Hom}_G(\pi_\nu, \mathrm{Ind}_N^G(\eta))$, and the spherical vector $v \in \pi_\nu$, $\Phi(v)$ becomes a nonzero element in $\mathrm{Wh}(\nu, \eta)$.

1.2. Fundamental Whittaker functions.

Let us recall Hashizume's construction of a basis of the space $\mathrm{Wh}(\nu, \eta)$ in our situation ([3, §4]).

For a set of nonnegative integers $\mathbf{m} = (m_1, \ldots, m_n)$, we determine complex numbers $c_{n,\mathbf{m}}(\nu)$ by the initial condition $c_{n,(0,\ldots,0)}(\nu) = 1$ and the recurrence relation

$$q_n(\mathbf{m}, \nu)c_{n,\mathbf{m}}(\nu) = \sum_{i=1}^{n-1} c_{n,\mathbf{m}-\mathbf{e}_i}(\nu) + \frac{1}{2}c_{n,\mathbf{m}-\mathbf{e}_n}(\nu), \qquad (1.1)$$

where $\mathbf{e}_i$ $(1 \le i \le n)$ is the i-th standard basis in $\mathbf{R}^n$ and q_n is defined by

$$q_n(\mathbf{m}, \nu) \equiv q_n\big((m_1, \ldots, m_n), (\nu_1, \ldots, \nu_n)\big)$$

$$:= \sum_{i=1}^{n-1} m_i^2 + \frac{1}{2}m_n^2 - \sum_{i=1}^{n-1} m_i m_{i+1} + \sum_{i=1}^{n-1}(\nu_i - \nu_{i+1})m_i + \nu_n m_n.$$

Hereafter we sometimes use the same symbol $\mathbf{e}_i$ for the i-th standard basis in $\mathbf{R}^{n-1}$. If $q_n(\mathbf{m}, \nu)$ does not vanish for all nonzero $\mathbf{m}$, we can uniquely determine $c_{n,\mathbf{m}}(\nu)$.

Definition 1.2. Define a power series $M_{\nu,\eta}^n(y) = y^{\rho_n}\widetilde{M}_{\nu,\eta}^n(y)$ on A by

$$\widetilde{M}_{\nu,\eta}^n(y) := \sum_{m_1,\ldots,m_n=0}^{\infty} c_{n,(m_1,\ldots,m_n)}(\nu)$$

$$\cdot \prod_{i=1}^{n-1}(\pi\eta_i y_i)^{2(m_i+\nu_1+\cdots+\nu_i)} \cdot (\sqrt{2}\pi\eta_n y_n)^{2(m_n+\nu_1+\cdots+\nu_n)}$$

and extend it to a function on G by

$$M_{\nu,\eta}^n(g) = \eta(n(g))M_{\nu,\eta}^n(a(g)),$$

where we denote by

$$g = n(g)a(g)k(g), \quad n(g) \in N, \, a(g) \in A, \, k(g) \in K$$

the Iwasawa decomposition of g. We call the function $M_{\nu,\eta}^n$ the *fundamental Whittaker function* on G.

It is known that ([3, Lemma 4.6]) the power series $M_{\nu,\eta}^n(y)$ converges absolutely and uniformly as functions of $y \in A$ and $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ (cf. Lemma 3.2).

Definition 1.3. An element ν of $\mathfrak{a}_{\mathbf{C}}^*$ is called *regular* if the following two conditions are satisfied:

- $q_n(\mathbf{m}, w\nu) \ne 0$ for all $\mathbf{m} \ne (0, \ldots, 0)$ and $w \in \mathcal{W}_n$,

- $w\nu - w'\nu \notin \{\sum_{i=1}^n m_i\alpha_i \mid m_i \in \mathbf{Z}\}$ for all pairs (w, w') in $\mathcal{W}_n$ with $w \ne w'$.

We denote by $'\mathfrak{a}_{\mathbf{C}}^*$ the subset of regular elements in $\mathfrak{a}_{\mathbf{C}}^*$.

Hashizume proved the following.

Theorem 1.4. *([3, Theorem 5.4]) If ν is a regular element, then the set*

$$\{M_{w\nu,\eta}^n \mid w \in \mathcal{W}_n\}$$

forms a basis of $\mathrm{Wh}(\nu, \eta)$.

1.3. Jacquet integral.

It is known that the subspace $\mathrm{Wh}(\nu,\eta)^{\mathrm{mod}}$ of moderate growth functions ([17]) of $\mathrm{Wh}(\nu,\eta)$ is at most one dimensional ([12], [11]). Jacquet [10] introduced an integral representation of the unique element in $\mathrm{Wh}(\nu,\eta)^{\mathrm{mod}}$:

$$J_{\nu,\eta}(g) = \int_N \eta^{-1}(n)\, a(w_0^{-1}ng)^{\nu+\rho_n}\, dn,$$

where w_0 is the longest element in $\mathcal{W}_n$ and dn is a normalized Haar measure on N as in [3, §1].

Proposition 1.5. ([2, Proposition 4.2]) *Let D be a subset of $\mathfrak{a}_{\mathbf{C}}^*$ defined by*

$$D = \{\nu \in \mathfrak{a}_{\mathbf{C}}^* \mid \mathrm{Re}(\nu_i \pm \nu_j) > 0 \ (1 \le i < j \le n), \ \mathrm{Re}(\nu_i) > 0 \ (1 \le i \le n)\}.$$

Then the Jacquet integral $J_{\nu,\eta}$ converges absolutely and uniformly on $(g,\nu) \in G \times D$ and gives a holomorphic function on $\nu \in D$. Moreover, as a function on ν, $J_{\nu,\eta}$ can be continued to an entire function on $\mathfrak{a}_{\mathbf{C}}^*$ and satisfies a functional equation

$$J_{\nu,\eta}(g) = \gamma(w,\nu,\eta) J_{w\nu,\eta}(g)$$

for $w \in \mathcal{W}_n$ and $g \in G$. Here $\gamma(w,\nu,\eta)$ is defined as follows. For the simple reflection w_i,

$$\gamma(w_i,\nu,\eta) = \begin{cases} (\pi\eta_i)^{2(\nu_i-\nu_{i+1})}\dfrac{\Gamma(-\nu_i-\nu_{i+1}+1/2)}{\Gamma(\nu_i+\nu_{i+1}+1/2)} & \text{if } 1 \le i \le n-1; \\[2ex] (\sqrt{2}\pi\eta_n)^{4\nu_n}\dfrac{\Gamma(-2\nu_n+1/2)}{\Gamma(2\nu_n+1/2)} & \text{if } i = n. \end{cases}$$

For $w \in \mathcal{W}_n$ with $l(w_i w) = l(w) + 1$,

$$\gamma(w_i w,\nu,\eta) = \gamma(w,\nu,\eta)\gamma(w_i,w\nu,\eta),$$

where $l(w)$ means the length of w.

Definition 1.6. We call the Jacquet integral $J_{\nu,\eta}$ (and its constant multiple) the *class one Whittaker function* on G.

As in the way of Harish-Chandra, Hashizume expressed the class one Whittaker function $J_{\nu,\eta}$ as a linear combination of the fundamental Whittaker functions $M^n_{w\nu,\eta}$ $(w \in \mathcal{W}_n)$.

Theorem 1.7. ([3, Theorem 7.8]) *If ν is a regular element, then we have*

$$J_{\nu,\eta}(g) = \sum_{w\in\mathcal{W}_n} \gamma(w_0 w,\nu,\eta) c(w_0 w\nu) M^n_{w\nu,\eta}(g),$$

where

$$c(\nu) = \int_N a(w_0^{-1}n)^{\nu+\rho_n}\, dn$$

$$= 2^{n/2}\{(4n-2)\pi\}^{n(n+1)/4} \prod_{1\le i<j\le n} \frac{\Gamma(\nu_i-\nu_j)\Gamma(\nu_i+\nu_j)}{\Gamma(\nu_i-\nu_j+1/2)\Gamma(\nu_i+\nu_j+1/2)}$$

$$\cdot \prod_{1 \leq i \leq n} \frac{\Gamma(2\nu_i)}{\Gamma(2\nu_i + 1/2)}$$

is the Harish-Chandra c-function on G. *Equivalently, if we put*

$$W^n_{\nu,\eta}(g) := 2^{-n/2}\{(4n-2)\pi\}^{-n(n+1)/4} \prod_{1 \leq i \leq n} \pi^{-2(\nu_1 + \cdots + \nu_i)}$$

$$\cdot \prod_{1 \leq i < j \leq n} \Gamma(\nu_i - \nu_j + 1/2)\Gamma(\nu_i + \nu_j + 1/2) \prod_{1 \leq i \leq n} \Gamma(2\nu_i + 1/2) \cdot J_{\nu,\eta}(g),$$

then we have

$$W^n_{\nu,\eta}(g) = \sum_{w \in \mathcal{W}_n} w \left[\prod_{1 \leq i < j \leq n} \Gamma(-\nu_i + \nu_j)\Gamma(-\nu_i - \nu_j) \prod_{1 \leq i \leq n} \Gamma(-2\nu_i) \cdot M^n_{\nu,\eta}(g) \right].$$

From now on we assume $\eta_i = 1$ for $1 \leq i \leq n-1$ and $\eta_n = 1/\sqrt{2}$ for simplicity, and denote by $M^n_\nu = M^n_{\nu,\eta}$, $W^n_\nu = W^n_{\nu,\eta}$ omitting the symbol η.

2. Explicit formulas for fundamental Whittaker functions

In this section we solve the recurrence relation (1.1) to find an explicit formula for the fundamental Whittaker function. Similar formulas for $SL_n(\mathbf{R})$ are given in [6] and [9]. We use the Pochhammer symbol $(a)_n = \Gamma(a+n)/\Gamma(a)$ for $n \in \mathbf{Z}$.

Theorem 2.1. *For* $\nu = (\nu_1, \ldots, \nu_n) \in {}'\mathfrak{a}^*_{\mathbf{C}}$, *put* $\tilde{\nu} = (\nu_1, \ldots, \nu_{n-1})$. *Then we have* $c_{1,m_1}(\nu) = 1/\{m_1!(2\nu_1 + 1)_{m_1}\}$ *and*

$$c_{n,(m_1,\ldots,m_n)}(\nu) = \sum_{\substack{\{l_1,\ldots,l_{n-1}\} \\ \{k_1,\ldots,k_{n-1}\}}} \frac{c_{n-1,(k_1,\ldots,k_{n-1})}(\tilde{\nu})}{\prod_{i=1}^{n-1}(m_i - l_i)! \cdot (m_n - k_{n-1})! \prod_{i=1}^{n-1}(l_i - k_i)!}$$

$$\cdot \frac{1}{\prod_{i=1}^{n}(\nu_i + \nu_n + 1)_{m_i - l_{i-1}} \prod_{i=1}^{n-1}(\nu_i - \nu_n + 1)_{l_i - k_{i-1}}}, \tag{2.1}$$

where the indexing sets $\{k_i\}$ *and* $\{l_i\}$ *are the sets of nonnegative integers satisfying*

$$0 \leq k_i \leq l_i \leq m_i \ (1 \leq i \leq n-1), \quad 0 \leq k_{n-1} \leq m_n$$

and we promise $k_0 = l_0 = 0$.

Proof.   The idea of proof is similar to [6] and [9]. We will check the right hand side of (2.1) satisfies (1.1) by dividing into two steps.

For a set of nonnegative integers $\mathbf{l} = (l_1, \ldots, l_n)$ with $0 \leq l_n \leq l_{n-1}$, define $b_\mathbf{l}(\nu) \equiv b_{(l_1,\ldots,l_n)}(\nu)$ by

$$b_\mathbf{l}(\nu) := (-1)^{l_n} \sum_{\{k_1,\ldots,k_{n-2}\}} \frac{c_{n-1,(k_1,\ldots,k_{n-2},l_n)}(\tilde{\nu})}{\prod_{i=1}^{n-2}(l_i - k_i)! \cdot (l_{n-1} - l_n)! \prod_{i=1}^{n-1}(\nu_i - \nu_n + 1)_{l_i - k_{i-1}}},$$

where $\{k_1, \ldots, k_{n-2}\}$ means $0 \le k_i \le l_i$ for $1 \le i \le n-2$. The term $(-1)^{l_n}$ is included for our later convenience. Then the formula (2.1) is equivalent to

$$c_{n,(m_1,\ldots,m_n)}(\nu) = \sum_{\{l_1,\ldots,l_n\}} \frac{(-1)^{l_n} b_{(l_1,\ldots,l_n)}(\nu)}{\prod_{i=1}^{n}(m_i - l_i)! \prod_{i=1}^{n}(\nu_i + \nu_n + 1)_{m_i - l_{i-1}}}, \qquad (2.2)$$

where $\{l_1, \ldots, l_n\}$ means $0 \le l_i \le m_i$ for $1 \le i \le n$.

We first show that $b_{\mathbf{l}}(\nu)$ satisfies the recurrence relation

$$q_n(\mathbf{l}, \nu) b_{\mathbf{l}}(\nu) = \sum_{i=1}^{n-1} b_{\mathbf{l}-\mathbf{e}_i}(\nu) + \frac{1}{2}(-l_{n-1} + l_n - 1) b_{\mathbf{l}-\mathbf{e}_n}(\nu). \qquad (2.3)$$

Set

$$P_{\mathbf{l},\mathbf{k}}(\nu) = \prod_{i=1}^{n-2}(l_i - k_i)! \cdot (l_{n-1} - l_n)! \prod_{i=1}^{n-1}(\nu_i - \nu_n + 1)_{l_i - k_{i-1}},$$

the denominator of the summand in $b_{\mathbf{l}}(\nu)$. The key identity is

$$\sum_{i=1}^{n-1} \frac{P_{\mathbf{l},\mathbf{k}}(\nu)}{P_{\mathbf{l}-\mathbf{e}_i,\mathbf{k}}(\nu)} - \sum_{i=1}^{n-2} \frac{P_{\mathbf{l},\mathbf{k}}(\nu)}{P_{\mathbf{l},\mathbf{k}+\mathbf{e}_i}(\nu)} = q_n\big((l_1,\ldots,l_n),\nu\big) - q_{n-1}\big((k_1,\ldots,k_{n-2},l_n),\tilde{\nu}\big). \qquad (2.4)$$

This is an easy algebra since the left hand side of (2.4) can be written as

$$\sum_{i=1}^{n-2}(l_i - k_i)(l_i - k_{i-1} + \nu_i - \nu_n) - \sum_{i=1}^{n-2}(l_i - k_i)(l_{i+1} - k_i + \nu_{i+1} - \nu_n)$$
$$+ (l_{n-1} - l_n)(l_{n-1} - k_{n-2} + \nu_{n-1} - \nu_n).$$

Let us compute the right hand side of (2.3). Since

$$b_{\mathbf{l}-\mathbf{e}_i}(\nu) = (-1)^{l_n} \sum_{\{k_1,\ldots,k_{n-2}\}} \frac{P_{\mathbf{l},\mathbf{k}}(\nu)}{P_{\mathbf{l}-\mathbf{e}_i,\mathbf{k}}(\nu)} \cdot \frac{c_{n-1,(k_1,\ldots,k_{n-2},l_n)}(\tilde{\nu})}{P_{\mathbf{l},\mathbf{k}}(\nu)}$$

for $1 \le i \le n-1$ and

$$\frac{1}{2}(-l_{n-1} + l_n - 1) b_{\mathbf{l}-\mathbf{e}_n}(\nu)$$
$$= \frac{1}{2}(-l_{n-1} + l_n - 1) \cdot (-1)^{l_n - 1} \sum_{\{k_1,\ldots,k_{n-2}\}} \frac{P_{\mathbf{l},\mathbf{k}}(\nu)}{P_{\mathbf{l}-\mathbf{e}_n,\mathbf{k}}(\nu)} \cdot \frac{c_{n-1,(k_1,\ldots,k_{n-2},l_n-1)}(\tilde{\nu})}{P_{\mathbf{l},\mathbf{k}}(\nu)}$$
$$= \frac{1}{2}(-l_{n-1} + l_n - 1) \cdot (-1)^{l_n - 1} \sum_{\{k_1,\ldots,k_{n-2}\}} \frac{1}{l_{n-1} - l_n + 1} \cdot \frac{c_{n-1,(k_1,\ldots,k_{n-2},l_n-1)}(\tilde{\nu})}{P_{\mathbf{l},\mathbf{k}}(\nu)}$$
$$= (-1)^{l_n} \sum_{\{k_1,\ldots,k_{n-2}\}} \frac{\frac{1}{2} c_{n-1,(k_1,\ldots,k_{n-2},l_n-1)}(\tilde{\nu})}{P_{\mathbf{l},\mathbf{k}}(\nu)},$$

the identity (2.4) implies that the right hand side of (2.3) can be written as a sum of the following four terms:

$$(-1)^{l_n} \sum_{i=1}^{n-2} \sum_{\{k_1,\dots,k_{n-2}\}} \frac{P_{\mathbf{l},\mathbf{k}}(\nu)}{P_{\mathbf{l},\mathbf{k}+\mathbf{e}_i}(\nu)} \cdot \frac{c_{n-1(k_1,\dots,k_{n-2},l_n)}(\tilde{\nu})}{P_{\mathbf{l},\mathbf{k}}(\nu)}, \qquad (2.5)$$

$$(-1)^{l_n} \sum_{\{k_1,\dots,k_{n-2}\}} \frac{\frac{1}{2}c_{n-1,(k_1,\dots,k_{n-2},l_n-1)}(\tilde{\nu})}{P_{\mathbf{l},\mathbf{k}}(\nu)}, \qquad (2.6)$$

$$q_n\big((l_1,\dots,l_n),\nu\big) \cdot (-1)^{l_n} \sum_{\{k_1,\dots,k_{n-2}\}} \frac{c_{n-1,(k_1,\dots,k_{n-2},l_n)}(\tilde{\nu})}{P_{\mathbf{l},\mathbf{k}}(\nu)} = q_n(\mathbf{l},\nu)b_{\mathbf{l}}(\nu), \qquad (2.7)$$

and

$$-q_{n-1}\big((k_1,\dots,k_{n-2},l_n),\tilde{\nu}\big) \cdot (-1)^{l_n} \sum_{\{k_1,\dots,k_{n-2}\}} \frac{c_{n-1,(k_1,\dots,k_{n-2},l_n)}(\tilde{\nu})}{P_{\mathbf{l},\mathbf{k}}(\nu)}. \qquad (2.8)$$

In (2.5), we substitute $k_i \to k_i - 1$ to rewrite

$$(-1)^{l_n} \sum_{i=1}^{n-2} \sum_{\{k_1,\dots,k_{n-2}\}} \frac{c_{n-1,(k_1,\dots,k_{n-2},l_n)-\mathbf{e}_i}(\tilde{\nu})}{P_{\mathbf{l},\mathbf{k}}(\nu)}.$$

Thus, in view of the recurrence relation for $c_{n-1,(k_1,\dots,k_{n-2},l_n)}(\tilde{\nu})$, we find (2.5) + (2.6) + (2.8) = 0 and finish the proof of (2.3).

In the next step, we prove the right hand side of (2.2) satisfies the recurrence relation (1.1) for $c_{n,\mathbf{m}}(\nu)$. As in the first step, if we put

$$Q_{\mathbf{m},\mathbf{l}}(\nu) = \prod_{i=1}^{n}(m_i - l_i)! \prod_{i=1}^{n}(\nu_i + \nu_n + 1)_{m_i - l_{i-1}}$$

then the identity

$$\sum_{i=1}^{n-1} \frac{Q_{\mathbf{m},\mathbf{l}}(\nu)}{Q_{\mathbf{m}-\mathbf{e}_i,\mathbf{l}}(\nu)} + \frac{1}{2}\frac{Q_{\mathbf{m},\mathbf{l}}(\nu)}{Q_{\mathbf{m}-\mathbf{e}_n,\mathbf{l}}(\nu)} - \sum_{i=1}^{n-1} \frac{Q_{\mathbf{m},\mathbf{l}}(\nu)}{Q_{\mathbf{m},\mathbf{l}+\mathbf{e}_i}(\nu)} - \frac{1}{2}(l_{n-1} - l_n)(m_n - l_n)$$
$$= q_n(\mathbf{m},\nu) - q_n(\mathbf{l},\nu)$$

holds. By means of (2.3) our claim follows, and therefore we complete the proof of Theorem 2.1. ∎

3.   Explicit formulas for class one Whittaker functions

In this section we will show a recursive integral representation of the class one Whittaker function.

Theorem 3.1. *For $\nu = (\nu_1, \ldots, \nu_n) \in \mathfrak{a}_{\mathbf{C}}^*$ and $y = (y_1, \ldots, y_n) \in A$, we inductively define a function $\widetilde{W}_\nu^n(y)$ on A by*

$$
\widetilde{W}_\nu^n(y) := \int_{(\mathbf{R}_+^\times)^n} \int_{(\mathbf{R}_+^\times)^{n-1}} \prod_{i=1}^{n} \exp\left\{-(\pi y_i)^2 t_i - \frac{1}{t_i}\right\} \prod_{i=1}^{n} (\pi y_i)^{2\nu_n}
$$
$$
\cdot \prod_{i=1}^{n-1} \exp\left\{-(\pi y_i)^2 \frac{t_i}{t_{i+1}} u_i - \frac{1}{u_i}\right\} \cdot t_1^{2\nu_n} \prod_{i=1}^{n-1} (t_{i+1} u_i)^{\nu_n}
$$
$$
\cdot \widetilde{W}_{\tilde{\nu}}^{n-1}\left(y_2 \sqrt{\frac{t_2 u_2}{t_3 u_1}}, \ldots, y_{n-1} \sqrt{\frac{t_{n-1} u_{n-1}}{t_n u_{n-2}}}, y_n \sqrt{\frac{t_n}{u_{n-1}}}\right) \prod_{i=1}^{n-1} \frac{du_i}{u_i} \prod_{i=1}^{n} \frac{dt_i}{t_i},
$$
(3.1)

and $\widetilde{W}_\nu^1(y) = \widetilde{W}_{\nu_1}^1(y_1) = 2K_{2\nu_1}(2\pi y_1)$. Here $\tilde{\nu} = (\nu_1, \ldots, \nu_{n-1})$. Then we have

$$
\widetilde{W}_\nu^n(y) = \sum_{w \in \mathcal{W}_n} w\left[\Gamma_n(\nu) \cdot \widetilde{M}_\nu^n(y)\right] \tag{3.2}
$$

with

$$
\Gamma_n(\nu) := \prod_{1 \le i < j \le n} \Gamma(-\nu_i - \nu_j)\Gamma(-\nu_i + \nu_j) \prod_{1 \le i \le n} \Gamma(-2\nu_i)
$$

and thus $W_\nu^n(y) = y^{\rho_n} \widetilde{W}_\nu^n(y)$.

We illustrate the outline of our proof of the expansion formula (3.2). It is done by induction on n, and as in the proof of Theorem 2.1, it consists of two steps. For $x = (x_1, \ldots, x_n) \in (\mathbf{R}_+^\times)^n$, let us define a function $V_\nu(x) = V_{(\nu_1, \ldots, \nu_n)}(x_1, \ldots, x_n)$ by

$$
V_\nu(x) := \int_{(\mathbf{R}_+^\times)^{n-1}} \prod_{i=1}^{n-1} \exp\left\{-(\pi x_i)^2 u_i - \frac{1}{u_i}\right\} \prod_{i=1}^{n} (\pi x_i)^{2\nu_n} \prod_{i=1}^{n-1} u_i^{\nu_n}
$$
$$
\cdot \widetilde{W}_{\tilde{\nu}}^{n-1}\left(x_2 \sqrt{\frac{u_2}{u_1}}, \ldots, x_{n-1} \sqrt{\frac{u_{n-1}}{u_{n-2}}}, x_n \sqrt{\frac{1}{u_{n-1}}}\right) \prod_{i=1}^{n-1} \frac{du_i}{u_i}. \tag{3.3}
$$

The induction hypothesis implies the rapid decay of the function $\widetilde{W}_\nu^{n-1}(y)$ and therefore the above integral converges absolutely for $x \in (\mathbf{R}_+^\times)^n$ and $\nu \in \mathbf{C}^n$. By using $V_\nu(x)$, we can write $\widetilde{W}_\nu^n(y)$ as

$$
\widetilde{W}_\nu^n(y) = \int_{(\mathbf{R}_+^\times)^n} \prod_{i=1}^{n} \exp\left\{-(\pi y_i)^2 t_i - \frac{1}{t_i}\right\} \prod_{i=1}^{n} t_i^{\nu_n}
$$
$$
\cdot V_\nu\left(y_1 \sqrt{\frac{t_1}{t_2}}, \ldots, y_{n-1} \sqrt{\frac{t_{n-1}}{t_n}}, y_n \sqrt{t_n}\right) \prod_{i=1}^{n} \frac{dt_i}{t_i}. \tag{3.4}
$$

We will first establish an expansion formula for $V_\nu(x)$ in Theorem 3.3 below. We notice that when $n = 3$, this computation is essentially the same as [5], which we expressed the generalized principal series Whittaker functions on $Sp_3(\mathbf{R})$ in

terms of the class one Whittaker functions on $SO_5(\mathbf{R})$. In the next step (subsection 3.2), by way of the results of Theorem 3.3 we will prove the relation (3.2).

As in [5], to justify interchange of the order of integrations and infinite sums in the computation in the next subsections, we need the following lemma. For the proof see [5, Lemma 5.2].

Lemma 3.2. *For complex numbers* $\{a_{ij}\}_{1\leq i\leq j\leq n}$ *with* $a_{ii}\geq 0$, $\{b_i\}_{1\leq i\leq n}$ *and* d, *put*

$$\Delta(\mathbf{m}) \equiv \Delta(\mathbf{m}, \{a_{ij}\}, \{b_i\}, d) = \sum_{1\leq i\leq n} a_{ii}m_i^2 + \sum_{1\leq i<j\leq n} a_{ij}m_im_j + \sum_{1\leq i\leq n} b_im_i + d$$

Let $\{p_{ij}\}_{1\leq i,j\leq n}$ *be complex numbers and* $\{q_i\}_{1\leq i\leq n}$ *nonzero complex numbers. We can inductively define complex numbers* $A_{\mathbf{m}} \equiv A_{(m_1,\ldots,m_n)}(\{a_{ij}\}, \{b_i\}, d)$ *by* $A_{(0,\ldots,0)} = 1$ *and the recurrence relation*

$$\Delta(\mathbf{m})A_{\mathbf{m}} = \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} p_{ij}m_j + q_i\Big)A_{\mathbf{m}-\mathbf{e}_i},$$

if $\Delta(\mathbf{m})$ *does not vanish for all* $(m_1,\ldots,m_n) \neq (0,\ldots,0)$. *Set*

$$X = \{(\{a_{ij}\}, \{b_i\}, d) \in \mathbf{C}^{n(n+3)/2+1} \mid \Delta(\mathbf{m}) \neq 0 \text{ for all } \mathbf{m} \in \mathbf{N}^n\backslash\{(0,\ldots,0)\}\}.$$

Let U *be any compact subset in* X. *There exists a positive constant* c_U *depending only on* U *such that*

$$|A_{\mathbf{m}}| \leq \frac{c_U^{m_1+\cdots+m_n}}{(m_1+\cdots+m_n)!} \tag{3.5}$$

for all $\mathbf{m} \in \mathbf{N}^n$ *and* $(\{a_{ij}\}, \{b_i\}, d) \in U$. *Thus the power series*

$$\sum_{m_1,\ldots,m_n=0}^{\infty} A_{\mathbf{m}}x_1^{m_1}\cdots x_n^{m_n}$$

converges absolutely and uniformly on compacta for $(x_1,\ldots,x_n) \in (\mathbf{R}_+^\times)^n$ *and* $(\{a_{ij}\}, \{b_i\}, d) \in X$.

3.1. The first step –expansion formula for V_ν–.

In this subsection we prove the following:

Theorem 3.3. *Let* $V_\nu(x)$ *be the function defined by (3.3). Then, for* $\nu \in {}'\mathfrak{a}_{\mathbf{C}}^*$, *we have*

$$V_\nu(x) = \sum_{w\in\mathcal{W}_{n-1}} \sum_{1\leq p\leq n} w\left[\Gamma^p(\nu) \sum_{\substack{l_1,\ldots,l_n=0 \\ l_{n-1}\geq l_n}}^{\infty} b_{(l_1,\ldots,l_n)}^p(\nu) \prod_{i=1}^{n} (\pi x_i)^{2(l_i+\nu_1^{(p)}+\cdots+\nu_i^{(p)})}\right], \tag{3.6}$$

where

$$\Gamma^p(\nu) := \prod_{1\leq i<j\leq n-1} \Gamma(-\nu_i-\nu_j)\Gamma(-\nu_i+\nu_j) \prod_{1\leq i\leq n-1} \Gamma(-2\nu_i)$$

$$\cdot \prod_{1 \le i \le p-1} \Gamma(-\nu_i + \nu_n) \prod_{p \le i \le n-1} \Gamma(\nu_i - \nu_n),$$

$$\nu^{(p)} \equiv (\nu_1^{(p)}, \ldots, \nu_n^{(p)}) := (\nu_1, \ldots, \nu_{p-1}, \nu_n, \nu_p, \ldots, \nu_{n-1}).$$

and

$$b_{(l_1,\ldots,l_n)}^p(\nu) := (-1)^{l_n} \sum_{\{k_1,\ldots,k_{n-2}\}} \frac{c_{n-1,(k_1,\ldots,k_{n-2},l_n)}(\tilde{\nu})}{\prod_{1 \le i \le \min(p-1,n-2)}(l_i - k_i)!(\nu_i - \nu_n + 1)_{l_i - k_{i-1}}}$$

$$\cdot \frac{1}{\prod_{p \le i \le n-2}(l_i - k_{i-1})!(-\nu_i + \nu_n + 1)_{l_i - k_i}}$$

$$\cdot \begin{cases} \frac{1}{(l_{n-1} - l_n)!(\nu_{n-1} - \nu_n + 1)_{l_{n-1} - k_{n-2}}} & \text{if } p = n; \\ \frac{1}{(l_{n-1} - k_{n-2})!(-\nu_{n-1} + \nu_n + 1)_{l_{n-1} - l_n}} & \text{if } p \ne n. \end{cases}$$

Here $\{k_1, \ldots, k_{n-2}\}$ *means* k_i *runs through such that*

$$0 \le k_i \le l_i \ (1 \le i \le p-2), \quad 0 \le k_{p-1} \le \min(l_{p-1}, l_p), \quad 0 \le k_i \le l_{i+1} \ (p \le i \le n-2).$$

Moreover $b_{\mathbf{l}}^p(\nu) = b_{(l_1,\ldots,l_n)}^p(\nu)$ *is uniquely determined by the initial condition* $b_{(0,\ldots,0)}^p(\nu) = 1$ *and the recurrence relation:*

$$q_n(\mathbf{l}, \nu^{(p)}) b_{\mathbf{l}}^p(\nu) = \sum_{i=1}^{n-1} b_{\mathbf{l}-\mathbf{e}_i}^p(\nu) + \frac{1}{2}(-l_{n-1} + l_n + \nu_n^{(p)} - \nu_n - 1) b_{\mathbf{l}-\mathbf{e}_n}^p(\nu).$$

Proof. We substitute the expansion formula for $W_{\tilde{\nu}}^{n-1}(y)$ to find

$$V_\nu(x) = \sum_{w \in \mathcal{W}_{n-1}} w \left[\Gamma_{n-1}(\tilde{\nu}) \int_{(\mathbf{R}_+^\times)^{n-1}} \prod_{i=1}^{n-1} \exp\left\{ -(\pi x_i)^2 u_i - \frac{1}{u_i} \right\} \prod_{i=1}^{n} (\pi x_i)^{2\nu_n} \prod_{i=1}^{n-1} u_i^{\nu_n} \right.$$

$$\left. \cdot \sum_{k_1,\ldots,k_{n-1}=0}^{\infty} c_{n-1,(k_1,\ldots,k_{n-1})}(\tilde{\nu}) \prod_{i=2}^{n} \left(\pi x_i \sqrt{\frac{u_i}{u_{i-1}}} \right)^{2(k_{i-1} + \nu_1 + \cdots + \nu_{i-1})} \prod_{i=1}^{n-1} \frac{du_i}{u_i} \right].$$

We change the order of the integration and the infinite sum, to get

$$V_\nu(x) = \sum_{w \in \mathcal{W}_{n-1}} w \left[\Gamma_{n-1}(\tilde{\nu}) \sum_{k_1,\ldots,k_{n-1}=0}^{\infty} c_{n-1,(k_1,\ldots,k_{n-1})}(\tilde{\nu}) \prod_{i=1}^{n} (\pi x_i)^{2(k_{i-1} + \nu_1 + \cdots + \nu_{i-1} + \nu_n)} \right.$$

$$\left. \cdot \prod_{i=1}^{n-1} \int_0^\infty \exp\left\{ -(\pi x_i)^2 u_i - \frac{1}{u_i} \right\} u_i^{k_{i-1} - k_i - \nu_i + \nu_n} \frac{du_i}{u_i} \right].$$

(3.7)

As in [5, §7] this interchange is justified by Lemma 3.2, and an analytic continuation argument implies that (3.7) is valid for all $(x_1, \ldots, x_n) \in (\mathbf{R}_+^\times)^n$.

In view of

$$\int_0^\infty \exp\left\{-(\pi x)^2 u - \frac{1}{u}\right\} u^s \frac{du}{u}$$
$$= 2(\pi x)^{-s} K_s(2\pi x)$$
$$= \frac{\pi}{\sin s\pi}(\pi x)^{-s}\left(I_{-s}(2\pi x) - I_s(2\pi x)\right) \tag{3.8}$$
$$= \frac{\pi}{\sin s\pi}\left(\sum_{l=0}^\infty \frac{(\pi x)^{2(l-s)}}{l!\,\Gamma(l-s+1)} - \sum_{l=0}^\infty \frac{(\pi x)^{2l}}{l!\,\Gamma(l+s+1)}\right),$$

$V_\nu(x)$ becomes

$$\sum_{w\in\mathcal{W}_{n-1}} w\left[\Gamma_{n-1}(\tilde\nu)\sum_{k_1,\ldots,k_{n-1}=0}^\infty c_{n-1,(k_1,\ldots,k_{n-1})}(\tilde\nu)\prod_{i=1}^n (\pi x_i)^{2(k_{i-1}+\nu_1+\cdots+\nu_{i-1}+\nu_n)}\right.$$
$$\cdot\prod_{i=1}^{n-1}\frac{\pi}{\sin(k_{i-1}-k_i-\nu_i+\nu_n)\pi}$$
$$\cdot\prod_{i=1}^{n-1}\left(\sum_{l_i=0}^\infty \frac{(\pi x_i)^{2(l_i-k_{i-1}+k_i+\nu_i-\nu_n)}}{l_i!\,\Gamma(l_i-k_{i-1}+k_i+\nu_i-\nu_n+1)}\right.$$
$$\left.\left.-\sum_{l_i=0}^\infty \frac{(\pi x_i)^{2l_i}}{l_i!\,\Gamma(l_i+k_{i-1}-k_i-\nu_i+\nu_n+1)}\right)\right]$$
$$= \sum_{w\in\mathcal{W}_{n-1}} w\left[\Gamma_{n-1}(\tilde\nu)\sum_{k_1,\ldots,k_{n-1}=0}^\infty (-1)^{k_{n-1}}c_{n-1,(k_1,\ldots,k_{n-1})}(\tilde\nu)(\pi x_n)^{2(k_{n-1}+\nu_1+\cdots+\nu_n)}\right.$$
$$\cdot\prod_{i=1}^{n-1}\left(\Gamma(-\nu_i+\nu_n)\sum_{l_i=0}^\infty \frac{(\pi x_i)^{2(l_i+k_i+\nu_1+\cdots+\nu_i)}}{l_i!(\nu_i-\nu_n+1)_{l_i-k_{i-1}+k_i}}\right.$$
$$\left.\left.+\Gamma(\nu_i-\nu_n)\sum_{l_i=0}^\infty \frac{(\pi x_i)^{2(l_i+k_{i-1}+\nu_1+\cdots+\nu_{i-1}+\nu_n)}}{l_i!(-\nu_i+\nu_n+1)_{l_i+k_{i-1}-k_i}}\right)\right].$$

We change the order of the sum and substitute $l_i \to l_i - k_i$ or $l_i \to l_i - k_{i-1}$ for $1 \le i \le n-1$ and $k_{n-1} \to l_n$. Then we have

$$V_\nu(x) = \sum_{w\in\mathcal{W}_{n-1}}\sum_{P\subset\{1,\ldots,n-1\}} w\left[\Gamma^P(\nu)\sum_{l_1,\ldots,l_n=0}^\infty b^P_{(l_1,\ldots,l_n)}(\nu)\right.$$
$$\left.\cdot\prod_{i\in P\cup\{n\}}(\pi x_i)^{2(l_i+\nu_1+\cdots+\nu_i)}\prod_{i\in P^c}(\pi x_i)^{2(l_i+\nu_1+\cdots+\nu_{i-1}+\nu_n)}\right], \tag{3.9}$$

where P ranges all the subset of $\{1,\ldots,n-1\}$ and P^c means the complement of P in $\{1,\ldots,n-1\}$. Here

$$\Gamma^P(\nu) := \Gamma_{n-1}(\tilde\nu)\prod_{i\in P}\Gamma(-\nu_i+\nu_n)\prod_{i\in P^c}\Gamma(\nu_i-\nu_n)$$

and

$$b^P_{(l_1,\ldots,l_n)}(\nu) := (-1)^{l_n} \sum_{\{k_1,\ldots,k_{n-2}\}} \frac{c_{n-1,(k_1,\ldots,k_{n-2},l_n)}(\tilde{\nu})}{\prod_{i\in P,1\leq i\leq n-2}(l_i - k_i)!(\nu_i - \nu_n + 1)_{l_i-k_{i-1}}}$$

$$\cdot \frac{1}{\prod_{i\in P^c,1\leq i\leq n-2}(l_i - k_{i-1})!(-\nu_i + \nu_n + 1)_{l_i-k_i}}$$

$$\cdot \begin{cases} \frac{1}{(l_{n-1}-l_n)!(\nu_{n-1}-\nu_n+1)_{l_{n-1}-k_{n-2}}} & \text{if } n-1 \in P; \\ \frac{1}{(l_{n-1}-k_{n-2})!(-\nu_{n-1}+\nu_n+1)_{l_{n-1}-l_n}} & \text{if } n-1 \in P^c, \end{cases}$$

for $l_{n-1} - l_n \geq 0$, and $b^P_{(l_1,\ldots,l_n)}(\nu) = 0$ for $l_{n-1} - l_n < 0$. Here the indexing set $\{k_1,\ldots,k_{n-2}\}$ runs through such that

$$0 \leq k_i \leq l_i \ (i \in P), \quad 0 \leq k_i \leq l_{i+1} \ (i+1 \in P^c), \quad 0 \leq k_i \ (i \in P^c \text{ and } i+1 \in P).$$

From now on we consider which P contributes to the summation in (3.9). We first derive a recurrence relation for $b^P_{(l_1,\ldots,l_n)}(\nu)$ and an explicit formula for the initial value $b^P_{(0,\ldots,0)}(\nu)$.

Lemma 3.4. *For* $P \subset \{1,2,\ldots,n-1\}$, *set* $\widetilde{P} = \{i \mid 1 \leq i \leq n-2, \ i \in P^c \text{ and } i+1 \in P\}$.

(i) $b^P_{\mathbf{l}}(\nu) = b^P_{(l_1,\ldots,l_n)}(\nu)$ *satisfies the recurrence relation*

$$\left(\sum_{i=1}^{n-1} l_i^2 + \frac{1}{2}l_n^2 - \sum_{i=1}^{n-1} l_i l_{i+1} + \sum_{i=1}^{n-1} \lambda_i^P l_i + \kappa^P\right) b^P_{\mathbf{l}}(\nu)$$

$$= \sum_{i=1}^{n-1} b^P_{\mathbf{l}-\mathbf{e}_i}(\nu) + \frac{1}{2}(-l_{n-1} + l_n + \lambda_n^P - \nu_n - 1) b^P_{\mathbf{l}-\mathbf{e}_n}(\nu)$$

Here

$$\kappa^P = \sum_{i\in\widetilde{P}} (\nu_i - \nu_n)(\nu_{i+1} - \nu_n)$$

and $\lambda^P = (\lambda_1^P,\ldots,\lambda_n^P)$ *is defined as follows:*

$$\lambda_i^P = \begin{cases} \nu_i - \nu_{i+1} & \text{if } i-1 \in P, i \in P, i+1 \in P; \\ \nu_i - \nu_n & \text{if } i-1 \in P, i \in P, i+1 \in P^c; \\ \nu_{i-1} + \nu_i - \nu_{i+1} - \nu_n & \text{if } i-1 \in P^c, i \in P, i+1 \in P; \\ \nu_{i-1} + \nu_i - 2\nu_n & \text{if } i-1 \in P^c, i \in P, i+1 \in P^c; \\ -\nu_i - \nu_{i+1} + 2\nu_n & \text{if } i-1 \in P, i \in P^c, i+1 \in P; \\ -\nu_i + \nu_n & \text{if } i-1 \in P, i \in P^c, i+1 \in P^c; \\ \nu_{i-1} - \nu_i - \nu_{i+1} + \nu_n & \text{if } i-1 \in P^c, i \in P^c, i+1 \in P; \\ \nu_{i-1} - \nu_i & \text{if } i-1 \in P^c, i \in P^c, i+1 \in P^c, \end{cases}$$

for $1 \le i \le n-2$,

$$\lambda_{n-1}^P = \begin{cases} \nu_{n-1} - \nu_n & \text{if } n-2 \in P, \ n-1 \in P; \\ \nu_{n-2} + \nu_{n-1} - 2\nu_n & \text{if } n-2 \in P^c, \ n-1 \in P; \\ -\nu_{n-1} + \nu_n & \text{if } n-2 \in P, \ n-1 \in P^c; \\ \nu_{n-2} - \nu_{n-1} & \text{if } n-2 \in P^c, \ n-1 \in P^c, \end{cases}$$

and

$$\lambda_n^P = \begin{cases} \nu_n & \text{if } n-1 \in P; \\ \nu_{n-1} & \text{if } n-1 \in P^c. \end{cases}$$

(ii) *We have*

$$b_{(0,\dots,0)}^P(\nu) = \prod_{i \in \widetilde{P}} \frac{\Gamma(\nu_i - \nu_{i+1} + 1)}{\Gamma(\nu_i - \nu_n + 1)\Gamma(-\nu_{i+1} + \nu_n + 1)},$$

and $b_{(0,\dots,0)}^P(\nu) = 1$ *if* $\widetilde{P} = \emptyset$.

Proof.   (i) The idea of proof is similar to the first step in the proof of Theorem 2.1 $(P = \{1, \dots, n-1\})$. Our claim follows from the identity

$$\sum_{i \in P,\, 1 \le i \le n-2} (l_i - k_i)(l_i - k_{i-1} + \nu_i - \nu_n) + \sum_{i \in P^c,\, 1 \le i \le n-2} (l_i - k_{i-1})(l_i - k_i - \nu_i + \nu_n)$$

$$+ \begin{cases} (l_{n-1} - l_n)(l_{n-1} - k_{n-2} + \nu_{n-1} - \nu_n) & \text{if } n-1 \in P; \\ (l_{n-1} - k_{n-2})(l_{n-1} - l_n - \nu_{n-1} + \nu_n) & \text{if } n-1 \in P^c \end{cases}$$

$$- \sum_{i \in P, i+1 \in P,\, 1 \le i \le n-2} (l_i - k_i)(l_{i+1} - k_i + \nu_{i+1} - \nu_n)$$

$$- \sum_{i \in P, i+1 \in P^c,\, 1 \le i \le n-2} (l_i - k_i)(l_{i+1} - k_i)$$

$$- \sum_{i \in P^c, i+1 \in P,\, 1 \le i \le n-2} (l_i - k_i - \nu_i + \nu_n)(l_{i+1} - k_i + \nu_{i+1} - \nu_n)$$

$$- \sum_{i \in P^c, i+1 \in P^c,\, 1 \le i \le n-2} (l_i - k_i - \nu_i + \nu_n)(l_{i+1} - k_i)$$

$$= \left(\sum_{i=1}^{n-1} l_i^2 + \frac{1}{2} l_n^2 - \sum_{i=1}^{n-1} l_i l_{i+1} + \sum_{i=1}^{n-1} \lambda_i^P l_i + \kappa^P \right) - q_{n-1}\big((k_1, \dots, k_{n-2}, l_n), \tilde{\nu}\big).$$

(ii) From the definition of $b_{(l_1,\dots,l_n)}^P(\nu)$, we have

$$b_{(0,\dots,0)}^P(\nu)$$

$$= \sum_{\substack{k_i = 0 \\ (i \in \widetilde{P})}}^{\infty} \frac{c_{n-1,(k_1,\dots,k_{n-2},0)}^{\widetilde{P}}(\nu)}{\prod_{i \in P \cap \{1,\dots,n-2\}} (\nu_i - \nu_n + 1)_{-k_{i-1}} \prod_{i \in P^c \cap \{1,\dots,n-2\}} (-\nu_i + \nu_n + 1)_{-k_i}}$$

$$\cdot \begin{cases} \dfrac{1}{(\nu_{n-1} - \nu_n + 1)_{-k_{n-2}}} & \text{if } n-2 \in \widetilde{P}; \\ 1 & \text{otherwise} \end{cases}$$

$$= \sum_{\substack{k_i = 0 \\ (i \in \widetilde{P})}}^{\infty} \left(\prod_{i \in \widetilde{P}} \frac{1}{(\nu_{i+1} - \nu_n + 1)_{-k_i} (-\nu_i + \nu_n + 1)_{-k_i}} \right) \cdot c_{n-1,(k_1,\ldots,k_{n-2},0)}^{\widetilde{P}}(\nu),$$

where

$$c_{n-1,(k_1,\ldots,k_{n-2},0)}^{\widetilde{P}}(\nu) = c_{n-1,(k_1,\ldots,k_{n-2},0)}(\nu)\big|_{k_i = 0 \ (i \notin \widetilde{P})}.$$

Since $i \in \widetilde{P}$ implies that $i \pm 1 \notin \widetilde{P}$, we can find a recurrence relation for $c_{n-1,(k_1,\ldots,k_{n-2},0)}^{\widetilde{P}}(\nu)$:

$$\left(\sum_{i \in \widetilde{P}} k_i^2 + \sum_{i \in \widetilde{P}} (\nu_i - \nu_{i+1}) k_i \right) c_{n-1,(k_1,\ldots,k_{n-2},0)}^{\widetilde{P}}(\nu) = \sum_{i \in \widetilde{P}} c_{n-1,(k_1,\ldots,k_{n-2},0)-\mathbf{e}_i}^{\widetilde{P}}(\nu).$$

We can easily solve this recurrence relation to find

$$c_{n-1,(k_1,\ldots,k_{n-2},0)}^{\widetilde{P}}(\nu) = \prod_{i \in \widetilde{P}} \frac{1}{k_i!(\nu_i - \nu_{i+1} + 1)_{k_i}}.$$

Thus we get

$$\begin{aligned} b_{(0,\ldots,0)}^{P}(\nu) &= \prod_{i \in \widetilde{P}} \left(\sum_{k_i = 0}^{\infty} \frac{1}{k_i!(\nu_i - \nu_{i+1} + 1)_{k_i}(\nu_{i+1} - \nu_n + 1)_{-k_i}(-\nu_i + \nu_n + 1)_{-k_i}} \right) \\ &= \prod_{i \in \widetilde{P}} {}_2F_1(-\nu_{i+1} + \nu_n, \nu_i - \nu_n; \nu_i - \nu_{i+1}; 1) \\ &= \prod_{i \in \widetilde{P}} \frac{\Gamma(\nu_i - \nu_{i+1} + 1)}{\Gamma(\nu_i - \nu_n + 1)\Gamma(-\nu_{i+1} + \nu_n + 1)}. \end{aligned}$$

Here we used $(a)_{-n} = (-1)^n / (1-a)_n$ and Gauss' formula

$$_2F_1(a, b; c; 1) = \frac{\Gamma(c - a - b)\Gamma(c)}{\Gamma(c - a)\Gamma(c - b)}$$

in the last step. Therefore we complete the proof of Lemma 3.4. ∎

Returning to the proof of Theorem 3.3, the next proposition implies our expansion formula for $V_\nu(x)$. Because, for $1 \le p \le n$, we have

$$\Gamma^{\{1,2,\ldots,p-1\}}(\nu) = \Gamma^p(\nu), \quad b_{(l_1,\ldots,l_n)}^{\{1,2,\ldots,p-1\}}(\nu) = b_{(l_1,\ldots,l_n)}^{p}(\nu),$$

from the definition.

Proposition 3.5. *The following P contributes in the right hand side of (3.9).*

- $P = \emptyset$,

- *P is of the form* $\{1, 2, \ldots, p-1\}$ *for some* $2 \leq p \leq n$.

More precisely, if there exists an element p_0 *in* $\widetilde{P}$ *then we have*

$$\sum_{w \in \{1, w_{p_0}\}} w \left[\Gamma^P(\nu) \sum_{l_1, \ldots, l_n = 0}^{\infty} b^P_{(l_1, \ldots, l_n)}(\nu) \right.$$

$$\left. \cdot \prod_{i \in P \cup \{n\}} (\pi x_i)^{2(l_i + \nu_1 + \cdots + \nu_i)} \prod_{i \in P^c} (\pi x_i)^{2(l_i + \nu_1 + \cdots + \nu_{i-1} + \nu_n)} \right] = 0.$$

Here $w_{p_0} \in \mathcal{W}_{n-1}$ *is the simple reflection which permutes* ν_{p_0} *and* ν_{p_0+1}.

Proof.   Fix $p_0 \in \widetilde{P}$. Since $p_0 \in P^c$ and $p_0 + 1 \in P$,

$$\prod_{i \in P \cup \{n\}} (\pi x_i)^{2(l_i + \nu_1 + \cdots + \nu_i)} \prod_{i \in P^c} (\pi x_i)^{2(l_i + \nu_1 + \cdots + \nu_{i-1} + \nu_n)}$$

is invariant under the permutation of ν_{p_0} and ν_{p_0+1}. Then it is enough to show

$$a^P_{(l_1, \ldots, l_n)}(\nu) := \sum_{w \in \{1, w_{p_0}\}} w \left[\Gamma^P(\nu) b^P_{(l_1, \ldots, l_n)}(\nu) \right] = 0.$$

In view of Lemma 3.4 (i), we can check that λ^P_i and κ^P are invariant under the action of w_{p_0}. Then $b^P_{(l_1, \ldots, l_n)}(\nu)$ and $b^P_{(l_1, \ldots, l_n)}(w_{p_0}\nu)$ satisfies the same recurrence relation and therefore $a^P_{(l_1, \ldots, l_n)}(\nu)$ also satisfies the same one. Thus, if we can say $a^P_{(0, \ldots, 0)}(\nu) = 0$ then $a^P_{(l_1, \ldots, l_n)}(\nu) = 0$ inductively follows.

From Lemma 3.4 (ii), we have

$$\Gamma^p(\nu) b^P_{(0, \ldots, 0)}(\nu) = \prod_{1 \leq i < j \leq n-1} \Gamma(-\nu_i - \nu_j)\Gamma(-\nu_i + \nu_j) \prod_{1 \leq i \leq n-1} \Gamma(-2\nu_i)$$

$$\cdot \prod_{i \in P} \Gamma(-\nu_i + \nu_n) \prod_{i \in P^c} \Gamma(\nu_i - \nu_n)$$

$$\cdot \prod_{i \in \widetilde{P}} \frac{\Gamma(\nu_i - \nu_{i+1} + 1)}{\Gamma(\nu_i - \nu_n + 1)\Gamma(-\nu_{i+1} + \nu_n + 1)}.$$

We pick up the terms which are not invariant under the action of w_{p_0}:

$$\Gamma(-\nu_{p_0} + \nu_{p_0+1})\Gamma(-\nu_{p_0+1} - \nu_n)\Gamma(\nu_{p_0} - \nu_n) \cdot \frac{\Gamma(\nu_{p_0} - \nu_{p_0+1} + 1)}{\Gamma(\nu_{p_0} - \nu_n + 1)\Gamma(-\nu_{p_0+1} + \nu_n + 1)}$$

$$= \frac{\pi}{\sin(-\nu_{p_0} + \nu_{p_0+1})\pi} \cdot \frac{1}{(\nu_{p_0} - \nu_n)(-\nu_{p_0+1} + \nu_n)}.$$

Therefore we get $a^P_{(0, \ldots, 0)}(\nu) = 0$, and complete the proof of Theorem 3.3.   ∎

3.2. The second step –expansion formula for $\widetilde{W}^n_\nu$–.

In the similar way to the previous subsection, we shall prove the linear relation (3.2). We need a little more complicated argument. We insert the expansion formula (3.6) for $V_\nu(x)$ to get

$$\widetilde{W}_\nu^n(y) = \sum_{w \in \mathcal{W}_{n-1}} \sum_{p=1}^n w\left[\Gamma^p(\nu) \int_{(\mathbf{R}_+^\times)^n} \prod_{i=1}^n \exp\left\{-(\pi y_i)^2 t_i - \frac{1}{t_i}\right\} \prod_{i=1}^n t_i^{\nu_n}\right.$$
$$\left. \cdot \sum_{\substack{l_1,\ldots,l_n=0 \\ l_{n-1} \geq l_n}}^\infty b_{(l_1,\ldots,l_n)}^p(\nu) \prod_{i=1}^n \left(\pi y_i \sqrt{\frac{t_i}{t_{i+1}}}\right)^{2(l_i + \nu_1^{(p)} + \cdots + \nu_i^{(p)})} \prod_{i=1}^n \frac{dt_i}{t_i}\right].$$

Changing the order of the integration and the infinite sum, we have

$$\widetilde{W}_\nu^n(y) = \sum_{w \in \mathcal{W}_{n-1}} \sum_{p=1}^n w\left[\Gamma^p(\nu) \sum_{\substack{l_1,\ldots,l_n=0 \\ l_{n-1} \geq l_n}}^\infty b_{(l_1,\ldots,l_n)}^p(\nu) \prod_{i=1}^n (\pi y_i)^{2(l_i + \nu_1^{(p)} + \cdots + \nu_i^{(p)})}\right.$$
$$\left. \cdot \prod_{i=1}^n \int_0^\infty \exp\left\{-(\pi y_i)^2 t_i - \frac{1}{t_i}\right\} t_i^{-l_{i-1} + l_i + \nu_i^{(p)} + \nu_n} \frac{dt_i}{t_i}\right].$$

We use (3.8) for the integral above to find

$$\widetilde{W}_\nu^n(y) = \sum_{w \in \mathcal{W}_{n-1}} \sum_{p=1}^n w\left[\Gamma^p(\nu) \sum_{\substack{l_1,\ldots,l_n=0 \\ l_{n-1} \geq l_n}}^\infty (-1)^{l_n} b_{(l_1,\ldots,l_n)}^p(\nu)\right.$$
$$\cdot \prod_{i=1}^n \left\{\Gamma(-\nu_i^{(p)} - \nu_n) \sum_{m_i=0}^\infty \frac{(\pi y_i)^{2(m_i + l_i + \nu_1^{(p)} + \cdots + \nu_i^{(p)})}}{m_i!(\nu_i^{(p)} + \nu_n + 1)_{m_i - l_{i-1} + l_i}}\right.$$
$$\left.\left. + \Gamma(\nu_i^{(p)} + \nu_n) \sum_{m_i=0}^\infty \frac{(\pi y_i)^{2(m_i + l_{i-1} + \nu_1^{(p)} + \cdots + \nu_{i-1}^{(p)} - \nu_n)}}{m_i!(-\nu_i^{(p)} - \nu_n + 1)_{m_i + l_{i-1} - l_i}}\right\}\right].$$

We substitute $m_i \to m_i - l_{i-1}$ or $m_i \to m_i - l_i$, and arrange the order of the sum. Then we get

$$\widetilde{W}_\nu^n(y) = \sum_{w \in \mathcal{W}_{n-1}} \sum_{p=1}^n \sum_{Q \subset \{1,2,\ldots,n\}} w\left[\Gamma^{p,Q}(\nu) \widetilde{M}_\nu^{p,Q}(y)\right], \qquad (3.10)$$

where

$$\Gamma^{p,Q}(\nu) := \Gamma^p(\nu) \prod_{i \in Q} \Gamma(-\nu_i^{(p)} - \nu_n) \prod_{i \in Q^c} \Gamma(\nu_i^{(p)} + \nu_n)$$

and

$$\widetilde{M}_\nu^{p,Q}(y) := \sum_{m_1,\ldots,m_n=0}^\infty c_{(m_1,\ldots,m_n)}^{p,Q}(\nu)$$
$$\cdot \prod_{i \in Q} (\pi y_i)^{2(m_i + \nu_1^{(p)} + \cdots + \nu_i^{(p)})} \prod_{i \in Q^c} (\pi y_i)^{2(m_i + \nu_1^{(p)} + \cdots + \nu_{i-1}^{(p)} - \nu_n)}$$

is the power series with the coefficient

$$c_{(m_1,\ldots,m_n)}^{p,Q}(\nu) := \sum_{\{l_1,\ldots,l_n\}} (-1)^{l_n} b_{(l_1,\ldots,l_n)}^p(\nu) \prod_{i\in Q} \frac{1}{(m_i - l_i)!(\nu_i^{(p)} + \nu_n + 1)_{m_i - l_{i-1}}}$$
$$\cdot \prod_{i\in Q^c} \frac{1}{(m_i - l_{i-1})!(-\nu_i^{(p)} - \nu_n + 1)_{m_i - l_i}}.$$

Here $\{l_1, \ldots, l_n\}$ means that

$$0 \leq l_i \leq m_i \ (i \in Q), \quad 0 \leq l_i \leq m_{i+1} \ (i+1 \in Q^c), \quad 0 \leq l_i \ (i \in Q^c \text{ and } i+1 \in Q).$$

As in the previous subsection, let us find a recurrence relation satisfied by $c_{(m_1,\ldots,m_n)}^{p,Q}(\nu)$, and an explicit formula for the initial value $c_{(0,\ldots,0)}^{p,Q}(\nu)$.

Lemma 3.6. *For $Q \subset \{1, 2, \ldots, n\}$, we set $\widetilde{Q} = \{i \mid 1 \leq i \leq n-1, \ i \in Q^c \text{ and } i+1 \in Q\}$.*

(i) $c_{\mathbf{m}}^{p,Q}(\nu) = c_{(m_1,\ldots,m_n)}^{p,Q}(\nu)$ *satisfies the recurrence relation*

$$\left(\sum_{i=1}^{n-1} m_i^2 + \frac{1}{2}m_n^2 - \sum_{i=1}^{n-1} m_i m_{i+1} + \sum_{i=1}^{n-1} \lambda_i^{p,Q} m_i + \kappa^{p,Q}\right) c_{\mathbf{m}}^{p,Q}(\nu)$$
$$= \sum_{i=1}^{n-1} c_{\mathbf{m}-\mathbf{e}_i}^{p,Q}(\nu) + \frac{1}{2} c_{\mathbf{m}-\mathbf{e}_n}^{p,Q}(\nu),$$

where

$$\kappa^{p,Q} = \sum_{i\in\widetilde{Q}} (\nu_i^{(p)} + \nu_n)(\nu_{i+1}^{(p)} + \nu_n) + \begin{cases} 0 & \text{if } n \in Q; \\ \frac{1}{2}(\nu_n^{(p)} + \nu_n)(-\nu_n^{(p)} + \nu_n) & \text{if } n \in Q^c, \end{cases}$$

and $\lambda^{p,Q} = (\lambda_1^{p,Q}, \ldots, \lambda_n^{p,Q})$ is defined as follows:

$$\lambda_i^{p,Q} = \begin{cases} \nu_i^{(p)} - \nu_{i+1}^{(p)} & \text{if } i-1 \in Q, \ i \in Q, \ i+1 \in Q; \\ \nu_i^{(p)} + \nu_n & \text{if } i-1 \in Q, \ i \in Q, \ i+1 \in Q^c; \\ \nu_{i-1}^{(p)} + \nu_i^{(p)} - \nu_{i+1}^{(p)} + \nu_n & \text{if } i-1 \in Q^c, \ i \in Q, \ i+1 \in Q; \\ \nu_{i-1}^{(p)} + \nu_i^{(p)} + 2\nu_n & \text{if } i-1 \in Q^c, \ i \in Q, \ i+1 \in Q^c; \\ -\nu_i^{(p)} - \nu_{i+1}^{(p)} - 2\nu_n & \text{if } i-1 \in Q, \ i \in Q^c, \ i+1 \in Q; \\ -\nu_i^{(p)} - \nu_n & \text{if } i-1 \in Q, \ i \in Q^c, \ i+1 \in Q^c; \\ \nu_{i-1}^{(p)} - \nu_i^{(p)} - \nu_{i+1}^{(p)} - \nu_n & \text{if } i-1 \in Q^c, \ i \in Q^c, \ i+1 \in Q; \\ \nu_{i-1}^{(p)} - \nu_i^{(p)} & \text{if } i-1 \in Q^c, \ i \in Q^c, \ i+1 \in Q^c, \end{cases}$$

for $1 \leq i \leq n-1$ and

$$\lambda_n^{p,Q} = \begin{cases} \nu_n^{(p)} & \text{if } n-1 \in Q, \ n \in Q; \\ \nu_{n-1}^{(p)} + \nu_n^{(p)} + \nu_n & \text{if } n-1 \in Q^c, \ n \in Q; \\ -\nu_n & \text{if } n-1 \in Q, \ n \in Q^c; \\ \nu_{n-1}^{(p)} & \text{if } n-1 \in Q^c, \ n \in Q^c. \end{cases}$$

(ii) *We have*

$$c^{p,Q}_{(0,\ldots,0)}(\nu) = \prod_{i\in\widetilde{Q}} \frac{\Gamma(\nu_i^{(p)} - \nu_{i+1}^{(p)} + 1)}{\Gamma(-\nu_{i+1}^{(p)} - \nu_n + 1)\Gamma(\nu_i^{(p)} + \nu_n + 1)}$$

$$\cdot \begin{cases} 1 & \text{if } n \in Q; \\ \dfrac{\Gamma(2\nu_n^{(p)} + 1)}{\Gamma(\nu_n^{(p)} + \nu_n + 1)\Gamma(\nu_n^{(p)} - \nu_n + 1)} & \text{if } n \in Q^c. \end{cases}$$

Proof. (i) Our claim follows from the identity

$$\sum_{i\in Q,\, 1\leq i\leq n-1} (m_i - l_i)(m_i - l_{i-1} + \nu_i^{(p)} + \nu_n)$$

$$+ \sum_{i\in Q^c,\, 1\leq i\leq n-1} (m_i - l_{i-1})(m_i - l_i - \nu_i^{(p)} - \nu_n)$$

$$+ \frac{1}{2} \begin{cases} (m_n - l_n)(m_n - l_{n-1} + \nu_n^{(p)} + \nu_n) & \text{if } n \in Q; \\ (m_n - l_{n-1})(m_n - l_n - \nu_n^{(p)} - \nu_n) & \text{if } n \in Q^c \end{cases}$$

$$- \sum_{i\in Q,\, i+1\in Q,\, 1\leq i\leq n-1} (m_i - l_i)(m_{i+1} - l_i + \nu_{i+1}^{(p)} + \nu_n)$$

$$- \sum_{i\in Q,\, i+1\in Q^c,\, 1\leq i\leq n-1} (m_i - l_i)(m_{i+1} - l_i)$$

$$- \sum_{i\in Q^c,\, i+1\in Q,\, 1\leq i\leq n-1} (m_i - l_i - \nu_i^{(p)} - \nu_n)(m_{i+1} - l_i + \nu_{i+1}^{(p)} + \nu_n)$$

$$- \sum_{i\in Q^c,\, i+1\in Q^c,\, 1\leq i\leq n-1} (m_i - l_i - \nu_i^{(p)} - \nu_n)(m_{i+1} - l_i)$$

$$- \frac{1}{2} \begin{cases} (m_n - l_n - \nu_n^{(p)} - \nu_n)(l_{n-1} - l_n - \nu_n^{(p)} + \nu_n) & \text{if } n \in Q; \\ (m_n - l_n)(l_{n-1} - l_n - \nu_n^{(p)} + \nu_n) & \text{if } n \in Q^c \end{cases}$$

$$= \left(\sum_{i=1}^{n-1} m_i^2 + \frac{1}{2}m_n^2 - \sum_{i=1}^{n-1} m_i m_{i+1} + \sum_{i=1}^{n-1} \lambda_i^{p,Q} m_i + \kappa^{p,Q} \right) - q_n(\mathbf{l}, \nu^{(p)}).$$

(ii) We can prove in the same way as Lemma 3.4 (ii). ∎

The following is immediate from the above lemma.

Corollary 3.7. *We have*

- $\widetilde{M}_\nu^{p,\{1,2,\ldots,n\}}(y) = \widetilde{M}_{\nu^{(p)}}^n(y)$ *and* $\Gamma^{p,\{1,2,\ldots,n\}}(\nu) = \Gamma_n(\nu^{(p)})$ *for* $1 \leq p \leq n$,

- $\widetilde{M}_\nu^{n,\{1,2,\ldots,q-1\}}(y) = \widetilde{M}_{\bar{\nu}^{(q)}}(y)$ *and* $\Gamma^{n,\{1,2,\ldots,q-1\}}(\nu) = \Gamma_n(\bar{\nu}^{(q)})$ *for* $1 \leq q \leq n$.

Here we write $\bar{\nu}^{(q)} := (\nu_1, \ldots, \nu_{q-1}, -\nu_n, \nu_q, \ldots, \nu_{n-1})$.

Proof. By Lemma 3.6 (i), we can verify that $c_{\mathbf{m}}^{p,\{1,2,\ldots,n\}}(\nu)$ $(1 \leq p \leq n)$ and $c_{\mathbf{m}}^{n,\{1,2,\ldots,q-1\}}(\nu)$ $(1 \leq q \leq n)$ satisfy the same recurrence relations as (1.1) with

$\nu = \nu^{(p)}$ and $\nu = \bar{\nu}^{(q)}$, respectively. Since $c_{(0,\ldots,0)}^{p,\{1,2,\ldots,n\}}(\nu) = c_{(0,\ldots,0)}^{n,\{1,2,\ldots,q-1\}}(\nu) = 1$ from Lemma 3.6 (ii), we have $c_{\mathbf{m}}^{p,\{1,2,\ldots,n\}}(\nu) = c_{n,\mathbf{m}}(\nu^{(p)})$ and $c_{\mathbf{m}}^{n,\{1,2,\ldots,q-1\}}(\nu) = c_{n,\mathbf{m}}(\bar{\nu}^{(q)})$. Thus get our claim for $\widetilde{M}_{\nu}^{*,*}(y)$.

The latter part of the corollary can be seen from the definition. Indeed we have

$$
\begin{aligned}
&\Gamma^{p,\{1,2,\ldots,n\}}(\nu) \\
&= \Gamma_{n-1}(\tilde{\nu}) \prod_{1\le i\le p-1} \Gamma(-\nu_i+\nu_n) \prod_{p\le i\le n-1} \Gamma(\nu_i-\nu_n) \prod_{1\le i\le n} \Gamma(-\nu_i^{(p)}-\nu_n) \\
&= \prod_{1\le i<j\le n-1} \Gamma(-\nu_i-\nu_j)\Gamma(-\nu_i+\nu_j) \prod_{1\le i\le n-1} \Gamma(-2\nu_i) \\
&\quad \cdot \prod_{1\le i\le p-1} \Gamma(-\nu_i+\nu_n) \prod_{p\le i\le n-1} \Gamma(\nu_i-\nu_n) \\
&\quad \cdot \prod_{1\le i\le p-1} \Gamma(-\nu_i-\nu_n) \cdot \Gamma(-\nu_n-\nu_n) \cdot \prod_{p+1\le i\le n} \Gamma(-\nu_{i-1}-\nu_n) \\
&= \prod_{1\le i<j\le n-1} \Gamma(-\nu_i-\nu_j)\Gamma(-\nu_i+\nu_j) \prod_{1\le i\le n} \Gamma(-2\nu_i) \\
&\quad \cdot \prod_{1\le i\le p-1} \Gamma(-\nu_i+\nu_n)\Gamma(-\nu_i-\nu_n) \prod_{p+1\le i\le n} \Gamma(\nu_{i-1}-\nu_n)\Gamma(-\nu_{i-1}-\nu_n) \\
&= \Gamma_n(\nu^{(p)})
\end{aligned}
$$

and

$$
\begin{aligned}
&\Gamma^{n,\{1,2,\ldots,q-1\}}(\nu) \\
&= \Gamma_{n-1}(\tilde{\nu}) \prod_{1\le i\le n-1} \Gamma(-\nu_i+\nu_n) \prod_{1\le i\le q-1} \Gamma(-\nu_i^{(n)}-\nu_n) \prod_{q\le i\le n} \Gamma(\nu_i^{(n)}+\nu_n) \\
&= \prod_{1\le i\le n-1} \Gamma(-2\nu_i) \cdot \Gamma(2\nu_n) \prod_{1\le i<j\le n-1} \Gamma(-\nu_i-\nu_j)\Gamma(-\nu_i+\nu_j) \\
&\quad \cdot \prod_{1\le i\le q-1} \Gamma(-\nu_i+\nu_n)\Gamma(-\nu_i-\nu_n) \prod_{q+1\le i\le n} \Gamma(-\nu_{i-1}+\nu_n)\Gamma(\nu_{i-1}+\nu_n) \\
&= \Gamma_n(\bar{\nu}^{(q)}).
\end{aligned}
$$

Thus we can finish our proof. ∎

Now we state cancelations in the sum in (3.10).

Proposition 3.8. *The following terms contribute to the right hand side of (3.10).*

- $1 \le p \le n$ *and* $Q = \{1, 2, \ldots, n\}$,

- $p = n$ *and* Q *is of the form* $\{1, 2, \ldots, q-1\}$ $(1 \le q \le n)$.

More precisely we have the following:

(i) *If $1 \le p \le n - 1$ and Q is of the form $\{1, 2, \ldots, q - 1\}$ $(1 \le q \le n)$, then*

$$\sum_{w \in \{1, w_{n-1}\}} w\left[\Gamma^{p,Q}(\nu) \widetilde{M}_\nu^{p,Q}(y)\right] = 0.$$

Here $w_{n-1} \in \mathcal{W}_{n-1}$ is the simple reflection, which permutes the sign of ν_{n-1}.

(ii) *If Q is not of the form $\{1, 2, \ldots, q\}$ $(0 \le q \le n)$, then there exists an element q_0 in $\widetilde{Q}$ and we fix such q_0.*

(a) For $p \ne q_0, q_0 + 1$, we have

$$\sum_{w \in \{1, w_{q_0}^{(p)}\}} w\left[\Gamma^{p,Q}(\nu) \widetilde{M}_\nu^{p,Q}(y)\right] = 0.$$

Here $w_{q_0}^{(p)} \in \mathcal{W}_{n-1}$ permutes $\nu_{q_0}^{(p)}$ and $\nu_{q_0+1}^{(p)}$ and fixes other ν_i 's.

(b) The terms $p = q_0$ and $p = q_0 + 1$ cancel each other:

$$\sum_{p = q_0, q_0 + 1} \Gamma^{p,Q}(\nu) \widetilde{M}_\nu^{p,Q}(y) = 0.$$

Proof. The idea of proof is the same as Proposition 3.5.

(i) Since $\nu_n^{(p)} = \nu_{n-1}$ does not appear in the characteristic exponents of the power series $\widetilde{M}_\nu^{p,Q}$, it is enough to show

$$\sum_{w \in \{1, w_{n-1}\}} w\left[\Gamma^{p,Q}(\nu) \, c_{(m_1, \ldots, m_n)}^{p,Q}(\nu)\right] = 0.$$

The recurrence relation for $c_{(m_1, \ldots, m_n)}^{p,Q}(\nu)$ given in Lemma 3.6 (i), is invariant under the action of w_{n-1}. Actually ν_{n-1} does not appear in $\lambda_i^{p,Q}$ and $\kappa^{p,Q} = \frac{1}{2}(\nu_{n-1} + \nu_n)(-\nu_{n-1} + \nu_n)$ (note that $\widetilde{Q} = \emptyset$). Then our task is reduced to confirm

$$\sum_{w \in \{1, w_{n-1}\}} w\left[\Gamma^{p,Q}(\nu) \, c_{(0, \ldots, 0)}^{p,Q}(\nu)\right] = 0. \tag{3.11}$$

From the definition of $\Gamma^{p,Q}(\nu)$ and Lemma 3.6 (ii), we have

$$\begin{aligned}
&\Gamma^{p,Q}(\nu) \, c_{(0, \ldots, 0)}^{p,Q}(\nu) \\
&= \prod_{1 \le i < j \le n-1} \Gamma(-\nu_i - \nu_j)\Gamma(-\nu_i + \nu_j) \prod_{1 \le i \le n-1} \Gamma(-2\nu_i) \\
&\quad \cdot \prod_{1 \le i \le p-1} \Gamma(-\nu_i + \nu_n) \prod_{p \le i \le n-1} \Gamma(\nu_i - \nu_n) \\
&\quad \cdot \prod_{i \in Q} \Gamma(-\nu_i^{(p)} - \nu_n) \prod_{i \in Q^c} \Gamma(\nu_i^{(p)} + \nu_n) \cdot \frac{\Gamma(2\nu_n^{(p)} + 1)}{\Gamma(\nu_n^{(p)} + \nu_n + 1)\Gamma(\nu_n^{(p)} - \nu_n + 1)}.
\end{aligned}$$

We pick up the terms containing $\nu_n^{(p)} = \nu_{n-1}$ in the above:

$$\prod_{1 \leq i \leq n-2} \Gamma(-\nu_i - \nu_{n-1})\Gamma(-\nu_i + \nu_{n-1}) \cdot \Gamma(-2\nu_{n-1})\Gamma(\nu_{n-1} - \nu_n)$$

$$\cdot \Gamma(\nu_{n-1} + \nu_n) \cdot \frac{\Gamma(2\nu_{n-1} + 1)}{\Gamma(\nu_{n-1} + \nu_n + 1)\Gamma(\nu_{n-1} - \nu_n + 1)}$$

$$= \frac{\pi}{\sin(-2\nu_{n-1}\pi)} \cdot \frac{1}{(\nu_{n-1} + \nu_n)(\nu_{n-1} - \nu_n)} \prod_{1 \leq i \leq n-2} \Gamma(-\nu_i - \nu_{n-1})\Gamma(-\nu_i + \nu_{n-1}).$$

Then we have (3.11).

(ii) (a) In view of

$$\begin{cases} \nu_{q_0}^{(p)} = \nu_{q_0-1}, \ \nu_{q_0+1}^{(p)} = \nu_{q_0} & \text{if } p < q_0, \\ \nu_{q_0}^{(p)} = \nu_{q_0}, \ \nu_{q_0+1}^{(p)} = \nu_{q_0+1} & \text{if } p > q_0 + 1, \end{cases}$$

and $q_0 \in Q^c$, $q_0 + 1 \in Q$, we can see that $\prod_{i \in Q}(\pi y_i)^{(\cdots)} \prod_{i \in Q^c}(\pi y_i)^{(\cdots)}$ is invariant under the action of $w_{q_0}^{(p)}$. As in the proof of (i), we can see the assertion from Lemma 3.6.

(ii) (b) By using $\nu_{q_0}^{(q_0)} = \nu_n$, $\nu_{q_0+1}^{(q_0)} = \nu_{q_0}$, $\nu_{q_0}^{(q_0+1)} = \nu_{q_0}$ and $\nu_{q_0+1}^{(q_0+1)} = \nu_n$, our claim follows from Lemma 3.6. Indeed we have

$$\frac{\Gamma^{q_0,Q}(\nu)c_{(0,\ldots,0)}^{q_0,Q}(\nu)}{\Gamma^{q_0+1,Q}(\nu)c_{(0,\ldots,0)}^{q_0+1,Q}(\nu)}$$

$$= \frac{\prod_{1 \leq i \leq q_0-1} \Gamma(-\nu_i + \nu_n) \prod_{q_0 \leq i \leq n-1} \Gamma(\nu_i - \nu_n)}{\prod_{1 \leq i \leq q_0} \Gamma(-\nu_i + \nu_n) \prod_{q_0+1 \leq i \leq n-1} \Gamma(\nu_i - \nu_n)}$$

$$\cdot \frac{\prod_{i \in Q} \Gamma(-\nu_i^{(q_0)} - \nu_n) \prod_{i \in Q^c} \Gamma(\nu_i^{(q_0)} + \nu_n)}{\prod_{i \in Q} \Gamma(-\nu_i^{(q_0+1)} - \nu_n) \prod_{i \in Q^c} \Gamma(\nu_i^{(q_0+1)} + \nu_n)}$$

$$\cdot \frac{\Gamma(\nu_{q_0}^{(q_0)} - \nu_{q_0+1}^{(q_0)} + 1)}{\Gamma(-\nu_{q_0+1}^{(q_0)} - \nu_n + 1)\Gamma(\nu_{q_0}^{(q_0)} + \nu_n + 1)} \cdot \frac{\Gamma(-\nu_{q_0+1}^{(q_0+1)} - \nu_n + 1)\Gamma(\nu_{q_0}^{(q_0+1)} + \nu_n + 1)}{\Gamma(\nu_{q_0}^{(q_0+1)} - \nu_{q_0+1}^{(q_0+1)} + 1)}$$

$$= \frac{\Gamma(\nu_{q_0} - \nu_n)}{\Gamma(-\nu_{q_0} + \nu_n)} \cdot \frac{\Gamma(-\nu_{q_0} - \nu_n)\Gamma(2\nu_n)}{\Gamma(-2\nu_n)\Gamma(\nu_{q_0} + \nu_n)}$$

$$\cdot \frac{\Gamma(\nu_n - \nu_{q_0} + 1)}{\Gamma(-\nu_{q_0} - \nu_n + 1)\Gamma(2\nu_n + 1)} \cdot \frac{\Gamma(-2\nu_n + 1)\Gamma(\nu_{q_0} + \nu_n + 1)}{\Gamma(\nu_{q_0} - \nu_n + 1)}$$

$$= -\frac{\sin(\nu_{q_0} + \nu_n)\pi \cdot \sin(-\nu_{q_0} + \nu_n)\pi}{\sin(\nu_{q_0} - \nu_n)\pi \cdot \sin(-\nu_{q_0} - \nu_n)\pi}$$

$$= -1.$$

Thus we are done.   ∎

To conclude our proof of Theorem 3.1, we rewrite (3.10) by using Corollary 3.7 and Proposition 3.8:

$$\widetilde{W}_\nu^n(y) = \sum_{w \in \mathcal{W}_{n-1}} \sum_{p=1}^{n} w \left[\Gamma_n(\nu^{(p)}) \sum_{m_1,\ldots,m_n=0}^{\infty} c_{n,(m_1,\ldots,m_n)}(\nu^{(p)}) \right.$$

$$\cdot \prod_{i=1}^{p-1} (\pi y_i)^{2(m_i+\nu_1+\cdots+\nu_i)} \prod_{i=p}^{n} (\pi y_i)^{2(m_i+\nu_1+\cdots+\nu_{i-1}+\nu_n)} \Bigg]$$

$$+ \sum_{w\in\mathcal{W}_{n-1}} \sum_{q=1}^{n} w \Bigg[\Gamma_n(\bar\nu^{(q)}) \sum_{m_1,\ldots,m_n=0}^{\infty} c_{n,(m_1,\ldots,m_n)}(\bar\nu^{(q)})$$

$$\cdot \prod_{i=1}^{q-1} (\pi y_i)^{2(m_i+\nu_1+\cdots+\nu_i)} \prod_{i=q}^{n} (\pi y_i)^{2(m_i+\nu_1+\cdots+\nu_{i-1}-\nu_n)} \Bigg]$$

$$= \sum_{w\in\mathcal{W}_n} w \Big[\Gamma_n(\nu) \widetilde{M}^n_\nu(y) \Big].$$

Thus we can finish our proof of Theorem 3.1. ∎

4. Mellin transforms of class one Whittaker functions

In this section we compute the Mellin transform of the class one Whittaker function. We first derive another recursive integral representation between $W^n_\nu(y)$ and $W^{n-1}_\nu(y)$. This is similar to the recursive formula for $GL_n(\mathbf{R})$ and $GL_{n-2}(\mathbf{R})$ Whittaker functions obtained by Stade [13, Theorm 2.1].

Theorem 4.1. *We have*

$$\widetilde{W}^n_\nu(y) = 2^n \int_{(\mathbf{R}^+)^{n-1}} \prod_{i=1}^{n} K_{2\nu_n}\Big(2\pi y_i \sqrt{(1+u_{i-1})(1+1/u_i)}\Big)$$

$$\cdot \widetilde{W}^{n-1}_{\tilde\nu}\Big(y_2\sqrt{\frac{u_1}{u_2}}, \ldots, y_{n-1}\sqrt{\frac{u_{n-2}}{u_{n-1}}}, y_n\sqrt{u_{n-1}}\Big) \prod_{i=1}^{n-1} \frac{du_i}{u_i}.$$

Here we promise $u_0 = 1/u_n = 0$.

Proof. Substituting $u_i \to t_{i+1}/u_i$ $(1 \le i \le n-1)$ into the integral representation (3.1) to find

$$\widetilde{W}^n_\nu(y) = \int_{(\mathbf{R}^\times_+)^n} \int_{(\mathbf{R}^\times_+)^{n-1}} \prod_{i=1}^{n} \exp\Big\{-(\pi y_i)^2 t_i - \frac{1}{t_i}\Big\} \prod_{i=1}^{n} (\pi y_i t_i)^{2\nu_n}$$

$$\cdot \prod_{i=1}^{n-1} \exp\Big\{-(\pi y_i)^2 \frac{t_i}{u_i} - \frac{u_i}{t_{i+1}}\Big\} \prod_{i=1}^{n-1} u_i^{-\nu_n}$$

$$\cdot \widetilde{W}^{n-1}_{\tilde\nu}\Big(y_2\sqrt{\frac{u_1}{u_2}}, \ldots, y_{n-1}\sqrt{\frac{u_{n-2}}{u_{n-1}}}, y_n\sqrt{u_{n-1}}\Big) \prod_{i=1}^{n-1} \frac{du_i}{u_i} \prod_{i=1}^{n} \frac{dt_i}{t_i}$$

$$= \int_{(\mathbf{R}^\times_+)^{n-1}} \prod_{i=1}^{n} \Big(\int_0^\infty \exp\Big\{-(\pi y_i)^2 t_i\Big(1+\frac{1}{u_i}\Big) - \frac{1}{t_i}(1+u_{i-1})\Big\} t_i^{2\nu_n} \frac{dt_i}{t_i}\Big)$$

$$\cdot \widetilde{W}^{n-1}_{\tilde\nu}\Big(y_2\sqrt{\frac{u_1}{u_2}}, \ldots, y_{n-1}\sqrt{\frac{u_{n-2}}{u_{n-1}}}, y_n\sqrt{u_{n-1}}\Big)$$

$$\cdot \prod_{i=1}^{n} (\pi y_i)^{2\nu_n} \prod_{i=1}^{n-1} u_i^{-\nu_n} \cdot \prod_{i=1}^{n-1} \frac{du_i}{u_i}.$$

By (3.8), the integration $\int dt_i$ becomes

$$2(1+u_{i-1})^{2\nu_n}\left\{(\pi y_i)\sqrt{(1+u_{i-1})(1+1/u_i)}\right\}^{-2\nu_n}$$
$$\cdot K_{2\nu_n}\left(2\pi y_i\sqrt{(1+u_{i-1})(1+1/u_i)}\right)$$

and we finish the proof.   ∎

Let $s = (s_1, \ldots, s_n) \in \mathbf{C}^n$ and

$$T_\nu^n(s) = \int_{(\mathbf{R}^+)^n} \widetilde{W}_\nu^n(y_1, \ldots, y_n) \prod_{i=1}^n (\pi y_i)^{2s_i} \frac{dy_i}{y_i}$$

be the multiple Mellin transform of the (ρ-shifted) class one Whittaker function $\widetilde{W}_\nu^n(y)$. In the same way as in [15, Theorem 3.1] for $GL(n, \mathbf{R})$-Whittaker functions, we can prove the following recursive formula for $T_\nu^n(s)$.

Theorem 4.2. *Let $n \geq 2$ and fix real numbers τ_j $(1 \leq j \leq n-1)$ such that*

$$\tau_j < \min\{\mathrm{Re}(\textstyle\sum_{i=1}^j \varepsilon_i \nu_{\sigma(i)}) \mid \varepsilon_i \in \{\pm 1\}, \sigma \in \mathfrak{S}_n\},$$

and also define $\tau_{-1} = +\infty$, $\tau_0 = 0$. Let

$$\eta_j = \max\{-\tau_{j-1} + \nu_n, \tau_{j-1} - \nu_n, -\tau_j, -\tau_{j-2}\},$$

for $1 \leq j \leq n-1$ and

$$\Omega = \{s \in \mathbf{C}^{n-1} \mid \mathrm{Re}(s_j) > \eta_j \text{ for } 1 \leq j \leq n-1\}.$$

Then for $s \in \Omega$, we have

$$T_\nu^n(s) = \frac{2^{-1}}{(2\pi\sqrt{-1})^{n-1}} \int_{\tau_1 - \sqrt{-1}\infty}^{\tau_1 + \sqrt{-1}\infty} \cdots \int_{\tau_{n-1} - \sqrt{-1}\infty}^{\tau_{n-1} + \sqrt{-1}\infty}$$
$$\cdot \prod_{i=1}^n \Gamma(s_i + t_{i-1} + \nu_n)\Gamma(s_i + t_{i-1} - \nu_n)$$
$$\cdot \prod_{i=1}^{n-1} \frac{\Gamma(s_i + t_i)\Gamma(s_{i+1} + t_{i-1})}{\Gamma(s_i + s_{i+1} + t_{i-1} + t_i)} \cdot T_{\tilde\nu}^{n-1}(-t_1, \ldots, -t_{n-1}) \prod_{i=1}^{n-1} dt_i.$$

Here $T_\nu^1(s) = 2^{-1}\Gamma(s_1 + \nu_1)\Gamma(s_1 - \nu_1)$ and we promise $t_0 = 0$.

Proof.   (sketch) We use induction on n and the proof is quite analogous to [15, §4]. By Mellin inversion and Theorem 4.1,

$$T_\nu^n(s) = \frac{2^{2n-1}}{(2\pi\sqrt{-1})^{n-1}} \int_{\tau_1 - \sqrt{-1}\infty}^{\tau_1 + \sqrt{-1}\infty} \cdots \int_{\tau_{n-1} - \sqrt{-1}\infty}^{\tau_{n-1} + \sqrt{-1}\infty} T_{\tilde\nu}^{n-1}(-t_1, \ldots, -t_{n-1})$$
$$\cdot \int_{(\mathbf{R}^+)^{n-1}} \int_{(\mathbf{R}_+^\times)^n} \prod_{i=1}^n K_{2\nu_n}\left(2\pi y_i\sqrt{(1+u_{i-1})(1+1/u_i)}\right)(\pi y_i)^{2s_i}$$

$$\cdot \prod_{i=1}^{n-1}\left(\pi y_{i+1}\sqrt{\frac{u_i}{u_{i+1}}}\right)^{2t_i} \prod_{i=1}^{n}\frac{dy_i}{y_i} \prod_{i=1}^{n-1}\frac{du_i}{u_i} \prod_{i=1}^{n-1} dt_i$$

$$= \frac{2^{2n-1}}{(2\pi\sqrt{-1})^{n-1}} \int_{\tau_1-\sqrt{-1}\infty}^{\tau_1+\sqrt{-1}\infty} \cdots \int_{\tau_{n-1}-\sqrt{-1}\infty}^{\tau_{n-1}+\sqrt{-1}\infty} T_{\tilde{\nu}}^{n-1}(-t_1,\ldots,-t_{n-1})$$

$$\cdot \int_{(\mathbf{R}_+^\times)^{n-1}} \prod_{i=1}^{n}\left(\int_0^\infty K_{2\nu_n}\left(2\pi y_i\sqrt{(1+u_{i-1})(1+1/u_i)}\right)(\pi y_i)^{2(s_i+t_{i-1})}\frac{dy_i}{y_i}\right)$$

$$\cdot \prod_{i=1}^{n-1} u_i^{-t_{i-1}+t_i} \prod_{i=1}^{n-1}\frac{du_i}{u_i} \prod_{i=1}^{n-1} dt_i$$

$$= \frac{2^{-1}}{(2\pi\sqrt{-1})^{n-1}} \int_{\tau_1-\sqrt{-1}\infty}^{\tau_1+\sqrt{-1}\infty} \cdots \int_{\tau_{n-1}-\sqrt{-1}\infty}^{\tau_{n-1}+\sqrt{-1}\infty} T_{\tilde{\nu}}^{n-1}(-t_1,\ldots,-t_{n-1})$$

$$\cdot \prod_{i=1}^{n} \Gamma(s_i+t_{i-1}+\nu_n)\Gamma(s_i+t_{i-1}-\nu_n)$$

$$\cdot \int_{(\mathbf{R}_+^\times)^{n-1}}\left\{\prod_{i=1}^{n-1} u_i^{-t_{i-1}+t_i} \prod_{i=1}^{n}\{(1+u_{i-1})(1+1/u_i)\}^{-s_i-t_{i-1}}\right\}\prod_{i=1}^{n-1}\frac{du_i}{u_i} \prod_{i=1}^{n-1} dt_i$$

$$= \frac{2^{-1}}{(2\pi\sqrt{-1})^{n-1}} \int_{\tau_1-\sqrt{-1}\infty}^{\tau_1+\sqrt{-1}\infty} \cdots \int_{\tau_{n-1}-\sqrt{-1}\infty}^{\tau_{n-1}+\sqrt{-1}\infty} T_{\tilde{\nu}}^{n-1}(-t_1,\ldots,-t_{n-1})$$

$$\cdot \prod_{i=1}^{n} \Gamma(s_i+t_{i-1}+\nu_n)\Gamma(s_i+t_{i-1}-\nu_n)$$

$$\cdot \prod_{i=1}^{n-1}\left(\int_0^\infty (1+u_i)^{-s_i-s_{i+1}-t_{i-1}-t_i} u_i^{s_i+t_i}\frac{du_i}{u_i}\right)\prod_{i=1}^{n-1} dt_i.$$

By using

$$\int_0^\infty (1+u)^{-(x+y)} u^y \frac{du}{u} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for $\mathrm{Re}(x) > 0$, $\mathrm{Re}(y) > 0$, we get the assertion. ∎

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Taku Ishii
Faculty of Science and Technology
Seikei University
3-3-1 Kichijoji-Kitamachi
Musashino, Tokyo, 180-8633, Japan,
ishii@st.seikei.ac.jp