

Induced $*$ -Representations and C^* -Envelopes of Some Quantum $*$ -Algebras

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Abstract. We consider three quantum algebras: the q -oscillator algebra, the Podleś sphere and the q -deformed enveloping algebra of $su(2)$. To each of these $*$ -algebras we associate certain partial dynamical system and perform the “Mackey analysis” of $*$ -representations developed in Yu. Savchuk and K. Schmüdgen, Unbounded induced representations of $*$ -algebras, *Algebr. Represent. Theory*, DOI: 10.1007/s10468-011-9310-6. As a result we get the description of “standard” irreducible $*$ -representations. Further, for each of these examples we show the existence of a “ C^* -envelope” which is canonically isomorphic to the covariance C^* -algebra of the partial dynamical system. Finally, for the q -oscillator algebra and the q -deformed $\mathcal{U}(su(2))$ we show the existence of “bad” representations.

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Introduction and preliminaries

The aim of this paper is to demonstrate a unified approach to the $*$ -representation theory of various quantum algebras based on the techniques developed in [13]. Most of the quantum $*$ -algebras (especially those related to non-compact quantum groups) possess unbounded $*$ -representations. The main problem in the theory of unbounded $*$ -representations is to define and classify the “well-behaved” $*$ -representations of a given $*$ -algebra. We recall two classical examples.

Example. Let \mathfrak{g} be a finite-dimensional real Lie algebra, G be the corresponding simply connected Lie group and $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ be the complex enveloping $*$ -algebra of \mathfrak{g} . A $*$ -representation π of \mathfrak{g} is called *integrable* if $\pi = dU$ for some unitary representation U of G . If $G \neq \mathbb{R}$ there exists a $*$ -representation of \mathfrak{g} which is not integrable and, moreover, cannot be extended to an integrable representation even

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in a larger Hilbert space, see [14]. Already in the case $G = \mathbb{R}^2$, $\mathfrak{g} = \mathbb{C}[x_1, x_2]$ the category of all $*$ -representations of \mathfrak{g} is in a certain sense “very large” as shown in [16, Section 9].

Example. Let W_n be the n -dimensional Weyl algebra. That is, W_n is a complex $*$ -algebra generated by self-adjoint elements p_i, q_i , $i = 1, \dots, n$, satisfying $[p_i, q_j] = -\delta_{ij}\mathbf{i}$, $[p_i, p_j] = [q_i, q_j] = 0$. A self-adjoint $*$ -representation π of W_n is called integrable if $P_i = \pi(p_i)$, $Q_j = \pi(q_j)$, $i, j = 1, \dots, n$, are self-adjoint and the one-parameter unitary groups $e^{it_i P_i}$, $e^{is_j Q_j}$ satisfy the Weyl commutation relations. Already for W_1 one can show the existence of “bad representations” and show that the category of all $*$ -representations is again “very large”, whereas the only integrable $*$ -representations are sums of copies of the Schrödinger representation.

We investigate the following three $*$ -algebras in detail: the q -oscillator algebra \mathcal{A}_q for $q > 0$, the q -deformed enveloping algebra $\mathcal{U}_q(\mathfrak{su}(2))$, $q > 0$, and the Podleś spheres $\mathcal{O}(S_{qr}^2)$, $q \in (0, 1)$, $r \in (0, \infty)$. The algebras \mathcal{A}_q and $\mathcal{U}_q(\mathfrak{su}(2))$ are deformations of W_1 and $\mathcal{U}_{\mathbb{C}}(\mathfrak{su}(2))$ respectively, however, for both these algebras the notion of “integrability” cannot be generalized in a direct way. Instead of this we use the approach from [13], which applies to all three algebras \mathcal{A}_q , $\mathcal{O}(S_{qr}^2)$, $\mathcal{U}(\mathfrak{su}(2))$ as well as to their classical analogues. Let \mathcal{A} denote one of these algebras. The basic idea is to find a natural \mathbb{Z} -grading \mathcal{A}_k , $k \in \mathbb{Z}$, for \mathcal{A} such that $\mathcal{A}_0 =: \mathcal{B}$ is commutative. Further, we define the “positive” spectrum $\widehat{\mathcal{B}}^+$ of \mathcal{B} as the set of those characters $\chi \in \widehat{\mathcal{B}}$ which satisfy $\chi(a^*a) \geq 0$ for all $a \in \mathcal{A}$ such that $a^*a \in \mathcal{B}$. The group grading of \mathcal{A} defines a structure of a $*$ -algebraic bundle in the sense of [5], and there is a canonical partial action α of \mathbb{Z} on $\widehat{\mathcal{B}}^+$. By means of the partial dynamical system $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$ we

- define well-behaved $*$ -representations,
- show that the irreducible ones naturally correspond to the orbits of $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$,
- construct the dual partial action β on $C_0(\widehat{\mathcal{B}}^+)$ and the partial crossed product C^* -algebra $C_0(\widehat{\mathcal{B}}^+) \times_{\beta} \mathbb{Z}$ in the sense of [3]; using the Woronowicz’s theory of affiliated operators, we establish a Morita equivalence between $C_0(\widehat{\mathcal{B}}^+) \times_{\beta} \mathbb{Z}$ and \mathcal{A} .

It turns out that every irreducible well-behaved $*$ -representation of \mathcal{A} is induced from a one-dimensional representation. This result can be viewed as an analogue of the following theorem by Kirillov (see [6]): Every irreducible unitary representation of a nilpotent Lie group is induced from a one-dimensional representation of a certain subgroup.

In each of three cases the constructed crossed product C^* -algebra is of a special kind. Namely, the partial action of \mathbb{Z} on $C_0(\widehat{\mathcal{B}}^+)$ is generated by a single partial automorphism Θ , see [3, 9] and Section 0.2. In this case the partial crossed product C^* -algebra coincides with the covariance C^* -algebra of the partial automorphism in the sense of [3, Definition 3.7]. In the case $\mathcal{A} = \mathcal{O}(S_{qr}^2)$ all $*$ -representations are bounded, hence well-behaved, and the crossed product C^* -algebra $C_0(\widehat{\mathcal{B}}^+) \times_{\alpha} \mathbb{Z}$ is isomorphic to the enveloping C^* -algebra of \mathcal{A} .

Finally, for \mathcal{A}_q and for $\mathcal{U}_q(\mathfrak{su}(2))$ we show the existence of “bad” representations. More precisely, we prove the existence of a $*$ -representation which is not

well-behaved and cannot be extended to a well-behaved $*$ -representation even in a larger Hilbert space. It generalizes the well-known results for W_1 and $\mathcal{U}(su(2))$.

Other examples which can be analyzed in the same spirit include various bounded and unbounded $*$ -algebras: quantum group algebras $SU_q(2)$, $SU_q(1, 1)$, q -deformed $\mathcal{U}(su(1, 1))$, different deformations of CAR and CCR, AF pre- C^* -algebras (see [4]) etc.

0.1. $*$ -Algebras and $*$ -representations. By a $*$ -algebra we mean a complex associative algebra \mathcal{A} equipped with a mapping $a \mapsto a^*$ of \mathcal{A} into itself, called the *involution* of \mathcal{A} , such that $(\lambda a + \mu b)^* = \bar{\lambda}a^* + \bar{\mu}b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$. The unit of \mathcal{A} (if it exists) will be denoted by $\mathbf{1}_{\mathcal{A}}$ or simply by $\mathbf{1}$. For every $*$ -algebra \mathcal{A} denote by $\sum \mathcal{A}^2$ the set of finite sums $\sum a_i^*a_i$, $a_i \in \mathcal{A}$.

Throughout this paper we use some terminology and results from unbounded representation theory in Hilbert space (see e.g. [16]). We repeat some basic notions and facts. If T is a Hilbert space operator, $\mathcal{D}(T)$, \bar{T} and T^* denote its domain, its closure and its adjoint, respectively. Let \mathcal{D} be a dense linear subspace of a Hilbert space \mathcal{H} with scalar product $\langle \cdot, \cdot \rangle$. A $*$ -representation of a $*$ -algebra \mathcal{A} on \mathcal{D} is an algebra homomorphism π of \mathcal{A} into the algebra $L(\mathcal{D})$ of linear operators on \mathcal{D} such that $\langle \pi(a)\varphi, \psi \rangle = \langle \varphi, \pi(a^*)\psi \rangle$ for all $\varphi, \psi \in \mathcal{D}$ and $a \in \mathcal{A}$. We call $\mathcal{D}(\pi) := \mathcal{D}$ the *domain* of π and write $\mathcal{H}_\pi := \mathcal{H}$. Two $*$ -representations π_1 and π_2 of \mathcal{A} are (*unitarily*) *equivalent* if there exists an isometric linear mapping U of $\mathcal{D}(\pi_1)$ onto $\mathcal{D}(\pi_2)$ such that $\pi_2(a) = U\pi_1(a)U^{-1}$ for $a \in \mathcal{A}$. The *direct sum representation* $\pi_1 \oplus \pi_2$ acts on the domain $\mathcal{D}(\pi_1) \oplus \mathcal{D}(\pi_2)$ by $(\pi_1 \oplus \pi_2)(a) = \pi_1(a) \oplus \pi_2(a)$, $a \in \mathcal{A}$. A $*$ -representation π is called *irreducible* if a direct sum decomposition $\pi = \pi_1 \oplus \pi_2$ is only possible when $\mathcal{D}(\pi_1) = \{0\}$ or $\mathcal{D}(\pi_2) = \{0\}$. For a $*$ -subalgebra $\mathcal{B} \subseteq \mathcal{A}$ we denote by $\text{Res}_{\mathcal{B}}\pi$ its restriction to \mathcal{B} . The *graph topology* of π is the locally convex topology on the vector space $\mathcal{D}(\pi)$ defined by the norms $\varphi \mapsto \|\varphi\| + \|\pi(a)\varphi\|$, where $a \in \mathcal{A}$. If $\overline{\mathcal{D}(\pi)}$ denotes the completion of $\mathcal{D}(\pi)$ in the graph topology of π , then $\bar{\pi}(a) := \pi(a) \upharpoonright \overline{\mathcal{D}(\pi)}$, $a \in \mathcal{A}$, defines a $*$ -representation of \mathcal{A} with domain $\overline{\mathcal{D}(\pi)}$, called the *closure* of π . In particular, π is *closed* if and only if $\mathcal{D}(\pi)$ is complete in the graph topology of π . A $*$ -representation π is called *non-degenerate* if $\pi(\mathcal{A})\mathcal{D}(\pi) := \text{lin} \{ \pi(a)\varphi \mid a \in \mathcal{A}, \varphi \in \mathcal{D}(\pi) \}$ is dense in $\mathcal{D}(\pi)$ in the graph topology of π . If \mathcal{A} is unital and π is non-degenerate, then we have $\pi(\mathbf{1}_{\mathcal{A}})\varphi = \varphi$ for all $\varphi \in \mathcal{D}(\pi)$. We say that π is *cyclic* if there exists a vector $\varphi \in \mathcal{D}(\pi)$ such that $\pi(\mathcal{A})\varphi$ is dense in $\mathcal{D}(\pi)$ in the graph topology of π . For a C^* -algebra \mathfrak{A} and a Hilbert space \mathcal{H} , denote by $\text{Rep}(\mathfrak{A}, \mathcal{H})$ the category of non-degenerate $*$ -representations of \mathfrak{A} on \mathcal{H} . By $\text{Rep}\mathfrak{A}$ denote the category of all non-degenerate $*$ -representations of \mathfrak{A} .

We recall the induction procedure for $*$ -representations of general $*$ -algebras developed in [13, Section 2] in a slightly more general context. However, we will not perform this procedure but use Proposition 1.2 to get the explicit formulas. Let $\mathcal{B} \subseteq \mathcal{A}$ be $*$ -algebras. A linear map $p : \mathcal{A} \rightarrow \mathcal{B}$ is called a *bimodule projection* if $p(a^*) = p(a)^*$, $p(b_1ab_2) = b_1p(a)b_2$, $p(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{B}}$, for all $a \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$. Let ρ be a $*$ -representation of \mathcal{B} . Denote by $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$ the quotient of $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{D}(\rho)$ by the linear span of vectors $ab \otimes \varphi - a \otimes \rho(b)\varphi$, $a \in \mathcal{A}$, $b \in \mathcal{B}$, $\varphi \in \mathcal{D}(\rho)$. We say that ρ

is *inducible* from \mathcal{B} to \mathcal{A} via p if the sesquilinear form

$$\left\langle \sum_k x_k \otimes \varphi_k, \sum_l y_l \otimes \psi_l \right\rangle_0 := \sum_{k,l} \langle \rho(p(y_l^* x_k)) \varphi_k, \psi_l \rangle, \tag{0.1}$$

is positive semi-definite on $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{D}(\rho)$. Denote by \mathcal{K}_ρ the kernel of $\langle \cdot, \cdot \rangle_0$. Then $\mathcal{D}_0 = \mathcal{A} \otimes_{\mathcal{B}} \mathcal{D}(\rho) / \mathcal{K}_\rho$ is an inner-product space. Define a $*$ -representation π on \mathcal{D}_0 via

$$\pi(a) \left(\sum_i [a_i \otimes \varphi_i] \right) := \sum_i [aa_i \otimes \varphi_i],$$

where $\sum_i [a_i \otimes \varphi_i] \in \mathcal{D}_0$ denotes the image of $\sum_i a_i \otimes \varphi_i$ under the quotient mapping. Finally define $\text{Ind} \rho$ to be the closure of π .

Our major application of the induction procedure will be in the following context. Let G be a discrete group, $e \in G$ be the identity element and \mathcal{A} be a G -graded $*$ -algebra. That is, \mathcal{A} is a direct sum of vector spaces \mathcal{A}_g , $g \in G$, such that

$$\mathcal{A}_g \cdot \mathcal{A}_h \subseteq \mathcal{A}_{g \cdot h} \text{ and } (\mathcal{A}_g)^* \subseteq \mathcal{A}_{g^{-1}} \text{ for } g, h \in G. \tag{0.2}$$

The elements of $\cup_{g \in G} \mathcal{A}_g$ are called *homogeneous*. For every subgroup $H \subseteq G$ the sum $\oplus_{g \in H} \mathcal{A}_g =: \mathcal{A}_H$ is a $*$ -subalgebra of \mathcal{A} and the canonical projection $p : \mathcal{A} \rightarrow \mathcal{A}_H$ is a bimodule projection. If \mathcal{A}_e is commutative then a character $\chi : \mathcal{A}_e \rightarrow \mathbb{C}$ is inducible (via $p_e : \mathcal{A} \rightarrow \mathcal{A}_e$) if and only if $\chi(a^*a) \geq 0$ for all homogeneous $a \in \mathcal{A}$.

0.2. Partial actions and partial crossed products. The constructions and results of this subsection are taken from [3, 9]. A *partial action* of a discrete group G on a set X is a pair

$$\alpha = (\{\mathcal{D}_g\}_{g \in G}, \{\alpha_g\}_{g \in G}),$$

where $\mathcal{D}_g \subseteq X$, $g \in G$, are subsets and $\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g$ are bijections such that

- (i) $\alpha_g(\mathcal{D}_{g^{-1}} \cap \mathcal{D}_h) = \mathcal{D}_{gh} \cap \mathcal{D}_g$, $g, h \in G$,
- (ii) $\alpha_{hg}(x) = \alpha_h(\alpha_g(x))$, $x \in \mathcal{D}_{g^{-1}} \cap \mathcal{D}_{g^{-1}h^{-1}}$,
- (iii) $\mathcal{D}_e = X$, $\alpha_e = \text{Id}_X$.

For a partial action $\alpha = (\{\mathcal{D}_g\}_{g \in G}, \{\alpha_g\}_{g \in G})$ on a topological space X we require in addition that \mathcal{D}_g are open sets and $\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g$, $g \in G$, are homeomorphisms. We call (X, G, α) a *partial dynamical system (p.d.s.)*.

For a partial action $\beta = (\{I_g\}_{g \in G}, \{\beta_g\}_{g \in G})$ of G on a C^* -algebra \mathfrak{B} we require in addition that I_g , $g \in G$, are closed two-sided ideals and $\beta_g : I_{g^{-1}} \rightarrow I_g$ are $*$ -isomorphisms. We call (\mathfrak{B}, G, β) a *partial C^* -dynamical system (C^* -p.d.s.)*. For a p.d.s. (X, G, α) where X is a locally compact Hausdorff space we define the *dual C^* -p.d.s.* as follows. Put $\mathfrak{B} = C_0(X)$, $I_g = C_0(\mathcal{D}_g)$ and define $\beta_g : I_{g^{-1}} \rightarrow I_g$ by

$$(\beta_g(f))(x) = f(\alpha_{g^{-1}}(x)), \quad x \in \mathcal{D}_g, f \in I_{g^{-1}}, g \in G.$$

Direct computations show that $\beta = (\{I_g\}_{g \in G}, \{\beta_g\}_{g \in G})$ is a partial action on \mathfrak{B} and that (\mathfrak{B}, G, β) is a C^* -p.d.s.

Let (\mathfrak{B}, G, β) , $\beta = (\{I_g\}_{g \in G}, \{\beta_g\}_{g \in G})$ be a C^* -p.d.s. The *partial crossed product C^* -algebra* $\mathfrak{A} = \mathfrak{B} \times_{\beta} G$ is the enveloping C^* -algebra of the $*$ -algebra $\mathfrak{B}G$ defined as follows. $\mathfrak{B}G \subseteq \mathfrak{B} \otimes \mathbb{C}[G]$ is the linear span of the set $\{a \otimes g \mid a \in \mathfrak{I}_g\}$, with multiplication and involution defined by

$$(a \otimes g)(b \otimes h) := \alpha_g(\alpha_{g^{-1}}(a)b) \otimes gh, \quad (a \otimes g)^* := \alpha_{g^{-1}}(a^*) \otimes g^{-1}.$$

The examples of C^* -p.d.s. which appear below are of a special kind. Recall [3], that a *partial automorphism* of a C^* -algebra \mathfrak{B} is a triple $\Theta = (\theta, I, J)$, where $I, J \subseteq \mathfrak{B}$ are closed two-sided ideals and $\theta : I \rightarrow J$ is a $*$ -isomorphism. Set $I_0 = \mathfrak{B}$ and define I_n , $n \in \mathbb{Z}$, by induction

$$I_{n+1} = \{a \in J \mid \theta^{-1}(a) \in I_n\}, \quad \text{for } n \geq 0, \\ I_{n-1} = \{a \in I \mid \theta(a) \in I_n\}, \quad \text{for } n \leq 0.$$

In particular, $I = I_{-1}$ and $J = I_1$. It can be checked, see [3, Section 3], that the triple $(\mathfrak{B}, \mathbb{Z}, \beta)$, where $\beta = (\{I_n\}_{n \in \mathbb{Z}}, \{\theta^n\}_{n \in \mathbb{Z}})$ is a C^* -p.d.s. The partial crossed product algebra $\mathfrak{B} \times_{\beta} \mathbb{Z}$ is called the *covariance algebra of (\mathfrak{B}, Θ)* and is denoted by $C^*(\mathfrak{B}, \Theta)$. As in the case of a crossed-product by a $*$ -automorphism, $*$ -representations of $C^*(\mathfrak{B}, \Theta)$ are in one-to-one correspondence with covariant representations of the pair (\mathfrak{B}, Θ) , see [3, Section 5]. In case of the C^* -p.d.s. defined by (\mathfrak{B}, Θ) a *covariant representation* $\pi \times u$ consists of a $*$ -representation $\pi : \mathfrak{B} \rightarrow B(\mathcal{H})$ and a partial isometry u , whose initial and final spaces are $\overline{\pi(I)\mathcal{H}}$ and $\overline{\pi(J)\mathcal{H}}$ respectively, so that

$$\pi(\theta(b)) = u\pi(b)u^*, \quad \text{holds for every } b \in I.$$

If the latter is satisfied, then $\pi \times u$ becomes a $*$ -representation of $\mathfrak{B}\mathbb{Z}$, hence of $C^*(\mathfrak{B}, \mathbb{Z})$, via

$$(\pi \times u)(f \otimes k) = \pi(f)u^k, \quad \text{for } f \otimes k \in \mathfrak{B}\mathbb{Z},$$

where $u^{-k} = u^{*k}$ for $k \in \mathbb{N}$.

0.3. Unbounded elements affiliated with C^* -algebras and C^* -envelopes.

The theory of unbounded elements affiliated with a C^* -algebra was developed in [18], see also [8]. Let \mathfrak{A} be a C^* -algebra and let T be a densely defined closed linear operator on \mathfrak{A} . Denote by $D(T) \subseteq \mathfrak{A}$ its domain¹. The adjoint operator T^* is defined as follows. For $y, z \in \mathfrak{A}$ write $y \in D(T^*)$, $T^*y = z$ if $\langle Tx, y \rangle = \langle x, z \rangle$ holds for all $x \in D(T)$. Following [8] we say that T is *affiliated² with \mathfrak{A}* and write $T \eta \mathfrak{A}$, if $D(T^*)$ and the range of $1 + T^*T$ are dense in \mathfrak{A} , see [8, Chapter 9].

Every non-degenerate $*$ -representation of a C^* -algebra \mathfrak{A} can be continued to the set \mathfrak{A}^η of all operators affiliated with \mathfrak{A} . Namely, for every $\pi \in \text{Rep}(\mathfrak{A}, \mathcal{H})$ and $T \eta \mathfrak{A}$, there exists a closed operator $\pi(T) \eta \pi(\mathfrak{A})$ with a core $\pi(D(T))\mathcal{H}$ such that

$$\pi(T)(\pi(a)\varphi) = \pi(Ta)\varphi, \quad \text{for all } \varphi \in \mathcal{H}, a \in D(T).$$

Moreover, if $D_0 \subseteq D(T)$ is a core of T , then $\pi(D_0)\mathcal{H}$ is a core of $\pi(T)$.

¹Recall that $\mathcal{D}(\cdot)$ is domain of a Hilbert space operator.

²In [8] the term *regular operator* on \mathfrak{A} is used.

Definition 0.1. Let \mathcal{A} be a $*$ -algebra with a given category of $*$ -representations $\text{Rep}\mathcal{A}$ and fixed generators a_1, \dots, a_n . We will say that a C^* -algebra \mathfrak{A} is a C^* -envelope of \mathcal{A} if there exist affiliated elements $A_1, \dots, A_n \in \mathfrak{A}$ such that

$$\pi(A_i) = \overline{\rho(a_i)}, \quad i = 1, \dots, n. \tag{0.3}$$

defines an equivalence functor $\rho \mapsto \pi$ between $\text{Rep}\mathcal{A}$ and $\text{Rep}\mathfrak{A}$.

Remarks 1. If every $*$ -representation of \mathcal{A} is bounded, then there exists the enveloping C^* -algebra $C_{env}^*(\mathcal{A})$, which is obviously a C^* -envelope of \mathcal{A} .

2. In the last definition, the isomorphism class of \mathfrak{A} depends a priori on the choice of the generators a_i and of the category $\text{Rep}\mathcal{A}$. However, we cannot provide any example, where \mathfrak{A} would depend on the generators a_i .

1. The orbit method

In this section we recall the orbit method developed in [13]. Throughout the section G is a countable discrete group and \mathcal{A} is a G -graded $*$ -algebra. We assume that the $*$ -subalgebra $\mathcal{B} := \mathcal{A}_e$ is commutative and denote by $\widehat{\mathcal{B}}$ the set of all characters of \mathcal{B} (i.e. nontrivial $*$ -homomorphisms $\chi : \mathcal{B} \rightarrow \mathbb{C}$). Further, we define the "positive" spectrum $\widehat{\mathcal{B}}^+ \subseteq \widehat{\mathcal{B}}$ to be the set of all characters $\chi \in \widehat{\mathcal{B}}$ which satisfy³

$$\chi(a^*a) \geq 0 \text{ for all homogeneous elements } a \in \mathcal{A}. \tag{1.1}$$

Lemma 1.1. Assume that for every $g \in G$ there exists an element $a_g \in \mathcal{A}_g$ such that $\mathcal{A}_g = a_g\mathcal{B}$. Then $\chi \in \widehat{\mathcal{B}}$ belongs to $\widehat{\mathcal{B}}^+$ if and only if $\chi(a_g^*a_g) \geq 0$ for all $g \in G$.

Proof. The "only if" part is clear. Assume that $\chi(a_g^*a_g) \geq 0$ for all $g \in G$. By assumption, if $c_g \in \mathcal{A}_g$, then $c_g = a_gb$ for some $b \in \mathcal{B}$. Hence

$$\chi(c_g^*c_g) = \chi(b^*a_g^*a_gb) = \chi(a_g^*a_g)\chi(b^*b) \geq 0. \quad \blacksquare$$

The set $\widehat{\mathcal{B}}^+$ consists of those characters which satisfy (0.1), i.e. are inducible from \mathcal{B} to \mathcal{A} via p_e .

Definition 1.1. For $g \in G$ define⁴

$$\mathcal{D}_{g^{-1}} = \left\{ \chi \in \widehat{\mathcal{B}}^+ \mid \chi(a_g^*a_g) \neq 0 \text{ for some } a_g \in \mathcal{A}_g \right\}. \tag{1.2}$$

If $\chi \in \mathcal{D}_{g^{-1}}$ and $\chi(a_g^*a_g) \neq 0$ set

$$(\alpha_g(\chi))(b) := \frac{\chi(a_g^*ba_g)}{\chi(a_g^*a_g)} \text{ for } b \in \mathcal{B}. \tag{1.3}$$

³The theory developed in [13] requires the additional condition $\chi(c^*d)\chi(d^*c) = \chi(c^*c)\chi(d^*d)$ for all $\chi \in \widehat{\mathcal{B}}^+, g \in G, c, d \in \mathcal{A}_g$, which holds automatically. It can be checked using the equation $(c^*cd^*d)^2 = (c^*cd^*d)(c^*dd^*c)$ which follows by commutativity of \mathcal{B} .

⁴In [13] the notation $\alpha_g : \mathcal{D}_g \rightarrow \mathcal{D}_{g^{-1}}$ was used.

Direct computations (see [13, Proposition 13]) show that the pair $\alpha = (\{\alpha_g\}_{g \in G}, \{\mathcal{D}_g\}_{g \in G})$ is a well-defined partial action of G on $\widehat{\mathcal{B}}^+$. We also use χ^g instead of $\alpha_g(\chi)$. For a character $\chi \in \widehat{\mathcal{B}}^+$ denote by $\text{Orb}\chi \subseteq \widehat{\mathcal{B}}^+$ its orbit under the partial action of G .

Proposition 1.2 (see Proposition 16 in [13]). *Let $\chi \in \widehat{\mathcal{B}}^+$ and $\pi = \text{Ind}\chi$ be the induced $*$ -representation. For every $g \in G$ such that $\chi \in \mathcal{D}_{g^{-1}}$ fix an element $a_g \in \mathcal{A}_g$ such that $\chi(a_g^*a_g) \neq 0$. Then there exists an orthonormal base $\{e_g \mid \chi \in \mathcal{D}_{g^{-1}}\}$ in $\mathcal{D}(\pi)$ such that for $h \in G$ and $b_h \in \mathcal{A}_h$ we have*

$$\pi(b_h)e_g = \frac{\chi(a_{hg}^*b_h a_g)}{\chi(a_{hg}^*a_{hg})^{1/2}\chi(a_g^*a_g)^{1/2}}e_{hg}, \text{ if } \chi \in \mathcal{D}_{g^{-1}h^{-1}}$$

and $\pi(b_h)e_g = 0$ otherwise. In particular, if $b \in \mathcal{B}$, we have $\pi(b)e_g = \chi^g(b)e_g$.

For an element $b \in \mathcal{B}$ define its "Gel'fand transform"

$$\widehat{b} : \widehat{\mathcal{B}} \rightarrow \mathbb{C}, \widehat{b}(\chi) = \chi(b), \chi \in \widehat{\mathcal{B}}.$$

The set $\widehat{\mathcal{B}}$ is equipped with the weak topology defined by $\{\widehat{b} \mid b \in \mathcal{B}\}$ and the Borel structure generated by the open sets. By definition of $\widehat{\mathcal{B}}^+$ it is a closed subset of $\widehat{\mathcal{B}}$. It can be checked, that the partial action of G is topological. That is, $\mathcal{D}_g, g \in G$, are open sets, and $\alpha_g : \mathcal{D}_{g^{-1}} \rightarrow \mathcal{D}_g$ are homeomorphisms. Since G is countable and the one-point sets are closed, the G -orbits are Borel subsets of $\widehat{\mathcal{B}}^+$.

Definition 1.2. A closed $*$ -representation π of \mathcal{A} is called *well-behaved* if:

- (i) there exists a spectral measure E_π on $\widehat{\mathcal{B}}^+$ such that

$$\overline{\pi(b)} = \int_{\widehat{\mathcal{B}}^+} \widehat{b}(\chi)dE_\pi(\chi) \text{ for } b \in \mathcal{B},$$

- (ii) for all $a_g \in \mathcal{A}_g, g \in G$, and all Borel subsets $\Delta \subseteq \widehat{\mathcal{B}}^+$, we have

$$\pi(a_g)E_\pi(\Delta) \supseteq E_\pi(\alpha_g(\Delta \cap \mathcal{D}_{g^{-1}}))\pi(a_g). \tag{1.4}$$

A well-behaved representation π is *associated with an orbit* $\text{Orb}\chi$ if E_π is supported on the set $\text{Orb}\chi$. Denote by $\text{Rep}\mathcal{A}$ the category of all well-behaved representations.

By [13, Proposition 17], relation (1.4) can be replaced with

$$u_g \int f(t)dE_\pi(t) \subseteq \int_{\mathcal{D}_g} f(\alpha_{g^{-1}}(t))dE_\pi(t) \cdot u_g. \tag{1.5}$$

where u_g is the partial isometry in the polar decomposition $\overline{\pi(a_g)} = u_g c_g$, and f is any measurable function on $\widehat{\mathcal{B}}^+$. If f is bounded, then " \subseteq " becomes an equality.

In the next proposition we collect some basic properties of well-behaved representations. For the proof see Propositions 18, 29 and Theorem 7 in [13].

Proposition 1.3. (i) *Every bounded $*$ -representation is well-behaved.*

(ii) *If the partial action of G on $\widehat{\mathcal{B}}^+$ possesses a measurable countably separated section, then every irreducible well-behaved representation is associated with an orbit.*

(iii) *Condition (i) in Definition 1.2 holds automatically if \mathcal{B} is countably generated, and the restriction of π on \mathcal{B} is integrable, that is $\overline{\pi(a)}$ is normal for all $b \in \mathcal{B}$.*

A measurable set Γ is *countably separated* if and only if there exist Borel sets $B_k, k \in \mathbb{N}, \Gamma \subseteq \bigcup_{k \in \mathbb{N}} B_k$ such that for arbitrary $x, y \in \Gamma, x \neq y$, we have $x \in B_{k_0}, y \notin B_{k_0}$ for some $k_0 \in \mathbb{N}$. A subset Γ containing exactly one point from each orbit is called a *section* of a partial dynamical system.

Recall, that for a subgroup $H \subseteq G, \oplus_{g \in H} \mathcal{A}_g$ is a $*$ -subalgebra of \mathcal{A} denoted by \mathcal{A}_H .

Theorem 1.4 (See Theorem 5 in [13]). *Let $\chi \in \widehat{\mathcal{B}}^+$ be a character and let $H = \text{St}\chi$ be its stabilizer group. Then the map*

$$\rho \mapsto \text{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}}(\rho) = \pi$$

is a bijection from the set of unitary equivalence classes of inducible $$ -representations ρ of \mathcal{A}_H for which*

$$\text{Res}_{\mathcal{B}} \rho \text{ corresponds to a multiple of the character } \chi \tag{1.6}$$

onto the set of unitary equivalence classes of well-behaved representations π of \mathcal{A} associated with $\text{Orb}\chi$. A $$ -representation ρ satisfying (1.6) is bounded and inducible. Moreover, π is irreducible if and only if ρ is irreducible.*

The last theorem suggests the following algorithm for the description of all irreducible well-behaved representations of \mathcal{A} :

- determine $\widehat{\mathcal{B}}, \widehat{\mathcal{B}}^+$, the partial action of G on $\widehat{\mathcal{B}}^+$ and a section $\Gamma \subseteq \widehat{\mathcal{B}}^+$,
- for each $\chi \in \Gamma$
 - if the stabilizer $\text{St}\chi$ is trivial, compute $\text{Ind}\chi$,
 - otherwise find all irreducible representations ρ of $\mathcal{A}_{\text{St}\chi}$ satisfying (1.6) and compute $\text{Ind}\rho$.

If Proposition 1.3 (ii) applies, then we obtain all irreducible well-behaved representations of \mathcal{A} .

2. The q -oscillator algebra

By the *quantum harmonic oscillator* (q -oscillator) we mean the relation

$$aa^* = \mathbf{1} + qa^*a, \quad q > 0. \tag{2.1}$$

In this section, we use the notation of the q -calculus $[[k]]_q = 1 + q + \dots + q^{k-1}$. Further, we put

$$F(t) := 1 + qt.$$

Clearly $F([[k]]_q) = [[k + 1]]_q$, $k \in \mathbb{N}_0$.

In [1] the authors have obtained the following representations of (2.1) by Hilbert space operators:

- For every $q > 0$ the Fock representation π_F acting on the orthonormal base $\{\mathbf{e}_k\}_{k \in \mathbb{N}_0}$ as

$$\pi_F(a)\mathbf{e}_k = [[k]]_q^{1/2}\mathbf{e}_{k-1}, \quad \pi_F(a^*)\mathbf{e}_k = [[k + 1]]_q^{1/2}\mathbf{e}_{k+1}, \quad \text{where } \mathbf{e}_{-1} := 0. \quad (2.2)$$

- For $q \in (0, 1)$ the series of unbounded $*$ -representations $\pi_\gamma, \gamma \in (0, 1]$ acting on the orthonormal base $\{\mathbf{e}_k\}_{k \in \mathbb{Z}}$ as

$$\pi_\gamma(a)\mathbf{e}_k = \left(\frac{1 + q^{\gamma+k}}{1 - q}\right)^{1/2} \mathbf{e}_{k+1}, \quad \pi_\gamma(a^*)\mathbf{e}_k = \left(\frac{1 + q^{\gamma+k+1}}{1 - q}\right)^{1/2} \mathbf{e}_{k-1}. \quad (2.3)$$

- For $q \in (0, 1)$ the series of one-dimensional $*$ -representations

$$\pi_\varphi(a) = e^{i\varphi}(1 - q)^{-1/2}, \quad \pi_\varphi(a^*) = e^{-i\varphi}(1 - q)^{-1/2}, \quad \varphi \in [0, 2\pi). \quad (2.4)$$

Using the orbit method described in the previous section, we classify all irreducible well-behaved representations of the q -oscillator algebra

$$\mathcal{A} = \mathbb{C}\langle a, a^* \mid aa^* = qa^*a + \mathbf{1} \rangle, \quad q > 0.$$

We will see that the formulas for the irreducible well-behaved representations of \mathcal{A} coincide with (2.2)–(2.4).

We now introduce the ingredients needed for the orbit method. Define the \mathbb{Z} -grading on \mathcal{A} by setting $a \in \mathcal{A}_1$, $a^* \in \mathcal{A}_{-1}$ and put $\mathcal{B} := \mathcal{A}_0$. It is easily checked that $\mathcal{B} = \mathbb{C}[N]$, where $N = a^*a$, and $\mathcal{A}_n = a^n\mathcal{B}$, $\mathcal{A}_{-n} = a^{*n}\mathcal{B}$ for every $n \in \mathbb{N}$. Using induction on $k \in \mathbb{N}$ we obtain the relations

$$\begin{aligned} a^k a^{*k} &= \prod_{j=1}^k (q^j N + [[j]]_q \mathbf{1}), \quad k \in \mathbb{N}. \\ a^{*k} a^k &= \prod_{j=0}^{k-1} (q^{-j} N + [[-j]]_q \mathbf{1}), \quad k \in \mathbb{N}. \end{aligned} \quad (2.5)$$

Since $\mathcal{B} = \mathbb{C}[N]$, every character on \mathcal{B} is of the form $\chi_t(N) = t \in \mathbb{R}$. In what follows we identify the space of all characters $\widehat{\mathcal{B}}$ with \mathbb{R} .

Proposition 2.1. (i) $\widehat{\mathcal{B}}^+ = \{[[k]]_q \mid k \in \mathbb{N}_0\}$ for $q \geq 1$,

$$\widehat{\mathcal{B}}^+ = \{[[k]]_q \mid k \in \mathbb{N}_0\} \cup [1/(1 - q), +\infty) \text{ for } q \in (0, 1).$$

(ii) The partial action $\alpha = (\{\mathcal{D}_n\}_{n \in \mathbb{Z}}, \{\alpha_n\}_{n \in \mathbb{Z}})$ is given as follows.

$$\mathcal{D}_{-n} = \{[[k]]_q \mid k \geq n\} \text{ if } q \geq 1,$$

$$\mathcal{D}_{-n} = \{[[k]]_q \mid k \geq n\} \cup [1/(1 - q), \infty) \text{ if } q \in (0, 1).$$

If $\chi_t \in \mathcal{D}_{-n}$, then $\chi_t^n = \chi_{F^{-n}(t)}$. In particular, $\chi_{[[k]]_q}^n = \chi_{[[k-n]]_q}$ for $n \leq k$.

Proof. (i) By Lemma 1.1 a character $\chi \in \widehat{\mathcal{B}}$ belongs to $\widehat{\mathcal{B}}^+$ if and only if $\chi(a^k a^{*k}) \geq 0$, $\chi(a^{*k} a^k) \geq 0$ for all $k \in \mathbb{N}$. Further, (2.5) implies that $a^n a^{*n} = \sum_{j=0}^n \alpha_j N^j$ for some $\alpha_j \geq 0$. Hence $\chi_t \in \widehat{\mathcal{B}}^+$ if and only if $\chi(a^{*k} a^k) = \chi\left(\prod_{j=0}^{k-1} (q^{-j} N + [[-j]]_q \mathbf{1})\right) \geq 0$ for all $k \in \mathbb{N}$. The last system of inequalities is equivalent to

$$\prod_{j=0}^k (t - [[j]]_q) \geq 0 \text{ for all } k \in \mathbb{N}_0. \tag{2.6}$$

Consider first $q \geq 1$. Then $[[k]]_q \rightarrow \infty$, $k \rightarrow \infty$, and (2.6) is satisfied if and only if $t = [[k]]_q$ for some $k \in \mathbb{N}_0$. If $q \in (0, 1)$, then $[[k]]_q \rightarrow \frac{1}{1-q}$, $k \rightarrow \infty$, and every $t \geq \frac{1}{1-q}$ satisfies (2.6). For $t \in \left[0, \frac{1}{1-q}\right)$, (2.6) holds if and only if $t = [[k]]_q$ for some $k \in \mathbb{N}_0$.

(ii) One can verify by induction on $n \in \mathbb{N}$ that $F^n(t) = q^n t + [[n]]_q$ for all $n \in \mathbb{Z}$. Using (2.5), we obtain

$$\chi_t^n(N) = \frac{\chi_t(a^{*n} N a^n)}{\chi_t(a^{*n} a^n)} = \chi_t(q^{-n} N + [[-n]]_q) = \chi_{F^{-n}(t)}(N),$$

for $\chi_t \in \mathcal{D}_{-n}$, $n \in \mathbb{N}$, and

$$\chi_t^{-n}(N) = \frac{\chi_t(a^n N a^{*n})}{\chi_t(a^n a^{*n})} = q^{-1} \chi_t(q^{n+1} N + [[n+1]]_q) - q^{-1} = \chi_{F^n(t)}(N).$$

for $\chi_t \in \mathcal{D}_n$, $n \in \mathbb{N}$. Inequalities (2.6) imply that for $q > 0$ and $t = [[k]]_q$, we have $\chi_t \in \mathcal{D}_{-n}$ if and only if $n \leq k$. In case $q \in (0, 1)$ and $t \geq \frac{1}{1-q}$, we have $\chi_t \in \mathcal{D}_{-n}$ for all $n \in \mathbb{Z}$. ■

Using Proposition 2.1, we conclude that the stabilizer St_{χ_t} of $\chi_t \in \widehat{\mathcal{B}}^+$ is trivial except for the case $t = 1/(1 - q)$, where the stabilizer is \mathbb{Z} . Define the subset $\Gamma \subseteq \widehat{\mathcal{B}}^+$ as

$$\Gamma = \{0\} \cup \left\{ \frac{1}{1-q} \right\} \cup \left\{ \frac{1+q^\gamma}{1-q} \mid \gamma \in (0, 1] \right\}, \text{ if } q \in (0, 1),$$

$$\Gamma = \{0\}, \text{ if } q \geq 1.$$

Direct computations using Proposition 2.1 show that each orbit under the partial action of \mathbb{Z} on $\widehat{\mathcal{B}}^+$ intersects Γ in exactly one point, i.e. Γ is a section of the partial action. The topology on $\widehat{\mathcal{B}}^+$ is induced from the standard topology on \mathbb{R} . Hence Γ is countably separated and measurable. By Proposition 1.3(ii) every irreducible well-behaved representation of \mathcal{A} is associated to some $\text{Orb}\chi$, $\chi \in \Gamma$. For $\chi \in \Gamma$ we consider three cases.

- (i) Case $\chi = \chi_0$. Since the stabilizer of χ is trivial, the only irreducible well-behaved representation associated to $\text{Orb}\chi$ is (up to unitary equivalence) $\pi_F := \text{Ind}\chi$. Using Proposition 1.2 and relations (2.5), we calculate the

action of π_F on the orthonormal basis $\{e_{-n}\}_{n \in \mathbb{N}_0}$ of the representation space \mathcal{H}_{π_F} .

$$\begin{aligned} \pi_F(a)e_{-n} &= \frac{\chi(a^{n-1}aa^{*n})}{\chi(a^{n-1}a^{*(n-1)})^{1/2}\chi(a^na^{*n})^{1/2}}e_{-n+1} = \frac{\chi(a^na^{*n})^{1/2}}{\chi(a^{n-1}a^{*(n-1)})^{1/2}}e_{-n+1} \\ &= q^{n/2}(\chi(N) - [[-n]]_q)^{1/2}e_{-n+1} = [[n]]_q^{1/2}e_{-n+1}, \\ \pi_F(a^*)e_{-n} &= \frac{\chi_0(a^{n+1}a^*a^{*n})}{\chi_0(a^{n+1}a^{*(n+1)})^{1/2}\chi_0(a^na^{*n})^{1/2}}e_{-n-1} = \frac{\chi_0(a^{n+1}a^{*(n+1)})^{1/2}}{\chi_0(a^na^{*n})^{1/2}}e_{-n-1} \\ &= q^{(n+1)/2}(\chi(N) - [[-n-1]]_q)^{1/2}e_{-n-1} = [[n+1]]_q^{1/2}e_{-n-1}, \end{aligned}$$

where $e_1 := 0$ and $n \in \mathbb{N}_0$. It exists for any $q > 0$ and is bounded if and only if $q \in (0, 1)$.

- (ii) Case $\chi = \chi_{\frac{1+q^\gamma}{1-q}}$, $\gamma \in (0, 1]$. The stabilizer of χ is again trivial, thus $\pi_\gamma := \text{Ind}\chi_{\frac{1+q^\gamma}{1-q}}$ is the only irreducible well-behaved representation associated to $\text{Orb}\chi$. We calculate the action of $\pi_\gamma(a)$ respectively $\pi_\gamma(a^*)$ using Proposition 1.2 and relations (2.5). For $n \in \mathbb{Z}$ we have

$$\begin{aligned} \pi_\gamma(a)e_n &= \frac{\chi(a^{*(n+1)}aa^n)}{\chi(a^{*(n+1)}a^{n+1})^{1/2}\chi(a^{*n}a^n)^{1/2}}e_{n+1} = \frac{\chi(a^{*(n+1)}a^{n+1})^{1/2}}{\chi(a^{*n}a^n)^{1/2}}e_{n+1} \\ &= \left(q^{-n}\frac{1+q^\gamma}{1-q} + \frac{1-q^{-n}}{1-q}\right)^{1/2}e_{n+1} = \left(\frac{1+q^{\gamma-n}}{1-q}\right)^{1/2}e_{n+1}. \end{aligned}$$

In the same way we obtain

$$\pi_\gamma(a^*)e_n = \left(\frac{1+q^{\gamma-n+1}}{1-q}\right)^{1/2}e_{n-1}, \text{ for } n \in \mathbb{Z}.$$

Note that π_γ is not bounded for every $\gamma \in (0, 1]$.

- (iii) Case $\chi = \chi_{\frac{1}{1-q}}$. The stabilizer group H of $\chi_{\frac{1}{1-q}}$ is \mathbb{Z} . Let ρ be an irreducible $*$ -representation of \mathcal{A} satisfying (1.6). Since $\chi(aa^* - a^*a) = 0$, we have $\rho(a)\rho(a^*) = \rho(a^*)\rho(a)$. By Schur's Lemma ρ is one-dimensional. For $\lambda \in \mathbb{C}$ such that $\lambda = \rho(a)$ we get $|\lambda|^2 = \rho(aa^*) = \rho(\mathbf{1} + qa^*a) = 1 + q|\lambda|^2$. Hence $\rho = \rho_\varphi$, for some $\varphi \in [0, 2\pi)$, where $\rho_\varphi(a) = e^{i\varphi}(1-q)^{-1/2}$. Since $\mathcal{A}_H = \mathcal{A}$, $\pi_\varphi := \text{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}}\rho_\varphi$ is equivalent to ρ_φ and we have

$$\pi_\varphi(a) = e^{i\varphi}(1-q)^{-1/2}, \pi_\varphi(a^*) = e^{-i\varphi}(1-q)^{-1/2}, \varphi \in [0, 2\pi).$$

By Theorem 1.4 these are all (up to unitary equivalence) irreducible well-behaved representations of \mathcal{A} . Moreover, putting $\mathbf{e}_k := e_{-k}$, $k \in \mathbb{Z}$, we see that the above formulas coincide with (2.2), (2.3) and (2.4) respectively. We have proved the following

Theorem 2.2. *Every irreducible well-behaved representation of the q -oscillator algebra, $q > 0$, is induced from a one-dimensional $*$ -representation.*

2.1. Existence of bad $*$ -representations. In this subsection we prove the existence of a $*$ -representation π of \mathcal{A} which is not well-behaved and which cannot be continued to a well-behaved representation in a possibly larger Hilbert space. The idea is similar to the proof of [14, Theorem 4.1].

Lemma 2.3. *The polynomial*

$$p := (N - 1)(N - (1 + q)) \in \mathbb{C}[N]$$

is positive in every well-behaved $$ -representation of \mathcal{A} and $p \notin \sum \mathcal{A}^2$.*

Proof. We first show that every element of $\sum \mathcal{A}^2 \cap \mathcal{B}$ is of the form

$$\sum_{k=0}^n a^{*k} a^k \cdot p_k^* p_k, \text{ where } p_k \in \mathbb{C}[N], \ n \in \mathbb{N}. \tag{2.7}$$

Indeed, an element $b \in \mathcal{B}$ belongs to $\sum \mathcal{A}^2$ if and only if $b = \sum b_j^* b_j$, where $b_j \in \mathcal{A}_{k_j}$, $k_j \in \mathbb{Z}$. Since $\mathcal{A}_k = a^k \cdot \mathcal{B}$ for every $k \in \mathbb{Z}$ (here $a^{-k} = a^{*k}$ for $k > 0$), we obtain $b = \sum a^{*k} a^k \cdot s_k$, where $s_k \in \sum \mathcal{B}^2$. It is a well-known fact, that every positive polynomial in $\mathbb{C}[N]$ is a single square $r^* r$. Hence $s_k = p_k^* p_k$, $p_k \in \mathbb{C}[N]$, for $k \in \mathbb{Z}$. Furthermore, relations (2.5) imply $a^n a^{*n} \in \sum \mathcal{B}^2 + a^* a \sum \mathcal{B}^2$, which proves (2.7).

Let π be a well-behaved representation of \mathcal{A} with associated spectral measure E_π . Since $\text{supp} E_\pi \subseteq \widehat{\mathcal{B}}^+$ and $p \geq 0$ on $\widehat{\mathcal{B}}^+$, we have $\pi(p) = \int_{\widehat{\mathcal{B}}^+} p(\lambda) dE_\pi(\lambda) \geq 0$.

Assume to the contrary that $p \in \sum \mathcal{A}^2$. Since the degree of $p(N)$ in $\mathbb{C}[N]$ is 2, we get by (2.7)

$$p = f^* f + a^* a \cdot g^* g + a^{*2} a^2 \cdot h^* h = f^* f + N g^* g + \frac{1}{q} N(N - 1) h^* h$$

for some polynomials $f, g, h \in \mathbb{C}[N]$, where $\deg f \leq 1$, $\deg g = 0$ and $\deg h = 0$, that is, g and h are constant. Setting $N := 1$, we obtain $|f(1)|^2 + |g|^2 = 0$, i.e. $g = 0$, $f(1) = 0$. Setting $N := 1 + q$, we get $|f(1 + q)|^2 + (1 + q) |h|^2 = 0$ which implies $h = f(1 + q) = 0$. Since $\deg f \leq 1$, $f \equiv 0$, i.e. $p \equiv 0$, a contradiction. ■

For the proof of the next theorem we will need the following technical result, see [15, Lemma 2].

Lemma 2.4. *Let \mathcal{A} be a unital $*$ -algebra which has a faithful $*$ -representation π (that is, $\pi(a) = 0$ implies that $a = 0$) and is a union of a sequence of finite dimensional subspaces E_n , $n \in \mathbb{N}$. Assume that for each $n \in \mathbb{N}$ there exists a number $k_n \in \mathbb{N}$ such that the following is satisfied: If $a \in \sum \mathcal{A}^2$ is in E_n , then we can write a as a finite sum $\sum a_j^* a_j$ such that all a_j are in E_{k_n} .*

Then the cone $\sum \mathcal{A}^2$ is closed in \mathcal{A} with respect to the finest locally convex topology on \mathcal{A} .

Theorem 2.5. *There exists a $*$ -representation π of the q -oscillator algebra \mathcal{A} which cannot be extended to a well-behaved representation in a possibly larger Hilbert space.*

Proof. Since $p \notin \sum \mathcal{A}^2$ and $\sum \mathcal{A}^2$ is closed by Lemma 2.4, there exists a linear functional $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ such that $\varphi(\sum \mathcal{A}^2) \geq 0$ and $\varphi(p) < 0$ by the Hahn-Banach Theorem. Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be its GNS-construction (see [16, Section 8.6.]). Assume to the contrary, that that π_φ has a well-behaved extension, say π . Then $\langle \pi(p)\xi_\varphi, \xi_\varphi \rangle = \langle \pi_\varphi(p)\xi_\varphi, \xi_\varphi \rangle = \varphi(p) < 0$. On the other hand, $\pi(p) \geq 0$ by Lemma 2.3, a contradiction. \blacksquare

2.2. C^* -envelope of the q -oscillator algebra. In this subsection we show that \mathcal{A} , considered with the category $\text{Rep}\mathcal{A}$ and generators a, a^* has a C^* -envelope \mathfrak{A} in the sense of Definition 0.1. For let $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ be the C^* -p.d.s. dual to $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$. More precisely, define the partial action $\beta = (\{I_k\}_{k \in \mathbb{Z}}, \{\beta_k\}_{k \in \mathbb{Z}})$ on $C_0(\widehat{\mathcal{B}}^+)$ by setting $I_k := C_0(\mathcal{D}_k)$ and

$$(\beta_k(f))(t) := f(\alpha_{-k}(t)) = f(F^k(t)), \text{ for } f \in I_{-k}, t \in \mathcal{D}_k.$$

Proposition 2.1 implies that the C^* -p.d.s. $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ is defined by the partial automorphism $\Theta = (\theta, I, J)$, where

$$I = I_{-1}, J = I_1 = C_0(\widehat{\mathcal{B}}^+), (\theta(f))(t) = (\beta_1(f))(t) = f(1 + qt), f \in I_{-1}.$$

We define

$$\mathfrak{A} := C^*(C_0(\widehat{\mathcal{B}}^+), \Theta) = C_0(\widehat{\mathcal{B}}^+) \times_\beta \mathbb{Z}.$$

Theorem 2.6. *Consider the q -oscillator algebra \mathcal{A} with generators a, a^* and the category of well-behaved representations $\text{Rep}\mathcal{A}$. Then \mathfrak{A} is a C^* -envelope of \mathcal{A} .*

Proof. Let \mathfrak{A}_0 be the linear hull of

$$\{f \otimes k \in \mathfrak{A} \mid k \in \mathbb{Z}, \text{supp} f \subseteq \mathcal{D}_k \text{ is compact}\}.$$

\mathfrak{A}_0 is obviously dense in \mathfrak{A} . For $f \otimes k \in \mathfrak{A}_0$ define

$$A(f(t) \otimes k) = \sqrt{1 + qt} f(1 + qt) \otimes (k + 1),$$

and

$$A^*(f(t) \otimes k) = \sqrt{t} f(q^{-1}t - q^{-1}) \otimes (k - 1)$$

Then A and A^* are densely defined linear operators on \mathfrak{A} and their closures, denoted again by A and A^* , are adjoint to each other. For $f \otimes k \in \mathfrak{A}_0$ we have

$$A^*A(f(t) \otimes k) = A^*(\sqrt{\alpha_{-1}(t)} f(\alpha_{-1}(t)) \otimes (k + 1)) = tf(t) \otimes k.$$

The last equation shows that the range of $I + A^*A$ is dense in $\mathfrak{A}_0 \subseteq \mathfrak{A}$, so that A is affiliated with \mathfrak{A} . By [18, Theorem 1.4.] the adjoint A^* is also affiliated with \mathfrak{A} . We show the correspondence (0.3) between the generators $a, a^* \in \mathcal{A}$ and affiliated elements $A, A^* \eta \mathfrak{A}$.

By [3, Theorem 5.6] every $*$ -representation of \mathfrak{A} is given by a covariant representation $\pi \times u$ of $(C_0(\widehat{\mathcal{B}}^+), \Theta)$. Here $\pi : C_0(\widehat{\mathcal{B}}^+) \rightarrow B(\mathcal{H}_\pi)$ is a $*$ -representation of $C_0(\widehat{\mathcal{B}}^+)$ and u is a partial isometry on \mathcal{H}_π satisfying $\pi(\theta(b)) = u\pi(b)u^*$ for every

$b \in I$. By the spectral theory of commutative C^* -algebras, there exists a unique spectral measure E_π on $\widehat{\mathcal{B}}^+$ such that

$$\pi(f) = \int_{\widehat{\mathcal{B}}^+} f(t) dE_\pi(t), \quad f \in C_0(\widehat{\mathcal{B}}^+).$$

By definition of θ for $f \in I_{-1} = C_0(\mathcal{D}_{-1})$ we have

$$u \left(\int f dE_\pi \right) u^* = u\pi(f)u^* = \pi(\theta(f)) = \int f(1 + qt) dE_\pi(t).$$

Multiplying the latter by u from the right and remembering that the initial space of u is $\pi(I_{-1})\mathcal{H}_\pi = E_\pi(\mathcal{D}_{-1})\mathcal{H}_\pi$ we get

$$u \int f dE_\pi = \int f(1 + qt) dE_\pi(t) \cdot u, \quad \text{for } f \in I_{-1}. \tag{2.8}$$

The extension of $(\pi \times u)$ to A and A^* is given by

$$(\pi \times u)(A) = u \int \sqrt{t} dE_\pi, \quad (\pi \times u)(A^*) = \int \sqrt{t} dE_\pi(t) \cdot u^* \tag{2.9}$$

Indeed, for every $f \otimes k \in \mathfrak{A}_0$ we have

$$\begin{aligned} u \int \sqrt{t} dE_\pi \cdot ((\pi \times u)(f \otimes k)) &= u \int \sqrt{t} dE_\pi \cdot \int f dE_\pi \cdot u^k \\ &= u \int \sqrt{t} f(t) dE_\pi \cdot u^k = \int \sqrt{1 + qt} f(1 + qt) dE_\pi \cdot u^{k+1} = (\pi \times u)(A(f \otimes k)). \end{aligned}$$

Since $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is a core of A , $\pi(\mathfrak{A}_0)\mathcal{H}_\pi$ is a core of A , and we get the first part of (2.9). The second part follows from $(\pi \times u)(A^*) = ((\pi \times u)(A))^*$.

Let ρ be a well-behaved representation of \mathcal{A} , and E_ρ be the corresponding spectral measure on $\widehat{\mathcal{B}}^+ \subseteq \mathbb{R}_+$. Further, let $\overline{\rho(a)} = u_1 c_1$ be the polar decomposition of $\overline{\rho(a)}$. Since a^*a is the generator of \mathcal{B} , E_ρ coincides with the spectral measure of $\overline{\rho(a^*a)} = \rho(a)^* \overline{\rho(a)}$. Hence

$$\overline{\rho(a)} = u_1 \int \sqrt{t} dE_\rho, \quad \text{and } \overline{\rho(a^*)} = \int \sqrt{t} dE_\rho \cdot u_1^*. \tag{2.10}$$

Since $\ker u_1 = \ker c_1$, the initial space of u_1 is the range of $E_\rho(\widehat{\mathcal{B}}^+ \setminus \{0\}) = E_\rho(\mathcal{D}_{-1})$. Further, we have $u_1 c_1^2 u_1^* = 1 + qc_1^2$, which implies that $\ker u_1^*$ is trivial, so that the final space of u_1 is $E_\rho(\widehat{\mathcal{B}}^+) = E_\rho(\mathcal{D}_1)$. Applying (1.5) to $f \in C_0(\mathcal{D}_{-1})$ we obtain

$$\begin{aligned} u_1 \rho(f) u_1^* &= u_1 \int f(t) dE_\rho(t) \cdot u_1^* = \int f(\alpha_{-1}(t)) dE_\rho(t) \cdot u_1^* u_1 \\ &= \int f(1 + qt) dE_\rho(t) = \rho(\theta(f)), \end{aligned}$$

i.e. $(\rho_{\mathcal{B}} \times u_1)$, where $\rho_{\mathcal{B}}$ is the restriction of ρ to \mathcal{B} , defines a covariant representation of $(C_0(\widehat{\mathcal{B}}^+), \Theta)$.

The correspondence (0.3) between π and ρ follows now by comparing (2.9) with (2.10). ■

Remark. In [19, Section 3] the author shows that the operators p, q of the Weyl algebra W_1 generate a C^* -algebra \mathfrak{A} in the sense of the Definition 3.1 therein, and that \mathfrak{A} is the algebra of compact operators. It corresponds to the fact that the C^* -envelope of q -CCR with $q = 1$ is isomorphic to the partial crossed product $C_0(\mathbb{N}_0) \times_\alpha \mathbb{Z} \simeq K(l^2(\mathbb{N}_0))$.

3. The Podleś sphere

In this section we investigate $*$ -representations of the Podleś sphere $\mathcal{O}(S_{qr}^2)$. We consider only the case $q \in (0, 1)$, $r \in (0, \infty)$. The cases $r = 0$, $r = \infty$ can be treated similarly. Recall [11] that $\mathcal{A} := \mathcal{O}(S_{qr}^2)$ is the unital $*$ -algebra generated by $a = a^*, b, b^*$ and defining relations

$$ab = q^{-2}ba, ab^* = q^2b^*a, b^*b = a - a^2 + r\mathbf{1}, bb^* = q^2a - q^4a^2 + r\mathbf{1}. \tag{3.1}$$

The defining relations imply that every $*$ -representation of \mathcal{A} is bounded and hence well-behaved by Proposition 1.3 (i). In [11] the following irreducible $*$ -representations of \mathcal{A} were obtained.

- Two infinite-dimensional $*$ -representations π_\pm which act on an orthonormal base $\{\mathbf{e}_k\}_{k \in \mathbb{N}_0}$ of the representation space \mathcal{H}_\pm by

$$\begin{aligned} \pi_\pm(a)\mathbf{e}_k &= q^{2k}\lambda_\pm\mathbf{e}_k, \quad \pi_\pm(b)\mathbf{e}_k = (q^{2k}\lambda_\pm - (q^{2k}\lambda_\pm)^2 + r)^{1/2}\mathbf{e}_{k-1}, \\ \pi_\pm(b^*)\mathbf{e}_k &= (q^{2(k+1)}\lambda_\pm - (q^{2(k+1)}\lambda_\pm)^2 + r)^{1/2}\mathbf{e}_{k+1}, \quad \mathbf{e}_{-1} := 0, \end{aligned}$$

where $\lambda_\pm := \frac{1}{2} \pm (r + \frac{1}{4})^{1/2}$.

- The series of one-dimensional $*$ -representations π_φ , $\varphi \in [0, 2\pi)$,

$$\pi_\varphi(a) = 0, \quad \pi_\varphi(b) = e^{i\varphi}r^{1/2}, \quad \pi_\varphi(b^*) = e^{-i\varphi}r^{1/2}.$$

Using (3.1) and induction on $n \in \mathbb{N}$ we obtain the relations

$$\begin{aligned} ab^n &= q^{-2n}b^n a, \quad ab^{*n} = q^{2n}b^{*n} a, \\ b^{*n}b^n &= \prod_{j=1}^n (q^{-2(j-1)}a - q^{-4(j-1)}a^2 + r), \quad b^n b^{*n} = \prod_{j=1}^n (q^{2j}a - q^{4j}a^2 + r). \end{aligned} \tag{3.2}$$

Define a \mathbb{Z} -grading on \mathcal{A} by setting $a \in \mathcal{A}_0$, $b \in \mathcal{A}_1$ and $b^* \in \mathcal{A}_{-1}$. Using the defining relations one easily derives

$$\mathcal{B} := \mathcal{A}_0 = \text{Lin}\{a^l b^{*m} b^m \mid l, m \in \mathbb{N}_0\}, \quad \mathcal{A}_n := b^n \mathcal{B}, \quad \mathcal{A}_{-n} := b^{*n} \mathcal{B},$$

where $n \in \mathbb{N}_0$. Further, relations (3.2) imply that $\mathcal{B} = \mathbb{C}[a]$, hence $\widehat{\mathcal{B}} = \{\chi_t \mid t \in \mathbb{R}\}$ where $\chi_t(a) = t$. As in the previous section we identify $\widehat{\mathcal{B}}$ with \mathbb{R} .

Proposition 3.1. (i) $\widehat{\mathcal{B}}^+ = \{\chi_{m,+}\}_{m \in \mathbb{N}_0} \cup \{\chi_{m,-}\}_{m \in \mathbb{N}_0} \cup \{\chi_\infty\}$, where $\chi_{m,\pm}$ denotes χ_t , $t = q^{2m}\lambda_\pm$ and χ_∞ denotes χ_t , $t = 0$.

(ii) The partial action $\alpha = (\{\mathcal{D}_n\}_{n \in \mathbb{Z}}, \{\alpha_n\}_{n \in \mathbb{Z}})$ is given as follows:

$$\mathcal{D}_{-n} = \{\chi_{m,\pm} \mid m \geq n\} \cup \{\chi_\infty\}, \quad \text{and } \chi_{m,\pm}^n = \chi_{m-n,\pm}, \quad \chi_\infty^n = \chi_\infty.$$

Proof. (i) Lemma 1.1 and relations (3.2) imply that χ_t , $t \in \mathbb{R}$, belongs to $\widehat{\mathcal{B}}^+$ if and only if the following inequalities are satisfied for all $n \in \mathbb{N}$:

$$\begin{aligned} \chi_t(b^{*n}b^n) &= \prod_{k=0}^{n-1} (q^{-2k}t - q^{-4k}t^2 + r) \geq 0, \\ \chi_t(b^n b^{*n}) &= \prod_{k=1}^n (q^{2k}t - q^{4k}t^2 + r) \geq 0. \end{aligned} \tag{3.3}$$

Assume $q^{-2k}t - q^{-4k}t^2 + r > 0$ for all $k \in \mathbb{N}_0$. Since $q^{-2k} \rightarrow +\infty$, $k \rightarrow +\infty$, it is possible only if $t = 0$, i.e. $\chi_t = \chi_\infty$. If $t \neq 0$ we get $q^{-2k}t - q^{-4k}t^2 + r = 0$ for some $k \in \mathbb{N}_0$, whence

$$t = \frac{-q^{-2k} \pm \sqrt{q^{-4k} + 4q^{-4k}r}}{-2q^{-4k}} = q^{2k} \lambda_{\mp}.$$

One can easily check that every $t = q^{2k} \lambda_{\pm}$, $k \in \mathbb{N}_0$, satisfies (3.3).

(ii) Relations (3.3) imply that $\mathcal{D}_{-n} = \{\chi_{m,\pm} \mid m \geq n\} \cup \{\chi_\infty\}$. Assume that $\chi_{m,\pm} \in \mathcal{D}_{-n}$, where $n \in \mathbb{N}_0$. Using relations (3.2) we obtain

$$\chi_{m,\pm}^n(a) = \frac{\chi_{m,\pm}(b^{*n}ab^n)}{\chi_{m,\pm}(b^{*n}b^n)} = \frac{\chi_{m,\pm}(b^{*n}b^n)\chi_{m,\pm}(q^{-2n}a)}{\chi_{m,\pm}(b^{*n}b^n)} = \chi_{m-n,\pm}(a).$$

For χ_∞ we have $\chi_\infty(b^{*n}b^n) = \chi_\infty(b^n b^{*n}) = r^n \neq 0$ for all $n \in \mathbb{Z}$ by equations (3.3). Hence $\chi_\infty \in \mathcal{D}_n$ for all $n \in \mathbb{Z}$ and $\chi_\infty^n(a) = 0$. ■

Let Γ be the subset $\{\chi_{0,+}, \chi_{0,-}, \chi_\infty\} \subseteq \widehat{\mathcal{B}}^+$. Obviously Γ is a measurable countably separated section of the p. d. s. $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$. We calculate all irreducible $*$ -representations associated with $\text{Orb}\chi$, $\chi \in \Gamma$.

(i) Case $\chi_{0,\pm}$. The stabilizer of $\chi_{0,\pm}$ is trivial by Proposition 3.1, (ii). Put $\pi_{\pm} := \text{Ind}\chi_{0,\pm}$. We use Proposition 1.2 to compute the action of π_{\pm} on the orthonormal base $\{e_{-k}\}_{k \in \mathbb{N}_0}$.

$$\begin{aligned} \pi_{\pm}(b)e_{-k} &= (q^{2k}\chi_{0,\pm}(a) - q^{4k}\chi_{0,\pm}(a^2) + r)^{1/2} e_{-k+1} \\ &= \left(q^{2k}\lambda_{\pm} - (q^{2k}\lambda_{\pm})^2 + r\right)^{1/2} e_{-k+1}, \\ \pi_{\pm}(b^*)e_{-k} &= \left(q^{2(k+1)}\lambda_{\pm} - (q^{2(k+1)}\lambda_{\pm})^2 + r\right)^{1/2} e_{-k-1}, \\ \pi_{\pm}(a)e_{-k} &= \chi_{0,\pm}^{-k}(a) = q^{2k}\lambda_{\pm}e_{-k}. \end{aligned}$$

(ii) Case χ_∞ . The stabilizer group H of χ is \mathbb{Z} . Let ρ be an irreducible $*$ -representation of \mathcal{A}_H satisfying (1.6). Since $\chi(bb^* - b^*b) = 0$, we have $\rho(b)\rho(b^*) = \rho(b^*)\rho(b)$. By Schur's Lemma ρ is one-dimensional. For $\lambda \in \mathbb{C}$ such that $\lambda = \rho(b)$ we get $|\lambda|^2 = \rho(bb^*) = r$. Hence $\rho = \rho_\varphi$, for some $\varphi \in [0, 2\pi)$, where $\rho_\varphi(b) = e^{i\varphi}r^{1/2}$. Since $\mathcal{A}_H = \mathcal{A}$, $\pi_\varphi := \text{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}} \rho_\varphi$ is equivalent to ρ_φ and we get

$$\pi_\varphi(a) = 0, \pi_\varphi(b) = e^{i\varphi}r^{1/2}, \pi_\varphi(b^*) = e^{-i\varphi}r^{1/2}, \varphi \in [0, 2\pi).$$

By Theorem 1.4 these are all, up to unitary equivalence, irreducible $*$ -representations of \mathcal{A} . Setting $\mathbf{e}_k := e_{-k}$, $k \in \mathbb{N}_0$, we see that these coincide with the ones found in [11]. In particular, we have the following

Theorem 3.2. *Every irreducible $*$ -representation of the Podleś sphere $\mathcal{O}(S_{qr}^2)$, $q \in (0, 1)$, $r \in [0, \infty]$, is induced from a one-dimensional $*$ -representation.*

In the remaining part of this section we describe the enveloping C^* -algebra of $\mathcal{O}(S_{qr}^2)$. For let $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ be the C^* -p.d.s. dual to $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$ as defined in the Subsection 0.2. Note, that the sets \mathcal{D}_k , $k \in \mathbb{Z}$, are compact, hence $I_k := C(\mathcal{D}_k)$. By definition of β we have

$$(\beta_k(f))(t) = f(\alpha_{-k}(t)) = f(q^{2k}t), \quad f \in I_{-k}, k \in \mathbb{Z}. \tag{3.4}$$

It is easily seen from the description of α in the Proposition 3.1 that the partial action $\beta = (\{I_k\}_{k \in \mathbb{Z}}, \{\beta_k\}_{k \in \mathbb{Z}})$ is defined by the partial automorphism $\Theta = (\theta, I, J)$, where $\theta = \beta_1$, $I = I_{-1}$, $J = I_1 = \mathcal{A}$.

Theorem 3.3. *The enveloping C^* -algebra \mathfrak{A} of \mathcal{A} is isomorphic to the covariance algebra $C^*(C(\widehat{\mathcal{B}}^+), \Theta) \simeq C(\widehat{\mathcal{B}}^+) \times_{\beta} \mathbb{Z}$*

Proof. The proof goes similarly to the proof of the Theorem 2.6 by replacing the η -relation with \in -relation.

We first define $*$ -homomorphism $\epsilon : \mathcal{A} \rightarrow C(\widehat{\mathcal{B}}^+) \times_{\beta} \mathbb{Z}$ by setting

$$\epsilon(a) = t \otimes 0, \quad \epsilon(b) = (q^2t - q^4t^2 + r)^{1/2} \otimes 1, \quad \epsilon(b^*) = (t - t^2 + r)^{1/2} \otimes (-1).$$

Direct computations using (3.4) show that $\epsilon(a), \epsilon(b), \epsilon(b^*)$ satisfy the defining relations of $\mathcal{O}(S_{qr}^2)$, that is, ϵ is well-defined. Every representation π of \mathcal{A} is bounded, hence well-behaved by Proposition 1.3. That is, π gives rise to a covariant representation $\pi|_{\mathcal{B}} \times u$, where u is the partial isometry in the polar decomposition $\pi(b) = uc$. On the other hand, every $*$ -representation of \mathfrak{A} is given by a covariant representation of the partial automorphism Θ . This proves the correspondence (0.3) for the representations of \mathcal{A} and \mathfrak{A} . ■

4. The quantum algebra $\mathcal{U}_q(su(2))$

In this section \mathcal{A} is the q -deformed enveloping $*$ -algebra $\mathcal{U}_q(su(2))$, $q > 0$, $q \neq 1$, which is generated by E, F, K, K^{-1} satisfying the defining relations

$$\begin{aligned} KK^{-1} &= K^{-1}K = \mathbf{1}, & KEK^{-1} &= q^2E, & KFK^{-1} &= q^{-2}F, \\ [E, F] &= EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \\ E^* &= FK, & F^* &= K^{-1}E, & K^* &= K. \end{aligned}$$

In this section, we use the standard notation $[n] \equiv [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, where $n \in \mathbb{Z}$ and $q \neq 0$. Further X^0 denotes $\mathbf{1}$ if X is one of the four generators E, F, K, K^{-1} .

Consider the $*$ -representations $\pi_{\omega, l}$, $\omega = \pm 1$, $l \in \frac{1}{2}\mathbb{N}_0$, of $\mathcal{U}_q(su(2))$, which

act on an orthonormal base $\{\mathbf{e}_m\}_{m=-l,\dots,l}$ as follows:

$$\begin{aligned} \pi_{\omega,l}(K)\mathbf{e}_m &= \omega q^{2m}\mathbf{e}_m, \\ \pi_{\omega,l}(E)\mathbf{e}_m &= q^{m+1}\sqrt{[l-m][l+m+1]}\mathbf{e}_{m+1}, \\ \pi_{\omega,l}(F)\mathbf{e}_m &= \omega q^{-m}\sqrt{[l+m][l-m+1]}\mathbf{e}_{m-1}, \end{aligned} \tag{4.1}$$

where we set $\mathbf{e}_{l+1} := 0$ and $\mathbf{e}_{-l-1} := 0$, cf. [7, Section 3.2.3] and [17]. We will show that every irreducible well-behaved representation of \mathcal{A} is unitarily equivalent to $\pi_{\omega,l}$ for some $l \in \frac{1}{2}\mathbb{N}_0$, $\omega = \pm 1$.

Define a \mathbb{Z} -grading of \mathcal{A} by setting $E \in \mathcal{A}_1$, $F \in \mathcal{A}_{-1}$ and $K, K^{-1} \in \mathcal{A}_0$. Then

$$\mathcal{B} := \mathcal{A}_0 = \text{lin} \{ E^l F^l K^m \mid l \in \mathbb{N}_0, m \in \mathbb{Z} \}.$$

The $*$ -subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is commutative and is equal to $\mathbb{C}[EF, K, K^{-1}] = \mathbb{C}[C_q, K, K^{-1}]$. For $n \in \mathbb{N}_0$, we have

$$\begin{aligned} \mathcal{A}_n &= E^n \mathcal{B} = \text{lin} \{ E^{n+l} F^l K^m \mid l \in \mathbb{N}_0, m \in \mathbb{Z} \}, \\ \mathcal{A}_{-n} &= F^n \mathcal{B} = \text{lin} \{ F^{n+l} E^l K^m \mid l \in \mathbb{N}_0, m \in \mathbb{Z} \}. \end{aligned}$$

One can verify by a direct computation that the *quantum Casimir element* C_q is a central element in \mathcal{A} , where

$$C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2}.$$

The following lemma can be easily proved by induction.

Lemma 4.1. *For every $n \in \mathbb{N}$ we have*

$$\begin{aligned} (i) \quad [E, F^n] &\equiv EF^n - F^n E = [n]F^{n-1}[K; 1 - n], \\ (ii) \quad [E^n, F] &\equiv E^n F - F E^n = [n]E^{n-1}[K; n - 1], \end{aligned}$$

where we set $[K; l] := (q^l K - q^{-l} K^{-1}) / (q - q^{-1})$ for $l \in \mathbb{Z}$.

This lemma implies the following relations:

$$\begin{aligned} E^n F^n &= \prod_{j=1}^n (EF + [j - 1][K; -j]), \\ F^n E^n &= \prod_{j=1}^n (EF - [j][K; j - 1]), \quad n \in \mathbb{N}. \end{aligned} \tag{4.2}$$

Since $\mathcal{B} = \mathbb{C}[C_q, K, K^{-1}]$, every character $\chi \in \widehat{\mathcal{B}}$ is equal to some $\chi_{st} \in \widehat{\mathcal{B}}$, $(s, t) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$ where

$$\chi_{st}(C_q) = s, \quad \chi_{st}(K) = t.$$

Proposition 4.2. (i) A character $\chi_{st} \in \widehat{\mathcal{B}}$ belongs to $\widehat{\mathcal{B}}^+$ if and only if

$$t = \pm q^{m-n} \text{ and } s = \frac{\pm q^{m+n+1} \pm q^{-m-n-1}}{(q - q^{-1})^2}, \text{ where } m, n \in \mathbb{N}_0.$$

In particular,

$$\widehat{\mathcal{B}}^+ = \{\chi_{m,n,+} \mid m, n \in \mathbb{N}_0\} \cup \{\chi_{m,n,-} \mid m, n \in \mathbb{N}_0\},$$

where $\chi_{m,n,\pm} = \chi_{st}$ with s, t from above.

(ii) The partial action $\alpha = (\{\mathcal{D}_n\}_{n \in \mathbb{Z}}, \{\alpha_n\}_{n \in \mathbb{Z}})$ is given as follows:

$$\mathcal{D}_{-k} = \{\chi_{m,n,\pm} \mid -m \leq k \leq n\}, \text{ and } \chi_{m,n,\pm}^k = \chi_{m+k,n-k,\pm}.$$

Proof. (i) Lemma 1.1 and equations (4.2) imply that $\chi \in \widehat{\mathcal{B}}^+$ if and only if the following inequalities are satisfied for arbitrary $k \in \mathbb{N}$:

$$\chi(E^k F^k K^{-k}) = \chi(K^{-1})^k \prod_{j=1}^k \chi(EF + [j-1][K; -j]) \geq 0, \tag{4.3}$$

$$\chi(F^k E^k K^k) = \chi(K)^k \prod_{j=1}^k \chi(EF - [j][K; j-1]) \geq 0. \tag{4.4}$$

We show that there exist $m, n \in \mathbb{N}_0$ such that

$$\chi(EF + [m][K; -m-1]) = 0, \tag{4.5}$$

$$\chi(EF - [n+1][K; n]) = 0. \tag{4.6}$$

Assume the contrary, i.e. $\chi(EF + [k][K; -k-1]) \neq 0$ for all $k \in \mathbb{N}_0$. Suppose $t > 0$. Then by (4.3) we have

$$\chi(EF) > -[k]\chi([K; -k-1]) = \frac{(q^{-2k-1} - q^{-1})t + (q^{2k+1} - q)t^{-1}}{(q - q^{-1})^2}, \quad k \in \mathbb{N}_0.$$

The right hand side of the last inequality tends to $+\infty$ as $k \rightarrow \infty$, which is a contradiction. In the same way one obtains a contradiction for $t < 0$. Thus, $\chi(EF + [m][K; -m-1]) = 0$ for some $m \in \mathbb{N}_0$. Similarly one can prove that $\chi(EF - [n+1][K; n]) = 0$ for some $n \in \mathbb{N}_0$, using inequalities (4.4). Subtracting (4.6) from (4.5) yields

$$\begin{aligned} & [m]\chi([K; -m-1]) = -[n+1]\chi([K; n]) \\ \iff & (q^{-1} - q^{-2m-1})t - (q^{2m+1} - q)t^{-1} \\ & = (q^{-1} - q^{2n+1})t - (q^{-2n-1} - q)t^{-1} \\ \iff & t^2 = \frac{q^{2m+1} - q^{-2n-1}}{q^{2n+1} - q^{-2m-1}} = \frac{q^{2m}(q - q^{-2m-2n-1})}{q^{2n}(q - q^{-2m-2n-1})} \\ \iff & t = \pm q^{m-n}. \end{aligned}$$

Further we obtain

$$\begin{aligned} \chi(C_q) &= [n + 1]\chi([K; n]) + \frac{q^{-1}\chi(K) + q\chi(K^{-1})}{(q - q^{-1})^2} \\ &= \frac{q^{2n+1}t + q^{-2n-1}t^{-1}}{(q - q^{-1})^2} = \frac{\pm q^{m+n+1} \pm q^{-m-n-1}}{(q - q^{-1})^2}. \end{aligned}$$

(ii) Observe that $\chi_{m,n,\pm}(E^k F^k K^{-k}) \neq 0$ if and only if $k \leq m$ by (4.3) and (4.5). Analogously, $\chi_{m,n,\pm}(F^k E^k K^k) \neq 0$ if and only if $k \leq n$ by (4.4) and (4.6). This implies that $\chi_{m,n,\pm} \in \mathcal{D}_{-k}$ if and only if $-m \leq k \leq n$. Now suppose $k \in \{0, 1, \dots, n\}$. Since C_q commutes with E, F , we have

$$\begin{aligned} \chi_{m,n,\pm}^k(K) &= \frac{\chi_{m,n,\pm}(E^{*k} K E^k)}{\chi_{m,n,\pm}(E^{*k} E^k)} = \frac{\chi_{m,n,\pm}(E^{*k} E^k q^{2k} K)}{\chi_{m,n,\pm}(E^{*k} E^k)} = q^{2k} \chi_{m,n,\pm}(K), \\ \chi_{m,n,\pm}^k(C_q) &= \frac{\chi_{m,n,\pm}(E^{*k} C_q E^k)}{\chi_{m,n,\pm}(E^{*k} E^k)} = \chi_{m,n,\pm}(C_q). \end{aligned}$$

The last equalities also hold for $k \in \{-m, -m + 1, \dots, 0\}$. Hence, if $\chi_{m,n,\pm}^k$ is defined, then $\chi_{m,n,\pm}^k(K) = \pm q^{(m+k)-(n-k)} = \chi_{m+k,n-k,\pm}(K)$ and $\chi_{m,n,\pm}^k(C_q) = \chi_{m,n,\pm}(C_q)$. ■

In particular, the previous proposition implies that for each $\chi \in \widehat{\mathcal{B}}^+$ the stabilizer $\text{St}\chi$ is trivial. We set

$$\Gamma := \{\chi_{0,n,+} \mid n \in \mathbb{N}_0\} \cup \{\chi_{0,n,-} \mid n \in \mathbb{N}_0\}.$$

As in Section 2, we conclude that Γ is a measurable countably separated section of the partial action. Using Proposition 4.2 we conclude that $\text{Orb}\chi_{0,n,\pm}$ consists of $n + 1$ elements and hence $\text{Ind}\chi_{0,n,\pm}$ has dimension $n + 1$ by Proposition 1.2, where $\chi_{0,n,\pm} \in \Gamma$. Now put $l := \frac{n}{2}$ and $\pi_{\omega,l} := \text{Ind}\chi_{0,n,\pm}$, where $\omega = \pm 1$. Let $\{e_{l+m}\}_{m=-l,-l+1,\dots,l}$ be an orthonormal base of the representation space $\mathcal{H}_{\pi_{l,\pm}}$ of $\pi_{l,\pm}$. For notational convenience, we put $e_{l+1} := 0$, and $e_{-l-1} := 0$.

Using Proposition 1.2, relations (4.2), Proposition 4.2 and the facts that $\chi_{0,n,\pm}(EF) = 0$, $\chi_{0,n,\pm}([K; l + m]) = -\omega[l - m]$, we obtain the action of $\pi_{\omega,l}$ on the base vectors e_{l+m} .

$$\begin{aligned} \pi_{\omega,l}(K)e_{l+m} &= \chi_{0,n,\pm}^{l+m}(K)e_{l+m} = \chi_{l+m,n-l-m,\pm}(K)e_{l+m} = \omega q^{2m} e_{l+m}, \\ \pi_{\omega,l}(E)e_{l+m} &= \frac{\chi_{0,n,\pm}(E^{*(l+m+1)} E^{l+m+1})}{(\chi_{0,n,\pm}(E^{*(l+m+1)} E^{l+m+1})\chi_{0,n,\pm}(E^{*(l+m)} E^{l+m}))^{1/2}} e_{l+m+1} \\ &= \left(\frac{\chi_{0,n,\pm}(E^{*(l+m+1)} E^{l+m+1})}{\chi_{0,n,\pm}(E^{*(l+m)} E^{l+m})} \right)^{1/2} e_{l+m+1} \\ &= (q^{(l+m+1)(l+m+2)-(l+m)(l+m+1)})^{1/2} \\ &\quad \times (\chi_{0,n,\pm}(EF - [l + m + 1][K; l + m])\chi_{0,n,\pm}(K))^{1/2} e_{l+m+1} \\ &= q^{m+1} \sqrt{[l - m][l + m + 1]} e_{l+m+1}, \\ \pi_{\omega,l}(F)e_{l+m} &= \pi_{\omega,l}(E^* K^{-1})e_{l+m} = \omega q^{-2m} \pi_{\omega,l}(E)^* e_{l+m} \\ &= \omega q^{-m} \sqrt{[l + m][l - m + 1]} e_{l+m-1}. \end{aligned}$$

Putting $e_m := e_{l+m}$, $m = -l, \dots, l$, we see that every irreducible well-behaved representation of $\mathcal{U}_q(\mathfrak{su}(2))$ is unitarily equivalent to $\pi_{\omega,l}$, given by the formulas (4.1), for some $\omega \in \{-1, +1\}$ and $l \in \frac{1}{2}\mathbb{N}_0$. Summarizing the above discussion, we obtain the following

Theorem 4.3. *Every irreducible well-behaved representation of $\mathcal{U}_q(\mathfrak{su}(2))$, $q \in \mathbb{R}_+ \setminus \{1\}$, is induced from a one-dimensional $*$ -representation.*

Similarly to Lemma 2.3 and Theorem 2.5 one can prove the following Lemma and Theorem.

Lemma 4.4. *The polynomial*

$$(EF - [2][K; 1])(EF - [3][K; 2]) \in \mathbb{C}[EF, K, K^{-1}]$$

is positive in every well-behaved representation of \mathcal{A} and is not of the form $\sum_{k=1}^n a_k^ a_k$ for $a_k \in \mathcal{A}$.*

Theorem 4.5. *There exists a $*$ -representation of $\mathcal{U}_q(\mathfrak{su}(2))$, $q \in \mathbb{R}_+ \setminus \{1\}$, which has no well-behaved extension in a possibly larger Hilbert space.*

Let $\text{Rep}\mathcal{A}$ denote the category of well-behaved non-degenerate representations of \mathcal{A} . Then \mathcal{A} , considered with $\text{Rep}\mathcal{A}$ and generators E, F, K, K^{-1} , has a C^* -envelope \mathfrak{A} in the sense of Definition 0.1. As in Section 2, let $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ be the C^* -p.d.s. dual to $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$. The description of the p.d.s. $(\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$ in Proposition 4.2 implies that the C^* -p.d.s. $(C_0(\widehat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ is defined by the partial automorphism $\Theta = (\theta, I, J)$, where $I = I_{-1}$, $J = I_1$, $(\theta(f))(t) = (\beta_1(f))(t)$, $f \in I_{-1}$.

The proof of the following theorem is completely analogous to the proof of Theorem 2.6.

Theorem 4.6. *Consider the q -deformed enveloping algebra $\mathcal{A} = \mathcal{U}_q(\mathfrak{su}(2))$ with generators E, F, K, K^{-1} and the category of well-behaved representations $\text{Rep}\mathcal{A}$. Then the covariance algebra $\mathfrak{A} := C^*(C_0(\widehat{\mathcal{B}}^+), \Theta)$ is a C^* -envelope of \mathcal{A} .*

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