

# Multiplicity-free Super Vector Spaces

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**Abstract.** Let  $V$  be a complex finite dimensional super vector space with an action of a connected semisimple group  $G$ . We classify those pairs  $(G, V)$  for which all homogeneous components of the super symmetric algebra of  $V$  decompose multiplicity-free.

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## 1. Introduction

One of the main questions of invariant theory is, given a Lie group  $G$  with a representation  $V$ , the determination of a set of generators of the invariant algebra  $\mathbb{C}[V]^G$  as well as finding all relations among them. Solutions to these questions are usually called a First and Second Fundamental Theorem of invariant theory of the underlying representation. In the most common examples,  $G$  is a classical group and  $V$  a number of copies of the defining representation (and maybe some copies of its dual). An important role in classical invariant theory was played by the famous identity of Capelli [3]

$$\det[\Delta_{ij} + \delta_{ij}(n - j)] = \det(x_{ij}) \cdot \Omega. \quad (1)$$

It describes a relation between the so-called “polarization operators”  $\Delta_{ij}$  on the variables  $x_{ij}$  and the  $\Omega$ -operator which was introduced by A. Cayley [4] and had also applications to invariant theoretical problems. Some applications of (1), such as a FFT for  $G = O_m$ , can be found in H. Weyl’s book [23].

In [6, 7, 8], R. Howe developed invariant theory on the basis of multiplicity-free actions. More precisely, he reproved well-known instances of FFT’s and SFT’s by using certain multiplicity-free actions, i.e., representation spaces  $V$  for which  $\mathbb{C}[V]$  is multiplicity-free. Moreover, he showed [9] that for a multiplicity-free action  $(G, V)$ , the algebra of  $G$ -invariant differential operators  $\mathbb{PD}(V)^G$  is a polynomial algebra and that such a pair naturally gives rise to an identity as (1). Namely, given the canonical homomorphism

$$\mathcal{Z}(\mathfrak{g}) \longrightarrow \mathbb{P}\mathbb{D}(V)^G \quad (2)$$

from the center of the enveloping algebra of the Lie algebra  $\mathfrak{g}$  of  $G$  into the  $G$ -invariant differential operators, one can ask for a preimage of such operator under this homomorphism. The classical example (1) comes from the action of  $\mathrm{GL}_n \times \mathrm{GL}_n$  on the space of  $n \times n$ -matrices [8].

In [7] one also finds FFT's for exterior invariant algebras  $(\bigwedge V)^G$ . It turns out that this "skew invariant theory" is, in some sense, very similar to its symmetric counterpart and some examples can be explained via multiplicity-free exterior algebras (or, as called by Howe, *skew multiplicity-free modules*).

In this article, we study the supersymmetric setting and therefore combine the above two types of multiplicity-free actions. Hence, let  $V$  be a (complex) *super vector space*, i.e., a finite dimensional complex vector space together with a fixed direct sum decomposition (or a  $\mathbb{Z}_2$  grading)  $V = V_0 \boxplus V_1$  of subspaces  $V_0$  and  $V_1$ . They are referred to as the *even* and *odd* part of  $V$ . If  $V_1 = 0$  (resp.  $V_0 = 0$ ) we call  $V$  *purely even* (resp. *purely odd*). In contrast, by a *proper super space* we mean that  $V_0$  and  $V_1$  are nonzero. A representation of  $G$  on a super vector space  $V$  is given by the direct sum of its representations on  $V_0$  and  $V_1$ .

Our main object of interest will be the supersymmetric algebra  $P(V)$  on  $V$ . This can be defined as the symmetric algebra on the even part of  $V$  tensored with the exterior algebra on the odd part,

$$P(V) = S(V_0) \otimes \bigwedge(V_1). \quad (3)$$

We assume that  $V$  is a module for a connected semisimple group  $G$ . Since both parts of a generic  $V$  can be reducible under  $G$ , we use the  $\boxplus$  notation to indicate the  $\mathbb{Z}_2$  grading. Thus, let

$$V_0 = U_1 \oplus \cdots \oplus U_k, \quad V_1 = W_1 \oplus \cdots \oplus W_l \quad (4)$$

be the decomposition of even and odd part into irreducible submodules of  $G$ . By basic multilinear algebra, the supersymmetric algebra (3) decomposes as a direct sum into  $G$  stable subspaces

$$P(V) = \bigoplus_{(\mathbf{i}, \mathbf{j})} S^{i_1} U_1 \otimes \cdots \otimes S^{i_k} U_k \otimes \bigwedge^{j_1} W_1 \otimes \cdots \otimes \bigwedge^{j_l} W_l. \quad (5)$$

Let us denote the direct summands on the right hand side of (5) by  $P^{(\mathbf{i}, \mathbf{j})}(V)$ . An irreducible representation of  $G$  of highest weight  $\lambda$  will be denoted by  $V(\lambda)$  or, if it is convenient, by  $V(\lambda)^{(n)}$ , indicating the rank of  $G$ .

**Definition 1.1.** The pair  $(G, V)$  is called *super multiplicity-free* (or *super MF*), if

$$\dim \mathrm{Hom}_G(P^{(\mathbf{i}, \mathbf{j})}(V), V(\lambda)) \leq 1$$

for all highest weights  $\lambda$  of  $G$  and all appropriate multiindices  $(\mathbf{i}, \mathbf{j})$ .

**Remark 1.2.** Assume that  $V$  decomposes as in (4). Consider the bigger group  $\tilde{G} = (\mathbb{C}^*)^{k+l} \times G$  and extend the action on  $V$  as follows: Let the  $i$ th copy of  $\mathbb{C}^*$  act trivially on all irreducible submodules except on  $U_i$  (resp.  $V_{i-k}$  if  $k < i \leq k+l$ ) where it should act by multiplication with scalars. (We call this particular action a *saturated action*.) We have a decomposition

$$P(V) = \bigoplus_{\lambda} m_{\lambda} \cdot V(\lambda), \tag{6}$$

where direct sums runs over all highest weights of  $\tilde{G}$ . Given a saturated action, it can be stated that  $(G, V)$  is super MF (according to Definition 1.1) if and only if the supersymmetric algebra  $P(V)$  is multiplicity-free as a  $\tilde{G}$  module, i.e., if

$$m_{\lambda} \leq 1 \text{ for all } \lambda. \tag{7}$$

In this case, we denote by  $\Xi$  the set of all highest weights such that  $m_{\lambda} = 1$  and hence,  $P(V) = \bigoplus_{\lambda \in \Xi} V(\lambda)$ .

The problem is that  $P(V)$  might or might not be multiplicity-free under a smaller group  $S \times G$ , where  $S$  is a subgroup of the  $(\mathbb{C}^*)^{k+l}$  torus. Since this is not easy to deal with, we shall classify only those representations that satisfy Definition 1.1 or, equivalently, multiplicity-free supersymmetric algebras for saturated actions.

The two extremal cases of purely even and odd super MF spaces have been investigated already. The classification of super MF spaces  $V = V_0$  was done by [10, 1, 16], while the odd case  $V = V_1$  was covered by [7] and [18].

At this point we shall make a further remark about our interest in general super MF actions. First, let  $V$  be a super vector space. The *parity* of a homogeneous element  $v \in V_i$  is defined by  $|v| = i$ . Write  $\tilde{V} = V \oplus V^*$  and consider the canonical supersymmetric pairing of  $V$  and  $V^*$  given by

$$\langle v + \lambda, w + \mu \rangle := \lambda(w) - (-1)^{|v||w|} \mu(v), \tag{8}$$

where  $v, w \in V$  and  $\lambda, \mu \in V^*$  are homogeneous with respect to the  $\mathbb{Z}_2$  grading on  $V$ , resp.  $V^*$ . In the free associative algebra over  $V$  with 1, define the super commutator of two elements  $x, y \in V$  by

$$[x, y] := x \cdot y - (-1)^{|x||y|} \cdot y \cdot x. \tag{9}$$

Then, we can consider the quantized algebra  $P(\tilde{V})_{\langle \cdot, \cdot \rangle}$  subject to the relations

$$[x, y] = \langle x, y \rangle \cdot 1 \tag{10}$$

for all  $x, y \in V$ . This algebra can be realized as an endomorphism algebra of  $P(V)$ . Furthermore, given an action of any *reductive* group  $G$  on  $V$ , the following is known [8, Theorem 3]:

**Proposition 1.3.** *There is a natural  $G$ -equivariant injective homomorphism  $\varphi : P(\tilde{V})_{\langle \cdot, \cdot \rangle} \rightarrow \text{End}(P(V))$ . Its image is isomorphic to  $\mathbb{P}\mathbb{D}(V)$ . ■*

**Remark 1.4.** Now assume that  $P(V)$  is multiplicity-free as in (7). (Although here,  $G$  is not necessarily the direct product of a torus and a semisimple group). Then, by Proposition 1.3, the subalgebra of  $\mathbb{P}\mathbb{D}(V)$  of  $G$ -invariant differential operators has the form

$$\mathbb{P}\mathbb{D}(V)^G = \bigoplus_{\lambda \in \Xi} (V(\lambda) \otimes V(\lambda)^*)^G. \quad (11)$$

By Schur's Lemma, this algebra has a canonical basis  $\{D_\lambda\}_{\lambda \in \Xi}$ . Applying this lemma yet another time, we find by the  $G$  invariance and by (7), that every  $D_\lambda$  acts on each irreducible subspace  $V(\mu)$  of  $P(V)$  as multiplication with a scalar  $c_\lambda(\mu)$ . For all irreducible super MF spaces  $V = V_0$ , [9] determined Capelli identities for the generators of  $\mathbb{P}\mathbb{D}(V)^G$ . In [13] one can find a characterization for the spectrum of these Capelli elements. On the other hand, for  $V = V_1$  some principal investigations concerning the spectral functions were done by [22]. It is an interesting problem to gain similar results in the supersymmetric setting.

This paper is organized follows: In Section 2 we introduce the notion of the representation diagram of a module  $V$  for a semisimple group  $G$  and state the classification of purely even and odd super MF spaces. Moreover, we recall decomposition formulas for symmetric and exterior algebras of some particular representations of linear groups. They will be used in some calculations in the preceding sections.

Our main result, the classification of saturated super multiplicity-free actions (Theorem 3.1) is given in Section 3. Beside this, we show two interesting properties of super MF spaces: first, we prove in Proposition 3.6 that for a given super MF representation  $(G, V)$  every smaller representation (which is given by a subgraph of the representation diagram of  $(G, V)$ ) is also super MF. This fact reduces the number of necessary calculations tremendously. But also the second result is labor-saving: It characterizes a super MF space  $(G, V)$  by the fact that its algebra of invariant differential operators is abelian (Corollary 3.4). With this, we deduce that the property of a module  $V$  to be super MF does not depend upon possible exchanges of some submodules by their duals (Corollary 3.5).

The rest of this article is concerned with establishing Theorem 3.1: in Section 4 we show that all modules that are listed in this theorem (except for some modules of simple groups) are indeed super MF. The remaining ones are treated in Section 5, which is mainly devoted to the completeness of part a) of Theorem 3.1. Here, we make use the decomposition formulas that are stated in Section 2. (However, in some preliminary calculations we also used the computer algebra packages LiE [15] and Schur [24].) In contrast, the considerations in Section 6 and Section 7 that deal with the completeness of part b) and c) of Theorem 3.1 mostly rely on the statement of Proposition 3.6 concerning subdiagrams.

## 2. Preliminaries

In the following, we will classify all pairs  $(G, V)$  satisfying Definition 1.1. Actually, for a representation  $\rho : G \rightarrow V$  the super MF property only depends on  $\rho(G)$ .

Hence, if  $\varphi : H \rightarrow G$  is a surjective homomorphism, then  $(H, V)$  is also super MF. Furthermore, it is obvious that  $(G, V)$  is super MF if and only if  $(G, V^*)$  has this property.

**Definition 2.1.** Two representations  $(G, \rho, V)$  and  $(G', \rho', V')$  are said to be *geometrically equivalent* if there exists an isomorphism  $\psi : V \xrightarrow{\sim} V'$  such that for the induced isomorphism  $\text{GL}(\psi) : \text{GL}(V) \rightarrow \text{GL}(V')$  one has  $\text{GL}(\psi)(\rho(G)) = \rho'(G')$ .

We write  $(G, \rho, V) \sim (G', \rho', V')$  for a pair of geometrically equivalent representations or, if the underlying homomorphisms are obvious,  $(G, V) \sim (G', V')$ .

This definition takes into account the two problems stated above: It is immediate that  $(G, \rho, V)$  and  $(H, \rho \circ \varphi, V)$  are geometrically equivalent. Moreover, if we fix a maximal torus  $T$  of  $G$ , then there is an automorphism  $\theta$  of  $G$  such that  $\theta(t) = t^{-1}$  for all  $t \in T$ . By this, also  $(G, V)$  and  $(G, V^*)$  are geometrically equivalent.

Since the two extremal cases of purely even and odd super MF spaces have been investigated already, we are left with the problem to find all instances of proper super spaces that are super MF. By definition of the supersymmetric algebra

$$P(V) = S(V_0) \otimes \bigwedge(V_1)$$

it is immediate that both, even and odd part of  $V$  have to be super MF according to Definition 1.1. Therefore, we recall the classification results for these two special cases. But before, we make some technical remarks. Note that there are infinitely many series of reducible super MF spaces since for every such pair  $(G_i, V_i)$  we can build the direct sum

$$(G_1 \times G_2, V_1 \oplus V_2) \tag{12}$$

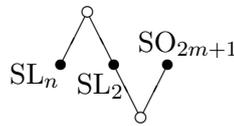
which is again super MF due to the fact  $P(V_1 \oplus V_2) = P(V_1) \otimes P(V_2)$ . We call a representation *decomposable* if it is geometrically equivalent to a representation as in (12) and *indecomposable* otherwise. Hence we will classify only indecomposable super MF representations.

Next, we introduce the notion of a representation diagram. This is a convenient tool for visualizing the action of  $G$  on  $V$ . Let  $G = G_1 \times \dots \times G_h$  be a group with  $h$  simple factors and  $V = U_1 \oplus \dots \oplus U_k \boxplus W_1 \oplus \dots \oplus W_l$  be a decomposition into irreducible subspaces. Draw three horizontal lines of dots, one upon the other, where the top row consists of  $k$  dots labeled by the  $U_p$ , the row in the middle consists of  $h$  dots, labeled by the  $G_r$  and the bottom row consists of  $l$  dots labeled by the  $W_q$ . These  $k + l + h$  dots form the vertices of the graph. For each pair of a simple group factor  $G_r$  and an irreducible submodule we draw an edge between the corresponding vertices if and only if  $G_r$  acts nontrivially on the submodule. As a further convention, we omit every label that stands for the natural representation.

**Definition 2.2.** The graph  $\mathcal{G} = \mathcal{G}(G, V)$  obtained by the above procedure is called the representation graph (or representation diagram) of  $(G, V)$ .

It follows immediately from this definition that indecomposable representations are in 1 – 1 correspondence with *connected* representation graphs.

**Example 2.3.** The representation graph of the action of  $SL_n \times SL_2 \times SO_{2m+1}$  on  $\mathbb{C}^n \otimes \mathbb{C}^2 \boxplus \mathbb{C}^2 \otimes \mathbb{C}^{2m+1}$  is given by



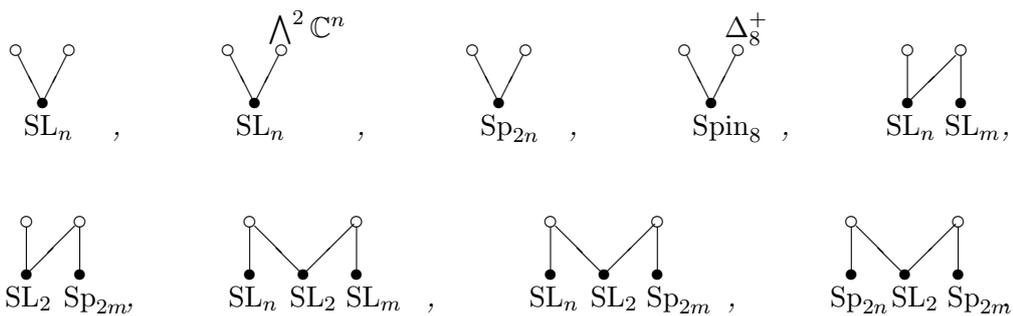
Now we can state the classification of super MF spaces for  $V = V_0$  and  $V = V_1$ . Note that we denote the natural representation of a group  $G$  by the same letter  $G$ . Plethysms of that representations, such as symmetric powers, are denoted by  $S^2G$ , etc. Similarly, the fundamental representations of  $G = Sp_{2n}$  are denoted by  $\bigwedge_0^k Sp_{2n}$ . Finally,  $\Delta_n$  stands for the spin representation of  $Spin_n$ . It is irreducible if  $n = 2k + 1$  and decomposes into the two half spin representations  $\Delta_n^+, \Delta_n^-$  if  $n = 2k$ .

**Theorem 2.4** (Symmetric case). *Let  $(G, V)$  be an indecomposable representation with  $V$  being a purely even super space.*

a) *If  $V$  is irreducible, then all instances of super MF spaces are given by the following list:*

$SL_n$ ( $n \geq 2$ )	$Sp_{2n}$ ( $n \geq 2$ )	$\Delta_7$	$G_2$	$SL_n \otimes SL_m$ ( $n, m \geq 2$ )
$S^2SL_n$ ( $n \geq 2$ )	$SO_{2n+1}$ ( $n \geq 2$ )	$\Delta_9$	$E_7$	$SL_k \otimes Sp_{2n}$ ( $k = 2, 3$ )
$\bigwedge^2 SL_n$ ( $n \geq 4$ )	$SO_{2n}$ ( $n \geq 4$ )	$\Delta_{10}$		$SL_n \otimes Sp_4$ ( $n \geq 2$ )

b) *If  $V$  is reducible, then all instances of super MF spaces (up to a possible exchange of an irreducible submodule by its dual) are given by:*



**Proof.** See [10, Theorem 3] for the irreducible case and [16, Theorem 2.5] or [1, Theorem 2] for the reducible. ■

**Theorem 2.5** (Skew-symmetric case). *Let  $(G, V)$  be an indecomposable representation with  $V$  being a purely odd super space.*

a) *If  $V$  is irreducible and  $G$  simple, then all instances of super MF spaces*

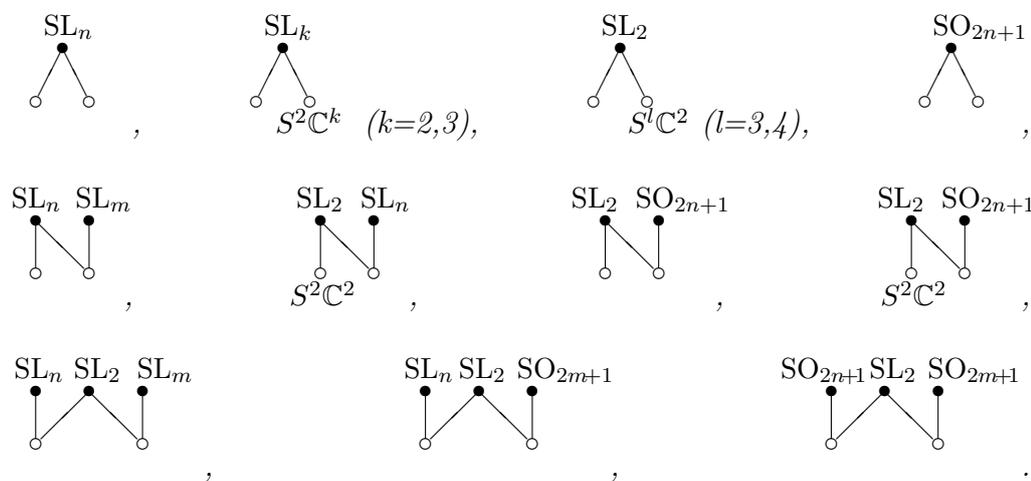
are given by the following list:

$SL_n$ ( $n \geq 2$ )	$SO_{2n+1}$ ( $n \geq 2$ )	$Sp_{2n}$ ( $n \geq 2$ )
$S^2SL_n$ ( $n \geq 2$ )	$\Delta_7$	$\Lambda_0^2 Sp_4$
$S^kSL_2$ ( $k = 3, \dots, 6$ )	$\Delta_9$	$\Lambda_0^3 Sp_6$
$S^3SL_3$	$SO_{2n}$ ( $n \geq 3$ )	$G_2$
$\Lambda^2 SL_n$ ( $n \geq 4$ )	$\Delta_{10}^+$	$E_6$
$\Lambda^3 SL_6$	$\Delta_{12}^+$	$E_7$

b) If  $V$  is as in a), but  $G$  not necessarily simple then:

$SL_n \otimes SL_m$ ,  $SL_n \otimes Sp_4$ ,  $SL_k \otimes SO_{2m+1}$  ( $k = 2, 3$ ),  $SL_2 \otimes SO_{2m}$ .

c) If  $V$  is reducible but indecomposable, then all instances of super MF spaces (up to a possible exchange of an irreducible submodule by its dual) are given by:



**Proof.** Part a) is proven in [7, Theorem 4.7.1], parts b) and c) in [18, Theorem 4.8]. ■

A very important instance of a super MF space is given by the action of  $GL_n \times GL_m$  on  $V = \mathbb{C}^n \otimes \mathbb{C}^m$ . This space is irreducible and we can interpret it either as a purely even or purely odd super space. Accordingly, we are dealing with the symmetric or exterior algebra on  $V$ . In both cases, the arising decomposition is not only multiplicity-free but also the isotypic components for  $GL_n$  and  $GL_m$  in this decomposition stand in a bijective correspondence. This is called *(skew) duality* and the explicit formulas for the homogeneous components are given by

$$S^k(\mathbb{C}^n \otimes \mathbb{C}^m) = \bigoplus_{|\lambda|=k, \ell(\lambda) \leq \min\{n,m\}} V(\lambda)^{(n)} \otimes V(\lambda)^{(m)} \tag{13}$$

and

$$\bigwedge^k(\mathbb{C}^n \otimes \mathbb{C}^m) = \bigoplus_{|\lambda|=k, \ell(\lambda) \leq n, \lambda_1 \leq m} V(\lambda)^{(n)} \otimes V(\lambda^t)^{(m)}. \tag{14}$$

There are two other series of modules for which symmetric and antisymmetric plethysms are easy to describe, namely the actions of  $GL_n$  on  $S^2\mathbb{C}^n$  and  $\bigwedge^2\mathbb{C}^n$ . For this reason it is convenient to introduce the following terms: A Young diagram of shape  $(s, 1^t)$  is called a  $(s, t)$ -hook. A sequence of  $(s_i, t_i)$ -hooks, where  $s_{i+1} \leq s_i - 1$  and  $t_{i+1} \leq t_i - 1$  can be combined to a *nested hook*, i.e., a regular Young diagram  $\lambda$  such that the boxes in the angle of each hook form the diagonal of  $\lambda$ . We have

$$S^k(S^2\mathbb{C}^n) = \bigoplus_{|\lambda|=2k, \lambda_i \text{ even}} V(\lambda), \tag{15}$$

$$S^k(\bigwedge^2\mathbb{C}^n) = \bigoplus_{|\lambda|=2k, \lambda_i \text{ even}} V(\lambda^t) \tag{16}$$

for the symmetric powers of  $S^2\mathbb{C}^n$  and  $\bigwedge^2\mathbb{C}^n$ , while their skew-symmetric powers decompose by

$$\bigwedge^k(S^2\mathbb{C}^n) = \bigoplus_{\lambda \in D} V(\lambda), \tag{17}$$

$$\bigwedge^k(\bigwedge^2\mathbb{C}^n) = \bigoplus_{\lambda \in E} V(\lambda). \tag{18}$$

Here,  $D$  is the set of all partitions that consist of a nested  $(r_i + 1, r_i - 1)$ -hook with  $\sum r_i = k$ ; and  $E$  is the set of all partitions that consist of a nested  $(r_i, r_i)$ -hook with  $\sum r_i = k$ . (Proofs for all these decomposition can be found in [6, 7, 8].)

We have no generalization of (17) to  $V = S^k\mathbb{C}^n$  for arbitrary  $k$ . But at least for the second exterior power one can easily show that

$$\bigwedge^2 S^k \text{SL}_2 = \bigoplus_{j=0, \dots, \lfloor \frac{k-1}{2} \rfloor} V(2k - 2 - 4j) \tag{19}$$

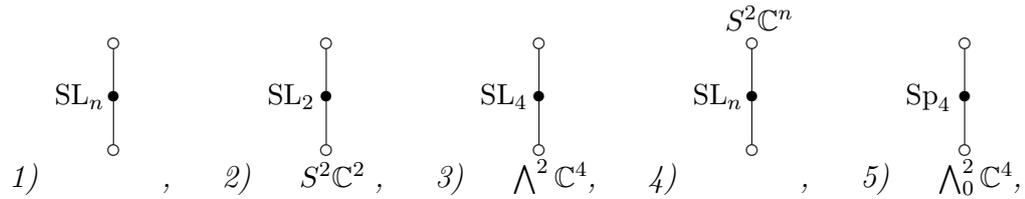
by writing down its character polynomial. Since  $V = V_1 = S^k \text{SL}_2$  is super MF for  $2 \leq k \leq 6$ , we need the above formula in some computations in Section 5.

### 3. Results

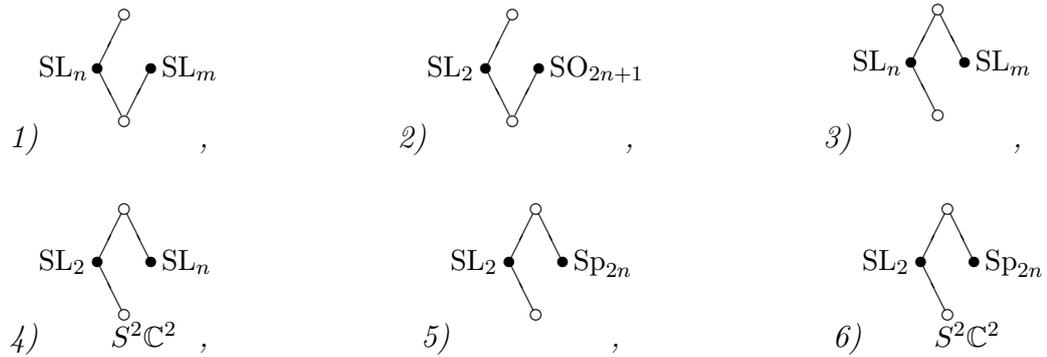
In this section, we start with our main result, the classification of saturated super MF spaces. Let  $V$  be the representation of a connected reductive group  $\tilde{G}$  on a super vector space. We can assume that  $\tilde{G} = (\mathbb{C}^*)^k \times G$  with  $G$  semisimple and  $k$  equal to the number of irreducible submodules in  $V$ .

**Theorem 3.1.** *Let  $(\tilde{G}, V)$  be an indecomposable super MF representation on a proper super space  $V$ , i.e.,  $V_0, V_1 \neq 0$ . Then,  $G$  has at most three simple factors. Up to geometric equivalence (and a possible exchange of irreducible submodules by their duals), all super MF are given by the following list.*

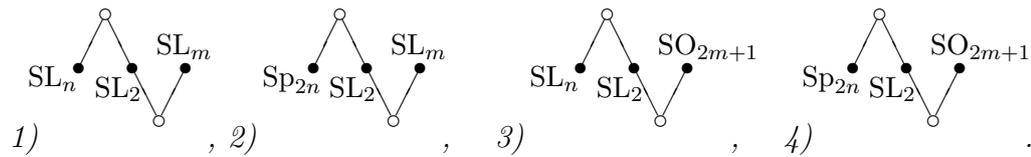
a) If  $G$  is simple:



b) If  $G = G_1 \times G_2$ :



c) If  $G = G_1 \times G_2 \times G_3$ :



Dropping the assumption that  $V$  is indecomposable and combining this theorem with the previous results, we get

**Corollary 3.2.** *If  $(G, V)$  is a saturated super MF representation of a connected reductive group  $G$ , then it is geometric equivalent to  $(\prod_{i=1}^r G_i, \bigoplus_{i=1}^r V^{(i)})$  with  $(G_i, V^{(i)})$  being a saturated indecomposable representation from Theorem 2.4 or 2.5 (if  $V^{(i)}$  is purely even or odd) or, if  $V^{(i)}$  is a proper super space, it occurs in Theorem 3.1.*

The rest of this section is devoted to some results that facilitate the proof of Theorem 3.1. Therefore, we give a characterization of super MF spaces in terms of the  $G$ -invariant differential operators  $\mathbb{P}\mathbb{D}(V)^G$ . Denote by  $\mathbb{C}[G]$  the group algebra of  $G$ .

**Theorem 3.3.** *Let  $G$  be a reductive group that acts on a super vector space  $V$ . Then, as a joint  $\mathbb{C}[G] \otimes \mathbb{P}\mathbb{D}(V)^G$  module,*

$$P(V) = \bigoplus_{\lambda \in \Xi} V(\lambda) \otimes V^\lambda, \tag{20}$$

where  $V^\lambda$  are pairwise non-isomorphic irreducible representations of  $\mathbb{P}\mathbb{D}(V)^G$ . ■

This theorem is a supersymmetric version of [5, Theorem 4.5.14] and is based on a generalization of Burnside's theorem. It has the following important corollary. Let  $G$  be a semisimple group acting on  $V = V_0 \boxplus V_1$ . Consider the extended action of  $\tilde{G} = (\mathbb{C}^*)^k \times G$  as described in Remark 1.2.

**Corollary 3.4.**  *$(G, V)$  is super MF if and only if  $\mathbb{PD}(V)^{\tilde{G}}$  is abelian.*

**Proof.** Let  $(G, V)$  be super MF, i.e.,  $P(V)$  is multiplicity-free with respect to  $\tilde{G}$ . Then, the commutativity of  $\mathbb{PD}(V)^{\tilde{G}}$  follows from Remark 1.4. On the other hand, if  $\mathbb{PD}(V)^{\tilde{G}}$  is abelian, all representations  $V^\lambda$  in (20) have dimension 1. ■

Although being an alternative characterization of super MF modules, this criterion is not easily read off from an arbitrary module  $V$ , since the task of determining the algebra structure of  $\mathbb{PD}(V)^G$  is at least as hard as determining the multiplicities of the  $G$  module  $P(V)$ . However, it makes it possible to deduce a labor-saving property of dual submodules: Recall that  $(G, V) \sim (G, V^*)$  for arbitrary representations. Unfortunately, it is not true in general that  $(G, V \oplus W) \sim (G, V \oplus W^*)$  for  $G$  modules  $V$  and  $W$ . Nevertheless, we will claim that a space  $(G, V \oplus W)$  is super MF whenever  $(G, V \oplus W^*)$  is super MF. Thus, in the classification of super MF spaces, we can ignore all extra cases coming from replacing a submodule by its dual.

**Corollary 3.5.** *Let  $G$  be a group that acts on super vector spaces  $V = V_0 \boxplus V_1$  and  $W = W_0 \boxplus W_1$ . Then,  $V \oplus W$  is super MF if and only if  $V \oplus W^*$  is super MF.*

**Proof.** By Proposition 1.3,  $\mathbb{PD}(V \oplus W) \simeq P(\tilde{V} \oplus \tilde{W})_{\langle \cdot, \cdot \rangle} \simeq \mathbb{PD}(V \oplus W^*)$ . Since this is also an algebra isomorphism,  $\mathbb{PD}(V \oplus W)^G$  is commutative if and only if  $\mathbb{PD}(V \oplus W^*)^G$  is commutative and the claim follows from Corollary 3.4. ■

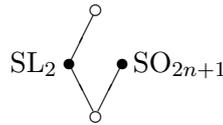
Another useful fact for super MF spaces is best described in terms of representation diagrams. Note that every subgraph  $\mathcal{G}'$  of a representation diagram  $\mathcal{G}$  is also a representation diagram.

**Proposition 3.6.** *If the representation described by  $\mathcal{G}$  is super MF, then the same is true for the underlying representation of  $\mathcal{G}'$ .*

**Proof.** This is proven in [18, Proposition 4.7] for purely odd super spaces. The proof makes use of  $(\mathrm{GL}_n, \mathrm{GL}_m)$  skew duality (14) and can be translated word-by-word to the general setting by simply exchanging skew duality (14) by the symmetric counterpart (13), if necessary. ■

As we shall see, the representation that is described by the graph in Exam-

ple 2.3 is super MF. By Proposition 3.6 it follows that, e.g.,



is also super MF.

#### 4. Decompositions of super MF modules

In this section we verify the super MF condition for all modules that are listed in Theorem 3.1b) and c). By the result on subgraphs in Proposition 3.6 this also shows that some modules of Theorem 3.1a) are super MF. We begin with a lemma that limits the possible cases of super MF modules.

**Lemma 4.1.**  $SL_n \otimes SL_p \boxplus SL_p \otimes SL_m$  is super MF if and only if  $p = 2$ .

**Proof.** First, let  $p = 2$ . There is an integer  $a$ , depending on  $l$  and  $m$  such that the decomposition of  $P^{(k,l)}$  is given by

$$\left( \bigoplus_{i=0, \dots, \lfloor \frac{k}{2} \rfloor} V(k-i, i) \otimes V(k-2i) \otimes \mathbb{C} \right) \otimes \left( \bigoplus_{j=a, \dots, \lfloor \frac{l}{2} \rfloor} \mathbb{C} \otimes V(l-2j) \otimes V(2^j, 1^{l-2j}) \right).$$

The tensor products of the occurring representations for  $SL_2$  are multiplicity-free and the representations for  $SL_n$  and  $SL_m$  are pairwise different. Hence, all homogeneous components  $P^{(k,l)}$  are multiplicity-free. Note that this case refers to representation 1) of Theorem 3.1c).

If  $p \geq 3$ ,  $P^{(3,3)} = (V(2, 1) \otimes V(2, 1) \otimes \mathbb{C} \oplus \dots) \otimes (\mathbb{C} \otimes V(2, 1) \otimes V(2, 1) \oplus \dots)$  is not multiplicity-free, since  $V(2, 1) \otimes V(2, 1) = 2 \cdot V(3, 2, 1) \oplus \dots$  is not. ■

In the following, it is convenient to use a different notation. For the irreducible representations of  $GL_n$  or  $SL_n$  we will sometimes write  $\{\lambda\}$  instead of  $V(\lambda)$ . Classically,  $\{\lambda\}$  refers to the character of  $V(\lambda)$  if  $\ell(\lambda) \leq n$ . But this symbol even makes sense for  $\ell(\lambda) > n$  if we set  $\{\lambda\} = 0$  in this case. Similarly, we use the notations  $[\lambda]$  and  $\langle \lambda \rangle$  for the groups  $O_n$  and  $Sp_{2m}$ . Also these symbols are well-defined even for  $\ell(\lambda) \geq n$ . But here, the modification rules are nontrivial and  $[\lambda]$  (resp.  $\langle \lambda \rangle$ ) can refer to a virtual representation. For a rigorous definition of these *universal characters* see [14].

One advantage of the universal characters is that the branching rules  $SL_n \downarrow O_n$  and  $SL_{2m} \downarrow Sp_{2m}$  for irreducible representations can be formulated in a simple way. The corresponding formulas are well-known (and proven, e.g., in [17]).

**Proposition 4.2** (Littlewood). *For any partition  $\lambda$  the following identities hold:*

- a)  $\{\lambda\} = \sum_{\mu \subseteq \lambda} \left( \sum_{\beta \text{ even}} c_{\mu, \beta}^\lambda \right) [\mu],$
- b)  $\{\lambda\} = \sum_{\mu \subseteq \lambda} \left( \sum_{\beta^t \text{ even}} c_{\mu, \beta}^\lambda \right) \langle \mu \rangle.$

The numbers  $c_{\mu, \beta}^\lambda$  are the Littlewood-Richardson coefficients. ■

Two particular cases are of special interest: If  $\lambda = (2^a, 1^b)$ , we have

$$\{\lambda\} = \sum_{i=0}^a [2^{a-i}, 1^b], \tag{21}$$

for the restriction  $SL_n \downarrow O_n$ , while for  $SL_{2m} \downarrow Sp_{2m}$  and  $\lambda^t = (k, l)$  with  $k \geq l$ , we have

$$\{\lambda\} = \sum_{i=0}^l \langle k - i, l - i \rangle. \tag{22}$$

Given an arbitrary  $\lambda$ , if we want to know which irreducible representations for  $O_n$  and  $Sp_{2m}$  actually occur in the restriction of  $V(\lambda)$ , we must apply the modification rules to the terms on the right hand side of Proposition 4.2 a) and b). However, in the special case (22) for  $Sp_{2m}$  this is not necessary, because  $\ell(k - i, l - i) \leq 2$  is inside the “stable range”. For  $O_n$  the modification rules for the special case (21) take the following form: Let  $\mu = (2^c, 1^d)$ . If  $\ell(\mu) > \lfloor \frac{n}{2} \rfloor$  then, depending on the parameter  $h = 2\ell(\mu) - n$ , one has

$$[\mu] = \begin{cases} [2^c, 1^{d-h}]^\sharp & : h \leq d \\ -[2^{c-(h-d-1)}, 1^{h-d-2}]^\sharp & : h > d + 1 \\ 0 & : h = d + 1 \end{cases} . \tag{23}$$

Here,  $[\lambda]^\sharp$  stands for  $[\lambda] \otimes \det$ . If the result still has length  $> \lfloor \frac{n}{2} \rfloor$ , one repeats the above process. Note that for the group  $SO_n$ , we have  $[\lambda]^\sharp = [\lambda]$ . Also, if  $n = 2m$  and  $\lambda$  has exactly  $m$  nonvanishing parts,  $[\lambda] = [\lambda]^+ + [\lambda]^-$  as irreducible  $SO_{2m}$  characters. But this does not affect the multiplicity-freeness and hence we do not care about this splitting. Modification rules for the general case can be found in [11] and [14].

**Lemma 4.3.** *Representations 2)  $Sp_{2n} \otimes SL_2 \boxplus SL_2 \otimes SL_m$ , 3)  $SL_n \otimes SL_2 \boxplus SL_2 \otimes SO_{2m+1}$  and 4)  $Sp_{2n} \otimes SL_2 \boxplus SL_2 \otimes SO_{2m+1}$  of Theorem 3.1c) are super MF. In particular, also representations 2)  $SL_2 \boxplus SL_2 \otimes SO_{2m+1}$  and 5)  $Sp_{2n} \otimes SL_2 \boxplus SL_2$  of Theorem 3.1b) are super MF.*

**Proof.** We show that all the  $Sp_{2n}$  (resp.  $SO_{2m+1}$ ) representations that occur in the decomposition of  $S^k(Sp_{2n} \otimes SL_2)$ , resp.  $\bigwedge^l(SL_2 \otimes SO_{2m+1})$  are pairwise different. This suffices to prove the claim for the representations 2), 3) and 4) of Theorem 3.1. First, by the skew duality (14) and the branching rule for  $Sp_{2n}$  we have for any  $k$

$$S^k(\mathrm{Sp}_{2n} \otimes \mathrm{SL}_2) = \bigoplus_{i=0, \dots, \lfloor \frac{k}{2} \rfloor} \left( \bigoplus_{j=0, \dots, i} \langle k-i-j, i-j \rangle \right) \otimes \{k-2i\} \tag{24}$$

and so the claim is obvious. For  $\mathrm{SO}_{2m+1}$  the statement is not immediate since we have to deal with modification rules in that case. Let  $0 \leq l \leq 2(2n+1)$ , so

$$\bigwedge^l \mathrm{SL}_2 \otimes \mathrm{SO}_{2n+1} = \bigoplus_{\lambda \in \mathcal{P}_l} \{\lambda\} \otimes \{\lambda^t\}, \tag{25}$$

where

$$\mathcal{P}_l = \{\lambda : |\lambda| = l, \ell(\lambda) \leq 2, \lambda_1 \leq 2n+1\} = \{(a, l-a), (a-1, l-a+1), \dots, (b, l-b)\}$$

for some integers  $a \geq b \geq 0$ . Thus, we have to show that

$$M_l := \bigoplus_{\lambda \in \mathcal{P}_l} \mathrm{Res}_{\mathrm{SO}_{2n+1}}^{\mathrm{SL}_{2n+1}}(V(\lambda))$$

is multiplicity-free. By the branching rule for universal characters, we have for  $\lambda = (s, l-s) \in \mathcal{P}_l$  with  $a \geq s \geq b$  that

$$\{\lambda^t\} = [(s, l-s)^t] + [(s-1, l-s-1)^t] + \dots + [(2s-l+1, 1)^t] + [(2s-l, 0)^t].$$

This results in a multiplicity-free decomposition

$$M_l = \sum_{s=a, \dots, b} \sum_{\nu=0, \dots, l-s} [s-\nu, l-s-\nu] =: \sum_{\tau \in I} [\tau].$$

In order to analyze the impact of the modification rules, we re-order this sum by setting  $\mathcal{S}_q = \{\tau \in I : \text{second column of } \tau \text{ has exactly } q \text{ boxes}\}$ , so that

$$\sum_{\tau \in I} [\tau] = \sum_{q \in \mathbb{N}_0} \sum_{\tau \in \mathcal{S}_q} [\tau]. \tag{26}$$

It suffices to consider only partitions  $\tau \in I$  with  $\ell(\tau) > n$  for which one needs only a single application of the modification rule and, furthermore, only the first column of  $\tau$  is affected.

First, note that for every  $q \in \mathbb{N}_0$  there exist  $p, r \in \mathbb{N}_0$  such that

$$\mathcal{S}_q = \{(p+q, q)^t, (p+q+2, q)^t, \dots, (p+q+2r, q)^t\}.$$

Since all the numbers  $\{\ell(\lambda) : \lambda \in \mathcal{S}_q\}$  are pairwise different and differ by at least 2, it cannot happen that two different elements in  $\mathcal{S}_q$  become equal after modification. Observe further that the numbers  $\{\lambda_1^t - \lambda_2^t : \lambda \in \mathcal{P}_i\}$  are either all even or all odd. So, modified elements from  $\mathcal{S}_q$  cannot equal any unmodified elements in  $\mathcal{S}_q$ , too. ■

**Lemma 4.4.** *Representations 1)  $SL_n \boxplus SL_n \otimes SL_m$  and 3)  $SL_n \otimes SL_m \boxplus SL_m$  of Theorem 3.1b) are super MF. In particular, representation 5) of Theorem 3.1a) is super MF.*

**Proof.** By the Pieri Rule (see, e.g., [5, p. 388]), all direct summands on the right hand side of

$$P^{(k,l)}(SL_n \boxplus SL_n \otimes SL_m) = \bigoplus_{|\lambda|=l, \ell(\lambda) \leq n, \lambda_1 \leq m} V(k)^{(n)} \otimes V(\lambda)^{(n)} \otimes V(\lambda^t)^{(m)}$$

are multiplicity-free. They can be distinguished by the representations of the  $SL_m$  factor, so 1) is super MF. The same argument applies to 3). ■

**Lemma 4.5.** *Representations 4)  $SL_2 \otimes SL_n \boxplus S^2SL_2$  and 6)  $SL_2 \otimes Sp_{2n} \boxplus S^2SL_2$  of Theorem 3.1b) are super MF. In particular, representation 2)  $SL_2 \boxplus S^2SL_2$  of Theorem 3.1a) is super MF.*

**Proof.** The exterior powers of  $S^2SL_2$  are either trivial, or isomorphic to  $V(2)$ . Hence  $P(SL_2 \otimes SL_n \boxplus S^2SL_2)$  is multiplicity-free. The same is true for  $P(SL_2 \otimes Sp_{2n} \boxplus S^2SL_2)$  by (24). ■

### 5. Representations of simple groups

In this section,  $G$  is a simple group. We show that Theorem 3.1a) contains all super MF representations of such  $G$ . In particular, we have to verify the super MF condition for representations 3) - 5) of this theorem. Since their representation diagrams do not occur as subdiagrams of bigger super MF representations, this has not been done yet.

So let  $V = V_0 \boxplus V_1$  be a proper super space, with  $V_0$  coming from Theorem 2.4a) and  $V_1$  from Theorem 2.5a). For each root type of  $G$  there is given one table with the different choices for  $V_0$  along the rows and for  $V_1$  along the columns. (A ★ indicates that the ranks of the groups that act on  $V_0$  and  $V_1$  cannot be equal.) This also yields the proof that representations 3) - 5) of Theorem 3.1a) are super MF. For type  $A$  we often use formulas (15) - (18).

Type <b>A</b>	$SL_n$	$S^2SL_n$	$\bigwedge^2 SL_n$	$S^3SL_2$	$S^4SL_2$	$S^5SL_2$	$S^6SL_2$	$S^3SL_3$	$\bigwedge^3 SL_6$
$SL_n$	1a)	1b)	1c)	1d)	1e)	1f)	1g)	1h)	1i)
$S^2SL_n$	2a)	2b)	2c)	2d)	2e)	2f)	2g)	2h)	2i)
$\bigwedge^2 SL_n$	3a)	3b)	3c)	★	★	★	★	★	3i)

1a) This is representation 1) of Theorem 3.1a).

1b)  $n \geq 4$ :  $P^{(2,3)}(SL_n \boxplus S^2SL_n) = \{2\} \cdot (\{4, 1^2\} + \{3^2\}) = 2 \cdot \{4, 3, 1\} + \dots$

1c)  $n \geq 5$ :  $P^{(2,5)} = \{2\} \cdot (\{4, 2^2, 1^2\} + \{3^2, 2^2\} + \dots) = 2 \cdot \{4, 3, 2^2, 1\} + \dots$

$n = 4$ :  $\bigwedge^l(\bigwedge^2 \text{SL}_4)$  is irreducible, except for  $l = 3$ . And in this case,  $P^{(k,3)}$  decomposes as

$$\{k\} \cdot (\{2\} + \{2^3\}) = \{k+2\} + \{k+1, 1\} + \{k, 2\} + \{k+2, 2^2\} + \{k, 1^2\} + \{k-2\}.$$

In particular, representation 3) of Theorem 3.1a) is super MF.

1d) - 1g) It follows from (19) that for  $p = 3, \dots, 6$  we have

$$P^{(2,2)}(\text{SL}_2 \boxplus S^p \text{SL}_2) = \{2\} \cdot (\sum_{k=0}^{\lfloor \frac{p}{2} \rfloor} \{2p - 4k\}) = 2 \cdot \{2p - 2\} + \dots$$

1h)  $P^{(2,2)}(\text{SL}_3 \boxplus S^3 \text{SL}_3) = \{2\} \cdot (\{3^2\} + \{5, 1\}) = 2 \cdot \{5, 3\} + \dots$

1i)  $P^{(2,5)}(\mathbb{C}^6 \boxplus \bigwedge^3 \text{SL}_6) = \{2\} \cdot (\{1^3\} + \{5, 3^2, 2^2\} + \dots) = 2 \cdot \{3, 1^2\} + \dots$

2a) It follows from (15) that for  $k \in \mathbb{N}$  and  $0 \leq l \leq n$ ,

$$P^{(k,l)}(S^2 \text{SL}_n \boxplus \text{SL}_n) = \left( \bigoplus_{|\lambda|=2k, \lambda_i \text{ even}} V(\lambda) \right) \otimes V(1^l).$$

In particular,  $\lambda_i \equiv \lambda'_i \pmod 2$  for all occurring partitions  $\lambda, \lambda'$ . By the Pieri Rule, the decomposition of  $V(\lambda) \otimes V(1^l)$  is a direct sum of those  $V(\mu)$  such that  $\mu$  is obtained from  $\lambda$  by adding  $l$  boxes, but no two in the same row. So,  $V(\lambda) \otimes V(1^l)$  and  $V(\lambda') \otimes V(1^l)$  share no isomorphic submodule if  $\lambda \neq \lambda'$ . (Even after considering the modification rule for rows with  $n$  boxes.) Thus, representation 4) of Theorem 3.1a) is super MF.

2b)  $n \geq 2$ :  $P^{(2,1)}(S^2 \text{SL}_n \boxplus S^2 \text{SL}_n) = (\{2^2\} + \{4\}) \cdot \{2\} = 2 \cdot \{4, 2\} + \dots$

2c)  $n \geq 4$ :  $P^{(2,2)}(S^2 \text{SL}_n \boxplus \bigwedge^2 \text{SL}_n) = (\{2^2\} + \{4\}) \cdot \{2, 1^2\} = 2 \cdot \{4, 2, 1^2\} + \dots$

2d) - 2i)  $P^{(1,k)}(S^2 \text{SL}_n \boxplus V) = P^{(2,k)}(\text{SL}_n \boxplus V)$ . See 1d) - 1i).

3a)  $n \geq 4$ :  $P^{(3,2)}(\bigwedge^2 \text{SL}_n \boxplus \text{SL}_n) = (\{3^2\} + \{2^2, 1^2\} + \dots) \cdot \{1^2\} = 2 \cdot \{3^2, 1^2\} + \dots$

3b)  $n \geq 3$ :  $P^{(5,3)}(\bigwedge^2 \text{SL}_n \boxplus S^2 \text{SL}_n) = \{5^2\} \cdot (\{4, 1^2\} + \{3^2\}) = 2 \cdot \{8, 6, 2\} + \dots$

3c)  $n \geq 4$ :  $P^{(6,3)} = (\{4^2, 2^2\} + \dots) \cdot (\{3, 1^3\} + \{2^3\} + \dots) = 2 \cdot \{6, 5, 4, 3\} + \dots$

3i)  $P^{(1,3)}(\bigwedge^2 \text{SL}_6 \boxplus \bigwedge^3 \text{SL}_6) = \{1^2\} \cdot (\{1^3\} + \{3^2, 1^3\} + \dots) = 2 \cdot \{2^2, 1\} + \dots$

Type <b>C</b>	$\text{Sp}_{2n}$	$\bigwedge_0^2 \text{Sp}_4$	$\bigwedge_0^3 \text{Sp}_6$
$\text{Sp}_{2n}$	1a)	1b)	1c)
$\bigwedge_0^2 \text{Sp}_4$	2a)	2b)	★

1a)  $P^{(2,2)} = \langle 2 \rangle \cdot (\langle 0 \rangle + \langle 1^2 \rangle) = 2 \cdot \langle 2 \rangle + \langle 2, 1^2 \rangle + \langle 1, 1 \rangle + \langle 3, 1 \rangle$

1b)  $P^{(k,1)} = P^{(k|4)} = \langle k \rangle \cdot \langle 1^2 \rangle = \langle k+1, 1 \rangle + \langle k-1, 1 \rangle + \langle k \rangle$ , and  $P^{(k|2)} = P^{(k,3)} = \langle k+2 \rangle + \langle k+1, 1 \rangle + \langle k \rangle + \langle k, 2 \rangle + \langle k-1, 1 \rangle + \langle k-2 \rangle$ .

In particular, representation 5) of Theorem 3.1a) is super MF.

1c)  $P^{(2,3)} = \langle 2 \rangle \cdot (\langle 1^3 \rangle + \langle 3, 2 \rangle) = 2 \cdot \langle 3, 2 \rangle + \dots$

2a)  $P^{(2,2)} = (\langle 0 \rangle + \langle 2^2 \rangle) \cdot (\langle 1^2 \rangle + \langle 0 \rangle) = 2 \cdot \langle 1^2 \rangle + \dots$

2b)  $P^{(2,1)} = (\langle 0 \rangle + \langle 2^2 \rangle) \cdot \langle 1^2 \rangle = 2 \cdot \langle 1^2 \rangle + \langle 3^2 \rangle + \langle 3, 1 \rangle$

For the remaining types of groups  $B$ ,  $D$ ,  $E$  and  $G$  we shall mainly label the highest weights relative to the basis of fundamental weights of the corresponding group. In those cases, we use round parentheses for the highest weight vectors.

Type <b>B</b>	$SO_{2n+1}$	$\Delta_7$	$\Delta_9$
$SO_{2n+1}$	1a)	1b)	1c)
$\Delta_7$	2a)	2b)	★
$\Delta_9$	3a)	★	2c)

1a)  $P^{(2,2)} = ([0] + [2]) \cdot [1^2] = 2 \cdot [1^2] + [2, 1^2] + [2] + [3, 1]$

1b)  $P^{(2,3)} = ((2, 0, 0) + (0, 0, 0)) \cdot ((0, 0, 1) + (1, 0, 1)) = 2 \cdot (1, 0, 1) + \dots$

1c)  $P^{(2,3)} = ((2, 0, 0, 0) + (0, 0, 0, 0)) \cdot ((0, 1, 0, 1) + (1, 0, 0, 1)) = 2 \cdot (1, 0, 0, 1) + \dots$

2a)  $P^{(3,2)} = ((0, 0, 1) + (0, 0, 3)) \cdot (0, 1, 0) = 2 \cdot (0, 1, 1) + \dots$

2b)  $P^{(2,3)} = ((0, 0, 2) + \dots) \cdot ((1, 0, 1) + \dots) = 2 \cdot (1, 0, 1) + \dots$

3a)  $P^{(3,2)} = ((0, 0, 0, 1) + (0, 0, 0, 3)) \cdot (0, 1, 0, 0) = 2 \cdot (0, 1, 0, 1) + \dots$

3c)  $P^{(2,3)} = ((0, 0, 0, 2) + \dots) \cdot ((1, 0, 0, 1) + \dots) = 2 \cdot (1, 0, 0, 1) + \dots$

For the half spin representations we use the convention  $\Delta_{2n}^+ = V_{\omega_{n-1}}$  and  $\Delta_{2n}^- = V_{\omega_n}$ . Note that  $(\Delta_{10}^+)^* = \Delta_{10}^-$  while  $\Delta_8^\pm$  and  $\Delta_{12}^\pm$  are self-dual.

Type <b>D</b>	$SO_{2n}$	$\Delta_8^+$	$\Delta_{10}^+$	$\Delta_{12}^+$	$\Delta_{12}^-$
$SO_{2n}$	1a)	1b)	1c)	1d)	1e)
$\Delta_8^+$	2a)	2b)	★	★	★
$\Delta_{10}^+$	3a)	★	3b)	★	★

1a) is not super MF, since it has the same decomposition as B 1a).

1b)  $P^{(3,4)} = ((1, 0, 0, 0) + (3, 0, 0, 0)) \cdot ((2, 0, 0, 0) + \dots) = 2 \cdot (1, 0, 0, 0) + \dots$

1c)  $P^{(2,5)} = ((2, 0, 0, 0, 0) + \dots) \cdot ((1, 1, 0, 0, 1) + \dots) = 2 \cdot (1, 1, 0, 0, 1) + \dots$

$$1d) P^{(2,5)} = ((2, 0, 0, 0, 0, 0) + \dots) \cdot ((0, 1, 1, 0, 0, 1) + \dots) = 2 \cdot (0, 1, 1, 0, 0, 1) + \dots$$

$$1e) P^{(2,5)} = ((2, 0, 0, 0, 0, 0) + \dots) \cdot ((0, 1, 1, 0, 1, 0) + \dots) = 2 \cdot (0, 1, 1, 0, 1, 0) + \dots$$

$$2a) P^{(3,4)} = ((0, 0, 1, 0) + (0, 0, 3, 0)) \cdot ((0, 0, 2, 0) + \dots) = 2 \cdot (0, 0, 1, 0) + \dots$$

2b)  $\Delta_8^+ \boxplus \Delta_8^+$  is geometrically equivalent to  $\mathbb{C}^8 \boxplus \mathbb{C}^8$ .

$$3a) P^{(3,3)} = ((1, 0, 0, 1, 0) + \dots) \cdot ((0, 0, 1, 0, 0) + \dots) = 2 \cdot (0, 1, 0, 1, 0) + \dots$$

$$3b) P^{(3,2)} = ((1, 0, 0, 1, 0) + \dots) \cdot (0, 0, 1, 0, 0) = 2 \cdot (1, 0, 0, 0, 1) + \dots$$

Last in line are the exceptional groups. According to Theorem 2.4a) and 2.5a), we have to check  $G_2 \boxplus G_2$  and  $E_6 \boxplus E_6$ . They are not super MF, since

$$P^{(1,2)}(G_2 \boxplus G_2) = (1, 0) \cdot ((1, 0) + (0, 1)) = 2 \cdot (1, 0) + \dots$$

$$\begin{aligned} P^{(2,1)}(E_6 \boxplus E_6) &= ((0, 0, 0, 0, 0, 1) + (2, 0, 0, 0, 0, 0)) \cdot (1, 0, 0, 0, 0, 0) \\ &= 2 \cdot (1, 0, 0, 0, 0, 1) + \dots \end{aligned}$$

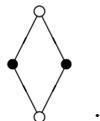
Let  $(G, V)$  be a saturated indecomposable super MF representation. Now we know that, if  $V$  has exactly two irreducible submodules, then there are no other equivalence classes than those in Theorem 3.1a). The next lemma (in connection with Proposition 3.6) shows that  $V$  cannot have more than two irreducible submodules. In particular, this concludes the proof of Theorem 3.1a).

**Lemma 5.1.** *A saturated indecomposable representation  $(G, V)$  with three irreducible submodules is never super MF.*

**Proof.** This follows from the fact that for any simple group, the triple tensor product of irreducible representations is never multiplicity-free and  $P^{(1,1,1)}(V_1 \oplus V_2 \oplus V_3) = V_1 \otimes V_2 \otimes V_3$ . ■

### 6. Representations of groups with two simple factors

Now we turn to the case where  $(\tilde{G}, V)$  is a saturated indecomposable representation and  $G$  has two simple factors  $G_1, G_2$ . Let  $V$  be a proper super space with  $V_0$  and  $V_1$  being irreducible. The next lemma tells us that there can be no super MF spaces related to diagrams of the form



**Lemma 6.1.** *If  $(\tilde{G}, V)$  is super MF, then it is impossible that both simple factors of  $G$  act nontrivially on  $V_0$  and  $V_1$ .*

**Proof.** By assumption,  $V_i = W_i^{(1)} \otimes W_i^{(2)}$  with  $W_i^{(j)}$  being an irreducible representation of  $G_j$ . For the moment we assume that

$$(\tilde{G}, V) = ((\mathbb{C}^*)^2 \times \mathrm{SL}_n \times \mathrm{SL}_m, \mathbb{C}^n \otimes \mathbb{C}^m \boxplus \mathbb{C}^n \otimes \mathbb{C}^m).$$

Then, the homogeneous components of  $P(V)$  are given by

$$P^{(k,l)} = \bigoplus_{\substack{|\lambda| = k, \ell(\lambda) \leq \min\{n, m\} \\ |\mu| = l, \ell(\mu) \leq n, \mu_1 \leq m}} (V(\lambda)^{(n)} \otimes V(\mu)^{(n)}) \otimes (V(\lambda)^{(m)} \otimes V(\mu)^{(m)}). \quad (27)$$

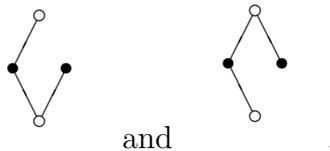
If  $n, m \geq 3$ , consider  $k = l = 3$  and  $\lambda = \mu = (2, 1)$ . Then,  $V(3, 2, 1)^{(n)}$  has multiplicity two inside  $V(2, 1)^{(n)} \otimes V(2, 1)^{(n)}$ . If  $n = 2$ , consider  $k = l = 2$ . So, we get terms that are labelled by  $\lambda, \mu \in \{(2), (1^2)\}$  and both,  $V(2)^{(2)} \otimes V(2)^{(2)}$  and  $V(1^2)^{(2)} \otimes V(1^2)^{(2)} \simeq \mathbb{C}$  contain the trivial representation of  $\mathrm{SL}_2$ .

Now, if the  $G_i$  are arbitrary simple groups, we get plethysms

$$G_1 \times G_2 \rightarrow \mathrm{GL}(W_i^{(1)}) \times \mathrm{GL}(W_i^{(2)}) \rightarrow \mathrm{GL}(W_i^{(1)} \otimes W_i^{(2)}). \quad (28)$$

Therefore, the multiplicities of the  $G_1 \times G_2$  module  $P(V_0 \boxplus V_1)$  are governed by (27) and hence they are always  $\geq 2$ . ■

We keep the above assumptions on  $V$  and assume that  $(G, V)$  is saturated indecomposable super MF. By Lemma 6.1, there is one part  $X$  of the super vector space on which both simple factors of  $G$  act nontrivially, while on the other part  $Y$  there is exactly one simple factor acting nontrivially. The respective representation diagrams look like



Let us consider the case  $X = V_1$  and  $Y = V_0$  first. Below we list all combinations  $V = V_0 \boxplus V_1$  with  $V_0$  being a representation of a simple group from Theorem 2.4a) and  $V_1 = W_1^{(1)} \otimes W_1^{(2)}$  being a representation of a group with two simple factors from Theorem 2.5b), such that  $V$  is not contained in Theorem 3.1b).

Note that by Proposition 3.6, we can avoid calculations for all  $V$  such that  $V_0 \boxplus W_1^{(i)}$  is not super MF. By this reasoning, we can omit computations for the following modules:

$$\begin{aligned} & \bigwedge^2 \mathrm{SL}_n \boxplus \mathrm{SL}_n \otimes \mathrm{SL}_m \quad (n \geq 4, m \geq 2), \quad S^2 \mathrm{SL}_2 \boxplus \mathrm{SL}_k \otimes S^2 \mathrm{SL}_2 \quad (k = 2, 3), \\ & \bigwedge^2 \mathrm{SL}_n \boxplus \mathrm{SL}_n \otimes \mathrm{Sp}_4 \quad (n \geq 4), \quad \mathrm{Sp}_4 \boxplus \mathrm{SL}_n \otimes \mathrm{Sp}_4, \quad \bigwedge_0^2 \mathrm{Sp}_4 \boxplus \mathrm{SL}_n \otimes \mathrm{Sp}_4, \quad \mathrm{SO}_{2n} \boxplus \mathrm{SL}_2 \otimes \mathrm{SO}_{2n}, \\ & \Delta_8^+ \boxplus \mathrm{SL}_2 \otimes \mathrm{SO}_8, \quad \Delta_{10}^+ \boxplus \mathrm{SL}_2 \otimes \mathrm{SO}_{10}, \quad \bigwedge^2 \mathrm{SL}_3 \boxplus \mathrm{SL}_3 \otimes \mathrm{SO}_{2n+1}, \quad \mathrm{SO}_{2n+1} \boxplus \mathrm{SL}_k \otimes \mathrm{SO}_{2n+1}, \\ & \Delta_7 \boxplus \mathrm{SL}_k \otimes \mathrm{SO}_7 \quad \text{and} \quad \Delta_9 \boxplus \mathrm{SL}_k \otimes \mathrm{SO}_9. \end{aligned}$$

For the remaining modules, we compute in each case a graded subspace

$P^{(i,j)}(V)$  of  $P(V)$  that contains multiplicities.

Representation	$(i, j)$	Multiplicity
$S^2\mathrm{SL}_n \boxplus \mathrm{SL}_n \otimes \mathrm{SL}_m$	$(5, 2)$	$2 \cdot \{8, 4\} \otimes \{1^2\}$
$\mathrm{SL}_2 \boxplus \mathrm{SL}_k \otimes S^2\mathrm{SL}_2$ ( $k = 2, 3$ )	$(2, 3)$	$2 \cdot \{3 - k\} \otimes \{4\}$
$\mathrm{SL}_4 \boxplus \mathrm{SL}_2 \otimes \bigwedge^2 \mathrm{SL}_4$	$(1, 4)$	$2 \cdot \{2\} \otimes \{2^2, 1\}$
$\mathrm{SL}_n \boxplus \mathrm{SL}_n \otimes \mathrm{Sp}_4$	$(1, 3)$	$2 \cdot \{3, 1\} \otimes \langle 1 \rangle$
$\mathrm{SL}_2 \boxplus \mathrm{SL}_2 \otimes \mathrm{SO}_{2n}$	$(2, n + 1)$	$2 \cdot \{n + 1\} \otimes [1^{n-1}]$
$\mathrm{SL}_3 \boxplus \mathrm{SL}_3 \otimes \mathrm{SO}_{2n+1}$	$(1, 3)$	$2 \cdot \{1\} \otimes [1]$
$\mathrm{Sp}_4 \boxplus \mathrm{SL}_3 \otimes \bigwedge_0^2 \mathrm{Sp}_4$	$(2, 3)$	$2 \cdot \{0\} \otimes \{3, 1\}$
$\Delta_5 \boxplus \mathrm{SL}_2 \otimes \mathrm{SO}_5$	$(2, 3)$	$2 \cdot \{1\} \otimes [2]$

Since the first representation in this list is not super MF, one can show that also  $S^2\mathrm{SL}_k \boxplus \mathrm{SL}_k \otimes \mathrm{SO}_m$  and  $S^2\mathrm{SL}_n \boxplus \mathrm{SL}_n \otimes \mathrm{Sp}_4$  are not super MF. This follows from an argument similar to that in (28).

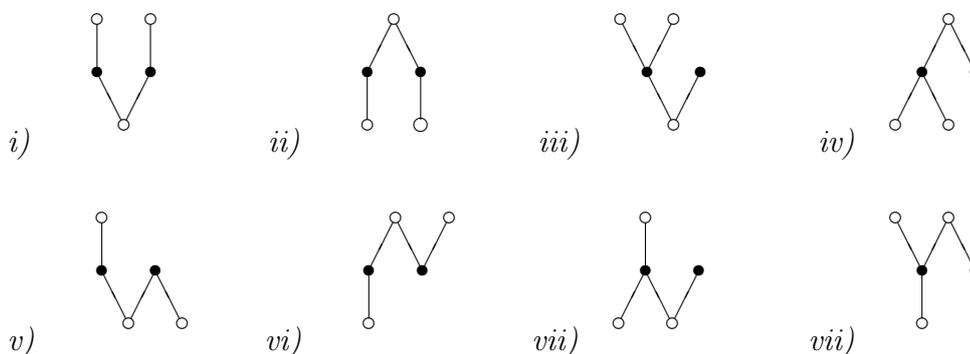
Now we interchange the role of even and odd part, i.e.  $X = V_0$  and  $Y = V_1$ . In this case we only have to consider the groups  $\mathrm{SL}_n \times \mathrm{SL}_m$  and  $\mathrm{SL}_n \times \mathrm{Sp}_4$ .

Representation	$(i, j)$	Multiplicity
$\mathrm{SL}_n \otimes \mathrm{SL}_m \boxplus \bigwedge^2 \mathrm{SL}_m$ ( $n \geq 2, m \geq 4$ )	$(3, 2)$	$2 \cdot \{2, 1\} \otimes \{3, 2, 1^2\}$
$\mathrm{SL}_n \otimes \mathrm{Sp}_{2m} \boxplus \mathrm{SL}_n$ ( $n > 3, m = 2$ or $n = 3, m > 2$ )	$(3, 1)$	$2 \cdot \{2, 1\} \otimes \langle 1 \rangle$
$\mathrm{SL}_n \otimes \mathrm{Sp}_4 \boxplus \bigwedge_0^2 \mathrm{Sp}_4$ ( $n \geq 2$ )	$(3, 2)$	$2 \cdot \{2, 1\} \otimes \langle 1^2 \rangle$

Here, we could omit calculations for the modules  $\mathrm{SL}_n \otimes \mathrm{SL}_2 \boxplus S^p\mathrm{SL}_2$  ( $p = 3, \dots, 6$ ),  $\mathrm{SL}_m \otimes \mathrm{SL}_n \boxplus S^2\mathrm{SL}_m$  ( $m \geq 3, n \geq 2$ ),  $\mathrm{SL}_n \otimes \mathrm{SL}_3 \boxplus S^3\mathrm{SL}_3$ ,  $\mathrm{SL}_n \otimes \mathrm{SL}_6 \boxplus \bigwedge^3 \mathrm{SL}_6$ ,  $\mathrm{SL}_n \otimes \mathrm{Sp}_4 \boxplus S^2\mathrm{SL}_n$  ( $n \geq 3$ ),  $\mathrm{SL}_m \otimes \mathrm{Sp}_4 \boxplus \bigwedge^2 \mathrm{SL}_m$ ,  $\mathrm{SL}_n \otimes \mathrm{Sp}_{2m} \boxplus \mathrm{Sp}_{2m}$  ( $n > 3, m = 2$  or  $n \leq 3, m > 2$ ) and  $\mathrm{SL}_k \otimes \mathrm{Sp}_6 \boxplus \bigwedge_0^3 \mathrm{Sp}_6$  ( $k = 2, 3$ ).

Now we turn to super vector spaces that decompose into three irreducible submodules for  $G$ . In particular, either  $V_0$  or  $V_1$  is irreducible. The possible representation diagrams are given in the lemma below. We show that  $(G, V)$  is never super MF in these cases.

**Lemma 6.2.** *Representations of the following type are never super MF:*



**Proof.** Assume for the moment that  $G = \mathrm{SL}_n \times \mathrm{SL}_m$ . Let  $U$  and  $W$  be the defining representations (or its symmetric square) of  $\mathrm{SL}_n$  or  $\mathrm{SL}_m$  and denote by  $V = U \oplus W \oplus \mathbb{C}^n \otimes \mathbb{C}^m$  a super vector space without deciding yet to which part

of  $V$ , even or odd, the direct summands belong. Observe that for fixed choice of  $U$  and  $W$  the decomposition of

$$P^1(U) \otimes P^1(W) \otimes P^2(\mathbb{C}^n \otimes \mathbb{C}^m)$$

depends only on the parity of  $\mathbb{C}^n \otimes \mathbb{C}^m$  in  $V$ , but not on those of  $U$  and  $W$ . For example, fix  $U = S^k \mathbb{C}^n$ ,  $W = S^l \mathbb{C}^m$  (with  $k, l = 1, 2$ ), then it follows that representations of type  $i$ ) and  $v$ ) are not super MF since  $P^{(1,1,2)}(U \oplus W \boxplus \mathbb{C}^n \otimes \mathbb{C}^m)$  (resp.  $P^{(1,1,2)}(U \boxplus W \oplus \mathbb{C}^n \otimes \mathbb{C}^m)$ ) is given by

$$V(k)^{(n)} \otimes V(l)^{(m)} \otimes (V(2)^{(n)} \otimes V(1^2)^{(m)} \oplus V(1^2)^{(n)} \otimes V(2)^{(m)}) \tag{29}$$

and contains  $V(k + 1, 1)^{(n)} \otimes V(l + 1, 1)^{(m)}$  with multiplicity two. The remaining cases can be ruled out by two completely analogous calculations.

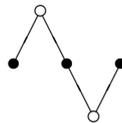
It follows by a branching argument that also for any other group  $G$  the diagrams  $i$ ) -  $vii$ ) do not give rise for a super MF representation: in this case, one has to restrict irreducible terms in (29) to  $SO_n$  or  $Sp_{2p}$ . Hence, all multiplicities for a  $SL_n$  factor also give rise to multiplicities for the subgroup. ■

Suppose we are given an indecomposable saturated action  $(\tilde{G}, V)$  with  $G = G_1 \times G_2$  and  $V$  consisting of three irreducible submodules or more. Its representation diagram has either a subdiagram which equals one of those in the above lemma, or any that describes the action of  $\tilde{G}$  on a purely even (or purely odd) super vector space with three irreducible submodules. In both cases,  $(G, V)$  fails to be super MF and thus the proof of Theorem 3.1b) is complete.

### 7. Representations of groups with more than two simple factors

Finally, let  $G = G_1 \times G_2 \times G_3$  be a group consisting of three simple factors. As above, we assume  $(\tilde{G}, V)$  to be saturated indecomposable. Note that part a) and b) of Theorem 3.1 are now completely proven. Since the representation diagram  $\mathcal{G}(G, V)$  has at least four vertices, we can find many different subdiagrams. That is why Proposition 3.6 becomes very useful at this point.

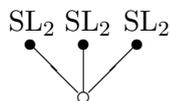
**Lemma 7.1.** *The representations listed in Theorem 3.1c) are the only super MF spaces of type*



**Proof.** Let  $V$  be a super MF space of the above type. Its representation diagram must contain subdiagrams corresponding to representations  $1)$ ,  $2)$ ,  $3)$  or  $5)$  in Theorem 3.1b). By Lemma 4.1, the group factor that acts nontrivially on both parts of  $V$  must be isomorphic to  $SL_2$ . This forces  $V$  to be equivalent to either representation  $1)$  -  $4)$  of Theorem 3.1c). ■

In the case of purely odd super spaces, we get an instance of an irreducible super MF representation for a group with three simple factors. This is given by  $SL_2 \otimes SO_4 \sim SL_2 \otimes SL_2 \otimes SL_2$  and we show that this is the only example of a super MF space where more than two simple factors act nontrivially on a single irreducible summand.

**Lemma 7.2.** *Let  $\mathcal{G}$  be a connected representation diagram that contains*



*as a proper subgraph. Then the action described by  $\mathcal{G}$  is never super MF.*

**Proof.** By assumption, there is a subdiagram  $\mathcal{G}'$  (with probably  $\mathcal{G}' = \mathcal{G}$ ) that corresponds to either a purely odd super vector space  $V = S^k SL_2 \oplus SL_2 \otimes SL_2 \otimes SL_2$  or to  $V' = S^k SL_2 \boxplus SL_2 \otimes SL_2 \otimes SL_2$ , where  $1 \leq k \leq 4$ . By restricting from  $GL_4$  to  $SO_4$  we can deduce from  $(GL_n, GL_m)$  skew duality (14) that  $\bigwedge^3(SL_2 \otimes SL_2 \otimes SL_2)$  contains

$$V(1)^{(2)} \otimes V(1)^{(2)} \otimes V(1)^{(2)} \text{ and } V(3)^{(2)} \otimes V(1)^{(2)} \otimes V(1)^{(2)}.$$

It follows that there are multiplicities in  $P^{(1,3)}(V')$ , resp.  $P^{(1,3)}(V)$  since the tensor product of  $P^1(S^k SL_2) = V(k)^{(2)} \otimes V(0)^{(2)} \otimes V(0)^{(2)}$  with one of the above modules contains  $V(k-1) \otimes V(1) \otimes V(1)$  in both cases. ■

Now let  $G = G_1 \times \dots \times G_k$  be a group with  $k \geq 3$  simple factors and consider a saturated indecomposable representation  $(\tilde{G}, V)$ . We conclude that Theorem 3.1c) is now proved since for the representation diagram of  $(\tilde{G}, V)$  one of the following is true:

1. it describes the action of  $G$  on a purely even (resp. odd) super vector space;
2. or it has a form as in Lemma 7.1 (if  $k = 3$  and  $V$  has exactly two irreducible submodules);
3. or it contains a connected subdiagram with two group vertices and three representation vertices (if  $V$  has more than two irreducible submodules);
4. or there is an irreducible submodules on which at least 3 simple factors act nontrivially.

In the first case, if  $(\tilde{G}, V)$  is super MF, it is geometrically equivalent to one of the representations 1) - 3) from Theorem 3.1c), while in the second case it is geometrically equivalent to one of 4) - 10). In cases three and four  $(\tilde{G}, V)$  can obviously not be super MF.

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