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Six-Dimensional Lie Algebras with a Five-Dimensional Nilradical

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Abstract. This paper attempts to correct and simplify an old paper by G. M. Mubarakzyanov which classifies the six-dimensional indecomposable solvable Lie algebras for which the nilradical is five-dimensional. Mubarakzyanov discerned nine classes of such algebras depending on the isomorphism class of the nilradical. Each of these nine classes is re-examined. Many errors are corrected and a few algebras are shown to be redundant. In many cases some parameters can be removed completely. In addition the range of values of the parameters that occur in many classes are reduced, an issue which Mubarakzyanov did not treat very systematically.
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1. Introduction

An old paper by Mubarakzyanov [Mub2] is concerned with classifying indecomposable six-dimensional solvable Lie algebras for which the nilradical is five-dimensional. The paper is frequently referred to but suffers from a number of serious defects:

- In some places it is almost illegible
- It contains many typographical and substantive errors
- The article is bereft of details
- Where details are given algebras listed do not always agree with the corresponding algebras derived in the body of the paper
- Several arguments depend on lemmas that are hard to find even now in the literature.
- The notation is not very systematic; for example in the parameters that are used to describe classes of Lie algebras, greek and roman letters are used interchangeably without any perceptible rationale.

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- Many of the algebras that appear in Mubarakzyanov's lists depend on parameters: (see next paragraph for our definition of *parameter*). Mubarakzyanov is inconsistent with respect to the way that he normalizes these parameters.

It has to be understood at the outset that classifying solvable Lie algebras is a different exercise from studying the semi-simple algebras, for example. It is probably impossible to classify solvable Lie algebras in general in arbitrary dimension. The algebras studied by Mubarakzyanov can depend on at most four parameters; some of the algebras involve constants that can assume discrete values, typically, $\delta = 0, 1$ or $\epsilon = 0, \pm 1$. In this case we do not consider δ and ϵ to be parameters because the corresponding algebras do not belong to a continuous family. There are many of Mubarakzyanov's algebras where one or more parameters can be eliminated.

Of the defects listed above we consider the third to be the most important. In many cases Mubarakzyanov provides in effect merely a summary for a particular class of algebras: it is impossible to reconstruct Mubarakzyanov's algebras from the outline that he gives. Accordingly, in order to engender some confidence in Mubarakzyanov's classification, it is necessary to redo the whole project again so as to provide a robust classification process that can be readily verified by other workers in the field. In this paper we go through each algebra case by case and explain how it is obtained. Even then it is not possible because of the confines of space to supply every single detail. For further details we refer to [Shab]. It is important to bear in mind that in the process of simplifying classes of algebras different normalizations are possible. At the end of this article we will provide a summary that significantly improves Mubarakzyanov's classification.

In the Sections below we shall have occasion to make various "Remarks": the purpose of these Remarks is to indicate that either there is an error in one of Mubarakzyanov's algebras or that a simplification in one of the algebras can be made by making a suitable change of basis, even to the extent of an algebra disappearing completely. Some Remarks pertain to work of other authors that are based also on [Mub2].

After some general considerations in Sections $2, 3$ and 4 we begin in Section 5 to consider classes of six-dimensional solvable Lie algebras for which the nilradical is five-dimensional according to the type of the nilradical. There are nine classes of such nilradical: see Section 4 below for details. Each of the succeeding sections is devoted to a particular kind of nilradical and the final section gives an amended list of Mubarakzyanov's algebras. In [Mub2] after a brief unnumbered introduction, Mubarakzyanov devotes a section to each of these nine cases. Despite the many defects we consider that in general Mubarakzyanov did a commendable job in all but classes four, six and seven, especially when we consider that he must have done all the calculations by hand. In truth it is almost impossible to carry out the classification without the use of symbolics and we make liberal use of the symbolic manipulation package Maple.

At the end of the paper we supply an amended list of Mubarakzyanov's algebras which makes it clear which classes of algebras are redundant, which are missing and which ones can be simplified. We mentioned above that all classes of algebras depend on at most four parameters; in fact precisely three have four parameters and apart from the first group of algebras having abelian nilradical, precisely four depend on three parameters. Thus "most" algebras depend on two, one or no parameters. Where a class of algebras does depend on parameters we have restricted their values as much as possible so as to make each algebra unique within its class. We shall now give a brief non-technical summary of exactly how our investigation improves upon [Mub2] based on each of the nine cases of the nilradical. We should mention that similar results have been obtained at least by implication in cases $1, 3, 5, 7, 8, 9$ by other authors [Ndogmo, Wang, Snob1, Rub, Snob2, Snob3], respectively and detailed comparisons are made below in the main body of the paper:

- The first case comprising 12 (classes of) algebras is substantially correct.
- In the second case one algebra is missing which belongs to an extended family of ${g_{6,17}}^{***}$ in the table for $(\delta, \epsilon, a) = (0, 1, 1)$.

  Also the description of several classes of algebras can be improved by either removing parameters or non-zero entry like in $g_{6,38}$ [Mub2] or restricting their range of values.
- The third case comprising 14 algebras is substantially correct.
- Case four comprises 19 classes of algebras; however, two are redundant and one belongs to an extended family that does not appear in [Mub2]. In addition several classes of algebras can be simplified by removing parameters.
- In the fifth case there are five classes and two of them depend on one parameter. In comparison with [Mub2] the single algebra $g_{6,73}$ properly belongs to a one-parameter family.
- For the sixth class comprising six algebras, two in [Mub2] are easily shown to be redundant but two more quite different classes of algebras are missing and of the six correct algebras, two depend on a single parameter.
- In case seven three of the 12 classes of algebras listed in [Mub2] are redundant; however, two other classes of algebras are missing.
- In the eighth case one of the classes of five algebras that appears in [Mub2] is but a single (or more precisely pair) of algebras and does not properly belong to a one-parameter family; thus only one of the five classes depends on a single parameter.
- The ninth case consisting of a single algebra is correct.

Our main purpose in supplying an amended list of Mubarakzyanov's algebras is to try achieve consensus on what the correct list should be. We welcome feedback from other researchers as to the veracity or otherwise of the updated list. However, we do not think that the bases chosen by Mubarakzyanov and necessarily by ourselves are optimal; for example, in the present work it would have been more convenient to have had e_1 as the basis vector not in the nilradical of the algebra rather than e_6. The reason is that in many cases it is easy to visualize an extended algebra as obtained by adding a single vector to the nilradical; with Mubarakzyanov's convention the numbering of the basis vectors in the extended algebra has to be completely changed. When a final list for the Mubarakzyanov's algebras is agreed upon, we hope to publish in some venue a further revised list in which the bases have been carefully thought through and also provide for each algebra an associated matrix Lie group.

We use the summation convention on repeated indices. We should also point out that the first author began her dissertation work on this topic in 2009. We thank the referee for a very helpful and thorough report and bringing to our attention several pertinent references and insisting that there was no ambiguity in algebra $g_{6,93}$ [Mub2] in Section 11: we had originally ± 1 in one term but the minus sign can actually be eliminated.

2. General Approach

The first step in classifying solvable Lie algebras in a specific dimension is to determine the possible nilradicals. A general theorem asserts that if $\mathfrak{g}$ is an *indecomposable* solvable Lie algebra the dimension of its nilradical $\mathfrak{nil}(\mathfrak{g})$ is at least $\frac{n}{2}+1$ [Mub3].

Once a specific nilradical has been chosen and one wants to construct a solvable Lie algebra of a certain dimension one satisifies the Jacobi identity in the most general possible way. After that we carry out the technique of "absorption". In our case we always use a basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ for which a basis for $\mathfrak{nil}(\mathfrak{g})$ is $\{e_1, e_2, e_3, e_4, e_5\}$. We make a change of basis of the form $e'_i = e_i \, (1 \leq i \leq 5), e'_6 = e_6 + \sum_i^5 a^i e_i$. We obtain $\overline{ad}(e'_6) = \overline{ad}(e_6) + \overline{ad}(\sum_i^5 a^i e_i)$ where $\overline{ad}$ denotes the restriction of the adjoint representation to $\mathfrak{nil}(\mathfrak{g})$. As such it may be possible to simplify $ad(e'_6)$ by using e_i's which do not belong to the center of $\mathfrak{g}$ and then solving some *linear* equations for the unknown a^i.

3. Suspension

There is a very simple way to extend by one dimension, a codimension one nilradical algebra $\mathfrak{g}$. We let a basis for $\mathfrak{g}$ be $\{e_1, e_2, ..., e_n\}$ where e_n is not in the nilradical. We let i, j, k range from 1 to $n-1$ and write this algebra in the basis as $[e_i, e_n] = C_i^j e_j$ and $[e_i, e_j] = C_{ij}^k e_k$, the latter terms representing the nilradical of $\mathfrak{g}$. We now introduce a new vector by taking a vector space direct sum $\mathfrak{h}$ of $\mathfrak{g}$ with $\mathbb{R}$ and let $\{e_0\}$ be a basis for $\mathbb{R}$. The brackets on $\mathfrak{h}$ are the same as for $\mathfrak{g}$ except that we add the bracket $[e_0, e_n] = ae_0$ where $a \in \mathbb{R}$ is non-zero. It is easy to check that $\mathfrak{h}$ is also a codimension one nilradical Lie algebra that we call a *suspension* of $\mathfrak{g}$. We note that a basis for the nilradical of $\mathfrak{h}$ consists of the $\{e_i\}$ together with $\{e_0\}$ and in fact that $nil(\mathfrak{h})$ is isomorphic to $nil(\mathfrak{g}) \oplus \mathbb{R}$. The suspension construction can be defined slightly more generally; for example in extending a nilpotent algebra to a codimension one nilradical algebra provided that a certain bracket condition is satisfied.

If we consider the problem of trying to classify (solvable) codimension one nilradical Lie algebras by extending one dimension at time it is apparent that at each stage every Lie algebra of dimension n gives rise to an algebra of dimension $n+1$ by using the suspension construction. In refining Mubarakzyanov's list of algebras, it is instructive to see which of the six-dimensional algebras are suspensions of five-dimensional algebras; such algebras are highlighted in the tables appearing at the end of the paper. Another advantage of suspensions, or more accurately algebras that can be seen to be suspensions after change of basis, is that it makes finding the range of values allowed for the parameters, so as to obtain mutually non-isomorphic algebras within a family, very simple. One simply uses the range of values on the quotient augmented by the new parameter a as described above, subject only to the restriction $a \neq 0$.

4. The nilradical

The nilradical $\mathfrak{nil}(\mathfrak{g})$ of an *indecomposable* six-dimensional solvable Lie algebra $\mathfrak{g}$ for which $\mathfrak{nil}(\mathfrak{g})$ is five-dimensional can be one of nine types. In his paper Mubarakzyanov devotes a separate paragraph corresponding to each of these nine possible nilradicals.

- algebras 1 to 12 have abelian nilradical $\mathbb{R}^5$ (section 1)
- algebras 13 to 38 have nilradical isomorphic to $\mathbb{R}^2 \oplus H$ (section 2)
- algebras 39 to 53 have nilradical isomorphic to $\mathbb{R} \oplus A_{4,1}$ (section 3)
- algebras 54 to 70 have nilradical isomorphic to $A_{5,1}$ (section 4)
- algebras 71 to 75 have nilradical isomorphic to $A_{5,2}$ (section 5);
- algebras 76 to 81 have nilradical isomorphic to $A_{5,3}$ (section 6);
- algebras 82 to 93 have nilradical isomorphic to $A_{5,4}$ (section 7)
- algebras 94 to 98 have nilradical isomorphic to $A_{5,5}$ (section 8);
- algebra 99 has nilradical isomorphic to $A_{5,6}$ (section 9).

We preferred to consider the nilradicals in different basis in sections three, six and eight to obtain an upper triangular adjoint representation, but the algebras in the tables at the end of the paper are given with the nilradical in the same basis as Mubarakzyanov's algebras.

Here $\mathbb{R}^n$ denotes the n-dimensional abelian Lie algebra, H denotes the three-dimensional Heisenberg algebra, $A_{4,1}$ and the algebras $A_{5,p}$ where $1 \leq p \leq 6$ are taken from [PSWZ] and denote nilpotent algebras, 4 and 5, respectively, being the dimensions of the algebra.

5. Abelian nilradical

In the case of an abelian codimension one nilradical it is easy to see that under change of basis that leaves $\mathfrak{nil}(\mathfrak{g})$ invariant, $ad(e_6)$ is acted on by conjugation apart from a single overall scaling. As such to reduce to canonical form, all we have to do is to put $ad(e_6)$ into Jordan normal form; however, we can further reduce $ad(e_6)$ by scaling. We find that Mubarakzyanov's analysis is substantially correct in Paragraph 1.

Remark 5.1. Solvable Lie algebras with abelian nilradicals not necessarily of codimension one were considered by J. C. Ndogmo and P. Winternitz in [Ndogmo].

6. Nilradical isomorphic to $\mathbb{R}^2 \oplus H$

6.1. Satisfying the Jacobi identity. In paragraph 2 of [Mub2] the nilradical is isomorphic to $H \oplus \mathbb{R}^2$. The only non-zero bracket is $[e_2, e_3] = e_1$. The non-zero brackets in the full algebra are $[e_2, e_3] = e_1, [e_i, e_6] = a_i^k e_k$ where $1 \leq i, k \leq 5$. Satisfying the Jacobi identity in the most general way and by the technique of absorption we can assume also that $a_2^1 = 0$ and $a_3^1 = 0$ (consider the transformation $e_6' = e_6 + a_3^1 e_2 - a_2^1 e_3$ keeping e_1, e_2, e_3, e_4, e_5 fixed), then we obtain the matrix A obtained from $-ad(e_6)$ by deleting the last column and row

$$A = \begin{bmatrix} a_2^2 + a_3^3 & 0 & 0 & a_4^1 & a_5^1 \\ 0 & a_2^2 & a_3^2 & 0 & 0 \\ 0 & a_2^3 & a_3^3 & 0 & 0 \\ 0 & a_2^4 & a_3^4 & a_4^4 & a_5^4 \\ 0 & a_2^5 & a_3^5 & a_4^5 & a_5^5 \end{bmatrix}.$$

The algebra has been reduced to

$$\begin{aligned} &[e_2, e_3] = e_1, [e_1, e_6] = (a_2^2 + a_3^3)e_1, [e_2, e_6] = a_2^2 e_2 + a_2^3 e_3 + a_2^4 e_4 + a_2^5 e_5, \\ &[e_3, e_6] = a_3^2 e_2 + a_3^3 e_3 + a_3^4 e_4 + a_3^5 e_5, [e_4, e_6] = a_4^1 e_1 + a_4^4 e_4 + a_4^5 e_5, \\ &[e_5, e_6] = a_5^1 e_1 + a_5^4 e_4 + a_5^5 e_5. \end{aligned} \tag{1}$$

Proposition 6.1. *The Lie algebra is invariant under a change of basis of the form*

$$\begin{bmatrix} p & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & R & Q & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where P, Q, R are 2×2 matrices and $\det(P) = p$.

The following Lemmas help to answer the question if it possible to eliminate the entries a_4^1 and a_5^1 in the six-dimensional algebra (1).

Lemma 6.2. *In the Lie algebra* (1) *we can remove by change of basis both the coefficients a_4^1 and a_5^1 provided that $(a_2^2 + a_3^3 - a_4^4)(a_2^2 + a_3^3 - a_5^5) - a_4^5 a_5^4 \neq 0$ or $(a_2^2 + a_3^3 - a_4^4)(a_2^2 + a_3^3 - a_5^5) - a_4^5 a_5^4 = 0$, $(a_2^2 + a_3^3 - a_4^4)a_5^1 + a_4^5 a_4^1 = 0$ and $(a_2^2 + a_3^3 - a_4^4)a_4^1 + a_5^4 a_5^1 = 0$.*

Proof. If we consider $P = \begin{bmatrix} 1&0&0&x&y&0\\ 0&1&0&0&0&-a_3^4x-a_3^5y\\ 0&0&1&0&0&a_2^4x+a_2^5y\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}$, which does not affect the nilradical, then the matrix A is transformed into

$$A' = \begin{bmatrix} a_2^2+a_3^3 & 0 & 0 & (a_2^2+a_3^3-a_4^4)x+a_4^1-a_4^5y & (a_2^2+a_3^3-a_5^5)y+a_5^1-a_5^4x & 0\\ 0 & a_2^2 & a_3^2 & 0 & 0 & 0\\ 0 & a_2^3 & a_3^3 & 0 & 0 & 0\\ 0 & a_2^4 & a_3^4 & a_4^4 & a_5^4 & 0\\ 0 & a_2^5 & a_3^5 & a_4^5 & a_5^5 & 0\\ 0&0&0&0&0&0 \end{bmatrix}.$$

This transformation only affects the $(1,4)$ and $(1,5)$ entries. The system

$$\begin{cases} (a_2^2+a_3^3-a_4^4)x-a_4^5y+a_4^1=0,\\ (a_2^2+a_3^3-a_5^5)y-a_5^4x+a_5^1=0, \end{cases}$$

has a solution provided either $(a_2^2+a_3^3-a_4^4)(a_2^2+a_3^3-a_5^5)-a_4^5a_5^4 \neq 0$ or $(a_2^2+a_3^3-a_4^4)(a_2^2+a_3^3-a_5^5)-a_4^5a_5^4=0$, $(a_2^2+a_3^3-a_4^4)a_5^1+a_5^4a_4^1=0$ and $(a_2^2+a_3^3-a_5^5)a_4^1+a_4^5a_5^1=0$. ∎

Lemma 6.3. *In the Lie algebra* (1) *we can always remove by change of basis at least one of the coefficients* a_4^1 *and* a_5^1 *and scale the remaining one to unity if it is non-zero.*

Proof. If the change of basis matrix is $P = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&x&0\\ 0&0&0&y&1&0\\ 0&0&0&0&0&1 \end{bmatrix}$, which does not affect the nilradical, then the matrix A above is transformed to

$$A' = \begin{bmatrix} a_2^2+a_3^3 & 0 & 0 & a_5^1y+a_4^1 & a_4^1x+a_5^1\\ 0 & a_2^2 & a_3^2 & 0 & 0\\ 0 & a_2^3 & a_3^3 & 0 & 0\\ 0 & \frac{a_2^4-a_2^5x}{1-xy} & \frac{a_3^4-a_3^5x}{1-xy} & \frac{a_4^4-a_4^5x+a_5^4y-a_5^5xy}{1-xy} & \frac{a_5^4+(a_4^4-a_5^5)x-a_4^5x^2}{1-xy}\\ 0 & \frac{a_2^5-a_2^4y}{1-xy} & \frac{a_3^5-a_3^4y}{1-xy} & \frac{a_4^5+(a_5^5-a_4^4)y-a_5^4y^2}{1-xy} & \frac{a_5^5-a_5^4y+a_4^5x-a_4^4xy}{1-xy} \end{bmatrix}.$$

The transformation does not affect the 2×2 upper diagonal block, 2×2 zero block above the diagonal and the $(1,1)$ - entry. Moreover, if the 2×2 block whose entries are $a_2^4, a_3^4, a_2^5, a_3^5$ is zero, it will remain zero after the transformation.

If both a_4^1 and a_5^1 are zero there is nothing to prove. If at least one of a_4^1 and a_5^1 is non-zero we can remove one of them by solving a linear inhomogeneous equation. Suppose for example that $a_4^1=0$ and $a_5^1 \neq 0$. Then replacing e_1 and e_2 by $a_5^1e_1$ and $a_5^1e_2$ has the effect of reducing the entry a_5^1 to unity. ∎

In practice when simplifying a six dimensional Lie algebra we first of all use Lemma 6.2. For certain exceptional values it will not be possible to eliminate the entries $i := a_4^1$ or $j := a_5^1$. In that case we appeal to Lemma 6.3 so as to eliminate either i or j and scale the non-zero entry to unity. We shall refer to the case where $i=j=0$ as the "split" case and the opposite case as the "non-split" case.

6.2. Finding six dimensional Lie algebras.
Following Mubarakzyanov let us define $B = \begin{bmatrix} a_2^2 & a_3^2 \\ a_2^3 & a_3^3 \end{bmatrix}, D = \begin{bmatrix} a_2^4 & a_3^4 \\ a_2^5 & a_3^5 \end{bmatrix}, C = \begin{bmatrix} a_4^4 & a_5^4 \\ a_4^5 & a_5^5 \end{bmatrix}$ so that the matrix $\widetilde{A}$ has the block form $\widetilde{A} = \left[\begin{smallmatrix} B & 0 \\ D & C \end{smallmatrix}\right]$. Under a change of basis of the type considered in Proposition 6.1 if initially we put $R = 0$ then it is clear that each of B and C will be conjugated by P and Q, respectively. There are three real Jordan normal forms for a 2×2 matrix and thus nine forms altogether for the matrix $\widetilde{A}$. The corresponding pairs (B, C) are given as follows where $bf \neq 0$:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & h \end{bmatrix}; \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} e & 0 \\ 1 & e \end{bmatrix}; \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} e & f \\ -f & e \end{bmatrix}; \begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & h \end{bmatrix}; \begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix} \begin{bmatrix} e & 0 \\ 1 & e \end{bmatrix};$$
$$\begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix} \begin{bmatrix} e & f \\ -f & e \end{bmatrix}; \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & h \end{bmatrix}; \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} e & 0 \\ 1 & e \end{bmatrix}; \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} e & f \\ -f & e \end{bmatrix}.$$

For each of these nine forms we can put $P = Q = I$, the identity, and use R to simplify D. For fixed B and C the matrix D changes according to $\overline{D} = D + CR - RB$; if we put $\overline{D} = 0$ we obtain a system of inhomogeneous linear equations. If we put $B = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], C = \left[\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right], R = \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]$ then the matrix of coefficients, where the unknowns are x, y, z, w, is given by

$$\begin{bmatrix} e-a & -c & f & 0 \\ -b & e-d & 0 & f \\ g & 0 & h-a & -c \\ 0 & g & -b & h-d \end{bmatrix}.$$

If we consider the nine cases of the pairs B, C above we find that we can reduce D to zero except in the following cases without loss of generality: $a = e; a = e$; never; $a = e; a = e$; never; never; never ; $a = e, b = f$. We consider next each of these nine cases in turn, the first case itself comprising nine subcases.

1.
   - If $a \neq e, d \neq e, a \neq h, d \neq h$ then D can be reduced to zero. If $D = 0$ then not all of a, d, e, h can be zero; as such we may reduce one of them to unity. According to Lemmas 6.2 and 6.3 we obtain in the split cases $\mathfrak{g}_{6,13}$ [Mub2]. If $e = a + d$ and $h = a + d$ we can remove neither i nor j if they are non-zero by using Lemma 6.2. But then replacing e_4 by $e_4 - \frac{i}{j} e_5$ reduces i to zero. Finally if $j \neq 0$ replacing e_5 by $\frac{1}{j} e_5$ reduces j to unity. Likewise if $e \neq a + d$ and $h = a + d$ we can again reduce to $i = 0$ and $j = 1$.

     For algebra $\mathfrak{g}_{6,13}$ neither e nor h can be zero or else the algebra will be decomposable. By permuting e_4 and e_5 if necessary we may assume that $|h| \leq |e|$ so by scaling e_6 we may assume that $0 < |h| \leq 1$. Furthermore not both a and d can be zero but we may assume by permuting e_2 and e_3 and changing the sign of e_1 if necessary, that $|a| \leq |d|$.

     In the case of algebra $\mathfrak{g}_{6,14}$ we can merely assume that $|a| \leq |d|$.
   - $a = e, d \neq a, a \neq h, d \neq h$ then D can be reduced to $\left[\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right]$. If $x = 0$ we are in the previous case and if $x \neq 0$ we can scale e_4 by x which has the effect of reducing x to unity. Suppose that $a = 0$. Then $h \neq 0$ because

in this subcase $a \neq h$. By scaling e_6 we may assume that $h = 1$. Now by using Lemma 6.2 we can reduce i to zero and j to zero because d cannot be unity or zero. We obtain thereby a subcase of $g_{6,17}$ in [Mub2] where in his notation $\alpha \neq 0$ and $\epsilon = 0$.

If $a \neq 0$ by scaling e_6 we may assume that $a = 1$. We can remove both i and j provided that $d \neq 0$ and $d+1 \neq h$ giving a subcase of the split case where $g_{6,18}$ in [Mub2]. If $d = 0$ we can remove j because in this subcase $h \neq 1$. If $i = 0$ we obtain a subcase of $g_{6,18}$; otherwise we may replace e_2, e_3, e_4 by $\frac{1}{i}e_2, ie_3, \frac{1}{i}e_4$ which reduces i to unity and produces $g_{6,20}$.

Suppose now that $a = 1$ and $h = d+1$. Then $d \neq 0$ since $a \neq h$ and therefore also $i = 0$. In this case we obtain either $g_{6,18}$ again or $g_{6,19}$ depending whether j is zero or not. In the latter case scaling e_5 by $\frac{1}{j}$ reduces j to unity.

- $d = a, e \neq a, h \neq a$ then D can be reduced to zero. This case merely produces special cases of algebras $g_{6,13}$ and $g_{6,14}$.
- $h = e, a \neq e, d \neq e$ then D can be reduced to zero. Again we obtain merely special cases of algebras $g_{6,13}$ and $g_{6,14}$.
- $a = e, d = h, a \neq d$ then D can be reduced to $\left[\begin{smallmatrix} x & 0 \\ 0 & w \end{smallmatrix}\right]$. Suppose that $w = 0$ and that $x \neq 0$. Then we can reduce x to unity by scaling e_2 by $\frac{1}{x}$ and e_3 by x. Now the entries a, d cannot both be zero or else $\mathfrak{g}$ will be nilpotent. If both are non-zero using Lemma 6.2 we may assume that $i = 0$ and $j = 0$; in that case we may normalize a to unity and we obtain a special case of algebra $g_{6,18}$ in [Mub2] for which $\beta = \alpha$.

On the other hand if $a = 0$ we can reduce i to zero using Lemma 6.2 and we may normalize j to unity if non-zero and d to unity. Then we permute e_2 and e_3 and change the sign of e_1 and e_5. We obtain the algebra

$$[e_2, e_3]{=}e_1, [e_1, e_6]{=}e_1, [e_2, e_6]{=}e_2, [e_3, e_6]{=}e_4, [e_5, e_6]{=}\delta e_1{+}e_5,\ \delta = 0, 1.$$

If $\delta = 0$ we obtain the special case of we algebra $g_{6,17}$ in [Mub2] for which $\alpha = 1$ and $\delta = 0$. If $\delta = 1$ we obtain an algebra not given in [Mub2], which belongs to an extended family of $g_{6,17}{}^{***}$ in the table for $(\delta, \epsilon, a) = (0, 1, 1)$.

The case where $x = 0$ and $w \neq 0$ is equivalent to the cases just considered after permuting e_2 and $e_3$and changing the sign of e_1 and simultaneously permuting e_4 and e_5.

Now assume that x and w are both non-zero and scale e_4 and e_5 by x and w, respectively, so as to reduce x and w to unity. Now the entries a, d cannot both be zero or else $\mathfrak{g}$ will be nilpotent. If both are non-zero using Lemma 6.2 we may assume that $i=0$ and $j=0$; then by scaling e_6 we may assume that $a=1$ and $|d|\leq 1$ and we obtain algebra $g_{6,15}$ in [Mub2]. The other possibility is that one of a or d is zero: it does not matter which one because we can permute e_2 and e_3 and e_4 and e_5, respectively, and change the sign of e_1 which has the effect of interchanging a and d. Thus we assume that $a=0$. The split case belongs to $g_{6,15}$ with $h=0$ and the non-split case gives $g_{6,16}$ in [Mub2].

- $d=a, h=e, a\neq e$ then D can be reduced to zero. Again we obtain merely special cases of algebras $g_{6,13}$ and $g_{6,14}$.
- $a=d=e\neq h$ then D can be reduced to $\left[\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right]$. In this case we make a transformation of the form

$$e_2' = \cos\theta e_2 - \sin\theta e_3, e_3' = -\sin\theta e_2 + \cos\theta e_3$$

and by choosing θ we can reduce y to zero if non-zero and hence to the previous case. Then we can replace e_4 and e_5 by xe_4 and xe_5 if $x\neq 0$ which has the effect of reducing x to unity. If $a\neq 0$ and $h\neq 2a$ we may assume that $i=0$ and $j=0$. Since $h\neq 0$ otherwise the algebra would be decomposable we may scale e_6 so as to reduce h to unity. We obtain the algebra

$$\begin{aligned}&[e_2,e_3]=e_1,[e_1,e_6]=2ae_1,[e_2,e_6]=ae_2,[e_3,e_6]=ae_3+\delta e_4,\\&[e_4,e_6]=ae_4,[e_5,e_6]=e_5,\delta=0,1.\end{aligned}$$

If $\delta=0$ then this algebra is $g_{6,13}$ with $\alpha=\beta=h$ in [Mub2]. If $\delta=1$ then we have $g_{6,18}$ with $\alpha=\beta=1$ in [Mub2].

If $a=0$ we can reduce j to zero and if also $i=0$ we obtain $g_{6,17}$ with $\alpha=0$ and $\epsilon=0$ in [Mub2]. If $i\neq 0$ then we make a change of basis

$$(e_1',e_2',e_3',e_4',e_5',e_6') = (-\frac{i}{h}e_1, he_3, \frac{i}{h^2}e_2, -e_4, e_5, \frac{1}{h}e_6).$$

This transformation will scale i to unity and will affect x if different from zero which can be scaled to unity applying the transformation given before. We obtain the algebra

$$[e_2,e_3]=e_1,[e_3,e_6]=\delta e_4,[e_4,e_6]=e_1,[e_5,e_6]=e_5,\delta=0,1.$$

If $\delta=1$ then it is $g_{6,17}$ with $\alpha=0,\epsilon=1$ in [Mub2]. If $\delta=0$ then we have $g_{6,14}$ with $\alpha=\beta=0$ [Mub2].

If $h=2a$ then a cannot be zero or else the algebra will be nilpotent. As such we can scale e_6 so as to reduce a to unity. Using Lemma 6.2 we can reduce i to zero. If $j=0$ we obtain algebra $g_{6,18}$ with $\alpha=1$ and $\beta=2$. We can scale j to unity and we obtain after permuting e_2 and e_3 and changing the sign of e_1 and e_5 algebra $g_{6,19}$ with $\alpha=1$.

- $a = e = h \neq d$ then D can be reduced to $\left[\begin{smallmatrix} x & 0 \\ z & 0 \end{smallmatrix}\right]$. In this case we make a transformation of the form

$$e_4' = \cos\theta e_4 - \sin\theta e_5, e_5' = -\sin\theta e_4 + \cos\theta e_5$$

and by choosing θ we can reduce either x or z to zero if non-zero. Suppose we decided to reduce z to zero. If $x = 0$ we will obtain either algebra $g_{6,13}$ or $g_{6,14}$ in [Mub2]. Suppose then that $x \neq 0$. Then scaling e_4 and e_5 reduces x to unity. If $d \neq 0$ then using Lemma 6.2 we can reduce i and j to zero. Furthermore $a \neq 0$ otherwise the algebra would be decomposable and we may scale e_6 so as to reduce a to unity which gives $g_{6,18}$ in [Mub2] with $\beta = 1$.

Finally if $d = 0$ and not both i and j are zero we may assume by permuting e_4 and e_5 that j is non-zero. If j is non-zero right away we choose to reduce x to zero if non-zero. If we had to permute e_4 and e_5 then we reduce to zero z and rename i and j after that. Then defining $e_4' = e_4 - \frac{i}{j}e_5$ we may reduce i to zero. Thereafter we reduce j to unity and we have $g_{6,19}$ with $\alpha = 0$ in [Mub2].

- $a = d = e = h$ then D cannot be reduced. However, in this case we have $B = aI, C = aI$. Since $a \neq 0$ so that $\mathfrak{g}$ is not decomposable we can scale e_6 so reduce to the case $a = 1$. We make a change of basis of the form $\left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right]$ where P and Q are 2×2 blocks. The only effect on $\mathfrak{g}$ is is to transform D into $Q^{-1}DP$. As such D can be reduced to $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Moreover we can clearly reduce a to unity and i and j to zero. For the first matrix we get a special case of $g_{6,13}$ in [Mub2]. For the second matrix we get $g_{6,18}$ in [Mub2] with $\alpha = \beta = 1$. Finally third matrix we get $g_{6,15}$ in [Mub2] with $h = 1$.

2. If $a \neq e$ and $d \neq e$ then D can be reduced to zero. Now not both a and d can be zero or else the algebra will be a direct sum of two three-dimensional algebras. By permuting e_2 and e_3 and changing the sign of e_1 if necessary and by scaling e_6 we may assume that $a = 1$ and $|d| \leq 1$. Using Lemma 6.2 we can always reduce i to zero. If $j = 0$ then we obtain algebra $g_{6,25}$ in [Mub2]. If $e = d + 1$ and $j \neq 0$ scaling e_4 and e_5 by $\frac{1}{j}$ reduces j to unity. Notice that $e = d + 1$ implies $d \neq 0$. In this case we obtain algebra $g_{6,26}$ with $h \neq 0$ [Mub2].

If $a = e$ and $d \neq a$ then D can be reduced to $\left[\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right]$. If $x = 0$ we have the limiting cases of the algebras found in the previous case, but otherwise we scale e_4 and e_5 by x which has the effect of reducing x to unity. Using Lemma 6.2 we can assume that $i = 0$. If $d \neq 0$ then according to Lemma 6.2 we can remove j and then scale d to unity. Then if we permute e_2 and e_3 and change the sign of e_1 we obtain a special case of algebra $g_{6,27}$ in [Mub2].

If $d = 0$ then a cannot be zero and we can reduce a to unity. If $j = 0$ we have a special case of $\mathfrak{g}_{6,27}$ and if $j \neq 0$ then mapping (e_1, e_3) to (je_1, je_3) reduces j to unity and does not change other entries. Then mapping $(e_2,\, e_3)$ to $(-e_3, e_2)$ we obtain a special case of algebra $\mathfrak{g}_{6,27}$ in [Mub2].

If $a \neq e$ and $d = e$ then D can be reduced to $\left[\begin{smallmatrix} 0 & w \\ 0 & 0 \end{smallmatrix}\right]$. However, by permuting e_2 and e_3 and changing the sign of e_1 we may reduce to the previous case.

If $a = e$ and $d = e$ then D can be reduced to $\left[\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right]$. Using Lemma 6.2 we can assume that $i = j = 0$. In this case we make a transformation of the form

$$e_2' = \cos\theta e_2 - \sin\theta e_3, e_3' = -\sin\theta e_2 + \cos\theta e_3$$

and by choosing θ we can reduce y to zero if non-zero and hence to a previous case.

3. D can be reduced to zero. By scaling e_6 we can reduce f to unity. Using Lemmas 6.2 and 6.3 we will be able to remove both i and j and we obtain immediately algebra $\mathfrak{g}_{6,35}$ in [Mub2]. Not both a and d can be zero or else the algebra will be decomposable. By permuting e_2 and e_3 and changing the sign of e_1 we may assume that $|a| \leq |d|$ and by changing the sign of e_6, permuting e_4 and e_5 if necessary we may assume that $e \geq 0$.

4. If $e \neq a$ and $h \neq a$ then D can be reduced to zero. If $D = 0$ then in order for the algebra not to be decomposable it is necessary that e and h must both be non-zero so we may assume by scaling that $e = 1$ and $0 < |h| \leq 1$. If $i = j = 0$ then we obtain a special case of $\mathfrak{g}_{6,21}$. According to Lemma 6.2 we will be able to remove at least one of i and j provided that not both $e = 2a$ and $h = 2a$ hold. In this case we can assume without loss of generality that $i = 0, h = 2a$ and $j = 1$ which gives a special case of algebra $\mathfrak{g}_{6,22}$. If $e = h = 2a$ we can use Lemma 6.3 to remove i and reduce j to unity. We can also scale a to $\frac{1}{2}$ which gives again a special case of algebra $\mathfrak{g}_{6,22}$.

If $e = a$ and $h \neq a$ then D can be reduced to $\left[\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right]$. If $y = 0$ we will obtain the limiting cases of algebras received in the previous case. If $y \neq 0$ then by scaling e_4 by y we may assume that $y = 1$. Now using Lemma 6.2 we may reduce i and j to zero provided that $a \neq 0$ and $h \neq 2a$ and we obtain a special case of algebra $\mathfrak{g}_{6,23}$. If $a = 0$ then $h \neq 0$ and we can reduce h to unity and remove j by using Lemma 6.3. If $i = 0$ then we obtain again a special case of $\mathfrak{g}_{6,23}$. If $i \neq 0$ then we replace (e_1, e_2, e_3, e_4) by $(i^2 e_1, ie_2, ie_3, ie_4)$ which has the effect of reducing i to unity and gives algebra $\mathfrak{g}_{6,24}$. The coefficient h that appears in [Mub2] can normalized to zero or unity and if $h = 0$ it is a special case of $\mathfrak{g}_{6,23}$. The algebra is a suspension of either $A_{5,5}$ or $A_{5,6}$ which are nilpotent. If $h = 2a$ by scaling we may assume that $a = 1$ and we may remove i using Lemma 6.2. By scaling e_5 by $\frac{1}{j}$ we may reduce j to unity if non-zero which gives us algebra $\mathfrak{g}_{6,23}$.

If $e \neq a$ and $h = a$ then D can be reduced to $\left[\begin{smallmatrix} 0 & 0 \\ 0 & w \end{smallmatrix}\right]$ but this case is equivalent to the subcase just considered by permuting e_4 and e_5.

If $e = a$ and $h = a$ then D can be reduced to $\left[\begin{smallmatrix} 0 & y \\ 0 & w \end{smallmatrix}\right]$. If we make a transformation of the form

$$e_4' = \cos\theta e_4 - \sin\theta e_5, e_5' = -\sin\theta e_4 + \cos\theta e_5$$

and by choosing θ appropriately we can reduce w to zero and hence obtain algebra $g_{6,18}$ in [Mub2] for which $\alpha = \beta = 1$.

5. If $e \neq a$ then D can be reduced to zero. Using Lemma 6.2 we can assume that $i = 0$. Suppose that $a \neq 0$ then by scaling we may assume that $a = 1$. We will be able to reduce j to zero if non-zero unless $e = 2$ in which case we can scale j to unity if non-zero. These cases correspond to algebras $g_{6,28}$ and $g_{6,29}$ in [Mub2].

   Suppose next that $a = 0$. Now $e \neq 0$ or else the algebra will be nilpotent. Mapping (e_1, e_2, e_4, e_6) to $(ee_1, ee_2, ee_4, \frac{e_6}{e})$ reduces e to unity. Using Lemma 6.2 we will be able to remove both i and j and we obtain immediately algebra $g_{6,30}$ in [Mub2].

   Now suppose $e = a$ then D can be reduced to $\left[\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right]$. Now $a \neq 0$ or else the algebra will be nilpotent. Mapping (e_1, e_2, e_4, e_6) to $(ae_1, ae_2, ae_4, \frac{e_6}{a})$ reduces a to unity and i and j may be removed by using Lemma 6.2. If $x = y = 0$ we well obtain the limiting cases of the algebra $g_{6,28}$ found in the previous case. If $x \neq 0$ then defining

$$e_4' = xe_4, e_5' = xe_5$$

   reduces x to unity. Then defining

$$e_3' = e_3 + e_4, e_4' = ye_4 + e_5, e_5' = ye_5$$

   reduces the algebra to

$$\begin{aligned}&[e_2, e_3] = e_1, [e_1, e_6] = 2e_1, [e_2, e_6] = e_2 + e_3, [e_3, e_6] = e_3 + e_4,\\ &[e_4, e_6] = e_4 + e_5, [e_5, e_6] = e_5\end{aligned}$$

   which is algebra $g_{6,31}$ in [Mub2]. Otherwise if $x = 0$ and $y \neq 0$ scaling e_4 and e_5 by y reproduces the algebra in the form above.

6. D can be reduced to zero. Mapping (e_1, e_2, e_6) to $(-fe_1, -fe_2, -\frac{e_6}{f})$ reduces f to minus unity. Using Lemma 6.2 we will be able to remove both i and j and we obtain immediately algebra $g_{6,36}$ in [Mub2]. By mapping (e_1, e_2, e_4, e_6) to $(-e_1, -e_2, -e_4, -e_6)$ if necessary we may assume that $e \geq 0$.

7. D can be reduced to zero. Then by scaling e_6 we can reduce b to minus unity. Using Lemma 6.2 we will be able to remove both i and j provided $e \neq 2a$ and $h \neq 2a$. If $e \neq 2a$ and $h = 2a$ we can reduce i to zero and j to unity if it is non-zero by scaling e_5 by $\frac{1}{j}$. If $h \neq 2a$ and $e = 2a$ we

can reduce to the case just considered by permuting e_4 and e_5. Finally if $e = h = 2a$ we make a transformation of the form

$$e_4' = \cos\theta e_4 - \sin\theta e_5, e_5' = -\sin\theta e_4 + \cos\theta e_5.$$

As such the vector (i, j) undergoes a rotation and it is possible to choose θ so that i is rotated to zero. Thereafter j can be reduced to unity if it is non-zero as before. Thus in case 7 we obtain the corrected form of algebras $g_{6,32}$ and $g_{6,33}$ in [Mub2]. In both algebras $e \neq 0$ otherwise decomposable and by mapping (e_1, e_3, e_6) to $(-e_1, -e_3, -e_6)$ if necessary we may assume that $a \geq 0$ and and in $g_{6,32}$ by permuting e_4 and e_5 we may assume that $0 < |e| \leq |h|$.

8. D can be reduced to zero. In this case we change the second Jordan form to $\left[\begin{smallmatrix} e & 0 \\ b & e \end{smallmatrix}\right]$. Since $b \neq 0$ we scale e_6 by $\frac{1}{b}$ so as to reduce to the case b is unity. Using Lemma 6.2 we can assume that $i = 0$. We will be able to remove j unless $e = 2a$ using Lemma 6.2 and obtain a special case of $g_{6,34}$. If $e = 2a$ and $j = 0$ we have a special case of $g_{6,34}$ as well. If $j \neq 0$ we can reduce j to unity by scaling e_4 and e_5 by $\frac{1}{j}$. Together these cases comprise algebra $g_{6,34}$ in [Mub2]. By changing the sign of e_5 and e_6 if necessary we may assume that $a \geq 0$.

9. In this case using Lemma 6.2 we can assume that $i = j = 0$. Unless $e = a$ and $f = b$ D can be reduced to zero. By changing the sign of e_6 if necessary we may assume that $a \geq 0$. We can also obtain that $f > 0$ by permuting e_4 and e_5 if necessary and we have algebra $g_{6,37}$ in [Mub2].

   If $e = a$ and $f = b$ then D can be reduced to $\left[\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right]$. Then by scaling e_6 by $-\frac{1}{b}$ we can reduce b to minus unity. If either x or y is zero we obtain $g_{6,38}$. If x and y are both nonzero we make a transformation of the form

$$e_2' = \cos\theta e_2 - \sin\theta e_3, e_3' = -\sin\theta e_2 - \cos\theta e_3$$

   and by choosing θ we can remove x. We can scale y to unity mapping (e_4, e_5) to (ye_4, ye_5) and may assume that $a \geq 0$ using the map defined by transforming $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ to $\{-e_1, e_2, -e_3, e_4, -e_5, -e_6\}$ if necessary and have algebra $g_{6,38}$.

Remark 6.4. Notice that Mubarakzyanov's algebra $g_{6,38}$ [Mub2]

$$[e_2, e_3] = e_1, [e_1, e_6] = 2ae_1, [e_2, e_6] = ae_2 + e_3 + e_4, [e_3, e_6] = ae_3 - e_2 + e_5,$$
$$[e_4, e_6] = ae_4 + e_5, [e_5, e_6] = ae_5 - e_4$$

can be simplified to

$$[e_2, e_3] = e_1, [e_1, e_6] = 2ae_1, [e_2, e_6] = ae_2 + e_3, [e_3, e_6] = ae_3 - e_2 + e_4,$$
$$[e_4, e_6] = ae_4 + e_5, [e_5, e_6] = ae_5 - e_4$$

applying the transformation $e_1' = e_1, e_2' = -e_3, e_3' = e_2 - e_5, e_4' = 2e_4, e_5' =$ $= 2e_5, e_6' = e_6$.

7. Nilradical isomorphic to $\mathbb{R} \oplus A_{4,1}$

7.1. Satisfying the Jacobi identity.

In paragraph three the nilradical is isomorphic to a direct sum of $\mathbb{R}$ and the four dimensional indecomposable nilpotent Lie algebra. As such Mubarakzyanov writes the non-zero brackets as $[e_1, e_5] = e_2, [e_4, e_5] = e_1$ and e_3 spans the abelian factor $\mathbb{R}$. We prefer $[e_2, e_5] = e_1, [e_4, e_5] = e_2$ because the adjoint matrices will then be upper triangular. We set $[e_i, e_6] = b_i^k e_k \, (1 \leq i, k \leq 5)$.

If we impose the Jacobi identity in the most general way and following Mubarakzyanov define $e_6' = e_6 - b_2^1 e_5 + b_5^1 e_2 + b_5^2 e_4$ keeping e_1, e_2, e_3, e_4, e_5 fixed: the effect is to remove the parameters b_2^1, b_5^1 and b_5^2, we find that B, which is obtained from $-ad(e_6)$ by eliminating the last row and column of zeros is of the following form depending on eight parameters:

$$B = \begin{bmatrix} b_4^4 + 2b_5^5 & 0 & b_3^1 & b_4^1 & 0 \\ 0 & b_4^4 + b_5^5 & 0 & 0 & 0 \\ 0 & 0 & b_3^3 & b_4^3 & b_5^3 \\ 0 & 0 & 0 & b_4^4 & b_5^4 \\ 0 & 0 & 0 & 0 & b_5^5 \end{bmatrix}.$$

Thus we have an algebra of the form:

$$\begin{aligned} &[e_2, e_5] = e_1, [e_4, e_5] = e_2, [e_1, e_6] = (b_4^4 + 2b_5^5)e_1, [e_2, e_6] = (b_4^4 + b_5^5)e_2, \\ &[e_3, e_6] = b_3^1 e_1 + b_3^3 e_3, [e_4, e_6] = b_4^1 e_1 + b_4^3 e_3 + b_4^4 e_4, [e_5, e_6] = b_5^3 e_3 + b_5^4 e_4 + b_5^5 e_5. \end{aligned} \tag{2}$$

We remark that the most general change of basis matrix that respects the nilradical can only be of the form:

$$P = \begin{bmatrix} p_4^4 (p_5^5)^2 & p_4^2 p_5^5 & p_3^1 & p_4^1 & p_5^1 & p_6^1 \\ 0 & p_4^4 p_5^5 & 0 & p_4^2 & p_5^2 & p_6^2 \\ 0 & 0 & p_3^3 & p_4^3 & p_5^3 & p_6^3 \\ 0 & 0 & 0 & p_4^4 & p_5^4 & p_6^4 \\ 0 & 0 & 0 & 0 & p_5^5 & p_6^5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Under such a change the lower 3×3 block of B $\begin{bmatrix} b_3^3 & b_4^3 & b_5^3 \\ 0 & b_4^4 & b_5^4 \\ 0 & 0 & b_5^5 \end{bmatrix}$ is acted on by conjugation.

Proposition 7.1. *The lower right* 3×3 *block of* $ad(e_6)$ *above is a relative invariant of this class of algebras.*

We can consider the algebra (2) in the following way: we note that $\langle e_1, e_2 \rangle$ forms a two-dimensional ideal: if we quotient by it we obtain an algebra which is four-dimensional with a three-dimensional abelian nilradical. Such algebras correspond to $g_{4,2} - g_{4,6}$ in [PSWZ]. Algebra $g_{4,1}$ is omitted because it is nilpotent and then the algebra (2) would also be nilpotent. Furthermore (2) is in *real* upper triangular form: therefore $g_{4,6}$ also does not occur.

After fixing the algebra (2) the question that remains is whether we can remove or simplify the entries b_3^1 and b_4^1 in the matrix B. The following lemma helps to answer this question.

Lemma 7.2. *(i) In the Lie algebra (2) we can remove by change of basis the coefficients* b_3^1 *provided that* $b_4^4 + 2b_5^5 - b_3^3 \neq 0$ *and* b_4^1 *provided that* $b_5^5 \neq 0$.
(ii) Changing basis by the matrix $P = \begin{bmatrix} a^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ *has the effect only of scaling both the coefficients* b_3^1 *and* b_4^1 *by* $\frac{1}{a^2}$. *Thus either but not both coefficients may be scaled to* ± 1 *if either is non-zero.*

Proof. If the change of basis matrix is $P = \begin{bmatrix} 1 & x & y & z & 0 & 0 \\ 0 & 1 & 0 & x & 0 & -b_5^3 y - b_5^4 z \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -b_5^4 x \\ 0 & 0 & 0 & 0 & 1 & -b_5^5 x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ then B is transformed to

$$B' = \begin{bmatrix} b_4^4 + 2b_5^5 & 0 & b_3^1 + (b_4^4 + 2b_5^5 - b_3^3)y & b_4^1 + 2b_5^5 z - b_4^3 y - b_5^5 x^2 & 0 \\ 0 & b_4^4 + b_5^5 & 0 & 0 & 0 \\ 0 & 0 & b_3^3 & b_4^3 & b_5^3 \\ 0 & 0 & 0 & b_4^4 & b_5^4 \\ 0 & 0 & 0 & 0 & b_5^5 \end{bmatrix}.$$

∎

7.2. Normal forms of the 3×3 upper triangular block.

Taking into account the possibility of scaling and the fact none of the matrices can be nilpotent, otherwise the entire algebra would be nilpotent, the 3×3 lower block can be reduced to one of the following:

$$b_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} b_2 = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} b_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} b_4 = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} b_5 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$b_6 = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} b_7 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} b_8 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} b_9 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} b_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

7.3. Corresponding algebras.

Now for each of the ten matrices above we apply Lemma 7.2. For simplicity we shall denote the entries b_3^1 and b_4^1 by d and e, respectively.

- In the first matrix b_1 we may assume that $d = e = 0$ unless $a = b + 2$ in which case $d = 0,\, 1,\, e = 0$. We obtain thereby algebras $g_{6,39}$ and $g_{6,40}$, respectively. For $g_{6,39}$ the value $a = 0$ has to be excluded, otherwise the algebra is decomposable.

- For b_2 we may assume that $d = e = 0$ giving $g_{6,41}$.

- For b_3 if $b \neq -1$, then $d = e = 0$ and if $b = -1$ then $e = 0$. If $d = 0$ when $b = -1$, then these two cases give $g_{6,42}$. If $d \neq 0$ then applying the transformation which fixes all but e_1, e_2 and e_4 such that $e_1' = de_1$, $e_2' = de_2$ and $e_4' = de_4$, we obtain $g_{6,43}$.

- For b_4 we may assume that $e = 0$ and if $a \neq 3$ that $d = 0$. Adding the values $a = 3, d = 0$ and excluding $a = 0$, otherwise the algebra is decomposable gives $g_{6,44}$. If $a = 3$ and $d \neq 0$ then applying the transformation, which fixes all but e_3 such that $e_3' = \frac{e_3}{d}$, we obtain $g_{6,45}$.

- For b_5 we may assume that $d = 0, e = 0$ which gives $g_{6,46}$.

- For b_6 we may assume if $a \neq 1$ that $d = 0$, $e = 0, \pm 1$. Adding the values $a = 1, d = 0$, $e = 0, \pm 1$ and excluding $a = 0$, otherwise the algebra is decomposable gives $g_{6,47}$. If $a = 1$ and $d \neq 0$ then defining $e_3' = \frac{e_3}{d}, e_4' =$ $= e_4 - \frac{e}{d} e_3$ and leaving the other e_i's unchanged gives $g_{6,48}$ essentially.

- For b_7 we may assume that $d = 0, e = 0, \pm 1$ which gives $g_{6,49}$.

- For b_8 we may assume that $e = 0$ since $b_4^3 \neq 0$ and furthermore that $d = 0, \pm 1$ giving $g_{6,50}$.

- For b_9 we may assume that $d = 0$, $e = 0, \pm 1$. However, if $e = 0$ the algebra would be decomposable. Altogether we obtain algebra $g_{6,51}$.

- For b_{10} we may assume that $d = 0, e = 0, \pm 1$ which gives $g_{6,52}$.

Remark 7.3. In [Wang] Y. Wang, J. Lin and S. Deng also considered extending the same nilradical, an example of a "quasi-filiform" algebra, to a six-dimensional solvable Lie algebra which is not necessarily indecomposable. However, they used a basis with brackets $[e_2, e_5] = -e_1, [e_3, e_5] = -e_2$ and where e_4 spans the abelian factor $\mathbb{R}$. However, they did not obtain the algebras $g_{6,41}$, $g_{6,46}$, $g_{6,48}$, and $g_{6,50}$ and some of the algebras are limiting cases of Mubarakzyanov's algebras and are decomposable. Below we give the correspondence between algebras that appear in [Wang] denoted by $g_{6,i}$ for $1 \leq i \leq 19$ and the Mubarakzyanov algebras denoted by $g_{6,i}$ for $39 \leq i \leq 52$: since the ranges do not overlap there should be no danger of confusion.

- $g_{6,1}$ is isomorphic to $g_{6,39}$ with $b \neq 0, -2$
- $g_{6,2}$ is isomorphic to $g_{6,39}$ with $b \neq 0, -2$ and $a = 0$ which is decomposable;
- $g_{6,3}$ is isomorphic to $g_{6,39}$ with $b = 0$;
- $g_{6,4}$ is isomorphic to $g_{6,39}$ with $a = b = 0$ (decomposable);
- $g_{6,5}$ is isomorphic to $g_{6,39}$ with $b = -2$;
- $g_{6,6}$ is isomorphic to $g_{6,39}$ with $a = 0$ and $b = -2$ (decomposable);
- $g_{6,7}$ is isomorphic to $g_{6,47}$ with $\epsilon = 0, a \neq 0$;

- $g_{6,8}$ is isomorphic to $g_{6,47}$ with $\epsilon = 0, a = 0$ (decomposable);
- $g_{6,9}$ is decomposable;
- $g_{6,10}$ is isomorphic to $g_{6,39}$ with $b \neq 0, -1, a \neq 0, b{+}2$; $g_{6,42}$ with $b \neq 0, -1, 1$; $g_{6,44}$ with $a \neq 0, 3, 1$; $g_{6,44}$ with $a = 1$; $g_{6,42}$ with $b = 1$; $g_{6,39}$ with $a = b = 1$;
- $g_{6,11}$ is isomorphic to $g_{6,42}$ with $b = 0$;
- $g_{6,12}$ is isomorphic to $g_{6,39}$ with $b = -1, a = 1$; $g_{6,43}$; $g_{6,40}$ with $b = -1$; $g_{6,42}$ with $b = -1$;
- $g_{6,13}$ is isomorphic to $g_{6,40}$ with $b \neq 0, -1, 1$; $g_{6,45}$;
- $g_{6,14}$ is isomorphic to $g_{6,40}$ with $b = 0$;
- $g_{6,15}$ is decomposable;
- $g_{6,17}$ (sic) is isomorphic to $g_{6,49}$;
- $g_{6,18}$ is decomposable;
- $g_{6,19}$ is isomorphic to $g_{6,51}$ with $\epsilon \neq 0$; $g_{6,52}$.

8. Nilradical isomorphic to $A_{5,1}$

8.1. Satisfying the Jacobi identity. In paragraph four the nilradical is isomorphic to $A_{5,1}$ [PSWZ] which is given by: $[e_3, e_5] = e_1, [e_4, e_5] = e_2$. The remaining brackets in the six-dimensional algebra are $[e_i, e_6] = a_i^k e_k (1 \leq i, k \leq 5)$. If we impose the Jacobi identity in the most general way and following Mubarakzyanov define $e_6' = e_6 - a_3^1 e_5 + a_5^1 e_3 + a_5^2 e_4$ keeping e_1, e_2, e_3, e_4, e_5 fixed: the effect is to remove the parameters a_3^1, a_5^1 and a_5^2. Also replacing $a_4^2 - a_3^1$ by a_4^2, we find that $-ad(e_6) := A$ is of the following form depending on twelve parameters:

$$A = \begin{bmatrix} a_3^3 + a_5^5 & a_4^3 & 0 & a_4^1 & 0 \\ a_3^4 & a_4^4 + a_5^5 & a_3^2 & a_4^2 & 0 \\ 0 & 0 & a_3^3 & a_4^3 & a_5^3 \\ 0 & 0 & a_3^4 & a_4^4 & a_5^4 \\ 0 & 0 & 0 & 0 & a_5^5 \end{bmatrix}.$$

Thus the algebra has been reduced to

$$\begin{aligned} &[e_3, e_5] = e_1, [e_4, e_5] = e_2, [e_1, e_6] = (a_3^3 + a_5^5)e_1 + a_3^4 e_2, [e_2, e_6] = a_4^3 e_1 + (a_4^4 + a_5^5)e_2, \\ &[e_3, e_6] = a_3^2 e_2 + a_3^3 e_3 + a_3^4 e_4, [e_4, e_6] = a_4^1 e_1 + a_4^2 e_2 + a_4^3 e_3 + a_4^4 e_4, \\ &[e_5, e_6] = a_5^3 e_3 + a_5^4 e_4 + a_5^5 e_5. \end{aligned} \tag{3}$$

We remark that a very general change of basis matrix that respects the nilradical is of the form

$$P = \begin{bmatrix} mw & nw & c & d & e & f \\ gw & sw & h & i & j & k \\ 0 & 0 & m & n & p & q \\ 0 & 0 & g & s & t & u \\ 0 & 0 & 0 & 0 & w & x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Under such a change the upper left 2×2 block $\left[\begin{smallmatrix} a_3^3+a_5^5 & a_4^3 \\ a_3^4 & a_4^4+a_5^5 \end{smallmatrix}\right]$ in A is acted on by conjugation.

Proposition 8.1. *The upper left 2×2 block of $ad(e_6)$ above is a relative invariant of this class of algebras.*

Accordingly it is possible to reduce the 2×2 block to one of the following Jordan normal forms:

$$a_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \quad a_3 = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \quad a_4 = \begin{bmatrix} b & 1 \\ -1 & b \end{bmatrix}.$$

8.2. Reduction of $ad(e_6)$ depending on the Jordan normal forms.

1. In case of $a_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ since the Lie algebra is not nilpotent we can assume that $a_5^5 \neq 0$. By following Mubarakzyanov we can divide every term by a_5^5 and then it is possible to remove $a_4^1, a_3^2, a_4^2, a_5^3$ and a_5^4 so as to obtain $g_{6,53}$.

2. For $a_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix}\right]$ we put $d = a_5^5$ so that $A = \left[\begin{smallmatrix} 1 & 0 & 0 & a_4^1 & 0 \\ 0 & b & a_3^2 & a_4^2 & 0 \\ 0 & 0 & 1-d & 0 & a_5^3 \\ 0 & 0 & 0 & b-d & a_5^4 \\ 0 & 0 & 0 & 0 & d \end{smallmatrix}\right]$. There is a plethora of cases to consider depending on the values of b and d. In $\mathbb{R}^2$ as the (b, d)-plane we have to consider the following five linear equations

$$d = 0, d = \frac{1}{2}, b + d = 1, b - d = 1, d = \frac{b}{2}.$$

There are seven points of intersection between the five linear equations which are numbered (*iii*), (*iv*), (*vi*), (*vii*), (*x*), (*xi*), (*xii*) below. We consider thirteen regions of $\mathbb{R}^2$; collectively their union is the whole of $\mathbb{R}^2$ corresponding to all possible ordered pairs (b, d). The first region (*i*) is the open subset of $\mathbb{R}^2$ for which all of the five equations are *not* satisified. The equation $d = 0$ corresponds to regions (*ii*), (*iii*), (*iv*); the equation $d = \frac{1}{2}$ corresponds to regions (*iii*), (*v*), (*vi*), (*vii*), (*x*); the equation $b + d = 1$ corresponds to regions (*iv*), (*ix*), (*x*), (*xi*); the equation $d = b - 1$ corresponds to regions (*iv*), (*vi*), (*xii*), (*xiii*); the equation $d = \frac{b}{2}$ corresponds to regions (*vii*), (*viii*), (*xi*), (*xiii*).

(i) If $b \neq 1-d, 1+d$ and $d \neq 0, \frac{1}{2}, \frac{b}{2}$ then we can reduce to $\mathfrak{g}_{6,54}$ but with the restriction $\lambda \neq 1-\gamma, 1+\gamma$ and $\gamma \neq 0, \frac{1}{2}, \frac{\lambda}{2}$.

(ii) Suppose $d = 0$ and $(b, d) \notin \{(0,0), (1,0)\}$.
1). If $a_4^2 = 0$, then we obtain a special case of $\mathfrak{g}_{6,54}$ with the restriction $\lambda \neq 0, 1$ and $\gamma = 0$.
2). If $a_4^2 \neq 0$, then the algebra is isomorphic to $\mathfrak{g}_{6,63}$ with $\lambda \neq 0, 1$.

(iii) Suppose $(b, d) = (0, 0)$.
1). If $a_4^2 = a_5^4 = 0$, then we obtain a special case of $\mathfrak{g}_{6,54}$ with $\lambda = 0$ and $\gamma = 0$.
2). If $a_4^2 = 0$ and $a_5^4 \neq 0$, then we have a special case of $\mathfrak{g}_{6,57}$ with $\gamma = 0$.
3). If $a_4^2 \neq 0$ and $a_5^4 = 0$, then we have a special case of $\mathfrak{g}_{6,59}$ with $h = 0$.
4). If $a_4^2 \neq 0$ and $a_5^4 \neq 0$, then we obtain a special case of $\mathfrak{g}_{6,59}$ with $h \neq 0$ and hence $\mathfrak{g}_{6,59}$ with h scaled to 0 or 1.

Remark 8.2. If we consider $\mathfrak{g}_{6,59}$:

$$[e_3, e_5]{=}e_1, [e_4, e_5]{=}e_2, [e_1, e_6]{=}e_1, [e_3, e_6]{=}e_3, [e_4, e_6]{=}e_2, [e_5, e_6]{=}he_4$$

and scale e_2 and e_4 by a the effect is only to change the bracket $[e_5, e_6] = he_4$ to $[e_5, e_6] = \frac{h}{a}e_4$ so we may assume without loss of generality that $h = 0, 1$.

(iv) Suppose $(b, d) = (1, 0)$.
1). If $a_3^2 \neq 0$ and $(a_4^2)^2 \neq -4a_4^1a_3^2$, then we obtain $\mathfrak{g}_{6,64}$ with $h \neq 0$ and if $h = 0$, then this algebra is a special case of $\mathfrak{g}_{6,55}$ with $\gamma = 0$. Combining all the special cases of $\mathfrak{g}_{6,64}$, we have $\mathfrak{g}_{6,64}$ with $h \neq 0$.
2). If $a_3^2 \neq 0$ and $(a_4^2)^2 = -4a_4^1a_3^2$, then we have a special case of $\mathfrak{g}_{6,56}$ with $\gamma = 0$.
3). Suppose $a_3^2 = 0$.
1. If $a_4^2 \neq 0$ then we have a special case of $\mathfrak{g}_{6,63}$ with $\lambda = 1$.
Altogether (ii)2). and (iv)3).1. give $\mathfrak{g}_{6,63}$ with $\lambda \neq 0$.

Remark 8.3. Mubarakzyanov's algebra $\mathfrak{g}_{6,63}$ with $\lambda = 0$ is isomorphic to $\mathfrak{g}_{6,59}$ with $h = 0$ [Mub2].

2. If $a_4^1 = a_4^2 = 0$ then we have a special case of $\mathfrak{g}_{6,54}$ with $\lambda = 1$ and $\gamma = 0$.
3. If $a_4^1 \neq 0$ and $a_4^2 = 0$ then we have a special case of $\mathfrak{g}_{6,55}$ with $\gamma = 0$.

Remark 8.4. (i). If $h = 0$ then $\mathfrak{g}_{6,64}$ is a special case of $\mathfrak{g}_{6,55}$ for which $\gamma = 0$.
(ii). The algebra $\mathfrak{g}_{6,64}$ can be reduced to

$$[e_3, e_5]{=}e_1, [e_4, e_5]{=}e_2, [e_1, e_6]{=}e_1, [e_2, e_6]{=}e_2, [e_3, e_6]{=}\epsilon e_2 + e_3,$$
$$[e_4, e_6] = e_1 + e_4, \ (\epsilon = 0, \pm 1) :$$

in other words in $g_{6,64}$, which is identical to the previous algebra except that ϵ is replaced by h, then h can be reduced to 0 or ±1. And due to (i), it suffices to take $\epsilon = \pm1$.

(v) Suppose $d = \frac{b}{2}$ and $(b,d) \notin \{(0,0), \left(\frac{2}{3}, \frac{1}{3}\right), \left(1, \frac{1}{2}\right), (2,1)\}$.
1). If $a_5^4 = 0$, then we obtain a special case of $g_{6,54}$ with $\lambda = 2b$ and $\gamma = b$.
2). If $a_5^4 \neq 0$, then we have a special case of $g_{6,57}$ with $\gamma \neq 0,\ \frac{1}{3},\ \frac{1}{2},\ 1$.

(vi) Suppose $(b,d) = (2,1)$.
1). If $a_4^1 = a_5^4 = 0$, then we have a special case of $g_{6,54}$ with $\lambda = 2$ and $\gamma = 1$.
2). If $a_4^1 = 0$ and $a_5^4 \neq 0$, then we have a special case of $g_{6,57}$ with $\gamma = 1$.
3). If $a_4^1 \neq 0$ and $a_5^4 = 0$, then we obtain a special case of $g_{6,55}$ with $\gamma = 1$.
4). If $a_4^1 a_5^4 \neq 0$, then we have $g_{6,60}$ with $\omega \neq 0$ and scaled to 1.

Remark 8.5. Algebra $g_{6,60}$ with $\omega = 0$ is a special case of $g_{6,55}$ with $\gamma = 1$.

(vii) Suppose $(b,d) = \left(1, \frac{1}{2}\right)$.
1). If $a_5^3 \neq 0$, and $a_5^4 = 0$, then we have a special case of $g_{6,61}$ with $\lambda = 1$.
2). If $a_5^3 = a_5^4 = 0$, then we have a special case of $g_{6,54}$ with $\lambda = 1$ and $\gamma = \frac{1}{2}$.
3). If $a_5^3 \neq 0$ and $a_5^4 \neq 0$, then we have a special case of $g_{6,57}$ with $\gamma = \frac{1}{2}$.
4). If $a_5^4 \neq 0$ and $a_5^3 = 0$, then we have a special case of $g_{6,61}$ with $\lambda = 1$.

Remark 8.6. Consider $g_{6,61}$:

$$[e_3,e_5] = e_1, [e_4,e_5] = e_2, [e_1,e_6] = 2e_1, [e_2,e_6] = 2\lambda e_2, [e_3,e_6] = e_3,$$
$$[e_4,e_6] = (2\lambda - 1)e_4, [e_5,e_6] = e_3 + e_5.$$

If $\lambda \neq 0$, then $g_{6,61}$ is isomorphic to $g_{6,57}$ with $\gamma \neq 0$. To see why it suffices to apply the transformation $e_1' = 2\lambda e_2, e_2' = 2\lambda e_1, e_3' = e_4, e_4' = e_3, e_5' = 2\lambda e_5, e_6' = \frac{1}{2\lambda} e_6$ and replace $\frac{1}{2\lambda}$ by γ. Therefore $g_{6,61}$ may be taken in the form:

$$[e_3,e_5] = e_1, [e_4,e_5] = e_2, [e_1,e_6] = 2e_1, [e_3,e_6] = e_3, [e_4,e_6] = -e_4,$$
$$[e_5,e_6] = e_3 + e_5.$$

(viii) Suppose $d = \frac{1}{2}$ and $(b,d) \notin \{\left(\frac{1}{2}, \frac{1}{2}\right), \left(1, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right)\}$.
1). If $a_5^3 = 0$, then we have a special case of $g_{6,54}$ with $\lambda = b$ and $\gamma = \frac{1}{2}$.
2). If $a_5^3 \neq 0$, then we have a special case of $g_{6,57}$ with $\gamma \neq \frac{1}{3}, \frac{1}{2}$ and 1.

(ix) Suppose $d=1-b$ and $(b,d)\notin\{(1,0),\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{2}{3},\frac{1}{3}\right)\}$.
1). If $a_3^2=0$, then we have $g_{6,54}$ with $\lambda=1-\gamma$ and $\gamma\neq 0,\ \frac{1}{2},\ \frac{1}{3}$.
2). If $a_3^2\neq 0$, then we have a special case of $g_{6,56}$.
Therefore all the special cases of $g_{6,56}$ give $g_{6,56}$ with $\gamma\neq\frac{1}{2},\ \frac{1}{3}$.

Remark 8.7. Consider $g_{6,56}$ [Mub2]:

$$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=e_1,[e_2,e_6]=(1-\gamma)e_2,$$
$$[e_3,e_6]=e_2+(1-\gamma)e_3,[e_4,e_6]=(1-2\gamma)e_4,[e_5,e_6]=\gamma e_5.$$

If $\gamma\neq 1$, then $g_{6,56}$ is isomorphic to $g_{6,55}$ for $\gamma\neq -1$. To see how it suffices to apply the transformation $e_1'=e_2,e_2'=(1-\gamma)e_1,e_3'=e_4,e_4'= =(1-\gamma)e_3,e_5'=e_5,e_6'=\frac{1}{1-\gamma}e_6$, and replace $\frac{\gamma}{1-\gamma}$ with γ. Therefore in $g_{6,56}$ we may assume that $\gamma=1$.

(x) Suppose $(b,d)=\left(\frac{2}{3},\frac{1}{3}\right)$.
1) If $a_3^2=a_5^4=0$, then we have a special case of $g_{6,54}$ with $\lambda=\frac{2}{3}$ and $\gamma=\frac{1}{3}$.
2) If $a_3^2=0$ and $a_5^4\neq 0$, then we have a special case of $g_{6,57}$ with $\gamma=\frac{1}{3}$.
3) The cases $a_3^2\neq 0$ and $a_5^4=0$, and $a_3^2\neq 0$ and $a_5^4\neq 0$ give $g_{6,58}$.

(xi) Suppose $(b,d)=\left(\frac{1}{2},\frac{1}{2}\right)$.
1) If $a_3^2=a_5^3=0$, then we have a special case of $g_{6,54}$ with $\lambda=\frac{1}{2}$ and $\gamma=\frac{1}{2}$.
2) The cases $a_3^2\neq 0$ and $a_5^3=0$, or $a_3^2a_5^3\neq 0$ give $g_{6,62}$.

Remark 8.8. Notice that $g_{6,62}$ with $\omega=0$ in [Mub2],

$$[e_3,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=2e_1,[e_2,e_6]=e_2,[e_3,e_6]=e_2+e_3,$$
$$[e_5,e_6]=e_5,$$

is isomorphic to $g_{6,55}$ with $\gamma=1$. It suffices to apply the transformation which fixes all but e_1,e_2,e_3 and e_4 such that $e_1'=e_2,e_2'=e_1,e_3'=e_4$ and $e_4'=e_3$.

3) If $a_3^2=0$ and $a_5^3\neq 0$, then we obtain a special case of $g_{6,57}$ with $\gamma=1$.
All the special cases of $g_{6,57}$ give $g_{6,57}$ in full generality.

(xii) Suppose $d=b-1$ and $(b,d)\notin\{(1,0),\left(\frac{3}{2},\frac{1}{2}\right),(2,1)\}$.
1). If $a_4^1=0$, then we obtain a special case of $g_{6,54}$ with $\lambda=1+d$ and $\gamma=d$.
2). If $a_4^1\neq 0$, then we obtain a special case of $g_{6,55}$.
All the special cases of $g_{6,55}$ give $g_{6,55}$ with $\gamma\neq\frac{1}{2}$.

Remark 8.9. Notice if $\gamma = \frac{1}{2}$ in $\mathfrak{g}_{6,55}$ we have

$$[e_3, e_5] = e_1, [e_4, e_5] = e_2, [e_1, e_6] = e_1, [e_2, e_6] = \frac{3}{2}e_2, [e_3, e_6] = \frac{1}{2}e_3,$$
$$[e_4, e_6] = e_1 + e_4, [e_5, e_6] = \frac{1}{2}e_5,$$

which is isomorphic to $\mathfrak{g}_{6,58}$ with $\omega = 0$. To see why it suffices to apply the transformation which fixes e_5 such that $e_1' = e_2, e_2' = 2e_1, e_3' = = e_4, e_4' = 2e_3$ and $e_6' = 2e_6$.

(xiii) Suppose $(b, d) = \left(\frac{3}{2}, \frac{1}{2}\right)$.
1) If $a_4^1 = a_5^3 = 0$, then we have a special case of $\mathfrak{g}_{6,54}$ with $\lambda = \frac{3}{2}$ and $\gamma = \frac{1}{2}$.
All the special cases of $\mathfrak{g}_{6,54}$ give $\mathfrak{g}_{6,54}$ in the full generality.
2) The cases $a_4^1 \neq 0$ and $a_5^3 = 0$, or $a_4^1 = 0$ and $a_5^3 \neq 0$, or $a_4^1 \neq 0$ and $a_5^3 \neq 0$ give $\mathfrak{g}_{6,58}$ again.

3. If $a_3 = \left[\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right]$ then $A = \begin{bmatrix} a & 1 & 0 & a_4^1 & 0 \\ 0 & a & a_3^2 & a_4^2 & 0 \\ 0 & 0 & a-d & 1 & a_5^3 \\ 0 & 0 & 0 & a-d & a_5^4 \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$.

(i) If $a \neq 2d$ and $d \neq 0$, then we obtain a special case of $\mathfrak{g}_{6,65}$.

(ii) Suppose $a = 2d$ and $d \neq 0$.
1). If $a_5^4 = 0$, then we obtain a special case of $\mathfrak{g}_{6,65}$ with $\lambda = 2$ and $\gamma = 1$.
Combining all the special cases of $\mathfrak{g}_{6,65}$, we obtain $\mathfrak{g}_{6,65}$ with $\gamma \neq 0$.
2). If $a_5^4 \neq 0$, then we obtain $\mathfrak{g}_{6,66}$.

(iii) If $a = 2d$ and $d = 0$, then the algebra is nilpotent.

Remark 8.10. Consider $\mathfrak{g}_{6,67}$:

$$[e_3, e_5] = e_1, [e_4, e_5] = e_2, [e_1, e_6] = 2e_1 + e_2, [e_2, e_6] = 2e_2,$$
$$[e_3, e_6] = e_3 + e_4, [e_4, e_6] = e_4, [e_5, e_6] = he_4 + e_5.$$

Applying the transformation which fixes all but $e_5' = e_5 - he_3$, we have

$$[e_3, e_5] = e_1, [e_4, e_5] = e_2, [e_1, e_6] = 2e_1 + e_2, [e_2, e_6] = 2e_2,$$
$$[e_3, e_6] = e_3 + e_4, [e_4, e_6] = e_4, [e_5, e_6] = e_5$$

is a special case of $\mathfrak{g}_{6,65}$ with $\lambda = 2$ and $\gamma = 1$. Therefore $\mathfrak{g}_{6,67}$ is eliminated.

(iii) Suppose $a \neq 2d$ and $d = 0$.
1). If $a_3^2 \neq 0$ then we obtain $\mathfrak{g}_{6,68}$ with $c \neq 0$ and scaled to 1.
2). If $a_3^2 = 0$, then we obtain a special case of $\mathfrak{g}_{6,65}$ with $\gamma = 0$ and $\lambda \neq 0$ and scaled to unity or $\mathfrak{g}_{6,68}$ with $c = 0$.
Combining all the special cases of $\mathfrak{g}_{6,65}$, we obtain $\mathfrak{g}_{6,65}$ in full generality.

Remark 8.11. Note that in $g_{6,68}$

$$[e_3, e_5] = e_1, [e_4, e_5] = e_2, [e_1, e_6] = e_1 + e_2, [e_2, e_6] = e_2,$$
$$[e_3, e_6] = e_3 + e_4, [e_4, e_6] = ce_1 + e_4,$$

the parameter $c \neq 0$, otherwise it is isomorphic to $g_{6,65}$ with $\lambda \neq 0$ and $\gamma = 0$ [Mub2] and it is always possible to reduce to $c = 1$ by applying the transformation fixing all but e_1, e_2 and e_5 such that $e'_1 = ce_1, e'_2 = ce_2$ and $e'_5 = ce_5$.

Remark 8.12. In algebra $g_{6,65}$ in order to satisfy the Jacobi identity the fifth bracket has to read $[e_3, e_6] = (\lambda - \gamma)e_3 + e_4$ rather than $[e_3, e_6] = (1 - \gamma)e_3 + e_4$. However, we can always reduce to the cases $\gamma = 1$ or $\gamma = 0$ and $\lambda = 1$; to do so, assuming that $\gamma \neq 0$ keep e_2, e_4, e_5 fixed and define $e'_1 = \gamma e_1, e'_3 = \gamma e_3, e'_6 = \frac{1}{\gamma} e_6$. On the other hand, if $\gamma = 0$ keep e_2, e_4, e_5 fixed and define $e'_1 = \lambda e_1, e'_3 = \lambda e_3, e'_6 = \frac{1}{\lambda} e_6$.

Remark 8.13. Note that $g_{6,69}$:

$$[e_3, e_5] = e_1, [e_4, e_5] = e_2, [e_1, e_6] = e_1 + e_2, [e_2, e_6] = e_2,$$
$$[e_3, e_6] = e_3 + e_4, [e_4, e_6] = e_2 + e_4,$$

is isomorphic to $g_{6,65}$ with $\lambda \neq 0$ and $\gamma = 0$ or $g_{6,68}$ with $c = 0$. Therefore $g_{6,69}$ can be eliminated completely. To see why it suffices to apply the transformation fixing all but e_4 and e_6 such that $e'_4 = e_4 - \frac{1}{2} e_1, e'_6 = e_6 - \frac{1}{2} e_5$.

4. If $a_4 = \begin{bmatrix} b & 1 \\ -1 & b \end{bmatrix}$ then $A = \begin{bmatrix} b & 1 & 0 & a_4^1 & 0 \\ -1 & b & a_3^2 & a_4^2 & 0 \\ 0 & 0 & b-d & 1 & a_5^3 \\ 0 & 0 & -1 & b-d & a_5^4 \\ 0 & 0 & 0 & 0 & d \end{bmatrix}$.

1). If $d \neq 0$, then we have $g_{6,70}$ with $d \neq 0$.
2). Suppose $d = 0$.
1. If $a_4^1 = a_3^2$ then we have $g_{6,70}$ with $d = 0$.
Combining 1). and 2).1., we have $g_{6,70}$ in full generality.
2. If $a_4^1 \neq a_3^2$ then we have a class of algebras not obtained by Mubarakzyanov:

$$[e_3, e_5] = e_1, [e_4, e_5] = e_2, [e_1, e_6] = be_1 - e_2, [e_2, e_6] = e_1 + be_2,$$
$$[e_3, e_6] = e_2 + be_3 - e_4, [e_4, e_6] = e_3 + be_4. \tag{4}$$

In the table at the end of the paper this class is included in an expanded class which includes $g_{6,70}$ in [Mub2].

9. Nilradical isomorphic to $A_{5,2}$

9.1. Satisfying the Jacobi identity. Paragraph 5 in [Mub2] is located on pages 110 - 111 with summary on page 111. The nilradical is isomorphic to the five dimensional indecomposable nilpotent Lie algebra

$$[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3.$$

Hence the non-zero brackets in the six-dimensional algebra are

$$[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3, [e_i, e_6] = c_i^k e_k,$$

where $1 \leq i, k \leq 5$. Satisfying the Jacobi identity in the most general way, applying absorption, which is the transformation fixing all but e_6 such that $e_6' = e_6 - c_2^1 e_5 + c_5^1 e_2 + c_5^2 e_3 + c_5^3 e_4$ and denoting $c_4^4 = a$ and $c_5^5 = b$, we find that C, which is obtained from $-ad(e_6)$ by eliminating the last row and column of zeroes, is of the following form:

$$C = \begin{bmatrix} a+3b & 0 & c_3^1 & c_4^1 & 0 \\ 0 & a+2b & 0 & c_3^1 & 0 \\ 0 & 0 & a+b & 0 & 0 \\ 0 & 0 & 0 & a & c_5^4 \\ 0 & 0 & 0 & 0 & b \end{bmatrix},$$

and the algebra is reduced to

$$[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3, [e_1, e_6] = (a+3b)e_1, [e_2, e_6] = (a+2b)e_2,$$
$$[e_3, e_6] = c_3^1 e_1 + (a+b)e_3, [e_4, e_6] = c_4^1 e_1 + c_3^1 e_2 + a e_4, [e_5, e_6] = c_5^4 e_4 + b e_5.$$

Since we are considering solvable not nilpotent Lie algebras, C cannot be nilpotent and there are the following cases to consider.

9.2. Corresponding Lie algebras.

(i) Suppose $b \neq 0$.
1). If $h \neq 1$, then we have a special case of $g_{6,71}$ with $h \neq 1$.
2). If $h = 1$, then there are the following cases:
1. If $c_5^4 = 0$, then we obtain a special case of $g_{6,71}$ with $h = 1$.
Together these cases of $g_{6,71}$ give the algebra $g_{6,71}$ in full generality.
2. If $c_5^4 \neq 0$, then we obtain $g_{6,72}$.

(ii) If $a = 0$, then $b \neq 0$ and we have the algebra $g_{6,71}$ with $h = 0$.

(iii) If $b = 0$, then $a \neq 0$ and there are the following cases and subcases to consider:
1) Suppose $c_3^1 = 0$.

1. If $c_4^1 = 0$, then we have $g_{6,74}$.
2. If $c_4^1 \neq 0$, then we have $g_{6,75}$.

2) If $c_3^1 \neq 0$, then we have an algebra that contains $g_{6,73}$ as a special case not obtained by Mubarakzyanov and given by:

$$[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3, [e_1, e_6] = e_1, [e_2, e_6] = e_2,$$
$$[e_3, e_6] = \epsilon e_1 + e_3, [e_4, e_6] = d e_1 + \epsilon e_2 + e_4, (\epsilon = \pm 1).$$

Remark 9.1. L. Snobl and P. Winternitz in [Snob1] considered the problem of extending nilradicals of type $A_{5,2}$ to solvable algebras. In fact they considered much more generally starting with a "standard" filiform algebra of dimension n where $n \geq 4$ of which $A_{4,1}$ and $A_{5,2}$ are the first two in the series. The overlap with [Mub2] and the present paper occurs for $n = 5$ and is represented in Theorem 2 on page 10 (op. cit.). As such there are six classes of algebra $\mathfrak{s}_{6,i}$ for $1 \leq i \leq 6$ of which the first depends on a single parameter β and the last depends apparently on two denoted by a_3 and a_4. However, the first three collectively constitute precisely $g_{6,71}$; it is only necessary to replace e_6 by its negative and β by h in [Mub2] or a in the present paper. The division into separate cases is an artefact of the derivation but in retrospect seems to be artificial. Similarly $\mathfrak{s}_{6,4}$ and $\mathfrak{s}_{6,5}$ in [Snob1] correspond easily to $g_{6,74}$ and $g_{6,72}$, respectively. For $\mathfrak{s}_{6,6}(a_3, a_4)$ the algebra has brackets given by

$$[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3, [e_1, e_6] = e_1, [e_2, e_6] = e_2,$$
$$[e_3, e_6] = a_3 e_1 + e_3, [e_4, e_6] = a_4 e_1 + a_3 e_2 + e_4.$$

Now it is asserted in [Snob1] that it may be assumed that $a_4 = 1$ if different from zero leaving the single parameter a_3. If we make a diagonal change of basis by mapping $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ to $\{\lambda^3 e_1, \lambda^2 e_2, \lambda e_3, e_4, \lambda e_5, e_6\}$ the only effect is to change a_3 and a_4 to $\frac{a_3}{\lambda^2}$ and $\frac{a_4}{\lambda^3}$, respectively. We chose to normalize a_3 if different from zero to ± 1 and leave a_4 as a free parameter if $a_3 \neq 0$, if not normalize it to one obtaining this way the algebras $g_{6,73}{}^{***}$ and $g_{6,75}$: otherwise we are in agreement with [Snob1] allowing for slightly different but inessential normalizations.

10. Nilradical isomorphic to $A_{5,3}$

10.1. Satisfying the Jacobi identity. Paragraph 6 in [Mub2] is on page 111. The nilradical is isomorphic to the five dimensional nilpotent indecomposable Lie algebra $A_{5,3}$ for which the non-zero brackets are:

$$[e_2, e_4] = e_3, [e_2, e_5] = e_1, [e_4, e_5] = e_2.$$

We prefer to take it in the form:

$$[e_1, e_4] = e_3, [e_1, e_5] = e_2, [e_4, e_5] = e_1.$$

Hence the non-zero brackets are $[e_1, e_4] = e_3, [e_1, e_5] = e_2, [e_4, e_5] = e_1, [e_i, e_6] =$ $= a_i^k e_k$ where $1 \leq i, k \leq 5$.

Satisfying the Jacobi identity in the most general way and denoting $a_4^4 = a$ and $a_5^5 = b$, applying the transformation $e_6' = e_6 + a_5^2 e_1 - a_1^3 e_4 - a_1^2 e_5$ which keeps e_1, e_2, e_3, e_4 and e_5 fixed, and replacing $-a_5^2 + a_4^3$ by a_4^3, we find that A, which is obtained from $-ad(e_6)$ by eliminating the last row and column of zeroes, is

$$A = \begin{bmatrix} a+b & 0 & 0 & 0 & 0 \\ 0 & a+2b & a_3^2 & a_4^2 & 0 \\ 0 & a_2^3 & 2a+b & a_4^3 & a_5^3 \\ 0 & 0 & 0 & a & a_2^3 \\ 0 & 0 & 0 & a_3^2 & b \end{bmatrix},$$

and the algebra is reduced to

$$\begin{aligned}
&[e_1,e_4]=e_3,[e_1,e_5]=e_2,[e_4,e_5]=e_1,[e_1,e_6]=(a+b)e_1,\\
&[e_2,e_6]=(a+2b)e_2+a_2^3e_3,[e_3,e_6]=a_3^2e_2+(2a+b)e_3,\\
&[e_4,e_6]=a_4^2e_2+a_4^3e_3+ae_4+a_3^2e_5,[e_5,e_6]=a_5^3e_3+a_2^3e_4+be_5.
\end{aligned}\tag{5}$$

Proposition 10.1. *Let P be the change of basis matrix* $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1+yz}{x} & z & 0 & 0 & 0\\ 0 & y & x & 0 & 0 & 0\\ 0 & 0 & 0 & x & y & 0\\ 0 & 0 & 0 & z & \frac{1+yz}{x} & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$ *acting on* (5)*. Then P preserves the nilradical and acts by conjugation on* $B_1 = \begin{bmatrix} a+2b & a_3^2\\ a_2^3 & 2a+b\end{bmatrix}$ *and* $B_2=\begin{bmatrix} a & a_2^3\\ a_3^2 & b\end{bmatrix}$*, which are diagonal blocks of*

$$A=\begin{bmatrix} a+b & 0 & 0 & 0 & 0\\ 0 & a+2b & a_3^2 & a_4^2 & 0\\ 0 & a_2^3 & 2a+b & a_4^3 & a_5^3\\ 0 & 0 & 0 & a & a_2^3\\ 0 & 0 & 0 & a_3^2 & b\end{bmatrix}.$$

It is necessary to modify e_6 by adding a multiple of e_1 in order to remove the e_2-component in $[e_5,e_6]$ so as to add no new terms to brackets in the Lie algebra under such a transformation.

Since B_1 and B_2 depend on the same four numbers, to obtain the Lie algebras it suffices to consider all possible Jordan 2×2 normal forms of B_2 and obtain corresponding forms of B_1. Let us take those forms in a lower triangular form with some entries scaled to unity using the fact that we can scale e_6: note that B_2 and hence B_1 cannot be nilpotent or else we will have a nilpotent Lie algebra. Thus we have the following three cases:

$$(i)\, d_1=\begin{bmatrix} a & 0\\ 0 & 1\end{bmatrix},\ (ii)\, d_2=\begin{bmatrix} 1 & 0\\ 1 & 1\end{bmatrix},\ (iii)\, d_3=\begin{bmatrix} a & -1\\ 1 & a\end{bmatrix}.$$

10.2. Corresponding Lie algebras.

(i) If $d_1=\begin{bmatrix} a & 0\\ 0 & 1\end{bmatrix}$ then $A=\begin{bmatrix} a+1 & 0 & 0 & 0 & 0\\ 0 & a+2 & 0 & a_4^2 & 0\\ 0 & 0 & 2a+1 & a_4^3 & a_5^3\\ 0 & 0 & 0 & a & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}.$

1). If $a(a+1)\neq 0$ then we obtain a special case of $g_{6,76}$ where we have used a instead of h in [Mub2].
2). Suppose $a=0$.
1. If $a_5^3\neq 0$ then we obtain a Lie algebra isomorphic to $g_{6,77}$.

Remark 10.2. The Lie algebra $g_{6,77}$ in Mubarakzyanov's paper is isomorphic to $g_{6,81}$ which is easy to see if we apply the transformation, which fixes e_6 and such that $e_1'=-e_3$, $e_2'=-e_2$, $e_3'=-e_1$, $e_4'=e_5$ and $e_5'=e_4$.

2. If $a_5^3=0$ then we obtain a special case of $g_{6,76}$ for which $a=0$.

Remark 10.3. In algebra $g_{6,76}$ we may assume that $|a| \leq 1$. Indeed if $|a| > 1$ define $e_1' = -e_3, e_2' = -e_2, e_3' = -e_1, e_4' = e_5, e_5' = e_4, e_6' = \frac{1}{a}e_6$. We end up with $g_{6,76}$ in exactly the same form but with a replaced by $\frac{1}{a}$.

Remark 10.4. Algebra $g_{6,80}$ is isomorphic to a special case of $g_{6,76}$ with a (or h) $= 0$. In order to see this, apply the transformation which fixes e_6 and such that $e_1' = -e_3$, $e_2' = -e_2$, $e_3' = -e_1$, $e_4' = e_5$ and $e_5' = e_4$, which gives

$$[e_2, e_4] = e_3, [e_2, e_5] = e_1, [e_4, e_5] = e_2, [e_1, e_6] = e_1, [e_2, e_6] = e_2,$$
$$[e_3, e_6] = 2e_3, [e_4, e_6] = e_4.$$

3) Suppose $a = -1$.
1. If $a_4^3 \neq 0$ then we obtain algebra $g_{6,78}$.
2. If $a_4^3 = 0$ then we obtain a special case of $g_{6,76}$ for which $a = -1$. Collectively the special cases of $g_{6,76}$ give $g_{6,76}$ in [Mub2] in full generality.

(ii) If $d_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ applying a change of basis matrix which removes a_4^2, a_4^3 and a_5^3, we obtain $g_{6,79}$.

(iii) If $d_3 = \begin{bmatrix} a & -1 \\ 1 & a \end{bmatrix}$ and $a \neq 0$ then we obtain a new algebra which is not given in [Mub2] and that is denoted by $n_{6,83}$ in [Shab] or in the tables below by $g_{6,80}'$:

$$[e_1, e_4] = e_3, [e_1, e_5] = e_2, [e_4, e_5] = e_1, [e_1, e_6] = 2ae_1, [e_2, e_6] = 3ae_2 - e_3,$$
$$[e_3, e_6] = e_2 + 3ae_3, [e_4, e_6] = ae_4 + e_5, [e_5, e_6] = -e_4 + ae_5.$$

Suppose $a = 0$.
1. If $a_4^2 \neq a_5^3$, then we obtain a new algebra denoted by $n_{6,84}$ in [Shab], which is not obtained in [Mub2] or $g_{6,81}'$:

$$[e_1, e_4] = e_3, [e_1, e_5] = e_2, [e_4, e_5] = e_1, [e_2, e_6] = -e_3, [e_3, e_6] = e_2,$$
$$[e_4, e_6] = e_5, [e_5, e_6] = \epsilon e_3 - e_4, (\epsilon = \pm 1).$$

2. If $a_4^3 \neq 0$ and $a_4^2 = a_5^3$ then we have a special case of $n_{6,83}$ [Shab] so that we obtain $g_{6,80}'$ for all values of a.

11. Nilradical isomorphic to $A_{5,4}$

11.1. Satisfying the Jacobi identity. Paragraph 7 is located on pages 112 - 114 with summary on page 115. The nilradical is isomorphic to $[e_2, e_4] = e_1, [e_3, e_5] = e_1$. Hence the non-zero brackets are

$$[e_2, e_4] = e_1, [e_3, e_5] = e_1, [e_i, e_6] = c_i^k e_k, 1 \leq i, k \leq 5.$$

Satisfying the Jacobi identity in the most general way denoting c_1^1, c_2^2 and c_3^3 by α, β and γ, respectively, and applying the transformation which fixes all but e_6

such that $e_6' = e_6 + c_4^1 e_2 + c_5^1 e_3 - c_2^1 e_4 - c_3^1 e_5$, we find that $-ad(e_6)$ which we denote by C is of the following form:

$$C = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \beta & c_3^2 & c_4^2 & c_4^3 \\ 0 & c_2^3 & \gamma & c_4^3 & c_5^3 \\ 0 & c_2^4 & c_2^5 & \alpha-\beta & -c_2^3 \\ 0 & c_2^5 & c_3^5 & -c_3^2 & \alpha-\gamma \end{bmatrix}$$

where the lower 4×4 block is "almost Hamiltonian" and the algebra is reduced to

$$[e_2, e_4] = e_1, [e_3, e_5] = e_1, [e_1, e_6] = \alpha e_1, [e_2, e_6] = \beta e_2 + c_2^3 e_3 + c_2^4 e_4 + c_2^5 e_5,$$
$$[e_3, e_6] = c_3^2 e_2 + \gamma e_3 + c_2^5 e_4 + c_3^5 e_5, [e_4, e_6] = c_4^2 e_2 + c_4^3 e_3 + (\alpha - \beta) e_4 - c_3^2 e_5,$$
$$[e_5, e_6] = c_4^3 e_2 + c_5^3 e_3 - c_2^3 e_4 + (\alpha - \gamma) e_5. \tag{6}$$

Before proceeding let us note that the linear transformation R that changes the signs of e_1, e_2, e_3 induces an automorphism of the nilradical and changes the matrix C into C' where

$$C' = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \beta & c_3^2 & -c_4^2 & -c_4^3 \\ 0 & c_2^3 & \gamma & -c_4^3 & -c_5^3 \\ 0 & -c_2^4 & -c_2^5 & \alpha-\beta & -c_2^3 \\ 0 & -c_2^5 & -c_3^5 & -c_3^2 & \alpha-\gamma \end{bmatrix}$$

We shall make use of the map R later to normalize the algebras.

A *Hamiltonian* matrix M is any real $2n \times 2n$ matrix M that satisfies the condition that JM is symmetric, where J is the skew-symmetric matrix $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ and I_n is the $n \times n$ identity matrix. In other words, M is Hamiltonian if and only if $JM - M^T J^T = JM + M^T J = 0$. In the vector space of all $2n \times 2n$ matrices, Hamiltonian matrices form a subspace of dimension $2n^2 + n$. They are otherwise known as infinitesimal symplectic matrices. If M is a $2n \times 2n$ block matrix given by $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, and D are $n \times n$ matrices then M is a Hamiltonian matrix provided that the matrices B and C are symmetric and that $A + D^T = 0$.

Returning to the matrix C derived from $-ad(e_6)$ above, if we subtract from it the matrix $\frac{\alpha}{2} I$, where I is the 5×5 identity matrix, then we obtain

$$\widetilde{C} = \begin{bmatrix} \frac{\alpha}{2} & 0 & 0 & 0 & 0 \\ 0 & \beta - \frac{\alpha}{2} & c_3^2 & c_4^2 & c_4^3 \\ 0 & c_2^3 & \gamma - \frac{\alpha}{2} & c_4^3 & c_5^3 \\ 0 & c_2^4 & c_2^5 & -(\beta - \frac{\alpha}{2}) & -c_2^3 \\ 0 & c_2^5 & c_3^5 & -c_3^2 & -(\gamma - \frac{\alpha}{2}) \end{bmatrix}$$

where now the 4×4 lower block is Hamiltonian.

We refer to a paper [Will] by J. Williamson for the list of all possible Hamiltonian matrices of order four. It is important to understand that two matrices can be equivalent under the general linear group but not via the symplectic group: an easy example is given by the pair $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. In terms of finding the symplectic normal form of a matrix we remind the reader of the key fact that eigenvectors corresponding to eigenvalues λ and μ are symplectically orthogonal if $\lambda + \mu \neq 0$.

Below we list all the possible elementary divisors of a Hamiltonian matrix:
(i) $\lambda \pm a$, $\lambda \pm b$ (ii) $\lambda \pm a \pm \imath b$ (iii) $\lambda \pm \imath a$, $\lambda \pm \imath b$ (iv) $\lambda \pm a$, $\lambda \pm \imath b$
(v) $(\lambda \pm a)^2$ (vi) $(\lambda \pm \imath a)^2$ (vii) λ^2, $\lambda \pm \imath a$ $(viii)$ λ^2, $\lambda \pm a$ (ix) λ^4
(x) λ^2, λ^2.
In fact Williamson [Will] considered the equivalence of symmetric matrices under a symplectic change. If M is Hamiltonian and we define $g := JM$ then $g^t = M^t J^t = -M^t J = JM$ so that g is symmetric. Conversely, if g is symmetric then $M := -Jg$ will be Hamiltonian.

11.2. 4×4 **Hamiltonian matrices corresponding to the elementary divisors.**

We list below the Hamiltonian matrices that are dictated by the possible elementary divisors *up to a sign* since in the Lie algebras we have the freedom to scale e_6. In those matrices b is assumed to be different from zero and $\epsilon = \pm 1$. Case (x) appears as the limiting case of (vi) for which $a = 0$.

$$(i) \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -b \end{bmatrix} \quad (ii) \begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & -a & b \\ 0 & 0 & -b & -a \end{bmatrix} \quad (iii) \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ -a & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \end{bmatrix} (a \neq 0)$$

$$(iv) \begin{bmatrix} 0 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \end{bmatrix} \quad (v) \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & -1 & -a \end{bmatrix} \quad (vi) \begin{bmatrix} 0 & a & 1 & 0 \\ -a & 0 & 0 & \epsilon \\ 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \end{bmatrix}$$

$$(vii) \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} (a \neq 0) \quad (viii) \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (ix) \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Remark 11.1. Notice that the matrix (vi) with $\epsilon = -1$ could be eliminated, because if $a = 0$ then (vi) with $\epsilon = -1$ is symplectically equivalent to (v) with $a = 0$ applying the change of basis $e'_1 = e_1 + e_2, e'_2 = e_3 - e_4, e'_3 = \frac{e_3}{2} + \frac{e_4}{2}, e'_4 = = \frac{e_2}{2} - \frac{e_1}{2}$. If $a \neq 0$ then (ii) with $a = 0$ and $b := a$ is symplectically equivalent to (vi) with $a \neq 0$ and $\epsilon = -1$ applying the change of basis $e'_1 = -2ae_2, e'_2 = = 2ae_1, e'_3 = e_1 - \frac{e_4}{2a}, e'_4 = \frac{e_3}{2a} - e_2$.

11.3. Corresponding Lie algebras.

(i) Then $C = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{2}+a & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha}{2}+b & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha}{2}-a & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha}{2}-b \end{bmatrix}$, $(\alpha^2 + a^2 + b^2 \neq 0)$. This case is $g_{6,82}$ with

$\lambda = a$ and $\lambda_1 = b$. The following maps determined by their effect on basis vectors $\{e_2, e_3, e_4, e_5\} \mapsto \{e_2, -e_5, e_4, e_3\}$ and $\{e_2, e_3, e_4, e_5\} \mapsto \{e_3, e_2, e_5, e_4\}$ enable us to assume that $0 \leq a \leq b$ and if $\alpha = 0$ that $b = 1, 0 \leq a \leq 1$.

(ii) Then $C = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{2}+a & b & 0 & 0 \\ 0 & -b & \frac{\alpha}{2}+a & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha}{2}-a & b \\ 0 & 0 & 0 & -b & \frac{\alpha}{2}-a \end{bmatrix}$. By scaling we may also assume that $b = 1$. Thus we obtain $g_{6,88}$ where $\mu_0 = a$ and $\nu_0 = -1$. Using the map determined by its effect on basis vectors $\{e_2, e_3, e_4, e_5\} \mapsto \{e_4, e_5, -e_2, -e_3\}$ we may further reduce to the case $a \geq 0$.

(iii) Then $C = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{2} & 0 & a & 0 \\ 0 & 0 & \frac{\alpha}{2} & 0 & b \\ 0 & -a & 0 & \frac{\alpha}{2} & 0 \\ 0 & 0 & -b & 0 & \frac{\alpha}{2} \end{bmatrix}$, $(ab \neq 0)$.

1). If $\alpha \neq 0$ then again we may assume that $\alpha = 2$ and so

$$[e_2, e_4]{=}e_1, [e_3, e_5]{=}e_1, [e_1, e_6]{=}2e_1, [e_2, e_6]{=}e_2{-}ae_4, [e_3, e_6]{=}e_3{-}be_5,$$
$$[e_4, e_6] = ae_2 + e_4, [e_5, e_6] = be_3 + e_5, \tag{7}$$

which is a Lie algebra not obtained by Mubarakzyanov. Using the map determined by $\{e_2, e_3, e_4, e_5\} \mapsto \{e_3, -e_2, e_5, -e_4\}$ we may assume that $0 < < a \leq |b|$.

2). If $\alpha = 0$ then we may assume that $a = 1$ by scaling and so

$$[e_2, e_4] = e_1, [e_3, e_5] = e_1, [e_2, e_6] = -e_4, [e_3, e_6] = -be_5, [e_4, e_6] = e_2,$$
$$[e_5, e_6] = be_3,$$

which is another algebra not obtained by Mubarakzyanov. Again we may assume that $0 < |b| \leq 1$.

(iv) Then $C = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{2} & 0 & b & 0 \\ 0 & 0 & \frac{\alpha}{2}+a & 0 & 0 \\ 0 & -b & 0 & \frac{\alpha}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha}{2}-a \end{bmatrix}$, $(b \neq 0)$.

The map R that follows eqn(11.1) and the linear map for which $\{e_3, e_5\} \mapsto \{-e_5, e_3\}$ change the signs of b and a, respectively so we may assume that $a \geq 0$ and $b > 0$. Therefore we can reduce to the cases $\alpha = 2, a \geq 0, b > 0$ or $\alpha = 0$ and $a = 1$, $b > 0$ or $a = 0$, $b = 1$. We obtain in all cases an algebra isomorphic to $g_{6,89}$ with $s = a$ and $\nu_0 = b$.

(v) Then $C = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{2}+a & 1 & 0 & 0 \\ 0 & 0 & \frac{\alpha}{2}+a & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha}{2}-a & 0 \\ 0 & 0 & 0 & -1 & \frac{\alpha}{2}-a \end{bmatrix}$, $(\alpha^2 + a^2) \neq 0$ and we have $g_{6,83}$. We note also that the transformation determined by mapping $\{e_2, e_3, e_4, e_5\}$ to $\{e_5, -e_4, -e_3, e_2\}$ allows us to assume that $a \geq 0$. Therefore we can reduce to the cases $\alpha = 2, a \geq 0$ and $\alpha = 0, a = 1$.

Remark 11.2. Algebra $g_{6,86}$ in [Mub2] is a special case of $g_{6,83}$ with $\lambda = 0$ and $\alpha = 2$.

(vi) Then $C = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{2} & a & 1 & 0 \\ 0 & -a & \frac{\alpha}{2} & 0 & 1 \\ 0 & 0 & 0 & \frac{\alpha}{2} & a \\ 0 & 0 & 0 & -a & \frac{\alpha}{2} \end{bmatrix}$, $(\alpha^2 + a^2 \neq 0)$. The transformation given by changing the signs of e_3 and e_5 has the effect of changing a into $-a$; therefore we may assume that $a \geq 0$.

We can reduce to the cases $\alpha = 2, a \geq 0$ by replacing e_3 and e_5 by their negatives if necessary and $\alpha = 0, a = 1$ by scaling.

The map determined by mapping

$$\{e_1, e_2, e_3, e_4, e_5, e_6\} \text{ to } \{-e_1, e_4, e_3, e_2, -e_5, e_6\}$$

produces the amended form of algebra $g_{6,93}$: see the following remark.

Remark 11.3. The algebra $g_{6,93}$ in [Mub2] differs from ours as follows. Instead of the brackets $[e_4, e_6] = e_2 - \nu_0 e_3 + \frac{\alpha}{2} e_4, [e_5, e_6] = -\nu_0 e_2 + \frac{\alpha}{2} e_5$ we prefer $[e_4, e_6] = -\nu_0 e_3 + \frac{\alpha}{2} e_4, [e_5, e_6] = \frac{\alpha}{2} e_5 - \nu_0 e_2 - e_3$. Note the difference in the location of the term "1" in the C-matrix. In [Mub2] the eigenvalues of the A matrix entail that it could belong to several of the ten types of Hamiltonian matrices listed above whereas in our version A is unambiguously of type (vi). The position of the 1 in the matrix C affects the spectrum and with Mubarakzyanov's choice does not necessarily lead to repeated pairs of pure imaginary eigenvalues.

(vii) If $A = \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ then $C = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{2} & 0 & a & 0 \\ 0 & 0 & \frac{\alpha}{2} & 0 & 1 \\ 0 & -a & 0 & \frac{\alpha}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha}{2} \end{bmatrix}$ and $a \neq 0$. Thus we obtain the amended form of $g_{6,90}$ (see remark immediately following) in which, without loss of generality, it may be assumed that $\alpha = 2$. If $\alpha = 0$ then it may be assumed that $a = \pm 1$ and we obtain the amended form of $g_{6,91}$ as the special case and $g_{6,90}$ in full generality. Again by using a diagonal transformation it is possible to preserve the "1" in matrix C but it is not possible to simultaneously normalize a so as make it non-negative. In both $g_{6,90}$ and $g_{6,91}$ it is necessary to permute e_2 and e_3 and e_4 and e_5, respectively.

Remark 11.4. When Mubarakzyanov considered this case he made an error. His matrix six lines from the bottom on page 113 [Mub2] should read $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \nu_0 \\ 0 & 0 & 0 & 0 \\ 0 & -\nu_0 & 0 & 0 \end{bmatrix}$ rather than $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \nu_0 \\ 1 & 0 & 0 & 0 \\ 0 & -\nu_0 & 0 & 0 \end{bmatrix}$. As a result the algebra $g_{6,90}$ is incorrect as too is $g_{6,91}$ which is a limiting case of $g_{6,90}$. In addition in going from his matrix to $g_{6,90}$ Mubarakzyanov changes the sign of ν_0. The key point is that the bracket $[e_2, e_6]$ should not contain an e_4-term.

(viii) If $A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ then $C = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{2}+a & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha}{2} & 0 & 1 \\ 0 & 0 & 0 & \frac{\alpha}{2}-a & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha}{2} \end{bmatrix}$ and $\alpha^2 + a^2 \neq 0$.

1. If $\alpha = 0$ then $a \neq 0$, otherwise the Lie algebra is nilpotent, and after scaling a to unity, we obtain $g_{6,84}$.

2. If $\alpha \neq 0$ then α may be reduced to 2 and we obtain $g_{6,85}$. Using the map determined by its effect on basis vectors $\{e_2, e_4\} \mapsto \{-e_4, e_2\}$ we may further reduce to the case $a \geq 0$.

(ix) If $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ then $C = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{\alpha}{2} & 0 & 1 \\ 0 & 0 & 0 & \frac{\alpha}{2} & 0 \\ 0 & 0 & 0 & -1 & \frac{\alpha}{2} \end{bmatrix}$ and $\alpha \neq 0$ otherwise the Lie algebra is nilpotent. We note that although there is an ambiguity in the symplectic Jordan normal form in case (ix), we can reduce to the case $\epsilon = 1$ using the map R that appears just after eqn.(6). This case reduces to the algebra $g_{6,87}$. We obtain the form given in [Mub2] by permuting e_2 and e_4 and then changing the sign of e_2.

Remark 11.5. The algebra $g_{6,92}$ in [Mub2]

$$[e_2,e_4]{=}e_1, [e_3,e_5]{=}e_1, [e_1,e_6]{=}\alpha e_1, [e_2,e_6]{=}\frac{\alpha}{2}e_2{+}\nu_0 e_3, [e_3,e_6]{=}-\mu_0 e_2+\frac{\alpha}{2}e_3,$$
$$[e_4,e_6]=\frac{\alpha}{2}e_4+\mu_0 e_5, [e_5,e_6]=-\nu_0 e_4+\frac{\alpha}{2}e_5,\ (\alpha^2+\mu_0^2+\nu_0^2\neq 0)$$

is isomorphic to a special case of $g_{6,88}$ with $\mu_0 = 0$ provided that $\mu_0\nu_0 > 0$ or a special case of $g_{6,82}$ provided that $\mu_0\nu_0 < 0$ or $\mu_0 = \nu_0 = 0$ or $g_{6,83}$ provided that only one of μ_0 and ν_0 is non-zero. To see that, assuming that $\mu_0\nu_0 > 0$, apply the transformation, which fixes all but e_3 and e_5 such that $e_3' = \sqrt{\frac{\nu_0}{\mu_0}}e_3$ and $e_5' = \sqrt{\frac{\mu_0}{\nu_0}}e_5$.

We have

$$[e_2,e_4]=e_1, [e_3,e_5]=e_1, [e_1,e_6]=\alpha e_1, [e_2,e_6]=\frac{\alpha}{2}e_2+\sqrt{\mu_0\nu_0}e_3,$$
$$[e_3,e_6]=-\sqrt{\mu_0\nu_0}e_2+\frac{\alpha}{2}e_3, [e_4,e_6]{=}\frac{\alpha}{2}e_4+\sqrt{\mu_0\nu_0}e_5, [e_5,e_6]{=}-\sqrt{\mu_0\nu_0}e_4+\frac{\alpha}{2}e_5.$$

If, however, $\mu_0\nu_0 < 0$ we may assume that $\nu_0 > 0$ and $\mu_0 < 0$ by changing the signs of e_3 and e_5 simultaneously, if necessary. Then we map $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ to $\{2\sqrt{-\mu_0\nu_0}e_1, -\sqrt{-\mu_0}e_2{+}\sqrt{\nu_0}e_3, -\sqrt{\nu_0}e_4{-}\sqrt{-\mu_0}e_5, -\sqrt{\nu_0}e_4 +\sqrt{-\mu_0}e_5, \sqrt{-\mu_0}e_2 + \sqrt{\nu_0}e_3, e_6\}$ and we obtain algebra $g_{6,82}$ with $\lambda = \lambda_1 = -\sqrt{-\mu_0\nu_0}$.

If $\mu_0 = 0$ and $\nu_0 \neq 0$ put $e_3' = \nu_0 e_3, e_5' = \frac{1}{\nu_0}e_5$ so as to obtain algebra $g_{6,83}$. The case $\mu_0 \neq 0$ and $\nu_0 = 0$ is similar and in case $\mu_0 = \nu_0 = 0$ we have a special case of $g_{6,82}$ immediately.

Remark 11.6. The idea of extending the same nilradical to a six-dimensional solvable indecomposable Lie algebras by adding one basis element was considered by J.L. Rubin and P. Winternitz in [Rub]. The authors distinguish nineteen types of Hamiltonian matrices over $\mathbb{R}$, which are symplectically equivalent to the Hamiltonian matrices $(i)-(ix)$ given above: to be specific following the notation of [Rub], $F_1(1,b,c)\&F_2(0,1,c) \simeq (i); R_{15}(1,b,c)\&R_{16}(0,1,c) \simeq (ii); R_{13}(1,b,c)\&R_{14}(0,1,c) \simeq (iii); R_9(1,b,c)\&R_{10}(0,b,1) \simeq (iv); F_6(1,b)\&F_7(0,1) \simeq (v); F_3(1) \simeq (vi)(a=0,\epsilon=1); R_{17}(1,b)\&R_{18}(0,1) \simeq (vi)(a\neq 0,\epsilon=1); R_{11}(1,b)$

$\&R_{12}(0,1) \simeq (vii); F_4(1,b)\&F_5(0,1) \simeq (viii); F_8(1) \simeq (ix)\,(\epsilon = -1)$. Notice $R_{19}(1)$ is actually symplectically equivalent to $F_6(1,0)$. Finally we achieve an improvement in the range of the parameters in $g_{6,89}$ as compared with [Rub].

12. Nilradical isomorphic to $A_{5,5}$

12.1. Satisfying the Jacobi identity. Paragraph 8 in [Mub2] is on page 114. The nilradical is isomorphic to the five dimensional nilpotent indecomposable Lie algebra given by

$$[e_3,e_4]=e_1,[e_2,e_5]=e_1,[e_3,e_5]=e_2,$$

which does not give an upper triangular form for the adjoint representation. To make it upper triangular we prefer to take the nilradical in the form:

$$[e_3,e_4]=e_1,[e_2,e_5]=e_1,[e_4,e_5]=e_2.$$

Therefore the non-zero brackets in the six-dimensional algebra are

$$[e_3,e_4]=e_1,[e_2,e_5]=e_1,[e_4,e_5]=e_2,[e_i,e_6]=d_i^k e_k\,(1\le i,k\le 5).$$

Satisfying the Jacobi identity in the most general way, denoting d_4^4 by a and d_5^5 by b, respectively, applying the transformation, which fixes all but e_6 such that $e_6' = e_6 + d_5^1 e_2 + d_4^1 e_3 - d_3^1 e_4 + (d_5^3 - d_4^2)e_5$ and replacing $d_3^1 + d_5^2$ by d_5^2 we find that D, obtained from $-ad(e_6)$ by eliminating the last row and column, is of the following form:

$$D = \begin{bmatrix} a+2b & 0 & 0 & 0 & 0 \\ 0 & a+b & -d_5^4 & d_5^3 & d_5^2 \\ 0 & 0 & 2b & d_4^3 & d_5^3 \\ 0 & 0 & 0 & a & d_5^4 \\ 0 & 0 & 0 & 0 & b \end{bmatrix},$$

and the algebra has been reduced to

$$[e_3,e_4]=e_1,[e_2,e_5]=e_1,[e_4,e_5]=e_2,[e_1,e_6]=(a+2b)e_1,[e_2,e_6]=(a+b)e_2,$$
$$[e_3,e_6]=-d_5^4e_2+2be_3,[e_4,e_6]=d_5^3e_2+d_4^3e_3+ae_4,[e_5,e_6]=d_5^2e_2+d_5^3e_3+d_5^4e_4+be_5.$$

Note that not both a and b can be zero or else D and the entire algebra will be nilpotent.

12.2. Corresponding Lie algebras.

(i) If $b(a-b)(a-2b)\neq 0$ then we obtain a special case of $g_{6,94}$ with $\lambda \neq 1,2$.

(ii) Suppose $b\neq 0$ and $a=b$.
1. If $d_5^4=0$ then we obtain a special case of $g_{6,94}$ with $\lambda=1$.
2. If $d_5^4\neq 0$ then we have an algebra isomorphic to $g_{6,96}$.

(iii) Suppose $b\neq 0$ and $a=2b$.
1. If $d_4^3=0$ then we obtain a special case of $g_{6,94}$ with $\lambda=2$.
Together the special cases of $g_{6,94}$ give $g_{6,94}$ in full generality.
2. If $d_4^3\neq 0$ then we have the algebra isomorphic to $g_{6,97}$.

(iv) Suppose $a=0$ and $b\neq 0$.
1. If $2d_5^3d_5^4b+d_4^3\left(d_5^4\right)^2+d_5^2b^2=0$ then we obtain a special case of $g_{6,94}$ with $\lambda=0$.
2. If $2d_5^3d_5^4b+d_4^3\left(d_5^4\right)^2+d_5^2b^2\neq 0$ then we have an algebra isomorphic to $g_{6,95}$.

(v) If $b=0$ and $a\neq 0$, then we have an algebra isomorphic to $g_{6,98}$ with h reduced to zero or unity.

Remark 12.1. Consider $g_{6,98}$:

$$[e_3,e_4]=e_1,[e_2,e_5]{=}e_1,[e_3,e_5]{=}e_2,[e_1,e_6]{=}e_1,[e_2,e_6]{=}he_1+e_2,[e_3,e_6]=e_3,\\ [e_5,e_6]=he_4.$$

Assuming that $h\neq 0$ then the transformation that fixes e_3 and e_6, scales e_2 and e_5 by h and scales e_1 and e_4 by h^2 reduces h to unity in $g_{6,98}$. Thus in $g_{6,98}$ we can also assume that h is either zero or unity.

Remark 12.2. The problem of extending the same nilradical to six dimensional solvable Lie algebras but in a different basis, precisely, $[e_2,e_5]=e_1,[e_3,e_4]=e_1,[e_4,e_5]=-e_3$, by adding one basis element was considered by L. Snobl and D. Karasek in [Snob2]. Actually they considered much more general extensions but the one just mentioned is the only one of relevance to the Mubarakzyanov's algebras. It turns out that all their algebras are isomorphic to Mubarakzyanov's algebras, precisely, $\mathfrak{s}_{6,1}(\beta)$, $\beta\in\mathbb{R}\backslash\{0,-\frac{1}{2}\}\simeq g_{6,94}$ with $\lambda\neq -2,0$; $\mathfrak{s}_{6,2}\simeq g_{6,98}$ with $h=0$; $\mathfrak{s}_{6,4}\simeq g_{6,94}$ with $\lambda=-2$; $\mathfrak{s}_{6,5}\simeq g_{6,94}$ with $\lambda=0$; $\mathfrak{s}'_{6,6}\simeq g_{6,97}$; $\mathfrak{s}'_{6,7}\simeq g_{6,98}$ with $h\neq 0$; $\mathfrak{s}'_{6,8}\simeq g_{6,95}$; $\mathfrak{s}'_{6,9}\simeq g_{6,96}$.
However in order to make Mubarakzyanov's algebras look simpler we changed in the table parameter λ to a.

13. Nilradical isomorphic to $A_{5,6}$

13.1. Satisfying the Jacobi identity. Paragraph 9 in [Mub2] is on pages 114 and 116 in
Mubarakzyanov's paper. The nilradical is isomorphic to the five-dimensional, nilpotent indecomposable Lie algebra:

$$[e_3,e_4]=e_1,[e_2,e_5]=e_1,[e_3,e_5]=e_2,[e_4,e_5]=e_3.$$

Hence the non-zero brackets in the six-dimensional algebra are

$$[e_3,e_4]=e_1,[e_2,e_5]=e_1,[e_3,e_5]=e_2,[e_4,e_5]=e_3,[e_i,e_6]=a_i^ke_k,\ (1\leq i,k\leq 5).$$

Satisfying the Jacobi identity in the most general way and applying the transformation which fixes all basis vectors but e_6 and such that $e_6'=e_6+(a_5^3-a_4^2)e_4-$ $-a_3^2e_5+a_5^1e_2+a_5^2e_3$, we find that A, which is obtained from $-ad(e_6)$ by eliminating the last row and column, is of the following form:

$$A = \begin{bmatrix} 5a_5^5 & a_5^4 & 0 & a_4^1 - a_5^2 & 0 \\ 0 & 4a_5^5 & 0 & a_4^2 & 0 \\ 0 & 0 & 3a_5^5 & 0 & a_4^2 \\ 0 & 0 & 0 & 2a_5^5 & a_5^4 \\ 0 & 0 & 0 & 0 & a_5^5 \end{bmatrix},$$

and the algebra has been reduced to

$$[e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3, [e_1, e_6] = 5a_5^5 e_1,$$
$$[e_2, e_6] = a_5^4 e_1 + 4a_5^5 e_2, [e_3, e_6] = 3a_5^5 e_3, [e_4, e_6] = (a_4^1 - a_5^2)e_1 + a_4^2 e_2 + 2a_5^5 e_4,$$
$$[e_5, e_6] = a_4^2 e_3 + a_5^4 e_4 + a_5^5 e_5.$$

Since the Lie algebra is not nilpotent then $a_5^5 \neq 0$, and applying a change of basis matrix we obtain $\mathfrak{g}_{6,99}$.

Remark 13.1. In [Snob3] L. Snobl and P. Winternitz considered extending a certain nilpotent Lie algebra to a solvable algebra. In the lowest dimension their results correspond to extending the nilpotent algebra $A_{5,6}$ to $\mathfrak{g}_{6,99}$ as described above and in [Mub2].

14. Corrected table of Mubarakzyanov algebras

In this Section we give an amended version of Mubarakzyanov's list of algebras. We have simplified the notation using for the parameters a, b, c, d. Also ϵ can only assume the values of 0 or ± 1 and δ can only be 0 or 1 and α can only be 0 or 2 except in $\mathfrak{g}_{6,88}$. We do not consider ϵ to be a parameter since it only assumes discrete values. At the start of each table we supply the brackets for the nilradical and in the table the brackets of the basis vectors with e_6. We do not highlight algebras where we merely improve the range of values of the parameters.

- Algebras below that have no asterisks are considered to be essentially correct and differ from Mubarakzyanov's algebras mutatis mudandis, that is, only by small notational differences
- Algebras that have one asterisk denote algebras in which the parameters in Mubarakzyanov's algebra can be simplified or an entry can be removed completely
- Algebras that have two asterisks denote algebras in which there is a serious computational or merely typographical mistake
- Algebras that have three asterisks denote algebras which do not appear at all in Mubarakzyanov's list
- Algebras that have four asterisks denote algebras which are redundant
- Algebras that are marked with † are suspensions in the sense of Section 3
- Algebras that are marked with $'$ supersede a Mubarakzyanov algebra that is problematic.

Lie algebras having nilradical isomorphic to $\mathbb{R}^5$.

Algebra	$[e_1, e_6]$	$[e_2, e_6]$	$[e_3, e_6]$	$[e_4, e_6]$	$[e_5, e_6]$	Conditions on parameters
$\dagger\mathfrak{g}_{6,1}$	e_1	ae_2	be_3	ce_4	de_5	$0 < \lvert d\rvert \le \lvert c\rvert \le \lvert b\rvert \le \lvert a\rvert \le 1$
$\dagger\mathfrak{g}_{6,2}$	ae_1	$e_1 + ae_2$	e_3	be_4	ce_5	$0 < \lvert c\rvert \le \lvert b\rvert \le 1$
$\dagger\mathfrak{g}_{6,3}$	ae_1	$e_1 + ae_2$	$e_2 + ae_3$	e_4	be_5	$0 < \lvert b\rvert \le 1$
$\dagger\mathfrak{g}_{6,4}$	ae_1	$e_1 + ae_2$	$e_2 + ae_3$	$e_3 + ae_4$	e_5	
$\mathfrak{g}_{6,5}$	e_1	$e_1 + e_2$	$e_2 + e_3$	$e_3 + e_4$	$e_4 + e_5$	
$\dagger\mathfrak{g}_{6,6}$	e_1	ae_2	$e_2 + ae_3$	be_4	$e_4 + be_5$	$a \le b$
$\mathfrak{g}_{6,7}$	ae_1	$e_1 + ae_2$	$e_2 + ae_3$	be_4	$e_4 + be_5$	$a = 0$, $b = 1$ or $a = 1$
$\dagger\mathfrak{g}_{6,8}$	ae_1	be_2	ce_3	$de_4 + e_5$	$-e_4 + de_5$	$0 < \lvert c\rvert \le \lvert b\rvert \le \lvert a\rvert$
$\dagger\mathfrak{g}_{6,9}$	ae_1	be_2	$e_2 + be_3$	$ce_4 - e_5$	$e_4 + ce_5$	$a \neq 0$
$\mathfrak{g}_{6,10}$	ae_1	$e_1 + ae_2$	$e_2 + ae_3$	$be_4 - e_5$	$e_4 + be_5$	
$\dagger\mathfrak{g}_{6,11}$	e_1	$ae_2 - be_3$	$be_2 + ae_3$	$ce_4 - de_5$	$de_4 + ce_5$	$bd \neq 0$
$\dagger\mathfrak{g}_{6,12}$	ae_1	$be_2 - e_3$	$e_2 + be_3$	$e_2 + be_4 - e_5$	$e_3 + e_4 + be_5$	$a \neq 0$

Lie algebras having nilradical isomorphic to $H \oplus \mathbb{R}^2$: $[e_2, e_3] = e_1$.

Algebra	$[e_1,e_6]$	$[e_2,e_6]$	$[e_3,e_6]$	$[e_4,e_6]$	$[e_5,e_6]$	Remarks
$\mathfrak{g}_{6,13}$	$(a+b)e_1$	ae_2	be_3	e_4	ce_5	$a^2+b^2 \neq 0$, $\lvert a\rvert \leq \lvert b\rvert$, $0 < \lvert c\rvert \leq 1$
$\mathfrak{g}_{6,14}$	$(a+b)e_1$	ae_2	be_3	e_4	$e_1+(a+b)e_5$	$\lvert a\rvert \leq \lvert b\rvert$
$\mathfrak{g}_{6,15}$	$(a+1)e_1$	e_2+e_4	ae_3+e_5	e_4	ae_5	$\lvert a\rvert \leq 1$
$\mathfrak{g}_{6,16}$	e_1	e_2+e_4	e_5	e_1+e_4	0	
$\dagger\mathfrak{g}_{6,17}{}^{***}$	ae_1	ae_2	e_4	δe_1	ϵe_1+e_5	$(\delta,\epsilon,a) = (0,0,a)$ or $(1,0,a)$ (together $\mathfrak{g}_{6,17}$ in [Mub2]) or $(0,1,1)$ (new algebra)
$\dagger\mathfrak{g}_{6,18}$	$(a+1)e_1$	ae_2	e_3+e_4	e_4	be_5	$b \neq 0$
$\mathfrak{g}_{6,19}$	$(a+1)e_1$	ae_2	e_3+e_4	e_4	$e_1+(a+1)e_5$	
$\dagger\mathfrak{g}_{6,20}$	e_1	0	e_3+e_4	e_1+e_4	be_5	$b \neq 0$
$\dagger\mathfrak{g}_{6,21}$	$2ae_1$	ae_2+e_3	ae_3	e_4	be_5	$0 < \lvert b\rvert \leq 1$
$\mathfrak{g}_{6,22}$	$2ae_1$	ae_2+e_3	ae_3	e_4	e_1+2ae_5	
$\mathfrak{g}_{6,23}{}^{*}$	$2\delta e_1$	δe_2+e_3	δe_3+e_4	δe_4	$\epsilon e_1+(2\delta+a)e_5$	$(\delta,\epsilon,a) = (1,0,\neq -2)$ or $(0,0,1)$ (suspensions) or $(1,1,0)$
$\dagger\mathfrak{g}_{6,24}{}^{*}$	0	e_3	e_4	e_1	e_5	
$\mathfrak{g}_{6,25}$	$(a+1)e_1$	e_2	ae_3	be_4+e_5	be_5	$\lvert a\rvert \leq 1$
$\mathfrak{g}_{6,26}$	$(a+1)e_1$	e_2	ae_3	$(a+1)e_4+e_5$	$e_1+(a+1)e_5$	$\lvert a\rvert \leq 1$
$\mathfrak{g}_{6,27}{}^{*}$	$(a+b)e_1$	ae_2	be_3+e_4	be_4+e_5	δe_1+be_5	$a=1, \delta=0$ or $a=0, b=1, \delta=0,1$
$\mathfrak{g}_{6,28}$	$2e_1$	e_2+e_3	e_3	be_4+e_5	be_5	
$\mathfrak{g}_{6,29}$	$2e_1$	e_2+e_3	e_3	$2e_4+e_5$	e_1+2e_5	
$\mathfrak{g}_{6,30}$	0	e_3	0	e_4+e_5	e_5	
$\mathfrak{g}_{6,31}$	$2e_1$	e_2+e_3	e_3+e_4	e_4+e_5	e_5	
$\mathfrak{g}_{6,32}{}^{****}$	$2ae_1$	ae_2+e_3	ae_3-e_2	$\delta e_1+(2a+b)e_4$	ce_5	$(\delta,b) = (0,\neq -2a)$ or $= (1,0)$, $c \neq 0$, $2a+b \geq c$
$\dagger\mathfrak{g}'_{6,32}$	$2ae_1$	ae_2+e_3	ae_3-e_2	be_4	ce_5	$a \geq 0$, $0 < \lvert b\rvert \leq \lvert c\rvert$
$\mathfrak{g}_{6,33}{}^{****}$	$2ae_1$	ae_2+e_3	ae_3-e_2	be_4	e_1+2ae_5	$b \neq 0, b \leq 2a$
$\dagger\mathfrak{g}'_{6,33}$	$2ae_1$	ae_2+e_3	ae_3-e_2	e_1+2ae_4	be_5	$a \geq 0, b \neq 0$
$\mathfrak{g}_{6,34}{}^{*}$	$2ae_1$	ae_2+e_3	ae_3-e_2	$(2a+b)e_4+e_5$	$\delta e_1+(2a+b)e_5$	$a \geq 0$, $\delta=0$ or $\delta=1$, $b=0$
$\mathfrak{g}_{6,35}$	$(a+b)e_1$	ae_2	be_3	ce_4+e_5	ce_5-e_4	$a^2+b^2 \neq 0$, $\lvert a\rvert \leq \lvert b\rvert$, $c \geq 0$
$\mathfrak{g}_{6,36}$	$2ae_1$	ae_2+e_3	ae_3	be_4+e_5	be_5-e_4	$b \geq 0$
$\mathfrak{g}_{6,37}$	$2ae_1$	ae_2+e_3	ae_3-e_2	be_4+ce_5	be_5-ce_4	$a \geq 0, c > 0$
$\mathfrak{g}_{6,38}{}^{*}$	$2ae_1$	ae_2+e_3	$ae_3-e_2+e_4$	ae_4+e_5	ae_5-e_4	$a \geq 0$

Lie algebras having nilradical isomorphic to $\mathbb{R} \oplus A_{4,1}$: $[e_1, e_5] = e_2, [e_4, e_5] = e_1$.

Algebra	$[e_1, e_6]$	$[e_2, e_6]$	$[e_3, e_6]$	$[e_4, e_6]$	$[e_5, e_6]$	Remarks
$\dagger g_{6,39}$	$(b+1)e_1$	$(b+2)e_2$	ae_3	be_4	e_5	$a \neq 0$
$g_{6,40}$	$(a+1)e_1$	$(a+2)e_2$	$e_2 + (a+2)e_3$	ae_4	e_5	
$g_{6,41}$	$(a+1)e_1$	$(a+2)e_2$	ae_3	$e_3 + ae_4$	e_5	
$g_{6,42}$	$(a+1)e_1$	$(a+2)e_2$	e_3	ae_4	$e_3 + e_5$	
$g_{6,43}$	0	e_2	$e_2 + e_3$	$-e_4$	$e_3 + e_5$	
$\dagger g_{6,44}$	$2e_1$	$3e_2$	ae_3	e_4	$e_4 + e_5$	$a \neq 0$
$g_{6,45}$	$2e_1$	$3e_2$	$e_2 + 3e_3$	e_4	$e_4 + e_5$	
$g_{6,46}$	$2e_1$	$3e_2$	e_3	$e_3 + e_4$	$e_4 + e_5$	
$\dagger g_{6,47}$	e_1	e_2	ae_3	$\epsilon e_2 + e_4$	0	$a \neq 0, \epsilon = 0, \pm 1$
$g_{6,48}$	e_1	e_2	$e_2 + e_3$	e_4	0	
$g_{6,49}$	e_1	e_2	0	$\epsilon e_2 + e_4$	e_3	$\epsilon = 0, \pm 1$
$g_{6,50}$	e_1	e_2	$\epsilon e_2 + e_3$	$e_3 + e_4$	0	$\epsilon = 0, \pm 1$
$\dagger g_{6,51}$	0	0	e_3	ϵe_2	0	$\epsilon = \pm 1$
$\dagger g_{6,52}$	0	0	e_3	ϵe_2	e_4	$\epsilon = 0, \pm 1$

Lie algebras having nilradical isomorphic to $A_{5,1}$: $[e_3, e_5] = e_1$ $[e_4, e_5] = e_2$.
In $g_{6,54} - g_{6,65}$, a and b are used in place of λ and γ in [Mub2].

Algebra	$[e_1, e_6]$	$[e_2, e_6]$	$[e_3, e_6]$	$[e_4, e_6]$	$[e_5, e_6]$	Remarks
$g_{6,53}$	0	0	e_3	e_4	$-e_5$	
$g_{6,54}$	e_1	ae_2	$(1-b)e_3$	$(a-b)e_4$	be_5	$\lvert a\rvert \leq 1$
$g_{6,55}$	e_1	$(1+a)e_2$	$(1-a)e_3$	$e_1 + e_4$	ae_5	
$g_{6,56}{}^*$	e_1	0	e_2	$-e_4$	e_5	
$g_{6,57}$	e_1	$2ae_2$	$(1-a)e_3$	ae_4	$e_4 + ae_5$	
$g_{6,58}$	$3e_1$	$2e_2$	$e_2 + 2e_3$	e_4	$e_4 + e_5$	
$g_{6,59}{}^*$	e_1	0	e_3	e_2	δe_4	$\delta = 0, 1$
$g_{6,60}{}^*$	e_1	$2e_2$	0	$e_1 + e_4$	$e_4 + e_5$	
$g_{6,61}{}^*$	$2e_1$	0	e_3	$-e_4$	$e_3 + e_5$	
$g_{6,62}{}^*$	$2e_1$	e_2	$e_2 + e_3$	0	$e_3 + e_5$	
$g_{6,63}$	e_1	ae_2	e_3	$e_2 + ae_4$	0	
$g_{6,64}{}^*$	e_1	e_2	$\epsilon e_2 + e_3$	$e_1 + e_4$	0	$\epsilon = \pm 1$
$g_{6,65}{}^{**}$	$ae_1 + e_2$	ae_2	$(a-b)e_3 + e_4$	$(a-b)e_4$	be_5	$a = 1$ or $b = 1$
$g_{6,66}$	$2e_1 + e_2$	$2e_2$	$e_3 + e_4$	e_4	$e_3 + e_5$	
$g_{6,67}{}^{****}$	$2e_1 + e_2$	$2e_2$	$e_3 + e_4$	e_4	$ae_4 + e_5$	equivalent to $g_{6,65}(a = 2, b = 1)$
$g_{6,68}{}^*$	$e_1 + e_2$	e_2	$e_3 + e_4$	$e_1 + e_4$	0	
$g_{6,69}{}^{****}$	$e_1 + e_2$	e_2	$e_3 + e_4$	$e_2 + e_4$	0	equivalent to $g_{6,65}(b = 0)$
$g_{6,70}{}^{***}$	$ae_1 + e_2$	$-e_1 + ae_2$	$\delta e_2 + (a-b)e_3 + e_4$	$-e_3 + (a-b)e_4$	be_5	$\delta = 0$ ($g_{6,70}$ in [Mub2]) or $\delta = 1$ and $b = 0$ (new algebra)

Lie algebras having nilradical isomorphic to $A_{5,2}$: $[e_2, e_5] = e_1, [e_3, e_5] = e_2, [e_4, e_5] = e_3$.

Algebra	$[e_1, e_6]$	$[e_2, e_6]$	$[e_3, e_6]$	$[e_4, e_6]$	$[e_5, e_6]$	Remarks
$g_{6,71}$	$(a+3)e_1$	$(a+2)e_2$	$(a+1)e_3$	ae_4	e_5	
$g_{6,72}$	$4e_1$	$3e_2$	$2e_3$	e_4	$e_4 + e_5$	
$g_{6,73}{}^{***}$	e_1	e_2	$\epsilon e_1 + e_3$	$ae_1 + \epsilon e_2 + e_4$	0	$\epsilon = \pm 1$, (if $a = 0$, then $g_{6,73}$ in [Mub2])
$g_{6,74}$	e_1	e_2	e_3	e_4	0	
$g_{6,75}$	e_1	e_2	e_3	$e_1 + e_4$	0	

Lie algebras having nilradical isomorphic to $A_{5,3}$: $[e_2, e_4] = e_3, [e_2, e_5] = e_1$, $[e_4, e_5] = e_2$.

Algebra	$[e_1, e_6]$	$[e_2, e_6]$	$[e_3, e_6]$	$[e_4, e_6]$	$[e_5, e_6]$	Remarks
$g_{6,76}$	$(2a+1)e_1$	$(a+1)e_2$	$(a+2)e_3$	e_4	ae_5	$\|a\| \leq 1$
$g_{6,77}$	e_1	e_2	$2e_3$	$\epsilon e_1 + e_4$	0	$\epsilon = \pm 1$
$g_{6,78}$	$-e_1$	0	e_3	$e_3 + e_4$	$-e_5$	
$g_{6,79}$	$3e_1$	$2e_2$	$e_1 + 3e_3$	$e_4 + e_5$	e_5	
$g'_{6,80}{}^{***}$	$3ae_1 + e_3$	$2ae_2$	$3ae_3 - e_1$	$ae_4 - e_5$	$e_4 + ae_5$	
$g'_{6,81}{}^{***}$	e_3	0	$-e_1$	$\epsilon e_1 - e_5$	e_4	$\epsilon = \pm 1$
$g_{6,80}{}^{****}$	$2e_1$	e_2	e_3	0	e_5	equivalent to $g_{6,76}(a = 0)$
$g_{6,81}{}^{****}$	$2e_1$	e_2	e_3	0	$\epsilon e_3 + e_5$	$\epsilon = \pm 1$ equivalent to $g_{6,77}$

Lie algebras having nilradical isomorphic to $A_{5,4}$: $[e_2,e_4]=e_1, [e_3,e_5]=e_1$.

Algebra	$[e_1,e_6]$	$[e_2,e_6]$	$[e_3,e_6]$	$[e_4,e_6]$	$[e_5,e_6]$	Remarks
$g_{6,82}$ $\lambda=a,\lambda_1=b$	$2\delta e_1$	$(\delta+a)e_2$	$(\delta+b)e_3$	$(\delta-a)e_4$	$(\delta-b)e_5$	$\delta=1, 0\leq a\leq b$ or $\delta=0$, $0\leq a\leq 1, b=1$
$g_{6,83}$ $\lambda=a$	$2\delta e_1$	$(\delta+a)e_2+e_3$	$(\delta+a)e_3$	$(\delta-a)e_4$	$(\delta-a)e_5-e_4$	$\delta=1, 0\leq a$ or $\delta=0, a=1$
$g_{6,84}$	0	e_2	0	$-e_4$	e_3	
$g_{6,85}$ $\lambda=a$	$2e_1$	$(a+1)e_2$	e_3	$(1-a)e_4$	e_3+e_5	$a\geq 0$
$g_{6,86}$****	$2e_1$	e_2+e_3	e_3	e_4	e_5	equivalent to $g_{6,83}$ for $\alpha=2,\lambda=0$
$g_{6,87}$**	$2e_1$	e_2+e_5	e_3+e_4	e_4	e_3+e_5	
$g_{6,88}$* $\mu_0=a,\nu_0=1$	αe_1	$(\frac{\alpha}{2}+a)e_2+e_3$	$(\frac{\alpha}{2}+a)e_3-e_2$	$(\frac{\alpha}{2}-a)e_4+e_5$	$(\frac{\alpha}{2}-a)e_5-e_4$	$a\geq 0$
$g_{6,89}$* $s=a,\nu_0=b$	$2\delta e_1$	$(\delta+a)e_2$	δe_3+be_5	$(\delta-a)e_4$	δe_5-be_3	$\delta=1$, $a\geq 0$ and $b>0$ or $\delta=0$ and $a=1, b>0$ or $a=0, b=1$
$g_{6,90}$** $\nu_0=a$	$2\delta e_1$	δe_2	δe_3+ae_5	$e_2+\delta e_4$	δe_5-ae_3	$\delta=1$, $a\neq 0$ or $\delta=0, a=\pm 1$
$g_{6,91}$**,****	0	0	e_5	e_2	$-e_3$	included in $g_{6,90}$ as $\delta=0, a=1$
$g'_{6,92}$*,*** $s=a,\nu_0=b$	$2\delta e_1$	δe_2+ae_4	δe_3+be_5	δe_4-ae_2	δe_5-be_3	$\delta=1$, $0<a\leq\lvert b\rvert$ or $\delta=0$, $a=1$, $0<\lvert b\rvert\leq 1$ are new algebras
$g_{6,92}$**** $\nu_0=a,\mu_0=b$	$2\delta e_1$	δe_2+ae_3	δe_3-be_2	δe_4+be_5	δe_5-ae_4	equivalent to cases of $g_{6,82}$, $g_{6,83}$ or $g_{6,88}$
$g_{6,93}$** $\nu_0=a$	$2\delta e_1$	$\delta e_2+e_4+ae_5$	δe_3+ae_4	δe_4-ae_3	$\delta e_5-ae_2-e_3$	$\delta=1$, $a\geq 0$ or $\delta=0, a=1$

Lie algebras having nilradical isomorphic to $A_{5,5}$: $[e_3,e_4]=e_1, [e_2,e_5]=e_1, [e_3,e_5]=e_2$.

Algebra	$[e_1,e_6]$	$[e_2,e_6]$	$[e_3,e_6]$	$[e_4,e_6]$	$[e_5,e_6]$	Remarks
$g_{6,94}$	$(a+2)e_1$	$(a+1)e_2$	ae_3	$2e_4$	e_5	
$g_{6,95}$	$2e_1$	e_2	0	e_1+2e_4	e_5	
$g_{6,96}$	$3e_1$	$2e_2$	e_3	e_2+2e_4	e_3+e_5	
$g_{6,97}$	$4e_1$	$3e_2$	$2e_3+e_4$	$2e_4$	e_5	
$g_{6,98}$*	e_1	δe_1+e_2	e_3	0	δe_4	$\delta=0,1$

Lie algebras having nilradical isomorphic to $A_{5,6}$.

Algebra	$[e_3,e_4]$	$[e_2,e_5]$	$[e_3,e_5]$	$[e_4,e_5]$	$[e_1,e_6]$	$[e_2,e_6]$	$[e_3,e_6]$	$[e_4,e_6]$	$[e_5,e_6]$	Remarks
$g_{6,99}$	e_1	e_1	e_2	e_3	$5e_1$	$4e_2$	$3e_3$	$2e_4$	e_5	

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