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Lie Hypergroups

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Abstract. We define Lie hypergroups and study their embedded and immersed subhypergroups. In particular we investigate the properties of the connected component of the identity, the universal covering and fundamental group of a Lie hypergroup. We also study the quotients and orbits in a Lie hypergroup.
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1. Introduction

Lie groups are (real or complex) manifolds which are groups with smooth operations. Many classical examples of topological groups are indeed Lie groups. In the theory of Lie groups, many aspects of harmonic analysis become simpler and more natural. For instance, the existence of Haar measure is easily proved by considering the wedge product of a basis of one forms on the corresponding Lie algebra.

Hypergroups are generalizations of (topological) groups with a fairly well-developed harmonic analysis. Roughly speaking, a hypergroup is a (locally compact, Hausdorff) space K with a convolution and involution on the measure space $M(K)$ making it a Banach $*$-algebra. In contrast with the group case, the convolution of point masses is not a point mass, but is a probability measure. The pioneering works on hypergroups are due to Dunkl [5], Spector [18], and Jewett [10]. A typical reference is the monograph by Bloom and Heyer [3]. Many basic notions and facts from harmonic analysis and representations theory of groups carry over to hypergroups only with minor changes, but there are certain topics where the differences are substantial.

A locally compact hypergroup is a locally compact space whose bounded Radon measures form a (convolution) unital Banach $*$-algebra with certain properties similar to the convolution measure algebra of a locally compact group. The main difference with groups is that for each pair x, y in a hypergroup, the convolution $\delta_x * \delta_y$ of Dirac measures δ_x and δ_y is not in general a Dirac measure. It is a

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probability measure with compact support. Hypergroups have unit element e such that δ_e is the identity of the measure algebra and each element x has an inverse $\bar{x}$, such that e is in the support of $\delta_x * \delta_{\bar{x}}$. Important examples of hypergroups are double coset spaces of locally compact groups by compact subgroups, and orbit spaces of actions of a group of automorphisms on a locally compact group. The dual space of a compact group is also a discrete hypergroup.

This paper introduces and studies the notion of Lie hypergroups. As the multiplication map in a hypergroup sends a pair of elements into a probability measure, the notion of smooth multiplication is more involved here and should be handled with care. We show that an appropriate weak topology on the algebra of bounded Borel measures could be employed to give the right definition of a hypergroup with smooth multiplication and ensure that Lie hypergroups share some of the basic properties of Lie groups. We show that many classical hypergroups are indeed Lie hypergroups. This paper is the first work in this direction and we only deal with very basic properties such as that of the connected component of the identity, the universal covering and fundamental group. We also briefly study the quotients and orbits in Lie hypergroups. We postpone the study of the fundamental notion of the Lie hyperalgebra of a Lie hypergroup to a forthcoming paper.

2. Hypergroups

A (locally compact) *hypergroup* is a locally compact, Hausdorff space K with an involution $^-$ and a binary operation $*$, called a *convolution*, on the Banach space $M(K)$ of all (complex) bounded Radon measures on K such that $(M(K), *)$ is an algebra and for each $x, y \in G$,

(i) $\delta_x * \delta_y$ is a probability measure on G with compact support,

(ii) the map $(x, y) \in G^2 \mapsto \delta_x * \delta_y \in M(G)$ is continuous,

(iii) the map $(x, y) \in G^2 \mapsto \operatorname{supp}(\delta_x * \delta_y) \in \mathcal{C}(G)$ is continuous with respect to the Michael topology on the space $\mathcal{C}(G)$ of nonvoid compact subsets of G,

(iv) G admits an *identity* element e satisfying $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$,

(v) $(\delta_x * \delta_y)^- = \delta_{\bar{y}} * \delta_{\bar{x}}$,

(vi) $e \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $x = \bar{y}$.

Some remarks on the topology used in (iii) are in order. The Michael topology [14, Def. 1.7] is a topology on the space $\mathcal{C}(X)$ of compact subsets of a locally compact, Hausdorff space X. It has a sub-basis consisting of all sets of the form

$$\mathcal{C}_U(V) = \{A \in \mathcal{C}(X) : A \cap U \neq \varnothing,\ A \subseteq V\}$$

where U and V range over open subsets of X (c.f. [10, X2.5]). When X is metrisable, with metric d, the induced Hausdorff metric on $\mathcal{C}(X)$ defined by

$$d(A, B) = \inf\{r > 0 : A \in V_r(B),\ B \in V_r(A)\}$$

where

$$V_r(A) = \{x \in X : d(x,a) < r \ (a \in A)\}$$

for $A \in \mathcal{C}(X)$, coincides with the Michael topology [13, Lemma 4.1].

A hypergroup K is called *commutative* if $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in K$. A locally compact group is a hypergroup with the usual convolution $\delta_x * \delta_y = \delta_{xy}$ and the inverse map $x \mapsto x^{-1}$ as the involution. Another typical example is the double coset space

$$H\backslash G/H = \{HxH : x \in G\}$$

where G is a locally compact group and H is a compact subgroup [3, 1.1.9]. The space $H\backslash G/H$ is considered with the quotient topology. The identity element is $H = HeH$ and the involution and convolution are given by $(HxH)^{-} = Hx^{-1}H$ and

$$\delta_{HxH} * \delta_{HyH} = \int_H \delta_{HxtyH} dt,$$

where the integral is taken against the (normalized) Haar measure of H. Note that the double coset space is usually denoted by $G//H$ in the hypergroup literature.

Let K be a hypergroup with involution $^{-}$ and identity element e. The maximal subgroup of K is the (locally compact) group

$$G(K) = \{x \in K : \delta_x * \delta_{\bar{x}} = \delta_{\bar{x}} * \delta_x = \delta_e\}.$$

The convolution of two subsets $A, B \subseteq K$ is defined by

$$A * B = \bigcup \{\mathrm{supp}\,(\delta_a * \delta_b) : a \in A, b \in B\}.$$

For $x, y \in K$ and $A \subseteq K$, we abbreviate $A * \{x\}$ to $A * x$ and $\{x\} * \{y\}$ to $x * y$, and denote by A^n the n-fold convolution of A and by $cl(A)$ the closure of A in K. Also we write $\bar{A} = \{\bar{a} : a \in A\}$. For $f \in C_c(K)$ and $x, y \in K$, we put

$$f(x * y) = \int_K f(t) d(\delta_x * \delta_y)(t).$$

Note that $M(K)$ is a Banach $*$-algebra under the convolution

$$\int_K f d(\mu * \nu) = \int_K \int_K f(x * y) d\mu(x) d\nu(y),$$

and involution $\mu^*(A) = \overline{\mu(\bar{A})}$, for $f \in C_c(K), \mu, \nu \in M(K)$, and Borel set $A \subseteq K$. A nonempty closed subset H of K is called a *subhypergroup* if $H^{-} = H$ and $H * H \subset H$. A subhypergroup H is called *normal* if $x * H = H * x$ for each $x \in K$, and *supernormal* if $x * H * \bar{x} \subseteq H$, for each $x \in K$. The two notions coincide when K is a group. Given a compact normal subhypergroup or a supernormal subhypergroup H of K, the right coset space

$$K/H = \{H * x : x \in K\}$$

is a hypergroup with convolution product

$$\delta_{H*x} * \delta_{H*y} := \int_K \delta_{H*t} d(\delta_x * \delta_y)(t)$$

and the quotient map $q : K \to K/H$ is an open hypergroup homomorphism [3, Theorem 1.5.22]. If H is supernormal in K, then $(K/H, *)$ is a group [23].

Given a subset A of K, $[A]$ denotes the smallest subhypergroup in K containing A [3, Definition 1.5.3]. It is easily seen that

$$[A] = \bigcup\{a_1 * \cdots * a_n : n \in \mathbb{N}, a_1, \ldots, a_n \in A \cup A^{\bar{}}\} \cup \{e\}.$$

Similarly if $\langle A\rangle$ is the smallest supernormal subhypergroup in K containing A, then $\langle A\rangle = [N(A)]$, where

$$N(A) = \bigcup\{b_1 * \cdots * b_n * a * \bar{b}_n * \cdots * \bar{b}_1 : n \in \mathbb{N}, a \in A, b_1, \ldots, b_n \in K\}.$$

In particular, $K_e = cl(\langle\{e\}\rangle)$ is the smallest closed supernormal subhypergroup in K.

If K is a locally compact hypergroup, $M(K)$ is a Banach space in norm topology. However in the next section K is also a real C^∞ (or complex analytic) manifold and we need to consider $M(K)$ as a flat (infinite dimensional) manifold and give it the structure of a topological vector space. The appropriate topology on $M(K)$ is the w^*_∞-topology induced by the space $C_c^\infty(K)$ of C^∞-functions of compact support on K. A net (μ_α) converges to μ in the w^*_∞-topology if and only if

$$\int_K f d\mu_\alpha \to \int_K f d\mu \quad (f \in C_c^\infty(K))$$

as $\alpha \to \infty$. We use the notation $M_\infty(K)$ to denote the locally convex topological vector space $M(K)$ with w^*_∞-topology and save the notation $M(K)$ for the Banach space with the norm topology. Note that $M_\infty(K)^* = C_c^\infty(K)$ whereas $M(K)^* = C_0(K)^{**}$.

3. Lie hypergroups and subhypergroups

This section is devoted to the study of the topology of Lie hypergroups. In particular we study the induced topology on Lie subhypergroups and recover some of the basic classical results in our general setting.

Definition 3.1. A Lie hypergroup is a hypergroup which is also a C^∞ (or real analytic) manifold K (possibly with boundary) such that the convolution map $m : K \times K \longrightarrow M_\infty(K);$, $(x, y) \mapsto \delta_x * \delta_y$, and involution map $i : K \longrightarrow K;$ $x \mapsto \bar{x}$ are C^∞ (real analytic, in the analytic case).

In the above definition, the requirement on the smoothness of the convolution could be equivalently rephrased as the map

$$(x, y) \mapsto \int_K f d(\delta_x * \delta_y)$$

being C^∞ on $K \times K$, for each $f \in C^\infty(K)$. Also note that since K may be a manifold with boundary, the C^∞ maps on K are defined based on the local charts using half space

$$H^n := \{(x_1, \ldots, x_n) : x_n \geq 0\}$$

(in the real case) where the boundary points are those mapped to $x_n = 0$ by a chart.

We begin this section by giving a typical example of Lie hypergroups. We show that the double coset hypergroup $H\backslash G/H$ of a Lie group G by a compact subgroup H is a Lie hypergroup. The main ingredient of the proof is the following result which is of interest in its own.

Lemma 3.2. *Let X be a locally convex topological vector space, H a compact Lie group, and M a smooth manifold. If $\Lambda : H \times M \longrightarrow X$ is smooth with the first Fréchet derivative*

$$D_{(t,m)}\Lambda : T_tH \times T_mM \longrightarrow X,$$

and $\tilde{\Lambda} : M \longrightarrow X$ is obtained by averaging Λ over H, namely

$$\tilde{\Lambda}(m) = \int_H \Lambda(t,m)dt,$$

where the integral is taken against the normalized Haar measure of H, then $\tilde{\Lambda}$ is smooth with the first Fréchet derivative

$$D_m\tilde{\Lambda} = \int_H D_{(t,m)}\Lambda(0,\cdot)dt.$$

Proof. Fix $m \in M$ and a local chart (V,ψ) of M at m such that V is relatively compact. Let $(t,m) \in H \times M$ and $(U_t \times V, \varphi_t \times \psi)$ be the local chart of $H \times M$ at (t,m) such that

$$\varphi_t \times \psi : U_t \times V \longrightarrow \mathbb{R}^n \times \mathbb{R}^k$$

is a diffeomorphism. Then the compact space $H \times cl(V)$ is covered by the charts $U_t \times V$, hence there is $N \geq 1$ and $t_1, \cdots, t_N \in H$ such that $H = U_{t_1} \cup \cdots \cup U_{t_N}$. If we identify $T_tH \times T_mM$ with $\mathbb{R}^n \times \mathbb{R}^k$ endowed with the norm $|(u,v)| = |u| + |v|$, then for $a_t = \varphi(t) \in \mathbb{R}^n$ and $b = \psi(m) \in \mathbb{R}^k$, and $\varepsilon > 0$ there is $\eta_t > 0$ such that for each $\xi \in X^*$ and each $(u,v) \in \mathbb{R}^n \times \mathbb{R}^k$ with $|u| + |v| < \eta_t$,

$$\frac{\big\langle \Lambda\big(\varphi^{-1}(a_t+u), \psi^{-1}(b+v)\big) - \Lambda\big(\varphi^{-1}(a_t), \psi^{-1}(b)\big) - D_{(t,m)}\Lambda(u,v), \xi\big\rangle}{|u|+|v|} < \varepsilon.$$

Let $\eta_0 = \min\{\eta_{t_1}, \cdots, \eta_{t_N}\}$ and fix $v \in \mathbb{R}^k$ with $|v| < \eta_0$. Let D_{η_0} be the closed unit disk around $0 \in \mathbb{R}^n$ of radius η_0, then the map

$$w \mapsto \big\langle \Lambda\big(\varphi^{-1}(w), \psi^{-1}(b+v)\big), \xi\big\rangle$$

is uniformly continuous on D_{η_0}, hence there is $\eta_1 > 0$ such that for each $w, w' \in D_{\eta_0}$ with $|w - w'| < \eta_1$,

$$\big\langle \Lambda\big(\varphi^{-1}(w), \psi^{-1}(b+v)\big) - \Lambda\big(\varphi^{-1}(w'), \psi^{-1}(b+v)\big), \xi\big\rangle < \varepsilon|v|/N.$$

Similarly, there is $\eta_2 > 0$ such that for each $u \in D_{\eta_0}$ with $|u| < \eta_2$,

$$\int_H \big\langle D_{(t,m)}\Lambda(u,v) - D_{(t,m)}\Lambda(0,v), \xi\big\rangle dt < \varepsilon|v|/N.$$

Now put $\eta = \min\{\eta_0, \eta_1, \eta_2\}$ and let $u \in \mathbb{R}^n$ be such that $|u| + |v| < \eta_0$ and $|u| < \min\{|v|, \eta\}$, then

$$\begin{aligned}
&\big\langle \tilde\Lambda\big(\psi^{-1}(b+v)\big) - \tilde\Lambda\big(\psi^{-1}(b)\big) - D_m\tilde\Lambda(v), \xi\big\rangle \\
&\quad\le \int_H \big\langle \Lambda\big(t, \psi^{-1}(b+v)\big)\big) - \Lambda\big(t, \psi^{-1}(b)\big)\big) - D_{(t,m)}\Lambda(0,v), \xi\big\rangle dt \\
&\quad\le \sum_{i=1}^N \int_{U_{t_i}} \big\langle \Lambda\big(t, \psi^{-1}(b+v)\big)\big) - \Lambda\big(t, \psi^{-1}(b)\big)\big) - D_{(t,m)}\Lambda(0,v), \xi\big\rangle dt \\
&\quad\le \sum_{i=1}^N \int_{U_{t_i}} \big\langle \Lambda\big(\varphi^{-1}(a_t), \psi^{-1}(b+v)\big)\big) - \Lambda\big(\varphi^{-1}(a_t+u), \psi^{-1}(b)\big)\big), \xi\big\rangle dt \\
&\quad+ \sum_{i=1}^N \int_{U_{t_i}} \big\langle \Lambda\big(\varphi^{-1}(a_t+u), \psi^{-1}(b+v)\big)\big) - \Lambda\big(\varphi^{-1}(a_t), \psi^{-1}(b)\big)\big) - D_{(t,m)}\Lambda(u,v), \xi\big\rangle dt \\
&\quad+ \sum_{i=1}^N \int_{U_{t_i}} \big\langle D_{(t,m)}\Lambda(u,v) - D_{(t,m)}\Lambda(0,v), \xi\big\rangle dt \\
&\quad\le \varepsilon|v| + N\varepsilon(|u|+|v|) + \varepsilon|v| < (2N+2)\varepsilon|v|. \qquad \blacksquare
\end{aligned}$$

Theorem 3.3. *(i) If G is a Lie group and H a compact Lie subgroup of G, then $H\backslash G/H$ is a Lie hypergroup.*

(ii) If G is a Lie group and H is a compact Lie group acting smoothly on G then the orbit space G^H is a Lie hypergroup.

Proof. (i). We only need to show that the convolution map is smooth. The rest are either easy to check or already contained in [3, 1.1.9]. Consider the smooth manifolds $M = (H\backslash G/H) \times (H\backslash G/H)$ and $N = H\backslash G/H$ and locally convex topological vector space $X = M_\infty(H\backslash G/H)$. Then the map $\Gamma : H \times M \longrightarrow N$; $(t, \dot x, \dot y) \mapsto (xty)^{\cdot}$ is smooth, where $\dot x = HxH$ for $x \in G$. We claim that the map $\Omega : N \longrightarrow X$; $\dot x \mapsto \delta_{\dot x}$ is also smooth. If this is proved, then applying the above lemma to $\Lambda = \Omega \circ \Gamma : H \times M \to X$; $(t, \dot x, \dot y) \mapsto \delta_{(xty)^{\cdot}}$, the map $\tilde\Lambda : M \to X$; $(\dot x, \dot y) \mapsto \int_H \delta_{(xty)^{\cdot}}\, dt = \delta_{\dot x} * \delta_{\dot y}$ is smooth and we are done.

To prove the claim, fix $\dot x \in N$ and let (U, φ) be a local chart at $\dot x$, where $\varphi : U \longrightarrow \mathbb{R}^d$ is a diffeomorphism. Let $u = \varphi(\dot x)$ and define

$$\langle D_{\varphi^{-1}(u)}\Omega, f\rangle = D_u(f \circ \varphi^{-1}),$$

for $f \in C^\infty(N)$. Then proving the claim amounts to showing that

$$\frac{1}{|h|}\big\langle \delta_{\varphi^{-1}(u+h)} - \delta_{\varphi^{-1}(u)} - D_{\varphi^{-1}(u)}\Omega, f\big\rangle \to 0$$

as $h \to 0$ in $\mathbb{R}^d$. This is to say that

$$\lim_{h\to 0} \frac{f \circ \varphi^{-1}(u+h) - f \circ \varphi^{-1}(u) - D_u(f \circ \varphi^{-1})(h)}{|h|} = 0,$$

which clearly holds.

(ii). This is proved similar to (i) as the convolution product of G^H is defined by

$$\delta_{\dot{x}} * \delta_{\dot{y}} = \int_H \delta_{(x^t y)^\cdot}\, dt,$$

where $\dot{x} = \{x^s : s \in H\}$ is the orbit of $x \in G$.

■

As a concrete example, Let S^2 be the unit sphere in $\mathbb{R}^3$ and $G = SO(3)$, let $n = (0,0,1)$ be the north pole in S^2 and $H = \{g \in G : gn = n\}$, then G is a Lie group and H is a compact Lie subgroup, topologically isomorphic to the compact group $\mathbb{T}$, and two elements of G are H-conjugate if and only if they rotate S^2 through the same angle. Hence G^H is a compact commutative Lie hypergroup, homeomorphic to $[0,\pi]$. The support of $\delta_{\dot{x}} * \delta_{\dot{y}}$ is either a closed interval or a singleton [3, 1.1.16]. On the other hand, for each $x \in [-1,1]$, let

$$D_x = \{g \in G : gn = (s,t,x) \text{ for some } s,t \in \mathbb{R}\},$$

then each double coset of H is of the form D_x for some $x \in [-1,1]$ and $H\backslash G/H$ is a compact commutative Lie hypergroup, homeomorphic to $[-1,1]$ [3, 1.1.17]. Since the only topological involution of $[-1,1]$ leaving 1 fixed is the identity map, this is a Hermitian hypergroup (each element is its own inverse). Note that in G^H we have $supp(\delta_\pi * \delta_\pi) = [0,\pi]$ where as in $H\backslash G/H$, $\delta_{-1} * \delta_{-1} = \delta_1$, hence the two hypergroups are not topologically isomorphic.

To build more concrete examples we need to introduce the class of polynomial hypergroups and their dual objects [3, 3.1.1]. For more details on the dual object $\hat{K}$ of a commutative hypergroup K and conditions for $\hat{K}$ to be a hypergroup, we refer the reader to [3, Chapter 2]. Let K be a countable (discrete) space and $\{Q_x : x \in K\}$ be a set of polynomials on $\mathbb{C}^d$, for some $d \geq 1$, such that for each $n \geq 0$, the set $\{Q_x : deg(Q_x) \leq n\}$ form a basis for the vector space $\mathcal{P}_n$ of polynomials of degree at most n in $\mathbb{C}[z_1, \ldots, z_n]$. Then

$$Q_x Q_y = \sum_{w \in K} c(x,y,w) Q_w$$

with $c(x,y,w) \in \mathbb{C}$. If K is a (discrete) hypergroup under the convolution

$$\delta_x * \delta_y(\{w\}) = c(x,y,w)$$

then K is called the *polynomial hypergroup* (in d variables) and is denoted by $(K, *(Q_x))$.

Example 3.4. We give several concrete examples of polynomial hypergroups whose dual objects are Lie hypergroups.

(i) Let $J = (\mathbb{Z}_+, *(Q_n^{\alpha,\beta}))$ where

$$Q_n^{\alpha,\beta}(x) = \frac{(\frac{-1}{2})^n}{(\alpha+1)_n}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\big((1-x)^{n+\alpha}(1+x)^{n+\beta}\big)$$

are normalized Jacobi polynomials and (α, β) ranges over an appropriate subset V of $\mathbb{R}^2$ [3, 3.1.10]. If V is chosen properly, the dual space $K = \hat{J}$ is the hypergroup $K = [-1, 1]$ with convolution

$$Q_n^{\alpha,\beta}(x)Q_n^{\alpha,\beta}(y) = \int_{-1}^{1} Q_n^{\alpha,\beta}(t)d(\delta_x * \delta_y)(t)$$

which is called the dual Jacobi polynomial hypergroup of type (F), studied by Gasper [8] and Igari-Uno [9]. For $\alpha = \beta = \frac{1}{2}(d-3)$, $d \geq 2$, the spherical hypergroup J [3, 3.3.3] has a dual Lie hypergroup $K = \hat{J} = SO(d-1)\backslash SO(d)/SO(d-1)$.

(ii) The disc polynomial hypergroup is $J_\alpha = (\mathbb{Z}_+^2, *(Q_{m,n}^\alpha))$ where $\alpha > -1$ and

$$Q_{m,n}^\alpha(z_1, z_2) = \begin{cases} Q_m^{\alpha,n-m}(2z_1z_2 - 1)z_2^{n-m}, & n \geq m \\ Q_{n,m}^\alpha(z_2, z_1), & \text{otherwise.} \end{cases}$$

It is shown by Koornwinder that J_α is a hypergroup [12] and the diagonal

$$H = \{(n,n) : n \in \mathbb{Z}_+\} \simeq (\mathbb{Z}_+, *(Q_n^{\alpha,0}))$$

is a supernormal subhypergroup [1]. The dual hypergroup $K_\alpha = \hat{J}_\alpha$ is equal to the closure $\bar{D}$ of the open disc $D = \{(z, \bar{z}) \in \mathbb{C}^2 : |z| < 1\}$ with convolution

$$Q_{m,n}^\alpha(z)Q_{m,n}^\alpha(w) = \int_{\bar{D}} Q_{m,n}^\alpha(u)d(\delta_z * \delta_w)(u)$$

for $z, w \in \bar{D}$, which is called the disc hypergroup of type (G). For the compact Lie group $G = U(d)$ and closed subgroup $H = U(d-1)$ the double coset Lie hypergroup $K = H\backslash G/H$ is identified with $\bar{D}$ as the dual of $J_{d-2} = (\mathbb{Z}_+^2, *(Q_{m,n}^{d-2}))$ [1], [3, 3.1.14]. Let's check this directly for the special case $\alpha = 0$. For $\alpha = 0$, $\bar{D}$ is the hypergroup with convolution

$$\int_{\bar{D}} fd(\delta_{z_1} * \delta_{z_2}) = \frac{1}{2\pi}\int_0^{2\pi} f\big(z_1\bar{z}_2 + e^{it}(1-|z_1|^2)^{\frac{1}{2}}(1-|z_2|^2)^{\frac{1}{2}}\big)dt$$

for $f \in C(\bar{D})$. It is clear that if $f \in C^\infty(\bar{D})$ then the map

$$(z_1, z_2) \mapsto \int_{\bar{D}} fd(\delta_{z_1} * \delta_{z_2})$$

is smooth on $\bar{D}^2$. Hence $\bar{D}$ is a Lie hypergroup under the above convolution and involution $z \mapsto \bar{z}$.

(iii) Let $\{Q_{\mathbf{n}}\}$ be the class of Chebychev polynomials in d variables indexed by

$$\mathbb{Z}_{+\uparrow}^d = \{\mathbf{n} = (n_1, \ldots, n_d) : n_i \in \mathbb{Z}_+, n_1 \geq \cdots \geq n_d\}$$

then $J = (\mathbb{Z}_{+\uparrow}^d, *(Q_{\mathbf{n}}))$ is the Chebychev hypergroup in d variables [3, 3.1.15] whose dual $K = \hat{J}$ is the open unit disc $\mathbb{D}^\circ$ [6]. The Chebychev hypergroup of the second kind (for $d = 1$) identifies with $\mathbb{Z}_+$ with convolution

$$\delta_m * \delta_n = \sum_{k=0}^{m\wedge n} \frac{|m-n| + 2k + 1}{(m+1)(n+1)}\delta_{|m-n|+2k}$$

whose dual object is the Lie hypergroup $SO(3)\backslash SO(4)/SO(3)$.

(iv) The Koornwinder class III hypergroup $J=(\mathbb{Z}^2_{+\uparrow},*(Q^{\alpha,\beta}_{m,n}))$, $\alpha,\beta>-1$, with

$$Q^{\alpha,\beta}_{m,n}(x_1,x_2)=Q^{\alpha,\beta+n+\frac{1}{2}}_{m-n}(2x_1-1)x_1^{\frac{n}{2}}Q^{\beta,\beta}_{n}(\frac{x_2}{\sqrt{x_1}})$$

defined on

$$E=\{(x_1,x_2)\in\mathbb{R}^2:|x_1|-1<x_2<\frac{1}{4}x_1^2,|x_1|<2\}$$

is shown to be a hypergroup for the following choices of α and β [3, 3.1.20-23], and in each case the dual Lie hypergroup is $K=\hat{J}=H\backslash G/H$ for some Gelfand pair (G,H):

- For $\alpha\in 2\mathbb{Z}_+ +1$ and $\beta=\frac{1}{2}$, for instance for $\alpha=2d-1$ and $\beta=\beta=\frac{1}{2}$ we get $G=Sp(d)\times Sp(1)$ and $H=Sp(d-1)\times Sp(1)$.
- For $\alpha=3$ and $\beta=\frac{5}{2}$, we get $G=Sp(9)$ and $H=Sp(7)$.
- For $\alpha\in\mathbb{Z}_+$ and $\beta=\frac{-1}{2}$, for instance for $\alpha=d-2$ and $\beta=\beta=\frac{-1}{2}$ we get $G=S(U(d)\times U(1))$ and $H=S(U(d-1)\times D(U(1)))$, where $D(U(1))$ is the diagonal of $U(1)\times U(1)$ and

$$S(U(d)\times U(1))=\Big\{\begin{pmatrix}A&0\\0&B\end{pmatrix}:A\in U(d),B\in U(1),det(A)det(B)=1\Big\}.$$

(v) The one dimensional hypergroup $\mathbb{R}_+$ (of non compact type) with convolution $\delta_r*\delta_s=\frac{1}{2}(\delta_{|r-s|}+\delta_{r+s})$ is a Lie hypergroup as the double coset hypergroup with $G=\mathbb{R}\rtimes\mathbb{Z}_2$ and $H=\mathbb{Z}_2$ [3, 3.4.5]. The same holds for the one dimensional hypergroup $[0,1]$ (of compact type) under $\delta_r*\delta_s=\frac{1}{2}(\delta_{|r-s|}+\delta_{|1-|1-r-s||})$ with $G=\mathbb{T}\rtimes\mathbb{Z}_2$ and $H=\mathbb{Z}_2$ [3, 3.4.6]. More generally, for any one dimensional normalized hypergroup $K=([0,1],*)$ we have $G(K)=\{0\}$ or $\{0,1\}$ and if $x*y$ contains at least two points, for each $0\le x,y\le 1$, then $K\simeq\mathbb{Z}_2\backslash\mathbb{T}\rtimes\mathbb{Z}_2/\mathbb{Z}_2$ [3, 3.4.20-21]. A concrete example is the Bingham hypergroup [2], [3, 3.4.23].

Next we consider the connected component of identity $K_\circ$ in K. It is known that for a locally compact hypergroup K, $K_\circ$ is a normal subhypergroup of K and the connected component of $x\in K$ is $x*K_\circ$ [23, 1.3]. Also $K/K_\circ$ is a totally disconnected Hausdorff topological space [23, 1.4]. We show that more is true when K is a Lie hypergroup.

Lemma 3.5. *Let K be a Lie hypergroup and $K_\circ$ the connected component of identity, then $K_\circ$ is a normal Lie subhypergroup of K and is a Lie hypergroup. Also $K_\circ$ is the intersection of all open subhypergroups of K and the quotient space $K/K_\circ$ is a discrete space. When $K_\circ$ is compact, $K/K_\circ$ is a discrete hypergroup. Finally, if $x*\bar{x}$ is connected for each $x\in K$, then $K_\circ$ is a supernormal subhypergroup and K/K_0 is a discrete group.*

Proof. In general, $K_\circ$ is a normal subhypergroup of K [23, 1.3]. Since K (as a manifold) is locally pathwise connected, $K_\circ$ is also open. By the remark after [3, 1.2.11], $x*K_\circ\subseteq K$ is open, for each $x\in K$. Since the quotient map $q:K\to K/K_\circ$ is open [10], the singleton $\{x*K_\circ\}\subseteq K/K_\circ$ is open, and $K/K_\circ$ is discrete. If $K_\circ$ is compact, $K/K_\circ$ is a hypergroup [3]. Finally, if $x*\bar{x}$ is connected for each $x \in K$, then $x * K_\circ * \bar{x} = x * \bar{x} * K_\circ$ is connected by [23, 1.1] and contains e, hence $x * K_\circ * \bar{x} \subseteq K_\circ$, for each $x \in K$. Therefore

$$K_\circ \subseteq (\bar{x} * x) * K_\circ * (\bar{x} * x) = \bar{x} * (x * K_\circ * \bar{x}) * x \subseteq \bar{x} * K_\circ * x,$$

and changing x to $\bar{x}$, $K_\circ \subseteq x * K_\circ * \bar{x}$, hence $K_\circ = x * K_\circ * \bar{x}$, for each $x \in K$. Therefore $K_\circ$ is supernormal and $K/K_\circ$ is a discrete group by [23, 2.1]. ∎

In the classical case, it follows from the above lemma that every discrete normal subgroup of a connected Lie group is central. It would be desirable to find the analogous result for discrete (super) normal subhypergroups of a connected Lie hypergroup. The obstacle to prove such a result for hypergroups could be seen from the following argument. Let K be a connected Lie hypergroup and H be a discrete supernormal subhypergroup, then for $h \in H$ and $x \in K$, unlike the group case, $x * h * \bar{x} \subseteq H$ is not going to be connected (because then it should be a singleton). It is however plausible that still one could get $x * h = h * x$, using an undirect argument. On the other hand, in many applications the following weaker result is enough.

Lemma 3.6. *Every discrete normal subgroup of a connected Lie hypergroup is central.*

Proof. Let K be a connected Lie hypergroup and H be a discrete normal subgroup and fix $h \in H$. Then for each $x \in K$, $h * x$ is a singleton [10, 12.4B], hence with a slight abuse of notation we may write

$$h * x = hx \in H * x = x * H = \{xk : k \in H\}$$

therefore there is $h_x \in H$ such that $hx = xh_x$. Now the map $K \longrightarrow H$, $x \mapsto h_x$ is continuous and so constant. Hence $h_x = h_e = h$, that is $hx = xh$, therefore $h \in Z(K)$. ∎

Definition 3.7. A Lie subhypergroup H of a Lie hypergroup K is a subhypergroup which is also an embedded submanifold.

Since each embedded submanifold is locally compact in the induced topology of the manifold, the following lemma follows from the proof of [3, 1.5.2]. Here we give a proof for the sake of completeness.

Lemma 3.8. *Any Lie subhypergroup is closed.*

Proof. If H is a Lie subhypergroup of K, then so is its closure $cl(H)$ in K. We claim that H is an open subset of $cl(H)$. Since H is an embedded submanifold of K, for each $x \in H$ there is $k \leq n := dimK$ and a local chart (U_x, φ) at x such that

$$\varphi(y) = (\varphi^1(y), \ldots, \varphi^k(y), 0, \ldots, 0)$$

for each $y \in U_x \cap H$ and conversely each $y \in U_x$ which satisfies the above equality is in H. By continuity of φ, $U_x \cap cl(H) \subseteq H$ and the claim is proved. Now if $H \neq cl(H)$, then

$$U := \bigcup_{x \in cl(H) \setminus H} x * H$$

is a non empty open subset of $cl(H)$ by the remark after [3, 1.2.11]. But $H \cup U = cl(H)$ and $H \cap U = \emptyset$ by [10, 10.3A]. This contradicts the density of H in $cl(H)$. ■

Corollary 3.9. *Every connected Lie hypergroup is generated by any open neighborhood of e.*

Proof. If U is an open neighborhood of e, and $H = [U]$, then H is open in K, since for $h \in H$, $h * U$ is a neighborhood of h in K. Therefore H is a submanifold and hence closed by the above lemma. Thus $H = K$. ■

Theorem 3.10. *If K is a Lie hypergroup of dimension n and H is a Lie subhypergroup of dimension k, then the coset space K/H is a manifold of dimension $n-k$ and the quotient map $q : K \longrightarrow K/H$ is a fiber bundle, with fiber diffeomorphic to H. When H is a compact normal Lie subhypergroup, a supernormal Lie subhypergroup, or a normal closed Lie subhypergroup of K, then K/H is a Lie hypergroup.*

Proof. For $x \in K$ let $\dot{x} = q(x) \in K/H$. Then $x * H \subseteq K$ is a submanifold of dimension k as the image of diffeomorphism $K \longrightarrow \mathcal{C}_K$, $y \mapsto \delta_x * \delta_y \mapsto supp(\delta_x * \delta_y)$. Choose a submanifold $M \subseteq K$ such that $x \in M$ and M is transversal to $x * H$ (in particular, $dimM = dimK - dimH$). Let $U \subseteq M$ be a sufficiently small neighborhood of $x \in M$ such that $U * H \subseteq K$ is open (just apply the inverse mapping theorem to $U \times H \longrightarrow \mathcal{C}_K$, $(u, h) \mapsto supp(\delta_u * \delta_h)$.) Consider $\widetilde{U} = q(U) \subseteq K/H$ and note that $q^{-1}(\widetilde{U}) = U * H \subseteq K$ is open. Then $\widetilde{U}$ is an open neighborhood of $\dot{x} \in K/H$ and the map $U \longrightarrow \widetilde{U}$, $x \mapsto \dot{x}$ is a homeomorphism. This provides a local chart for K/H and shows that the quotient map is a fiber bundle with fiber H. It's not hard to see that the transition functions between charts are smooth (holomorphic) and the smooth (analytic) structure is independent of the choice x and M. The last assertion follows from [3, 1.5.22]. ■

Corollary 3.11. *If H is a connected Lie subhypergroup of K, then $\pi_0(K) = \pi_0(K/H)$. If moreover K/H is connected, then K is connected.*

Proof. Just note that $\pi_0(K)$ is the set of connected components of K. ■

The next corollary follows from the more general result on long exact sequence of homotopy groups associated with any fiber bundle.

Corollary 3.12. *If K is a connected Lie hypergroup and H is a connected Lie subhypergroup, then there is an exact sequence of fundamental groups*

$$\pi_2(K/H) \to \pi_2(H) \to \pi_1(K) \to \pi_1(K/H) \to \{e\}.$$

As in the group case [11, 2.3] the notion of Lie subhypergroup is too restrictive, the main drawback is that the image of a Lie hypergroup under a morphism is not a Lie subhypergroup. For Lie groups, the typical example is $\phi : \mathbb{R} \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$, $t \mapsto (t, \alpha t) + \mathbb{Z}^2$, where $\alpha \in \mathbb{R} - \mathbb{Q}$ [11, 2.13]. In this example, ϕ has dense range in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and so $Im(\phi)$ cannot be a Lie subgroup.

We use the concept of immersed submanifold [11, section 2.1] to give a more general (and suitable) notion of Lie subhypergroup. To distinguish this from the above notion, from now on we call Lie subhypergroups in the sense of Definition 3.8, "closed Lie subhypergroups". This is justified with Lemma 3.9.

Definition 3.13. A Lie subhypergroup of a Lie hypergroup is an immersed submanifold which is also a subhypergroup.

The following lemma is proved similar to the classical case [11].

Lemma 3.14. *If H is a Lie subhypergroup of Lie hypergroup K then H is a Lie hypergroup and the inclusion map $i : H \hookrightarrow K$ is a morphism of Lie hypergroups.*

Clearly every closed subhypergroup is a Lie subhypergroup (since every submanifold is an immersed submanifold). But the converse is not true (see the discussion after Corollary 3.13). Indeed we have the following result, whose proof is straightforward.

Proposition 3.15. *A Lie subhypergroup is a closed Lie subhypergroup if and only if it is closed.*

4. Orbits and Quotients

Lie groups are particularly important as symmetry groups of geometric objects. This is best explained by the notion of action of Lie groups on manifolds. In this section we define action of Lie hypergroups on manifolds and study the corresponding orbits and quotients.

Let K be a locally compact Hausdorff hypergroup with identity element e and K_e be the closed subhypergroup of K generated by $S_0 = \cup_{x \in K} x * \bar{x}$, that is the intersection of all closed subhypergroups of K containing S_0. The next lemma gives a characterization of K_e. When K is a Lie group, $K_e = \{e\}$.

Lemma 4.1. *K_e is a supernormal closed subhypergroups of K and is the intersection of all supernormal closed subhypergroups of K.*

Proof. If $H \leq K$ is a closed supernormal subhypergroup then $x * \bar{x} \subseteq x * H * \bar{x} \subseteq H$, hence $K_e \subseteq H$. Therefore it is enough to check that K_e is supernormal. Take $y \in S_0$, then $\{y\} \subseteq z * \bar{z}$, for some $z \in K$. Let $x \in K$, and put $A_x = x * z$, then

$$x * y * \bar{x} \subseteq x * z * \bar{z} * \bar{x} = (x * z) * (x * z)^{-} = \cup_{a \in A_x} a * \bar{a} \subseteq K_e.$$

■

Definition 4.2. An action of a Lie hypergroup K on a manifold $\mathfrak{M}$ is a smooth map $\alpha : K \times \mathfrak{M} \to \mathfrak{M}$; $(x, m) \mapsto \alpha(x, m) =: x \cdot m$ satisfying the following conditions:

$$\begin{aligned} (i) \qquad & \alpha(y, m) = m \quad (y \in K_e, m \in \mathfrak{M}), \\ (ii) \qquad & \alpha(\delta_x * \delta_y) = \alpha(x) \circ \alpha(y) \quad (x, y \in K), \end{aligned}$$

where in (ii), $\alpha(\mu)$ is defined for a probability measure μ by

$$\alpha(\mu) : \mathfrak{M} \to \mathfrak{M};\ m \mapsto \mu \cdot m := \int_K \alpha(x, m) d\mu(x).$$

For each measure μ of compact support, $\alpha(\mu)$ lifts to a bounded linear map

$$\alpha(\mu) : C^\infty(\mathfrak{M}) \to C^\infty(\mathfrak{M});\ f \mapsto \alpha(\mu)f := \int_K \alpha(x) f d\mu(x),$$

where $\alpha(x)f(m) = f(\alpha(x, m))$, for $x \in K, m \in \mathfrak{M}$, and $f \in C^\infty(\mathfrak{M})$. Also $\alpha(\delta_x) = \alpha(x)$ for $x \in K$.

Lemma 4.3. *The stabilizer K_m is closed Lie subhypergroup and there is an injective immersion $K/K_m \hookrightarrow \mathfrak{M}$ whose image is the orbit $\mathcal{O}_m$, namely $K/K_m \simeq \mathcal{O}_m$.*

Proof. If $x, y \in K_m$, $\alpha(\delta_x * \delta_y)(m) = \alpha(\delta_x)(y \cdot m) = \alpha(\delta_x)(m) = x \cdot m = m$, hence $x * y = \operatorname{supp}(\delta_x * \delta_y) \subseteq K_m$, i.e., $K_m * K_m \subseteq K_m$. Also $\bar{x} \cdot m = \bar{x} \cdot (x \cdot m) = (\delta_{\bar{x}} * \delta_x) \cdot m = \int_K t \cdot m \, d(\delta_{\bar{x}} * \delta_x)(t) = \int_K d(\delta_{\bar{x}} * \delta_x) = m(\delta_{\bar{x}} * \delta_x)(K) = m$. Hence K_m is a subhypergroup. Since K_m is a submanifold of K, K_m is a closed Lie subhypergroup by Lemma 3.9.

Next if $x, y \in K$ and $x * K_m = y * K_m$ then

$$\bar{x} * y \subseteq \bar{x} * y * K_m = \bar{x} * x * K_m \subseteq K_e * K_m \subseteq K_m * K_m \subseteq K_m$$

hence

$$x \cdot m \in (x * \bar{x} * y) \cdot m \subseteq (K_e * y) \cdot m = (y * K_e) \cdot m = \{y \cdot m\}.$$

Therefore $x \cdot m = y \cdot m$ and the immersion $K/K_m \hookrightarrow \mathfrak{M}$ defined by $x * K_m \mapsto x \cdot m$ is well-defined. Conversely, if $x \cdot m = y \cdot m$ then

$$(\bar{y} * x) \cdot m = (\bar{y} * y) \cdot m \subseteq K_e \cdot m = \{m\}$$

hence $\bar{y} * x \subseteq K_m$ therefore $x \in y * \bar{y} * x \subseteq y * K_m$ that is $x * K_m \subseteq y * K_m$. The reverse inclusion is proved similarly and the above map is also injective. The rest is now clear. ∎

Corollary 4.4. *The orbit space $\mathcal{O}_m$ is an immersed submanifold of $\mathfrak{M}$ and if it is a submanifold, then K/K_m is diffeomorphic to $\mathcal{O}_m$.*

Definition 4.5. We say that the action α of K on $\mathfrak{M}$ is transitive if for each $m \in \mathfrak{M}$, there is $\mu \in M_p(K)$ and $n \in \mathfrak{M}$ such that $\mu \cdot n = m$. A K-homogeneous space is a manifold $\mathfrak{M}$ with a transitive action of K.

It follows from above corollary that each homogeneous space is a coset space K/H. Note that K/H is not necessarily a hypergroup (it is a semiconvo in the sense of [10]).

Proposition 4.6. *If $\mathfrak{M}$ is a K-homogeneous space and $m \in \mathfrak{M}$, then the map $K \longrightarrow \mathfrak{M}$; $x \mapsto x \cdot m$ is a fiber bundle over $\mathfrak{M}$ with fiber K_m.*

The classical Lie group examples of such fiber bundles are $SO(n) \longrightarrow S^{n-1}$ with fiber $SO(n-1)$ and $SU(n) \longrightarrow S^{2n-1}$ with fiber $SU(n-1)$. It is not hard to construct an example of a transitive action whose fiber is a Lie hypergroup, but not a Lie group.

We end the paper with a discussion of what could be done afterwards. Transitive actions of a Lie hypergroup K on a set $\mathfrak{M}$ could lend a smooth structure on the set coming from the bijection of $\mathfrak{M}$ onto K/K_m. Then $\mathfrak{M}$ is a manifold of dimension $\dim K - \dim K_m$. A classical example of such phenomenon are flag manifolds: A flag on $\mathbb{R}^n$ is a sequence $0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{R}^n$ of subspace V_i with $\dim V_i = i$. The set $\mathcal{F}(\mathbb{R})$ of all flags in $\mathbb{R}^n$could be given a smooth structure via transitive action of $GL(n,\mathbb{R})$. The fact that the action is transitive is a result of the fact that by change of basis (which is the action of an element of $GL(n,\mathbb{R})$) each flag is transferred to the standard flag, where $V_i = \langle e_1, \cdots, e_i \rangle$. Thus $\mathcal{F}(\mathbb{R})$ identifies with the coset space $GL(n,\mathbb{R})/B(n,\mathbb{R})$, where $B(n,\mathbb{R})$ is the group of all invertible upper triangular matrices. Therefore $\mathcal{F}(\mathbb{R})$, called the flag manifold, is a smooth manifold of dimension $n^2 - n(n+1)/2 = n(n-1)/2$. It is desirable to give an example of a "hyper flag" manifold using transitive acting of a Lie hypergroup.

Next it is natural to consider the space of orbits. When a Lie group G acts on a manifold $\mathfrak{M}$, then $\mathfrak{M}/G$ is a topological space. This space could be non Hausdorff (for instance $G = GL(n,\mathbb{C})$ acting on $\mathfrak{M} = \mathbb{M}_n(\mathbb{C})$ by conjugation). There are several ways to deal with this pathological phenomenon in the classical case. If the action is proper, then $\mathfrak{M}/G$ is Hausdorff and even a manifold under same additional conditions [4]. The same situation could be considered for hypergroup actions.

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