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Algebraic Supergroups with Lie Superalgebras of Classical Type

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Abstract. We show that every connected affine algebraic supergroup defined over a field k, with diagonalizable maximal torus and whose tangent Lie superalgebra is a k–form of a complex simple Lie superalgebra of classical type is a Chevalley supergroup, as it is defined and constructed explicitly in [R. Fioresi, F. Gavarini, *Chevalley Supergroups*, Memoirs of the Amer. Math. Soc. **215** (2012), no. 1014].
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1. Introduction

In [7] we have given the supergeometric analogue of the classical Chevalley's construction (see [16]), which enabled us to build a supergroup out of data involving only a complex Lie superalgebra $\mathfrak{g}$ of classical type and a suitable complex faithful representation. Such a supergroup is affine connected, with associated classical subgroup being reductive k–split (i.e. it admits a diagonalizable maximal torus) and with tangent Lie superalgebra isomorphic to $\mathfrak{g}$: thus we obtained an *existence result* for such supergroups. In particular, this provided the first unified construction of affine algebraic supergroups with tangent Lie superalgebras of classical type; in particular, it was also (as far as we know) the very first explicit construction of algebraic supergroups corresponding to the simple Lie superalgebras of basic exceptional type.

In this paper we tackle the *uniqueness* problem, cast in the following form: "is any such supergroup isomorphic to a supergroup obtained via the Chevalley's construction"? Our answer is positive.

We start with an affine algebraic supergroup G, defined over a field k with associated classical subgroup G_0 which is k–split reductive, and with tangent Lie superalgebra a k–form of a complex Lie superalgebra of classical type (plus a consistency condition): then we prove that G is given by our Chevalley supergroup

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construction. Note that all the conditions we impose actually are necessary, as they do hold for Chevalley supergroups.

As G_0 is k–split reductive, by Chevalley-Demazure theory it can be realized via the Chevalley construction as a closed subgroup of some $\mathrm{GL}(\widetilde{V})$, where $\widetilde{V}$ is a suitable G_0–module. Let $\widetilde{V}^*$ be the dual G_0–module. Since G is an affine supergroup over a field k, it is linearizable, that is $G \subseteq \mathrm{GL}_{m|n}$ (for suitable m and n), hence we can build the induced $(\mathrm{GL}_{m|n})_0$–module $U := Ind_{G_0}^{(\mathrm{GL}_{m|n})_0}(\widetilde{V}^*)$ and its dual U^*, which both are naturally $(\mathfrak{gl}_{m|n})_0$–modules as well: note also that U^* contains a G_0–submodule isomorphic to $\widetilde{V}$. Inducing then for the Lie superalgebras we get the $\mathfrak{gl}_{m|n}$–module $W := Ind_{(\mathfrak{gl}_{m|n})_0}^{\mathfrak{gl}_{m|n}}(U^*) = \mathcal{U}(\mathfrak{gl}_{m|n}) \otimes_{\mathcal{U}((\mathfrak{gl}_{m|n})_0)} U^*$. Now W is also a $\mathrm{GL}_{m|n}$–module and (by restriction) a G–module: moreover, it contains the (finite-dimensional) G–submodule $V := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} \widetilde{V}$, where $\widetilde{V}$ is identified with a G_0–submodule of U^*. *N.B.:* for the sake of simplicity of exposition, we are hiding here several technicalities, to be specified later on in the main text.

The very construction of V allows us to build the Chevalley supergroup G_V associated with the $\mathfrak{g}$–representation V and to view both G and G_V as closed subgroups of the *same* $\mathrm{GL}(V)$. The last step is to note that both G and G_V are globally split — as any affine supergroup over a field, by Theorem 4.5 in [14]. Since the ordinary algebraic groups are the same, $G_0 = (G_V)_0$, we have that both supergroups are smooth as well. We conclude then $G = G_V$ by infinitesimal considerations, since they have the same Lie superalgebra.

In the last section we make some important remarks between the equivalence of categories of certain Super Harish-Chandra pairs and the algebraic supergroups we have studied in the present work.

Parallel constructions and results, concerning existence (by a Chevalley like construction) and uniqueness of algebraic supergroups associated with simple Lie superalgebras of Cartan type are presented in [9].

2. Chevalley supergroups

In this section we review briefly the construction of Chevalley supergroups (see [7], [8]) and then we discuss some of their properties. For all details about the construction we refer to [7]. The new property that we present here is that every Chevalley supergroup G_V, defined as a subgroup of some $\mathrm{GL}(V)$, is in fact *closed* in $\mathrm{GL}(V)$.

2.1. Definition of Chevalley supergroups.

Let $\mathfrak{g}$ be a complex Lie superalgebra of classical type and $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}_0$. Then we have the corresponding root system $\Delta = \Delta_0 \cup \Delta_1$, with Δ_0 and Δ_1 being the sets of even and of odd roots respectively: these roots are the non-zero eigenvalues of the (adjoint) action of $\mathfrak{h}$ on $\mathfrak{g}$, while the corresponding eigenspaces, resp. eigenvectors, are called *root spaces*, resp. *root vectors*. For root vectors, we adopt the simplified notation of the cases when $\mathfrak{g}$ is not of type $A(1,1)$, $P(3)$ or $Q(n)$ — cf. [13] — but *all what follows holds for those cases too*, and all our results hold for all complex Lie superalgebras of classical type, but for the cases $D(2,1;a)$ when $a \notin \mathbb{Z}$.

Like in the classical setting, one can define special elements $H_\alpha \in \mathfrak{h}$, called *coroots*, associated with the roots α.

A key notion in [7] is that of *Chevalley basis* of $\mathfrak{g}$. This is any $\mathbb{C}$–basis of $\mathfrak{g}$ of the form

$$B = \big\{H_1 \dots H_\ell\big\} \cup \big\{X_\alpha, \alpha \in \Delta\big\}$$

such that (cf. [7], Def. 3.3):

- the H_i's, called the *Cartan elements* of B, form a $\mathbb{C}$–basis of $\mathfrak{h}$ (with some additional properties);
- every X_α is a root vector associated with the root α;
- the structure coefficients for the Lie superbracket in $\mathfrak{g}$ with respect to these basis elements are integers with some special properties.

The very existence of Chevalley bases is proved in [7], sec. 3.

If B is a Chevalley basis of $\mathfrak{g}$ as above, we set $\mathfrak{g}_\mathbb{Z} := \mathit{Span}_\mathbb{Z}\{B\}\big(\subseteq \mathfrak{g}\big)$ for its $\mathbb{Z}$–span. Moreover, we define an important integral lattice inside $\mathcal{U}(\mathfrak{g})$, namely the *Kostant superalgebra*. This is the $\mathbb{Z}$–supersubalgebra $\mathcal{U}_\mathbb{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ generated by the following elements: all divided powers in the even root vectors of B, all odd root vectors of B, and all binomial coefficients in the Cartan elements of B (see [7], sec. 4.1).

We associate to $\mathcal{U}_\mathbb{Z}(\mathfrak{g})$ the notion of *admissible lattice* in a $\mathfrak{g}$–module:

Definition 2.1. Let $\mathfrak{g}$, $B = \big\{H_1 \dots H_\ell\big\} \cup \big\{X_\alpha, \alpha \in \Delta\big\}$ and $\mathcal{U}_\mathbb{Z}(\mathfrak{g})$ be as above. Let V be a complex finite dimensional $\mathfrak{g}$–module. We say that V is *rational* if the H_i's act diagonally on V with integral eigenvalues. We say that an integral lattice M in V — that is, a free $\mathbb{Z}$–submodule M of V such that $\mathit{rk}_\mathbb{Z}(M) = \mathit{dim}_\mathbb{C}(V)$ — is *admissible* if it is $\mathcal{U}_\mathbb{Z}(\mathfrak{g})$–stable.

Given a complex representation V of $\mathfrak{g}$ as above, there exists always an admissible lattice M and an integral form $\mathfrak{g}_V$ of $\mathfrak{g}$ keeping such a lattice stable (see [7], §5.1). This allows us to shift from the complex field $\mathbb{C}$ to any commutative unital ring k.

Definition 2.2. Let the notation be as above, and assume also that the representation V is *faithful*. For any fixed commutative unital ring k, define

$$\mathfrak{g}_k := k \otimes_\mathbb{Z} \mathfrak{g}_V, \qquad V_k := k \otimes_\mathbb{Z} M, \qquad \mathcal{U}_k(\mathfrak{g}) := k \otimes_\mathbb{Z} \mathcal{U}_\mathbb{Z}(\mathfrak{g})$$

Then we say that $\mathfrak{g}_k$, resp. M, is a *k–form* of $\mathfrak{g}$, resp. of V_k.

Remark 2.3. For any algebraic supergroup G, one can introduce the notion of *superalgebra of distributions* $\mathit{Dist}_k(G)$, by an obvious extension of the standard notion in the even setting; see [1], §4, for details. One can easily see — like in [1],

§4 — that $Dist_k(G) = \mathcal{U}_k(\mathfrak{g})$; in particular, this shows that $\mathcal{U}_k(\mathfrak{g})$ is independent of the choice of a specific Chevalley basis in $\mathfrak{g}$.

More important (for later use), is the fact that if $\varphi : G' \longrightarrow G''$ is a morphism between two supergroups, then it induces (functorially) a morphism $D_\varphi : Dist_k\big(G'\big) \longrightarrow Dist_k\big(G''\big)$, which is injective whenever φ is injective. If in addition G' and G'' satisfy the assumptions we gave above for G, so that $\mathcal{U}_k\big(\mathfrak{g}'\big) = Dist_k\big(G'\big)$ and $\mathcal{U}_k\big(\mathfrak{g}''\big) = Dist_k\big(G''\big)$, we have then $D_\varphi : \mathcal{U}_k\big(\mathfrak{g}'\big) \to \mathcal{U}_k\big(\mathfrak{g}''\big)$, which is an embedding if G' is subsupersgroup of G''.

Now we need to recall the notion of commutative superalgebras.

We call *k–superalgebra* any associative, unital k–algebra A which is $\mathbb{Z}_2$–graded (as a k–algebra): so A bears a $\mathbb{Z}_2$–splitting $A = A_0 \oplus A_1$ into direct sum of super-subvector spaces, with $A_a A_b \subseteq A_{a+b}$. We define the *parity* $|a| \in \mathbb{Z}_2$ of any $a \in \big(A_0 \cup A_1\big) \setminus \{0\}$ by the condition $a \in A_{|a|}$; the elements in A_0 are called *even*, those in A_1 *odd*. All k–superalgebras form a category, whose morphisms are those in the category of k–algebras which preserve the unit and the $\mathbb{Z}_2$–grading.

A k–superalgebra A is said to be *commutative* iff $xy = (-1)^{|x||y|} yx$ for all homogeneous x, $y \in A$ and $z^2 = 0$ for all odd $z \in A_1$. We denote by (salg) — or $(\mathrm{salg})_k$ — the category of commutative k–superalgebras.

As a matter of notation, we write (grps) for the category of groups.

Finally, we are ready to give the definition of Chevalley supergroup over the commutative ring k.

Definition 2.4. Let the notation be as above. We define *Chevalley supergroup* the supergroup functor $G_V : (\mathrm{salg})_k \longrightarrow (\mathrm{grps})$ defined as: $G_V(A) := \Big\langle G_{V,0}(A), 1 + \theta_\beta X_\beta \Big| \beta \in \Delta_1, \theta_\beta \in A_1 \Big\rangle \ \Big(\subseteq \mathrm{GL}\big(V_k\big)(A)\Big)$, for all $A \in (\mathrm{salg})_k$, where $G_{V,0}$ is the ordinary reductive group scheme associated via the Chevalley recipe with the $G_{V,0}$–module V_k (cf. [7], sec. 5). As usual $\mathrm{GL}(V_k)$ denotes the general linear supergroup scheme.

Let us fix a total order (with some mild conditions) in Δ_1, and let $G_{V,1}^{<}$ be the functor of points of the superscheme corresponding to ordered products of elements of the type $1 + \theta X \in G_V(A)$ where X is a positive root vector. We have that $G_{V,1}^{<} \cong \mathbb{A}^{0|N}$ where $N = dim_{\mathbb{C}}(\mathfrak{g}_1) = \big|\Delta_1\big|$ and $\mathbb{A}^{0|N}$ denotes the purely odd affine superspace (see [7], sec. 5, and [8], sec. 4 for details).

Theorem 2.5. *The group product $G_{V,0} \times G_{V,1}^{<} \to G_V$ induces an isomorphism of superschemes. In particular we have $G_V \cong G_{V,0} \times \mathbb{A}^{0|N}$ (with N as above), so that G_V is an affine supergroup scheme (it is representable).*

Theorem 2.5 is the main result in [7]: in particular, it states the representability of the supergroup functor G_V, so that the terminology *Chevalley supergroup* is fully justified. Furthermore, for k a field we have $Lie(G_V) = \mathfrak{g}_k$ as expected. Finally since by the classical theory $G_{V,0}$ is *connected*, G_V is connected.

2.2. The Chevalley supergroup G is closed inside $\mathrm{GL}(V_k)$.

Let k be a unital commutative ring. All our algebras and modules will now be over k unless otherwise specified.

We now wish to prove that G_V embeds naturally into the general linear supergroup $\mathrm{GL}(V_k)$ as a *closed subsuperscheme*. Note that, when k is a field, the affine supergroup G_V embeds into some $\mathrm{GL}(W)$ as a closed supergroup subscheme (see [3], ch. 11); we now want to show that we can always choose $W := V_k$, where V_k is the $\mathfrak{g}$–supermodule used to construct G_V itself.

Let us start with some observations.

Let $\mathfrak{gl}(V_k)$ be the Lie superalgebra of all the endomorphisms of the free module V_k: we denote with $\mathfrak{gl}(V_k)_0$ the set of all the endomorphisms preserving parity, and with $\mathfrak{gl}(V_k)_1$ the set of those reversing parity. Its functor of points $\mathfrak{gl}(V_k) : (\mathrm{salg}) \to (\mathrm{Lie})$ is Lie algebra valued (hereafter (Lie) denotes the category of Lie algebras) and it is given by:

$$\mathfrak{gl}(V_k)(A) := \big(A \otimes \mathfrak{gl}(V_k)\big)_0 = A_0 \otimes \mathfrak{gl}(V_k)_0 \oplus A_1 \otimes \mathfrak{gl}(V_k)_1$$

Notice that in this equality the symbol $\mathfrak{gl}(V_k)$ appears with two very different meanings: on the left hand side it is a Lie algebra valued functor, while on the right hand side it is just a free module over k. This is a most common abuse of notation in the literature. Hence $\mathfrak{gl}(V_k)(A)$ splits into direct sum of

$$\mathfrak{gl}(V_k)_0(A) = A_0 \otimes \mathfrak{gl}(V_k)_0, \qquad \mathfrak{gl}(V_k)_1(A) = A_1 \otimes \mathfrak{gl}(V_k)_1$$

corresponding respectively to the functor of points of the purely even Lie superalgebra $\mathfrak{gl}(V_k)_0$ — hence a Lie algebra — and to the functor of points of the purely odd superspace $\mathfrak{gl}(V_k)_1$. Now define the functor $\mathrm{GL}(V_k)_1 : (\mathrm{salg}) \to (\mathrm{sets})$ by

$$\mathrm{GL}(V_k)_1(A) = I + \mathfrak{gl}(V_k)_1(A) \qquad\qquad \forall A \in (\mathrm{salg})$$

where I denotes the identity in $\mathrm{GL}(V_k)_1(A)$. One can check immediately that this is a representable functor corresponding to the affine purely odd superspace $\mathbb{A}^{0|2mn}$, where $m|n$ is the dimension of V_k. One also sees easily that $\mathrm{GL}(V_k)_1$ is a subfunctor and a subscheme of $\mathrm{GL}(V_k)$. The reader must be warned that $\mathrm{GL}(V_k)_1$ has no natural supergroup structure.

The next proposition clarifies the relation between $\mathrm{GL}(V_k)_1$ and $\mathrm{GL}(V_k)$.

Proposition 2.6. *Let the notation be as above. Then the multiplication map* $\mathrm{GL}(V_k)_0 \times \mathrm{GL}(V_k)_1 \longrightarrow \mathrm{GL}(V_k)$ *induces an isomorphism of superschemes, where* $\mathrm{GL}(V_k)_0$ *denotes as usual the closed superscheme of* $\mathrm{GL}(V_k)$ *corresponding to the ordinary underlying affine group. In particular, both* $\mathrm{GL}(V_k)_0$ *and* $\mathrm{GL}(V_k)_1$ *are closed supersubschemes of* $\mathrm{GL}(V_k)$.

Proof. Given $A \in (\mathrm{salg})$, let us consider an A–point of $\mathrm{GL}(V_k)$, say

$$\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \in \mathrm{GL}(V_k)(A)$$

Then a, $d \in \mathrm{GL}(V_k)_0$ are invertible matrices and this immediately allows us to build the inverse morphism of the map $\mathrm{GL}(V_k)_0 \times \mathrm{GL}(V_k)_1 \to \mathrm{GL}(V_k)$ given by restriction of the multiplication, namely

$$\begin{array}{ccc} \mathrm{GL}(V_k) & \xrightarrow{\;\cong\;} & \mathrm{GL}(V_k)_0 \times \mathrm{GL}(V_k)_1 \\ \left(\frac{a|\beta}{\gamma|d}\right) & \mapsto & \left(\left(\frac{a|0}{0|d}\right), \left(\frac{I_m \;|a^{-1}\beta}{d^{-1}\gamma|\;\; I_n}\right) \right) \end{array}$$

where $m|n$ is the dimension of V_k and I_s is the identity matrix of size s.
The statement about $\mathrm{GL}(V_k)_0$ and $\mathrm{GL}(V_k)_1$ being closed is clear. ∎

Theorem 2.7. *Let G_V be the Chevalley supergroup associated with the complex Lie superalgebra $\mathfrak{g}$ and to a complex representation V of $\mathfrak{g}$. Then G_V is a closed supergroup subscheme in the general linear supergroup scheme* $\mathrm{GL}(V_k)$.

Proof. By the very definition of Chevalley supergroup and by Theorem 2.5 we have that

$$G_V \cong G_{V,0} \times G_{V,1}^{<} \subseteq \mathrm{GL}(V_k) \cong \mathrm{GL}(V_k)_0 \times \mathrm{GL}(V_k)_1$$

By the classical theory we have that $G_{V,0}$ is a closed subgroup (scheme) of $\mathrm{GL}(V_k)_0$, thus it is enough to show that $G_{V,1}^{<}$ is closed too — as a super-subscheme of $\mathrm{GL}(V_k)$.

Let us look closely at the embedding of $G_{V,1}^{<}$ inside $\mathrm{GL}(V_k)$. By Theorem 2.5 we have an isomorphism $\Psi : \mathbb{A}^{0|N} \to G_{V,1}^{<}$ given by

$$\Psi_A : \mathbb{A}^{0|N}(A) \to G_{V,1}^{<}(A) \quad , \qquad (\vartheta_1, \dots, \vartheta_N) \mapsto \prod_{i=1}^{N} x_{\gamma_i}(\vartheta_i)$$

where the product in right-hand side is ordered w.r.t. some total order on Δ_1 for which Δ_1^+ follows Δ_1^-, or viceversa. In particular, the point 0 in $\mathbb{A}^{0|N}$ corresponds to the identity I in $G_{V,1}^{<}$; thus the tangent superspace to $G_{V,1}^{<}$ at I corresponds to the tangent superspace to $\mathbb{A}^{0|N}$ at 0, naturally identified with $\mathbb{A}^{0|N}$ again.

Given $A \in (\mathrm{salg})$, we have for $g = \prod_{i=1}^{N} x_{\gamma_i}(\vartheta_i) \in G_{V,1}^{<}(A)$:

$$g = \prod_{i=1}^{N} x_{\gamma_i}(\vartheta_i) = I + \sum_{i=1}^{N} \vartheta_i X_{\gamma_i} + \mathcal{O}(2) \in \mathfrak{gl}\big(V_k(A)\big) \qquad (\star)$$

where $\mathcal{O}(2)$ is some element in $\mathfrak{gl}\big(V_k(A)\big) = A_0 \otimes_k \mathfrak{gl}(V_k)_0 + A_1 \otimes_k \mathfrak{gl}(V_k)_1$ whose (non-zero) coefficients in A_0 and A_1 actually belong to J_A^2, the ideal of A generated by $A_1^2 := A_1 \cdot A_1$.

Consider now the closed subscheme H in $\mathrm{GL}(V_k)_1$ whose functor of points is defined as

$$H(A) := I + \sum_i \vartheta_i X_{\gamma_i}$$

We have an invertible natural transformation ϕ

$$\begin{array}{rccc} \phi_A : & G_{V,1}^{<}(A) & \to & H(A) \quad \big(\subseteq \mathrm{GL}(V_k)(A)\big) \\ & \prod_{i=1}^{N} x_{\gamma_i}(\vartheta_i) & \mapsto & I + \sum_i \vartheta_i X_{\gamma_i} \end{array}$$

which maps $G_{V,1}^{<}$ isomorphically onto the closed subscheme H in $\mathrm{GL}(V_k)_1$, whence $G_{V,1}^{<}$ is a closed subsuperscheme of $\mathrm{GL}(V_k)_1$. ∎

3. Uniqueness Theorem

Hereafter, we assume k to be a *field*, with $\mathrm{char}(k) \neq 2, 3$.

In this section we prove the main result of our paper, which we summarize as follows. Let G be a connected affine algebraic supergroup, whose tangent Lie superalgebra $\mathfrak{g}_k$ is a k–form of a complex Lie superalgebra of classical type (see Def. 2.2); we assume also that its even subgroup G_0 is reductive and k–split, i.e. it admits a diagonalizable maximal torus. We assume further that $(\mathfrak{g}_k)_0$, the even part of $\mathfrak{g}_k$ is an ingredient in the recipe that allows us to realize the ordinary group G_0 as a Chevalley group.

We then show that such a G is isomorphic to a Chevalley supergroup G_V as we constructed in [7] according to the recipe described in the previous section.

We start with a result relative to the chosen admissible representation V of the complex Lie superalgebra $\mathfrak{g}$, inducing the embedding of G_V in $\mathrm{GL}(V_k)$.

3.1. Linearizing G.

Let G be a connected affine algebraic supergroup over k and let $\mathfrak{g}_k := \mathit{Lie}(G)$ be the tangent Lie superalgebra of G.

We assume $\mathfrak{g}_k$ to be a k–form of a complex Lie superalgebra $\mathfrak{g}$, that is $\mathfrak{g}_k = k \otimes \mathfrak{g}^{\mathbb{Z}}$ (cf. Definition 2.2), where here $\mathfrak{g}^{\mathbb{Z}}$ is any integral lattice inside the complex Lie superalgebra $\mathfrak{g}$. Moreover, we assume the complex Lie superalgebra $\mathfrak{g}$ to be simple of *classical type* (in the sense of Kac's terminology, see [13]). It follows that the even part $\mathfrak{g}_0$ of $\mathfrak{g}$ is a *reductive* Lie algebra.

Let G_0 be the ordinary subgroup underlying G: its tangent Lie algebra is $\mathit{Lie}(G_0) = \mathit{Lie}(G)_0 = (\mathfrak{g}_k)_0$. We assume that G_0 is reductive and *k–split*, i.e. it admits a diagonalizable maximal torus.

By the classical theory then G_0 can be realized via the classical Chevalley construction (see for example [12], part II, 1.1). In short, there exists a complex $\mathfrak{g}_0$–module $\widetilde{V}$ which is faithful, rational, finite-dimensional, so that G_0 is isomorphic to the affine group-scheme (over $\mathbb{Z}$) associated with $\mathfrak{g}_0$ and $\widetilde{V}$ by the classical Chevalley's construction (see also Demazure [4]), using some admissible lattice $\widetilde{M}$ in $\widetilde{V}$. Here such words as *rational* and *admissible* refer to the choice of any Chevalley basis B'_0 (in the classical sense) of the reductive Lie algebra $\mathfrak{g}_0$. It follows also that the tangent Lie algebra $\mathit{Lie}(G_0) = (\mathfrak{g}_k)_0$ has the form $(\mathfrak{g}_k)_0 = k \otimes_{\mathbb{Z}} (\mathfrak{g}_0)_{\widetilde{V}}$ where $(\mathfrak{g}_0)_{\widetilde{V}}$ is the stabilizer of $\widetilde{M}$ in $\widetilde{V}$: in turn, this $(\mathfrak{g}_0)_{\widetilde{V}}$ depends only on the lattice of weights of the $\mathfrak{g}_0$-representation $\widetilde{V}$ and not on $\widetilde{M}$ or on the choice of a Chevalley basis of $\mathfrak{g}_0$ (see [16] for more details on this classical construction).

We furthermore require a consistency condition between $\mathfrak{g}^{\mathbb{Z}}$ and G_0, as follows. As the complex Lie algebra $\mathfrak{g}$ is simple of classical type, we can fix inside it a Chevalley basis, as in Sec. 2, call it B. Then we assume that

— *(a)* $B \cap \mathfrak{g}_0 = B'_0$,

— *(b)* $\mathfrak{g}^{\mathbb{Z}} \cap \mathfrak{g}_0 = (\mathfrak{g}_0)_{\widetilde{V}}$, $\mathfrak{g}^{\mathbb{Z}} \cap \mathfrak{g}_1 = \mathit{Span}_{\mathbb{Z}}\big(B \cap \mathfrak{g}_1\big)$.

By [3], ch. 11, we have that $G \subseteq \mathrm{GL}^k_{m|n}$ for suitable m and n and consequently $\mathfrak{g}_k \subseteq \mathfrak{gl}^k_{m|n}$, where we denote with $\mathrm{GL}^k_{m|n}$ and $\mathfrak{gl}^k_{m|n}$ the general linear supergroup and the general linear superalgebra defined over k, that is $\mathrm{GL}^k_{m|n} = \mathrm{GL}(k^{m|n})$ and $\mathfrak{gl}^k_{m|n} = \mathit{Lie}(\mathrm{GL}^k_{m|n})$, where $k^{m|n}$ is the free k–supermodule of dimension $m|n$ (see [3], ch. 1, for details).

Our goal now is to pass from the G_0–module $\widetilde{V}_k = k \otimes_{\mathbb{Z}} \widetilde{V}$ to a G–module V_k which is obtained as an "induced representation" from G_0 to G (both $\widetilde{V}_k$ and V_k are k–modules). This will be achieved by another "linearization step", and an "induced representation construction" from $\big(\mathrm{GL}^k_{m|n}\big)_0$ to $\mathrm{GL}^k_{m|n}$.

Remark 3.1. The results in this section can be easily generalized to the case of k a unital commutative ring, provided we assume G to be *linearizable*. Notice that this is granted when k is a *field* (see [3], ch. 11, and [5], ch. 2, for the ordinary setting). One can check that this is also granted for k a PID and $\mathcal{O}(G)$ a free k–module.

We start with a general result on algebraic supergroups, that will be instrumental to our goal.

Proposition 3.2. *Let G be an affine algebraic supergroup with $G \subseteq \mathrm{GL}(\mathcal{V}_k)$, for $\mathcal{V}_k$ a super vector space. Then we have the following decomposition:*

$$G = G_0 \times G_1 \subseteq \mathrm{GL}(\mathcal{V}_k)_0 \times \mathrm{GL}(\mathcal{V}_k)_1$$

where G_1 is the subscheme defined by $G_1(A) := G(A) \cap \mathrm{GL}(\mathcal{V}_k)_1$.

Proof. Since $G \subseteq \mathrm{GL}(\mathcal{V}_k)$, we have that every $g \in G(A)$ decomposes in $\mathrm{GL}(\mathcal{V}_k)_0 \times \mathrm{GL}(\mathcal{V}_k)_1$ uniquely as $g = g_0 g_1$, with $g_0 \in \mathrm{GL}(\mathcal{V}_k)_0(A)$ and $g_1 \in \mathrm{GL}(\mathcal{V}_k)_1(A)$ (see 2.6). As $g_0 = \pi_A \circ g$, where $\pi_A : A \to A\big/J_A$, (as usual J_A denotes the ideal generated by A_1 in A), we have that g_0 factors via $\mathcal{O}(G)\big/J_{\mathcal{O}(G)}$ and consequently $g_0 \in G_0(A)$, from which $g_1 = g_0^{-1} g \in G(A)$. Therefore we have the result. ■

Definition 3.3. With notation as above, let $\widetilde{V}^*_k$ be the G_0–module dual to $\widetilde{V}_k$. We define $\widetilde{U}_k$ as

$$\widetilde{U}_k := \mathit{Ind}_{G_0}^{(\mathrm{GL}^k_{m|n})_0}\big(\widetilde{V}^*_k\big)$$

i.e. $\widetilde{U}_k$ is the $\big(\mathrm{GL}^k_{m|n}\big)_0$–module induced from the G_0–module $\widetilde{V}^*_k$.

Let $\widetilde{U}^*_k$ be the $(\mathrm{GL}^k_{m|n})_0$–module dual to $\widetilde{U}_k$; note that, as $\mathit{Ind}_{G_0}^{(\mathrm{GL}^k_{m|n})_0}\big(\widetilde{V}^*_k\big)$ maps onto $\widetilde{V}^*_k$, we have that $\widetilde{V}_k \cong \widetilde{V}^{**}_k$ embeds into $\widetilde{U}^*_k$, i.e. the latter contains as a G_0–submodule an isomorphic copy of $\widetilde{V}_k$.

As $\widetilde{U}^*_k$ is a $\big(\mathrm{GL}^k_{m|n}\big)_0$–module, it is also a module for the algebra of distributions on $\big(\mathrm{GL}^k_{m|n}\big)_0$, which identifies with $\mathcal{U}_k\big(\big(\mathfrak{gl}_{m|n}\big)_0\big) := k \otimes_{\mathbb{Z}} \mathcal{U}_{\mathbb{Z}}\big(\big(\mathfrak{gl}_{m|n}\big)_0\big)$, the classical Kostant algebra of $\mathit{Lie}\big(\big(\mathrm{GL}^k_{m|n}\big)_0\big) = \big(\mathfrak{gl}^k_{m|n}\big)_0$ (cf., for instance, [11], § I.7). So $\widetilde{U}^*_k$ is a $\mathcal{U}_k\big(\big(\mathfrak{gl}_{m|n}\big)_0\big)$–module, and we can perform on it the induction from $\mathcal{U}_k\big(\big(\mathfrak{gl}_{m|n}\big)_0\big)$ to $\mathcal{U}_k\big(\mathfrak{gl}_{m|n}\big)$: this yields next relevant object:

Definition 3.4.

$$W_k := Ind_{\mathcal{U}_k((\mathfrak{gl}_{m|n})_0)}^{\mathcal{U}_k(\mathfrak{gl}_{m|n})}\big(\widetilde{U}_k^*\big) = \mathcal{U}_k\big(\mathfrak{gl}_{m|n}\big) \otimes_{\mathcal{U}_k((\mathfrak{gl}_{m|n})_0)} \widetilde{U}_k^*$$

Proposition 3.5. *Let the notation be as above. Then W_k has a natural structure of $\mathrm{GL}_{m|n}^k$–module and of G–module.*

Proof. Clearly, if W_k is a $\mathrm{GL}_{m|n}^k$–module then it is a G–module as well, since G is a closed subsupergroup of $\mathrm{GL}_{m|n}^k$. Let now ρ be the representation map of $\mathfrak{gl}_{m|n}^k$ into $\mathrm{End}(W_k)$ and σ the representation map of $(\mathrm{GL}_{m|n}^k)_0$ into $\mathrm{Aut}(W_k)$. To give W_k a $\mathrm{GL}_{m|n}^k$–module structure, in view of Proposition 2.6 we need to extend σ by specifying the images of all the elements $I+\theta X$ in $(\mathrm{GL}_{m|n}^k)_1(A)$, of course in a way compatible with respect to the images of the elements in $(\mathrm{GL}_{m|n}^k)_0$. Let us define

$$\sigma(I+\theta X).w = w + \theta\rho(X)w \qquad\qquad \forall w \in W_k$$

We leave to the reader the check that this definition is compatible with the one on $(\mathrm{GL}_{m|n}^k)_0$. This is essentially a consequence of the fact that $d\sigma_0 = \rho_0$, where σ_0 and ρ_0 are the even parts of the representations σ and ρ.

From another point of view, note that our definition of $\sigma(I+\theta X)$ is exactly the one giving the unique action of $\mathrm{GL}_{m|n}^k$ on W_k, induced by restriction of the action of $\mathrm{GL}_{m|n}^k$, extending to the action of $\mathfrak{gl}_{m|n}^k$ (here we just need to recall that $\mathrm{GL}_{m|n}^k$ is naturally embedded into $\mathfrak{gl}_{m|n}^k$). In particular, an action of $\mathrm{GL}_{m|n}^k$ on W_k with such properties exists, it is unique and it is given exactly by the formula above. ∎

Now comes the main result of this subsection.

Theorem 3.6. *Let the notation be as above.*

(a) The subspace

$$V_k := \mathcal{U}_k(\mathfrak{g}) \otimes_{\mathcal{U}((\mathfrak{g}_k)_0)} \widetilde{V}_k \subseteq W_k$$

is a rational faithful finite dimensional G–module, and G embeds into $\mathrm{GL}(V_k)$ as a closed subsupergroup.

(b) There exists a Chevalley supergroup G_V such that $G_V \subseteq \mathrm{GL}(V_k)$ and $Lie(G_V) = \mathfrak{g}_k$. In other words, both G and the Chevalley supergroup G_V embed into the same general linear supergroup $\mathrm{GL}(V_k)$ and have the same Lie superalgebra. Moreover $G_0 = (G_V)_0$.

Proof. First of all, note that by Remark 2.3 we have that $\mathcal{U}_k(\mathfrak{g}) \subseteq \mathcal{U}_k\big(\mathfrak{gl}_{m|n}\big)$, hence V_k is a well-defined subspace of W_k: then by construction, it is also clear that the former is a G–submodule of the latter.

Since $\widetilde{V}_k$ is rational and faithful as a G_0–module, V_k in turn is rational and faithful as a G–module. This happens because G acts on W_k leaving V_k invariant. This is a straightforward application of Proposition 3.2. In particular, G embeds as a closed subsupergroup inside $\mathrm{GL}(V_k)$.

Now let $\widetilde{M}$ be an admissible lattice — in the complex $\mathfrak{g}_0$–module $\widetilde{V}$ — used to construct G_0 via a Chevalley construction. Then we see at once that $M := \mathcal{U}_{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathcal{U}_{\mathbb{Z}}(\mathfrak{g}_0)} \widetilde{M}$ is an admissible lattice for the (rational, faithful) complex $\mathfrak{g}$–module $V := \mathcal{U}_{\mathbb{C}}(\mathfrak{g}) \otimes_{\mathcal{U}_{\mathbb{C}}(\mathfrak{g}_0)} \widetilde{V}$, which is also finite dimensional because $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ is free of finite rank as a $\mathcal{U}_{\mathbb{C}}(\mathfrak{g}_0)$–module (cf. [7], sec. 4).

Altogether, the above means that we can use V and its lattice M to construct a Chevalley supergroup G_V over k, realized as a closed subsupergroup of $\mathrm{GL}(V_k)$. As the faithful action of $\mathfrak{g}_0$ onto $\widetilde{V}$ yields an embedding of $G_{V,0}$ into $\mathrm{GL}\big(V_k\big)$, the restriction to $\mathfrak{g}_0$ of the (faithful) action of $\mathfrak{g}$ onto V yields an embedding of $G_{V,0}$ into $\mathrm{GL}\big(V_k\big)$. By construction — including the fact that $V_k = \mathcal{U}_k(\mathfrak{g}) \otimes_{\mathcal{U}((\mathfrak{g}_k)_0)} \widetilde{V}_k = \bigwedge(\mathfrak{g}_k)_1 \otimes_k \widetilde{V}_k$ as a $\mathfrak{g}_0$–module is just $\widetilde{V}_k^{\oplus r}$ for $r := rank_{\mathcal{U}((\mathfrak{g}_k)_0)}(\mathcal{U}_k(\mathfrak{g}))$ — the $\mathfrak{g}_0$–action on V is just an r–fold diagonalization of the $\mathfrak{g}_0$–action on $\widetilde{V}$: as a consequence, the embedded copy of $G_{V,0}$ inside V_k is just an r–fold diagonalized copy of the group obtained from the $\mathfrak{g}_0$–action on $\widetilde{V}$ via the Chevalley construction. Hence $G_{V,0} = G_0$ inside $\mathrm{GL}(V_k)$. ■

3.2. G as a Chevalley supergroup.

We want to show that G and G_V are isomorphic. Since we shall make use of the fact that their Lie superalgebras are isomorphic, we need to make some observations on the differentials.

Lemma 3.7. *Let $f \in \mathcal{O}\big(\mathrm{GL}(V_k)\big)$ and let $X \in \mathfrak{gl}_1(V_k)(A)$, $A \in$ (salg) with as usual $\mathfrak{gl}(V_k) = Lie(\mathrm{GL}(V_k))$. Then*

$$f(1+\theta X) = f(1) + (df)_1 \theta X \qquad\qquad \forall \theta \in A_1$$

Proof. Clearly it is enough to check this for a monomial $f = x_{i_1 j_1} \cdots x_{i_r j_r}$, where x_{ij} denotes an even or odd generator of $\mathcal{O}\big(\mathrm{GL}(V_k)\big)$. Notice that the case of $f = x_{ij}$ is true: $x_{ij}(1+\theta X) = x_{ij}(1) + x_{ij}(\theta X) = x_{ij}(1) + (dx_{ij})_1 \theta X$. The general case reads

$$\begin{aligned}(x_{i_1 j_1} \dots x_{i_r j_r})(1+\theta X) &= x_{i_1 j_1}(1+\theta X) \cdots x_{i_r j_r}(1+\theta X) = \\ &= x_{i_1 j_1}(1) \cdots x_{i_r j_r}(1) + x_{i_1 j_1}(\theta X) x_{i_2 j_2}(1) \cdots x_{i_r j_r}(1) + \\ &\quad + x_{i_1 j_1}(1) x_{i_2 j_2}(\theta X) \cdots x_{i_r j_r}(1) + x_{i_1 j_1}(1) \cdots x_{i_{r-1} j_{r-1}}(1) x_{i_r j_r}(\theta X) = \\ &= 1 + d(x_{i_1 j_1} \cdots x_{i_r j_r})_1 (\theta X)\end{aligned}$$

which gives what we wanted. ■

Lemma 3.8. *Let the notation be as above. Then $G_V \subseteq G$, in other words $G_V(A) \subseteq G(A)$ for all $A \in$ (salg).*

Proof. As G_V is a *closed* subscheme of $\mathrm{GL}(V_k)$, (by Theorem 2.7), an element $z \in G_V(A) \subseteq \mathrm{GL}(V_k)(A)$ corresponds to a morphism $z \colon \mathcal{O}(\mathrm{GL}(V_k)) \to A$, factoring through I_{G_V}, the ideal defining G_V in $\mathcal{O}(\mathrm{GL}(V_k))$, that is $z \colon \mathcal{O}(\mathrm{GL}(V_k))\big/I_{G_V} = \mathcal{O}(G_V) \to A$ (by an abuse of notation we use the same letter). Hence to prove that

$z \in G(A)$ we need to show that z factors also via the ideal I_G of $\mathcal{O}(G)$, which is also closed in $\mathrm{GL}(V_k)$ (see Theorem 3.6).

If $z \in (G_{V,0})(A) \subseteq \mathrm{GL}(V_k)_0(A)$, then there is nothing to prove, since $G_0 = G_{V,0}$, so we assume $z \in G_{V,1}^{<}(A)$ (refer to 2.5 for the notation). It is not restrictive to assume $z = 1 + \theta X$ for a suitable $X \in \mathfrak{g}_1$ and $\theta \in A_1$, since such z's together with $G_{V,0}$ generate $G_V(A)$ as an abstract group. Now let $f \in I_G$: we need to prove that

$$z(f) = (1 + \theta X)(f) = f(1 + \theta X) = 0$$

By the previous lemma we have

$$f(1 + \theta X) = f(1) + (df)_1 \theta X$$

Certainly $f(1) = 0$ because the identity is a topological point belonging to both G and G_V. Moreover, $(df)_1 X = 0$ because of Proposition 10.6.15 in [3], since X is in the tangent space at the identity to both supergroups G and G_V. ■

Lemma 3.9. *Let X and Y two smooth superschemes (cf. [6]) globally split and such that:*
1. $X \subseteq Y$, $|X| = |Y|$;
2. $T_x X = T_x Y$ for all $x \in |X|$.
Then $X = Y$.

Proof. We have a morphism of superschemes given by the inclusion $X \hookrightarrow Y$. In order to prove this is an isomorphism it is enough to verify this on the stalks of the structure sheaves. The inclusion induces a surjective morphism on the sheaves, hence we have $\mathcal{O}_{Y,y} \twoheadrightarrow \mathcal{O}_{X,x}$. Since both X and Y are globally split and smooth, taking completions we have that $\mathcal{O}_{X,x} \subseteq \widehat{\mathcal{O}_{X,x}}$ and $\mathcal{O}_{Y,y} \subseteq \widehat{\mathcal{O}_{Y,y}}$; moreover, we can write the following commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_{Y,x} & \twoheadrightarrow & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{O}_{Y,x}} & \longrightarrow & \widehat{\mathcal{O}_{X,x}} \end{array}$$

The arrow $\widehat{\mathcal{O}_{Y,x}} \longrightarrow \widehat{\mathcal{O}_{X,x}}$ is an isomorphism, since both X and Y are smooth and they have the same tangent space. Hence we have that also the arrow $\mathcal{O}_{Y,x} \twoheadrightarrow \mathcal{O}_{X,x}$ is an isomorphism. ■

We are eventually ready for our main result:

Theorem 3.10. *Let G be an affine algebraic supergroup scheme over the field k, with G_0 being k–split, whose Lie superalgebra $\mathfrak{g}$ is a k–form of a complex Lie superalgebra of classical type. Then there exists a Chevalley supergroup G_V such that $G_V \cong G$.*

Proof. Both G and G_V described in the previous propositions embed into the same $\mathrm{GL}(V_k)$ and decompose inside the latter as $G = G_0 \times G_1$ and $G_V = G_{V,0} \times G_{V,1}$, with $G_0 = G_{V,0}$.

By the previous analysis, we are now left with the following situation: $G_V \subseteq G \subseteq \mathrm{GL}(V_k)$, $G_0 = G_{V,0}$ and $T_1 G_V = T_1 G$. Actually this happens for all points, not just the identity, so that $T_x G_V = T_x G$ for all $x \in |G| = |G_V|$ (notation of ch. 10, sec. 4, in [3]). Then by the lemma 3.9 we have the result, since both G and G_V are globally split (cf. [14]) and smooth (since $G_{V,0} = G_0$ is smooth). ∎

Remark 3.11. We want to remark that Theorem 3.10 can be applied in a different setting, that can be useful for the applications. Assume G to be a smooth affine algebraic supergroup scheme over a field k: then G is a closed subsupergroup scheme in some $\mathrm{GL}(V_k)$ — see [3], ch. 11. Assume now that V is a suitable representation of a complex Lie superalgebra $\mathfrak{g}$, such that we can construct the Chevalley supergroup G_V according to the recipe described in sec. 2. In [8] we have shown that such recipe can be suitably generalized to include Lie superalgebras not of classical type, for instance the Heisenberg superalgebra. Assume furtherly that $G_0 = G_{V,0}$ and that $Lie(G) = Lie(G_V)$, in other words G and G_V have the same underlying classical group scheme and have the same Lie superalgebra. Then, one can show easily following the arguments in Theorem 3.10 that $G \cong G_V$, that is, our smooth affine algebraic supergroup G can be realized via the Chevalley supergroup construction.

3.3. Chevalley Supergroups and Super Harish-Chandra pairs.

In super Lie theory there is an equivalence of categories between the category of Lie supergroups and the category of Super Harish-Chandra pairs (SHCP), that is the category consisting of pairs $(G_0, \mathfrak{g})$, where G_0 is an ordinary real or complex Lie group and $\mathfrak{g}$ is a real or complex Lie superalgebra with $Lie(G_0) = \mathfrak{g}_0$ and there is an action of G_0 on $\mathfrak{g}$ corresponding to the adjoint action when restricted to $\mathfrak{g}_0$. Morphisms of SHCP's are defined in a natural way and one can show a bijective functorial correspondence between the objects and the morphisms of the given two categories, hence realizing the equivalence of categories mentioned above (a full account of the theory is found for example in [3], where the origins of this theory are carefully discussed and references are given).

A natural question is whether it is possible to extend the theory of SHCP's to the category of algebraic supergroups.

When the algebraic supergroups are over fields of characteristic zero, the problem has been already treated and solved in [2]: this applies differential techniques, which cannot be employed instead for arbitrary characteristic.

Instead, more general results are obtained in [15], using a different approach, rather closer to the standard one in use for studying algebraic groups in positive characteristic. Roughly, one considers a dual version of SHCP where the first item of the pair is no longer a (classical) algebraic group but a "hyperalgebra" instead. Indeed (still very roughly speaking) if one starts with an algebraic supergroup G, then in the corresponding SHCP in the sense of [15] the even subgroup G_0 is replaced by the (classical) distribution algebra of G_0, the "correct" tool for studying G_0 in infinitesimal terms.

In the special case of Chevalley supergroups, we can directly prove a certain equivalence of categories based on the theory developed so far here and in [7]. As any Chevalley supergroup is built by means of a "distribution superalgebra" (namely the Kostant $\mathbb{Z}$–form) this result is fully consistent with those in [15].

Definition 3.12. Let k be an arbitrary field such that $\mathrm{char}(k) \neq 2,3$. We say that $(G_0, \mathfrak{g})$ is *Chevalley Super Harish-Chandra Pair* (CSHCP), if

(1) G_0 is an ordinary Chevalley group over k;

(2) $\mathfrak{g}$ is a Lie superalgebra of classical type, with $\mathfrak{g}_0 = \mathit{Lie}(G_0)$;

(3) there is a well defined action, called the *adjoint action* (with a slight abuse of notation) of G_0 on $\mathfrak{g}$, reducing to the adjoint action on $\mathfrak{g}_0$.

A *morphism* $(\rho_0, \psi) : (G_0, \mathfrak{g}) \to (H_0, \mathfrak{h})$ of CSHCPs consists of a morphism $\rho_0 : G_0 \to H_0$ of algebraic groups and a morphism $\psi : \mathfrak{g} \to \mathfrak{h}$ intertwining the adjoint action of G_0 and H_0.

We shall denote the category of CSHCP with (CSHCP).

Proposition 3.13. *There is a unique Chevalley supergroup associated to a given CSHCP.*

Proof. Given a CSHCP the recipe given in [7] allows us to produce a Chevalley supergroup associated with it. Section 5.4 in [7] proves uniqueness. ■

We now define (chesgrps) to be the category of algebraic supergroups satisfying the hypothesis carefully detailed at the beginning of section 3. It is very clear that given $G \in$ (chesgrps) there is a unique CSHCP associated with it. Next theorem establishes an equivalence of categories.

Theorem 3.14. *There exists an equivalence of categories between* (CSHCP) *and* (chesgrps)

Proof. The bijective correspondence on the objects is clear, as it is for the morphisms. ■

A. Chevalley basis

In this appendix we quickly recall the definition of Chevalley basis (see [7] for more details).

Assume $\mathfrak{g}$ to be a Lie superalgebra of classical type different from $A(1,1)$, $P(3)$, $Q(n)$ and $D(2,1;a)$, $a \notin \mathbb{Z}$. We prefer to leave out these cases to simplify our definitions, for a complete treatment see [7].

Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$: its adjoint action gives the *root space* decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus\nolimits_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\Delta = \Delta_0 \cup \Delta_1$ is the root system, with

$$\Delta_0 := \big\{ \alpha \in \mathfrak{h}^* \setminus \{0\} \big| \mathfrak{g}_\alpha \cap \mathfrak{g}_0 \neq \{0\} \big\} = \{\textit{even roots} \text{ of } \mathfrak{g}\}.$$

$$\Delta_1 := \big\{ \alpha \in \mathfrak{h}^* \big| \mathfrak{g}_\alpha \cap \mathfrak{g}_1 \neq \{0\} \big\} = \{\textit{odd roots} \text{ of } \mathfrak{g}\}.$$

If we fix a simple root system (see [13] for its definition) the root system splits into positive and negative roots, exactly as in the ordinary setting:

$$\Delta = \Delta^+ \coprod \Delta^-, \qquad \Delta_0 = \Delta_0^+ \coprod \Delta_0^-, \qquad \Delta_1 = \Delta_1^+ \coprod \Delta_1^-.$$

If $\mathfrak{g}$ is neither of type $P(n)$ nor $Q(n)$, there is an even non-degenerate, invariant bilinear form on $\mathfrak{g}$, whose restriction to $\mathfrak{h}$ is in turn an invariant bilinear form on $\mathfrak{h}$. On the other hand, if $\mathfrak{g}$ is of type $P(n)$ or $Q(n)$, then such a form on $\mathfrak{h}$ exists because $\mathfrak{g}_0$ is simple (of type A_n), though it does not come by restricting an invariant form on the whole $\mathfrak{g}$.

If $\big(x, y\big)$ denotes such form, we can identify $\mathfrak{h}^*$ with $\mathfrak{h}$, via $H'_\alpha \mapsto \big(H'_\alpha, \ \big)$. We can then transfer $\big(,\big)$ to $\mathfrak{h}^*$ in the natural way: $\big(\alpha, \beta\big) = \big(H'_\alpha, H'_\beta\big)$. Define $H_\alpha := 2 \frac{H'_\alpha}{\left(H'_\alpha, H'_\alpha\right)}$ when the denominator is non zero. When $\big(H'_\alpha, H'_\alpha\big) = 0$ such renormalization can be found in detail in [10]. We call H_α the *coroot* associated with α.

Definition A.1. We define a *Chevalley basis* of a Lie superalgebra $\mathfrak{g}$ as above any homogeneous basis $B = \big\{H_1 \dots H_l, X_\alpha, \alpha \in \Delta\big\}$ of $\mathfrak{g}$ as complex vector space, with the following requirements:

(a) $\big\{H_1, \dots, H_\ell\big\}$ is a basis of the complex vector space $\mathfrak{h}$. Moreover

$$\mathfrak{h}_\mathbb{Z} := \mathit{Span}_\mathbb{Z}\big\{H_1, \dots, H_\ell\big\} = \mathit{Span}_\mathbb{Z}\big\{H_\alpha \big| \alpha \in \Delta\big\}$$

(b) $\big[H_i, H_j\big] = 0, \quad \big[H_i, X_\alpha\big] = \alpha(H_i) X_\alpha, \qquad \forall i, j \in \{1, \dots, \ell\}, \alpha \in \Delta;$

(c) $\big[X_\alpha, X_{-\alpha}\big] = \sigma_\alpha H_\alpha \qquad \forall \alpha \in \Delta \cap (-\Delta)$

with H_α suitably defined exactly as in the ordinary setting, and $\sigma_\alpha := -1$ if $\alpha \in \Delta_1^-$, $\sigma_\alpha := 1$ otherwise;

(d) $\big[X_\alpha, X_\beta\big] = c_{\alpha,\beta} X_{\alpha+\beta} \qquad \forall \alpha, \beta \in \Delta : \alpha \neq -\beta$, with $c_{\alpha,\beta} \in \mathbb{Z}$. More precisely,

- If $(\alpha, \alpha) \neq 0$, or $(\beta, \beta) \neq 0$, then $c_{\alpha,\beta} = \pm(r+1)$ or (only if $\mathfrak{g} = P(n)$) $c_{\alpha,\beta} = \pm(r+2)$, where r is the length of the α–string through β.
- If $(\alpha, \alpha) = 0 = (\beta, \beta)$, then $c_{\alpha,\beta} = \beta(H_\alpha)$.

Notice that this definition clearly extends to direct sums of finitely many of the $\mathfrak{g}$'s under the above hypotheses.

Definition A.2. If B is a Chevalley basis of a Lie superalgebra $\mathfrak{g}$ as above, we set $\mathfrak{g}_\mathbb{Z} := \mathrm{span}_\mathbb{Z}\{B\} \quad \big(\subseteq \mathfrak{g}\big)$ and we call it the *Chevalley superalgebra* of $\mathfrak{g}$.

Observe that $\mathfrak{g}_\mathbb{Z}$ is a Lie superalgebra over $\mathbb{Z}$ inside $\mathfrak{g}$.

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