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Momentum map in the Bundle of Connections

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Abstract. We introduce the moment map for the p^*adP-valued symplectic structure cannonically defined on the bundle of connections $p\colon \mathcal{K} \to M$ of an arbitrary principal G-bundle $\pi\colon P \to M$.
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1. Gauge algebra and generalized symplectic structure

Given a principal G-bundle $\pi : P \to M$, its gauge group, $Gau(P)$, is the group formed by the equivariant diffeomorphisms $\phi : P \to P$, $\phi(u \cdot g) = \phi(u) \cdot g$, $\forall u \in P$, $\forall g \in G$, which stabilize each $G-$orbit in P. A vector field $X \in \mathcal{X}(P)$ is said to be $G-$invariant if $(R_g)_* X = X$, $\forall g \in G$. Let us denote with $V(P) \subset \mathcal{X}(P)$ the vertical bundle of the fibration. If ϕ_t is the flow of a vector field $X \in V(P)$ then X is $G-$invariant if and only if $\phi_t \in Gau(P)$, $\forall t \in \mathbb{R}$. We shall denote by $gau(P)$ the Lie algebra of all π-vertical G-invariant vector fields on P, which is called the *gauge algebra* of P.

Let us denote with $V(P)$ the vertical bundle of the fibration. A natural trivialization $V(P) \cong P \times \mathcal{G}$ is given by: $(u, B) \longmapsto B^*_u$, where B^* is the fundamental vector field on P associated with $B \in \mathcal{G}$. It is easy to see that the right action of G when restricted to $V(P)$ corresponds with the action of G on $P \times \mathcal{G}$ defined by $(u, A) \cdot g = (ug, ad(g^{-1})A)$ where ad denotes the adjoint representation of G into its Lie algebra $\mathcal{G}$. This means hat $V(P)/G$ is the adjoint bundle associated with P by the adjoint representation of G in its Lie algebra $\mathcal{G}$. Accordingly, we have an exact sequence of vector bundles on M

$$0 \to adP \to T(P)/G \to T(M) \to 0.$$

The global sections of adP can be naturally identified with $gau(P)$, *i.e.*, $gau(P) = \Gamma(M, adP)$.

Let us denote with $C(P, \mathcal{G})$ the set of automorphic functions on P with respect to the adjoint representation of G, *i.e.*, maps $\tau : P \to \mathcal{G}$ verifying

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$\tau(p \cdot g) = ad(g^{-1})\tau(p)$, $\forall g \in G$. Given $D \in gau(P)$ we define the automorphic function $\tau_D \in C(P, \mathcal{G})$ as follows: given $p \in P$, $\tau_D(p)$ is the unique $A \in \mathcal{G}$ such that A_p^* is equal to $D_p \in T_p(P)$. Each element in $C(P, \mathcal{G})$ corresponds biunivocally and canonically to a section of the adjoint bundle adP. In this way a vector field $D \in gau(P)$ will be denoted either by $s_D \in \Gamma(M, adP)$ or by $\tau_D \in C(P, \mathcal{G})$ depending on what we want to mean whether it is a section of the adjoint bundle or an automorphic function. We shall refer the correspondence

$$s_D \leftrightarrow \tau_D$$

as the Bleecker correspondence.

We define the bundle of connections $p : \mathcal{K} \to M$ of the principal bundle $\pi : P \to M$ as the sub-bundle $\mathcal{K} \subset \wedge^1(TP, \mathcal{G})$ formed by the $\mathcal{G}$-valued 1-forms $w : T_uP \to \mathcal{G}$ such that $w(A_u^*) = A$, $\forall A \in \mathcal{G}$, and $(R_g)_* w_{ug} = ad(g^{-1})w_u$, $\forall g \in G$.

Let $\pi_1 : J^1P \to M$ be the 1-jet bundle of local sections of $\pi : P \to M$. The group G acts on J^1P by setting $j_x s \cdot g = j_x^1(R_g \circ s)$, where s is a local section of π defined in a neighborhood of $x \in M$. The quotient J^1P/G exists as a fibred differentiable manifold over M and can be identified with $\mathcal{K}$. In fact, for every $j_x^1 s$ we define $q(j_x^1 s) \in \mathcal{K}$ as follows. Given $X \in T_{s(x)}P$ then $q(j_x^1 s)(X)$ is the only element $A \in \mathcal{G}$ such that $A_{s(x)}^* = X - s_* \pi_* X$ The map $q : J^1P \to \mathcal{K}$ is a surjective submersion whose fibers are the orbits of G in J^1P and one has a commutative diagram

$$\begin{array}{ccc} J^1P & \overset{\pi_{10}}{\to} & P \\ q \downarrow & & \downarrow \pi \\ \mathcal{K} & \underset{p}{\to} & M \end{array}$$

where π_{10} is the canonical morphism $J^1P \to P$.

We define the canonical connection on the principal bundle $q : J^1P \to \mathcal{K}$ as follows. For every $Y \in T_{j_x^1 s}(J^1P)$ we set

$$\theta_{j_x^1 s}(Y) = q(j_x^1 s)((\pi_{10})_* Y) \in \mathcal{G}.$$

It is easy to see that $(R_g)^* \theta = Ad(g^{-1})^* \circ \theta$, and that $\theta(B^*) = B$ for every $B \in \mathcal{G}$, where B^* denotes the fundamental vector field associated with B. Accordingly, θ is the connection form of a connection Γ on J^1P whose curvature 2-form will be denoted by Ω_θ. In this way, as Ω_θ is a tensorial form of the adjoint type (cf. [4]) it defines a 2-form Ω_2 on the base manifold $\mathcal{K}$ with values in the adjoint bundle $ad(J^1P)$.

In this way, the connection θ on the principal G-bundle $q : J^1P \to \mathcal{K}$ induces a covariant derivative $\bigtriangledown$ on the sections of the adjoint bundle $\pi_G : ad(J^1P) \to \mathcal{K}$. Let us denote with $d^\bigtriangledown$ the exterior differential of $ad(J^1P)-$valued forms with respect to this derivation law. Then Ω_2 defines a symplectic structure in the sense that $d^\bigtriangledown \Omega_2 = 0$ and if $X \in T(\mathcal{K})$ verifies $i_X \Omega_2 = 0$ then $X = 0$ (see [3]).

2. Gauge algebra and Momentum map

Now we can pose the Hamilton equation for this symplectic structure, writing

$$i_D\Omega_2 = d^\nabla s$$

where s is a section of the bundle $ad(J^1P)$ and D is a vector field on K. We shall say that a section of the bundle $ad(J^1P)$ is Hamiltonian if it verifies this Hamilton equation.

It is important to describe the elements of the gauge algebra $gau(P)$ in terms of the geometry of the bundle $q : J^1P \to \mathcal{K}$. First of all, every vector field $D \in gau(P)$ defines an element $D_\mathcal{K} \in T(\mathcal{K})$ as follows: if f_t is the flow of D, then we define a flow $(f_t)_\mathcal{K}$ on $\mathcal{K}$ by pulling back connections forms, that is $(f_t)_\mathcal{K} w = \left(f_t^{-1}\right)^* w,\ w \in \mathcal{K}$; the corresponding infinitesimal generator is denoted by $D_\mathcal{K}$ and this fact provides us with a representation $gau(P) \to \mathcal{X}(\mathcal{K})$.

On the other hand, as the adjoint bundle $ad(J^1P)$ can be identified with the vector bundle p^*adP, for every element $D \in gau(P)$ we shall denote by $\overline{s}_D$ the element of $\Gamma(\mathcal{K}, ad(J^1P))$ corresponding to p^*s_D where s_D is the section of the adjoint bundle defined by D.

Since the form θ defined above coincides with the vertical-valued canonical form associated with the 1-jet bundle of any fibred manifold, we know from the theory of jet spaces that given $D \in gau(P)$ there exists a vector field $\overline{D}$ on J^1P such that $\overline{D}$ is projectable onto D and verifies $L_{\overline{D}}\theta = 0$.

Theorem 2.1. *Given $D \in gau(P)$ its $1-$jet extension $\overline{D}$ verifies*

$$\overline{D} = \overline{s}_D + D^h_\mathcal{K}$$

where $\overline{s}_D = p^*s_D \in \Gamma(\mathcal{K}, ad(J^1P))$, $D_\mathcal{K}$ is the image of D by the representation $gau(P) \to \mathcal{X}(\mathcal{K})$ and $D^h_\mathcal{K}$ is the horizontal lift of $D_\mathcal{K}$ with respect to the canonical connection θ. Moreover, D determines a Hamiltonian vector field on $\mathcal{K}$ which verifies

$$i_{D_\mathcal{K}}\Omega_2 = -d^\nabla \overline{s}_D \tag{H}$$

Proof. See [5]. ∎

On the other hand $D \in gau(P)$ defines an element $\overline{\tau}_D \in C(J^1P, \mathcal{G})$ by considering the pullback of automorphic functions on P : $\overline{\tau}_D = (\pi_{10})^* \tau_D$, and we have the correspondence

$$\begin{array}{ccc} C(J^1P, \mathcal{G}) & \longleftrightarrow & \Gamma(\mathcal{K}, ad(J^1P)) \\ \overline{\tau}_D & \leftrightarrow & \overline{s}_D \end{array}$$

by the Bleecker isomorphism.

Definition 2.2. Let us denote with $\mathcal{O}$ the set of orbits of G with respect to the adjoint representation in $\mathcal{G}$. Let us use the notation $(gauP)^* = Map(gauP, \mathcal{O})$

to denote the set of mappings of $gauP$ on $\mathcal{O}$. We shall call momentum map with respect to the $ad(J^1P)$-valued symplectic form Ω_2, the mapping

$$\begin{aligned} \mu \quad &: \quad \mathcal{K} \to (gauP)^* \\ \mu(w)D \quad &= \quad \overline{\tau}_D(j^1_x s) \end{aligned}$$

where w is a connection in $P|_{\pi^{-1}(x)}$, $j^1_x s \in J^1P$ is such that $q(j^1_x s) = w$ and $\overline{\tau}_D = (\pi_{10})^* \tau_D$, being τ_D the element of $C(P, \mathcal{G})$ defined by $D \in gauP$.

In an equivalent way, we can define the momentum map associated with the Hamiltonian character of the gauge algebra (H) by using sections of the adjoint bundle as follows.

We shall denote by $[adJ^1P]$ the set of points of the bundle adJ^1P, that is, if $(j^1s, B) \in J^1P \times \mathcal{G}$ then $[(j^1s, B)]$ will denote its $G-$orbit in $J^1P \times \mathcal{G}$

$$\left[(j^1s, B)\right] = \left\{\left(j^1s \cdot g, ad(g^{-1})B\right),\ g \in G\right\}.$$

We define then

$$\begin{aligned} \mu \quad &: \quad \mathcal{K} \to Map(gauP, \left[adJ^1P\right]) \\ \mu(w)D \quad &= \quad \overline{s}_D(w). \end{aligned}$$

It is verified that $[j^1_x s, \overline{\tau}_D(j^1_x s)] = \overline{s}_D(w)$ which means the equivalence of both definitions.

Proposition 2.3. *The momentum mapping μ is invariant for the natural action of $Gau(P)$ on $\mathcal{K}$, that is*

$$\mu(f_{\mathcal{K}} w) = \mu(w)$$

where $f_{\mathcal{K}}$ is the image of $f \in Gau(P)$ by the group homomorphism $Gau(P) \to Diff(\mathcal{K})$.

Proof. If $w \in \mathcal{K}$ is such that $w = q(j^1_x s)$ for a certain point $x \in M$ and a section s defined in a neighborhood of x, then it is easy to see that $f_{\mathcal{K}} w = q(j^1_x (f \circ s))$, consequently, we have

$$\begin{aligned} \mu(f_{\mathcal{K}} w)(D) \quad &= \quad \overline{\tau}_D(j^1_x (f \circ s)) = \tau_D(f(s(x)) \\ &= \quad \tau_D(s(x)) = \overline{\tau}_D(j^1_x s) = \mu(w)(D). \end{aligned}$$ ∎

Proposition 2.4. *Given two Hamiltonian sections $\overline{s}_{D_1}$, $\overline{s}_{D_2}$ of $ad(J^1P)$ associated respectively with the gauge vector fields $D_1, D_2 \in gau(P)$, we define the bracket Poisson as the element of $\Gamma(K, ad(J^1P))$ given by*

$$\{\overline{s}_{D_1}, \overline{s}_{D_2}\} = \Omega_2(D_{1K}, D_{1K}).$$

Then the automorphic function corresponding to $\{\overline{s}_{D_1}, \overline{s}_{D_2}\}$ is given by the Lie bracket of automorphic functions

$$-2[\overline{\tau}_{D_1}, \overline{\tau}_{D_2}] : J^1P \to \mathcal{G}$$

.

Proof. We shall prove $\Omega_\theta(\overline{D}_1, \overline{D}_2) = -2[\overline{\tau}_{D_1}, \overline{\tau}_{D_2}]$. Now, since

$$i_{\overline{D}_1} d\theta = (L_{\overline{D}_1} - d i_{\overline{D}_1})\theta = -d\overline{\tau}_{D_1},$$

we have

$$\begin{aligned}\Omega_\theta(\overline{D}_1, \overline{D}_2) &= i_{\overline{D}_2} i_{\overline{D}_1}(d\theta + \frac{1}{2}[\theta, \theta]) = i_{\overline{D}_2}(-d\overline{\tau}_{D_1} - [\theta, \overline{\tau}_{D_1}]) \\ &= -(\overline{D}_2 \overline{\tau}_{D_1} + [\overline{\tau}_{D_2}, \overline{\tau}_{D_1}]) = -(L_{\overline{D}_2}\{\theta(D_1)\} + [\overline{\tau}_{D_2}, \overline{\tau}_{D_1}]) \\ &= -((L_{\overline{D}_2}\theta)(D_1) + \theta(L_{\overline{D}_2}\overline{D}_1) + [\overline{\tau}_{D_2}, \overline{\tau}_{D_1}]) \\ &= -2[\overline{\tau}_{D_1}, \overline{\tau}_{D_2}].\end{aligned}$$

■

Proposition 2.5. *Given $w \in \mathcal{K}$, the kernel $\mu_{*,w}$ is the symplectic orthogonal to the image of the representation $gau(P) \longrightarrow \mathcal{X}(\mathcal{K})$.*

Proof. Given $X \in T_w(\mathcal{K})$ and calling γ_t a flow of X, we can make act $\mu_*(X)$ on $C(P, \mathcal{G})$ as follows

$$\mu_{*,w}(X)s_D = \left.\frac{d}{dt}\right|_{t=0} \mu(\gamma_t)s_D = Xs_D = (d^\nabla s_D)(X) = \Omega_2(D_\mathcal{K}, X)$$

whereby the statement of the proposition. ■

References

[1] Bleecker, D., *Gauge Theory and Variational Principles*, Addison-Wesley Publishing Company, Boston, 1981.

[2] Ferreiro, R., *Equivariant characteristic forms on the bundle of connections*, J. Geom. and Physics **54** (2005), 197–212.

[3] Garcia, P. L., *Gauge algebras, curvature and symplectic structures*, J. Diff. Geom. **12** (1977), 209–227.

[4] Kobayashi, S., and N. Nomizu, *Foundations of Differential Geometry*, Volume I, John Wiley and Sons, New York, 1963.

[5] Perez, J., *The Kostant-Garca quantization of the bundle of connections*, Math. Z. **262** (2009), 17–26.

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