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Some Notes on the Schur Multiplier of a Pair of Lie Algebras

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Abstract. In this paper, we review some properties of the non-abelian exterior product and using them provide special properties of the Schur multiplier of a pair of Lie algebras. By constructing a cover for a pair of Lie algebras, we give a necessary and sufficient condition for the capability of a pair of Lie algebras. Also, for a perfect pair of Lie algebras we construct a cover which is a universal extension. Finally, we give some bounds for the dimension of the Schur multiplier of a pair of Lie algebras.
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1. Introduction

Let L be a Lie algebra over a field Λ of arbitrary characteristic and M be a left L-module. We recall that n-th homology group of L with values in M is defined as:

$$H_n(L,M) = \mathrm{Tor}_n^{UL}(M,\Lambda), \quad n = 0,1,2,\cdots,$$

where UL is the universal enveloping algebra of L and Tor_n is the n-th left derived functor of tensor product. For more details on this topic see [11, Chapter 7]. In the special case, if Λ is considered as a trivial L-module then $H_n(L) = H_n(L,\Lambda)$ is called the n-th homology group of L. Let $\wedge^n L$ be the n-th exterior product of L, which is the free Λ–module generated by monomials $x_1 \wedge \cdots \wedge x_n$ with $x_i \in L$. If M is a left L-module, it is well-known that the homology groups $H_*(L,M)$, are the homology modules of the chain complex $(M \otimes_\Lambda \wedge L, d)$ with the boundary map

$$\begin{aligned} d(m\otimes x_1\wedge\cdots\wedge x_n) = &\sum_{i=1}^{n}(-1)^{i+1}(x_i m)\otimes(x_1,\ldots,\hat{x}_i,\ldots,x_n)\\ &+\sum_{i<j}(-1)^{i+j} m\otimes[x_i,x_j]\wedge x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_n), \end{aligned}$$

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where $m \in M$ and $x_1, \dots, x_n \in L$. Assume N is an ideal of the Lie algebra L and $\pi_0 : (M \otimes_\Lambda \wedge L, d) \to (M \otimes_\Lambda \wedge L/N, d)$ is the morphism of chain complexes which is induced by the canonical homomorphism $\pi : L \to L/N$. According to [11, Theorem 1.3.1], if $\mathrm{Cone}(\pi_0)$ is the mapping cone of π_0 then we obtain the following long exact sequence of homology groups:

$$\begin{aligned} \cdots \to H_{n+1}(\mathrm{Cone}(\pi_0)) \to H_n(L, M) \to \\ \to H_n(L/N, M) \to H_n(\mathrm{Cone}(\pi_0)) \to H_{n-1}(L, M) \to \cdots . \end{aligned} \tag{1}$$

$H_{n+1}(\mathrm{Cone}(\pi_0))$ is called the n-th relative homology of L and is denoted by $H_n(L, N, M)$.

In a special case, $H_2(L, N, \Lambda)$ is called the Schur multiplier of the pair of Lie algebras (L, N) and is denoted by $\mathcal{M}(L, N)$. For suitable pairs of Lie algebras, one can consider $\mathcal{M}(-, -)$ as a bifunctor. We note that when N equals to the zero ideal, sequence (1) yields $\mathcal{M}(L, N) = 0$. Furthermore, $\mathcal{M}(L, L) \cong H_2(L)$ which is called the Schur multiplier of L and is denoted by $\mathcal{M}(L)$. For more information about the Schur multiplier of a pair of Lie algebras see [5], [6], [7], and to see a different introduction by non-abelian homology see [3].

There is an intimate relationship between concepts of the Schur multiplier and the non-abelian exterior product which was introduced by G. Ellis in [3]. If K, N are ideals of a Lie algebra L then the non-abelian exterior product $K \wedge N$ is the Lie algebra generated by the elements $k \wedge n$ with $(k, n) \in K \times N$, subject to the following relations:

$$\begin{array}{lll} c(k \wedge n) = ck \wedge n = k \wedge cn & , & [k, k'] \wedge n = k \wedge [k', n] - k' \wedge [n, k] \\ (k + k') \wedge n = k \wedge n + k' \wedge n & , & k \wedge [n, n'] = [n', k] \wedge k - [k, n] \wedge n' \\ k \wedge (n + n') = k \wedge n + k \wedge n' & , & [(k \wedge n), (k' \wedge n')] = [k, n] \wedge [k', n'] \\ x \wedge x = 0, & & \end{array}$$

for all $x \in K \cap N$, $c \in \Lambda, k, k' \in K$ and $n, n' \in N$. The exterior product $K \wedge N$ can also be defined by its universal property: given a Lie algebra T and an exterior Lie pairing $f : K \times N \to T$, there is a unique Lie homomorphism $h : K \wedge N \to T$ such that $f(k, n) = h(k \wedge n)$ for all $k \in K, n \in N$. Note that, if $[K, N] = 0$ then $K \wedge N$ is the usual exterior product of modules which is denoted by $K \wedge_\Lambda N$ in this paper. The commutator map $[\ ,\ ] : K \wedge N \to L$ is the Lie homomorphism defined on generators by $k \wedge n \longmapsto [k, n]$. An interesting case of the commutator mapping is $[\ ,\ ] : L \wedge N \to L$, in which the kernel and image of this homomorphism are $\mathcal{M}(L, N)$ and $[L, N]$, respectively. Throughout this paper, we identify $\mathcal{M}(L, N)$ by its image in $L \wedge N$. The non-abelian tensor product $K \otimes N$, is defined similarly by eliminating the relations $x \wedge x = 0$ in the non-abelian exterior product.

The following lemma is an immediate result of the definition of exterior product.

Lemma 1.1. *Let L be a Lie algebra and I, N, K be ideals of L such that I is contained in $K \cap N$, then the following sequence is exact:*

$$(K \wedge I) + (I \wedge N) \to K \wedge N \to \frac{K}{I} \wedge \frac{N}{I} \to 0.$$

2. Capability of a pair of Lie algebras

Our main goal in this section is to give a sufficient condition for the capability of a pair of Lie algebras. Also, by a constructive method, we show that any finite dimensional pair of Lie algebras have a cover.

In 1982, Kassel and Loday in [4], introduced the concept of Lie crossed module (of the pair of Lie algebras (L, N)) to be a Lie homomorphism $\delta : M \to L$ together with an action of L on M, which is denoted by ${}^{x}m$ for all $x \in L, m \in M$, satisfying the following conditions:

(i) $\delta({}^{x}m) = [x, \delta(m)]$ for all $x \in L,\ m \in M$,

(ii) ${}^{\delta(m)}(m') = [m, m']$, for all $m, m' \in M$ (Peiffer identity).

(iii) $\delta(M) = N$.

Crossed modules in Lie algebras are algebraic objects which can be viewed as a simultaneous generalization of the concepts of ideals and of modules over Lie algebras. There is a correspondence between the group of the equivalence classes of crossed modules and the third cohomology group of Lie algebras.

It is readily seen that the kernel of δ is an L–invariant ideal contained in the center of M. Clearly, the inclusion map $i : N \hookrightarrow L$ is a simple example of a crossed module of the pair (L, N), where the action of L on N is determined by the Lie multiplication of L.

We use the notations and terminologies of [7] for the L–center and commutator of M which are defined as:

$$Z(L, M) = \{m \in M \mid {}^{y}m = 0, \quad \text{for all} \quad y \in L\},$$
$$[L, M] = < {}^{y}m \mid y \in L, m \in M >$$

Definition 2.1. The pair (L, N) is called capable if it admits a crossed module $\delta : M \to L$ with $\ker \delta = Z(L, M)$. It is obvious that a Lie algebra L is capable (i.e. L is isomorphic to a factor Lie algebra $H/Z(H)$ for some Lie algebra H) if the pair (L, L) is capable.

Also, we define the relative tensor and exterior centers of the pair (L, N) as follows:

$$Z^{\otimes}(L, N) = \{x \in N \ : \ x \otimes y = 0 \ \text{ for all } \ y \in L\},$$
$$Z^{\wedge}(L, N) = \{x \in N \ : \ x \wedge y = 0 \ \text{ for all } \ y \in L\}.$$

It is easy to check that $Z^{\otimes}(L, N) \subseteq Z^{\wedge}(L, N) \subseteq Z(L, N)$ and $Z^{\otimes}(L, N)$ and $Z^{\wedge}(L, N)$ are central ideals of L.

The following proposition gives a condition for equality of the relative centers.

Proposition 2.2. *Let (L, N) be an arbitrary pair of Lie algebras. Then $Z^{\otimes}(L, N) = Z^{\wedge}(L, N)$ if $[L, N] = N$ and $Z^{\wedge}(L, N) = Z(L, N)$ if $\mathcal{M}(L, N) = 0$.*

Proof. Let $\Gamma(-)$ be the universal quadratic functor. By [3, Proposition 14], the following natural exact sequence of Lie algebras exists:

$$\Gamma(\frac{N}{[L,N]}) \to L \otimes N \xrightarrow{\pi} L \wedge N \to 0,$$

where $\pi(m \otimes n) = m \wedge n$. Hence, if $N = [N,L]$ then $\Gamma(N/[N,L]) = 0$, which concludes the injectivity of π. For the second assertion, consider the exact sequence

$$0 \to \mathcal{M}(L,N) \to L \wedge N \xrightarrow{[\,,\,]} [L,N] \to 0.$$

If $\mathcal{M}(L,N) = 0$ then $L \wedge N \cong [L,N]$, which gives the desired result. ■

The following proposition is similar to a result of [2] in group case.

Proposition 2.3. *Let L be an arbitrary Lie algebra then $Z^{\wedge}(L,N)$ is the smallest central ideal containing all the central ideals I for which the canonical homomorphism $\mathcal{M}(L,N) \to \mathcal{M}(L/I, N/I)$ is a monomorphism. Equivalently, the canonical homomorphism $L \wedge N \to L/I \wedge N/I$ is an isomorphism.*

Proof. To prove the first part, it is suffices to show that the following sequence is exact:

$$L \wedge I \xrightarrow{f} \mathcal{M}(L,N) \to \mathcal{M}(L/I, N/I),$$

where f is the restriction of natural homomorphism $\bar{f} : L \wedge I \to L \wedge N$. Since $\bar{f}(L \wedge I) = \ker(L \wedge I \to L/I \wedge N/I)$ so $f(L \wedge I) = \ker(\mathcal{M}(L,N) \to \mathcal{M}(L/I, N/I))$. The second part is the consequence of the exactness of sequence

$$I \wedge L \to L \wedge N \to L/I \wedge N/I \to 0.$$ ■

The relation between the capability of a Lie algebra and its epic-center was discussed in [9]. Using free presentation tools and the Hopf's formula, it was shown that the capability of a Lie algebra is equivalent to trivialness of its epic-center. By a completely different discussion, we give a similar result for the capability of a pair of Lie algebras. G.Ellis in [1], proved similar result in group theory by some free presentation tools, which we avoid from his technique here.

Theorem 2.4. *The pair (L,N) of finite dimensional Lie algebras is capable if and only if $Z^{\wedge}(L,N)$ is trivial.*

Proof. First, we remind that the Lie product of L give rise to an action of L on $L \wedge N$ defined by

$$^{l'}(l \wedge n) = [l', l] \wedge n + l \wedge [l', n] = l' \wedge [l, n],$$

for all $l, l' \in L$ and $n \in N$. Now, suppose that $Z^{\wedge}(L,N) = \{0\}$, thus the only element $n \in N$ such that for all $l \in L$, $n \wedge l = 0$ is the zero element. Set C to be a complement vector space of $[L,N]$ in N, i.e. $N = C \oplus [L,N]$. It is easily verified that the vector space $K = (L \wedge N) \oplus C$ with the Lie multiplication defined by

$$[(x_1 \wedge y_1, l_1), (x_2 \wedge y_2, l_2)] = (\,^{l_1}(x_2 \wedge y_2) - \,^{l_2}(x_1 \wedge y_1) + l_1 \wedge l_2 + [x_1, y_1] \wedge [x_2, y_2], 0),$$

for $l_i \in C, x_i \wedge y_i \in L \wedge N$, $i = 1, 2$ is a Lie algebra and L can act on K by setting

$$^{l}(x \wedge y, n) = l \wedge [x, y] + l \wedge n = l \wedge ([x, y] + n)$$

for all $l \in L, x \wedge y \in L \wedge N, n \in C$. We may identify $L \wedge N$ with its images in K. By the definition of product in K, if $x \wedge y \in \mathcal{M}(L, N)$ and $l \in L$, since

$$^{l}(x \wedge y) = [l, x] \wedge y + x \wedge [l, y] = l \wedge [x, y] = 0,$$

then $\mathcal{M}(L, N)$ is contained in $Z(L, K)$. As $Z^\wedge(L, N) = 0$, L cannot centralizes any other element of K and $Z(L, K) = \mathcal{M}(L, N)$. Now, consider the homomorphism $\delta : K \to L$ defined on generators of K by

$$\delta(x \wedge y, n) = [x, y] + n,$$

for all $x, y \in L$ and $n \in C$. A routine review shows that δ is a crossed module with $\delta(K) = N$. Also, $\ker \theta = \mathcal{M}(L, N) = Z^\wedge(L, N)$, thus (L, N) is capable.

Conversely, suppose that (L, N) is capable and $0 \to Z(L, M) \hookrightarrow M \xrightarrow{\delta} N \to 0$ is the relevant crossed module. Also, let $s : N \to M$ be a right inverse of δ and choose an arbitrary $z \neq 0$ in $Z(L, N)$. Since $s(z) \notin Z(L, M)$, there exists $x \in M$ such that $^{z}x = [s(z), x] \neq 0$. This yields that the preimage of $[s(z), x]$ under the commutator map is not vanished i.e. $s(z) \wedge x \neq 0$. This concludes $z \wedge \delta(x) \neq 0$. Hence $Z^\wedge(L, N) = \{0\}$. ■

Recall that a crossed module $\delta : M \to L$ of the pair (L, N) is called a cover of the pair (L, N) if $\ker \delta \subseteq Z(L, M) \cap [L, M]$ and $\ker \delta \cong \mathcal{M}(L, N)$. It is clear that any cover of the pair (L, L) is the usual covering algebra of L. (See [9] for more details.) The construction of $\delta : K \to L$ in the proof of Theorem 2.4 shows the following.

Corollary 2.5. *Any finite dimensional pair of Lie algebras admit at least a cover.*

Let $z \in Z(L, N)$ be arbitrary, the equality $z \wedge [l, n] = [n, z] \wedge m - [z, l] \wedge n = 0_{L \wedge L}$ for all $l \in L, n \in N$ deduces that the mapping $d_z : L \to \mathcal{M}(L, N)$ defined by $d_z(l) = l \wedge z$ is a Lie homomorphism. Hence there exists a natural homomorphism $d : z \mapsto d_z$.

Corollary 2.6. *A Lie algebra L is capable if and only if the mapping*

$$d : Z(L, N) \to Hom(L/L^2, \mathcal{M}(L, N)),$$

is injective. In particular, if a finite dimensional Lie algebra L is capable then $\dim(Z(L)) \leq \dim(Hom(L/L^2, \mathcal{M}(L)))$.

Example 2.7. Let $H(n) = \langle x_i, y_i, z \mid [x_i, y_i] = z, i = 1, \ldots, n \rangle$ be the Heisenberg algebra of dimension $2n + 1$. It is easily verified that $Z(H(n)) = (H(n))^2 = \langle z \rangle$.

If $n>1$ then for any $1\leq i\leq n$ there exists $j\neq i$ such that

$$d_z(x_i)=z\wedge x_i=[x_j,y_j]\wedge x_i=x_j\wedge[y_j,x_i]-y_j\wedge[x_i,x_j]=0,$$

similarly $d_z(y_i)=0$. This concludes that $H(n)$ is not capable.

If $n=1$ and the characteristic of ground field is not 2, due to $z\wedge x_1\neq 0$, $d_z\neq 0$ and we can conclude that $H(1)$ is capable.

Example 2.8. Let $F_n (n\geq 3)$ be the standard filiform Lie algebra with bases $\{x,y_1\dots,y_n\}$ such that $[x,y_i]=y_{i+1}$ for all $1\leq i\leq n-1$. We have $Z(F_n)=\langle y_n\rangle$ and $d_{y_n}(x)\neq 0$. Hence F_n is a capable Lie algebra.

3. Perfect pair of Lie algebras

A Lie algebra is called perfect if it coincides with its derived subalgebra. The Schur multiplier and cover of a perfect Lie algebra was studied in [8]. Also, it was shown that any perfect Lie algebra admits a (unique) universal central extension. In this section, we generalize these results to a perfect pair of Lie algebras.

Definition 3.1. A pair of Lie algebras (L,N) is called perfect if $[L,N]=N$. It is obvious that if N is equal to L then the concept of perfect pair is equivalent to the usual concept of perfect Lie algebra.

Theorem 3.2. *Let (L,N) be a perfect pair of Lie algebras then the crossed module $[\ ,\ ]:L\otimes N\to L$ is a cover for the pair (L,N). Furthermore, for any other crossed module $\delta:M\to L$ such that $\ker\delta\subseteq Z(L,M)$, there exists a unique L-homomorphism $f:L\otimes N\to M$ such that $f\circ\delta(l\otimes n)=[l,n]$.*

Proof. A routine computations and using the fact $[L,N]=N$, shows that the mapping $[\ ,\ ]:L\otimes N\to L$ is crossed module. Also, by Proposition 2.2, $L\otimes N\cong L\wedge N$ and so

$$\mathcal{M}(L,N)=\ker(L\wedge N\to L)=\ker(L\otimes N\to L).$$

So, it is enough to show that $\ker(L\otimes N\to L)\subseteq[L,L\otimes N]\cap Z(L,L\otimes N)$. If $l\otimes n\in\mathcal{M}(L,N)$ then for any $l'\in N$ we have

$$^{l'}(l\otimes n)=l'\otimes[l,n]=0$$

so $l\otimes n\in Z(L,N)$. Also, since $l\otimes n$ is a linear combination of elements in the form $l\otimes[l',n']$ for some $l'\in L, n'\in N$ and $l\otimes[l',n']={}^{l}(l'\otimes n')$, we have $l\otimes n\in[L,L\otimes N]$. For the second part, we first prove the existence of homomorphism f. For any pair $(l,n)\in L\times N$, consider $\bar{n}$ to be an element of M such that $\delta(\bar{n})=n$. Now, the mapping $(l,n)\mapsto{}^{l}\bar{n}$ is well-define Lie pairing, so by [3, Proposition 1], there exists a Lie homomorphism $f:L\otimes N\to M$ such that $f(l\otimes n)={}^{l}\bar{n}$. Also f is an L-homomorphism because

$$f({}^{l'}(l\otimes n))=f(l'\otimes[l,n])={}^{l'}\overline{[l,n]}={}^{l'}({}^{l}\bar{n}),$$

for all $l, l' \in L, n \in N$. The second equality holds because δ is crossed module and $\delta({}^{l}m) = [l, \delta(m)]$. Now, suppose f_1, f_2 are L-morphisms such that

$$f_1 \circ \delta(l \otimes n) f_2 \circ \delta(l \otimes n) = [l, n].$$

Hence $(f_1 - f_2)(l \otimes n) \in \ker \delta \subseteq Z(L, M)$. Since $[L, N] = N$ we have

$$(f_1 - f_2)(l \otimes n) = (f_1 - f_2)(l \otimes [l', n]) = {}^{l}(f_1 - f_2)(l' \otimes n) = 0,$$

and so $f_1 = f_2$ as required. ■

4. Some bounds on the dimension of $\mathcal{M}(L, N)$

In this section, we give some inequalities for the dimension of the Schur multiplier of a pair of finite dimensional Lie algebras and their factor Lie algebras. Our work is a slight generalizations of some theorems in [10],[6]. Let L be a Lie algebra and N be an ideal of L and I be an ideal of L contained in N. Using the functorial properties of non-abelian exterior product and the commutator mapping we are able to extract the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{M}(L,I) & & \mathcal{M}(L,N) & \longrightarrow & \mathcal{M}(L/I,N/I) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & L \wedge I & \longrightarrow & L \wedge N & \longrightarrow & L/I \wedge N/I & \longrightarrow & 0 \\
 & & {\scriptstyle [\ ,\ ]}\downarrow & & {\scriptstyle [\ ,\ ]}\downarrow & & {\scriptstyle [\ ,\ ]}\downarrow & & \\
0 & \longrightarrow & ([L,N] \cap I) & \longrightarrow & [L,N] & \longrightarrow & [L/I,N/I] & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \qquad (2)$$

In this diagram, rows and columns are exact but unfortunately only the right hand side squares are commutative. If L, N, I are such that the left hand side square is also commutative then the "Snake Lemma" of homological algebra contexts yields the following exact sequence of abelian Lie algebras:

$$0 \to \mathcal{M}(L, I) \to \mathcal{M}(L, N) \to \mathcal{M}(L/I, N/I) \to \frac{[L, N] \cap I}{[I, L]} \to 0. \qquad (3)$$

It can be checked that if I is a central ideal of L (in this case the commutator map is zero) or N admit a subalgebra complement in L, this conditions can occurs. This argumentation can strongly generalizes the conditions of main sequence in [6], where using some free presentation tools to give the above sequence in the case that N admits a subalgebra compliment in L. Sequence (3) and its existence conditions is the most general sequence which relates the second homology of a Lie algebra and its factor algebra. For instance, if we choose I to be a central ideal and $N = L$ we obtain the following sequence which is given in [10, Proposition 2.1]

$$0 \to \mathcal{M}(L, I) \to \mathcal{M}(L) \to \mathcal{M}(L/I) \to (I \cap L^2) \to 0.$$

The following lemma plays an important role in our remaining results. It can be obtained by taking $j = 1$ in [10, Corollary 1.5].

Lemma 4.1. *Let L be a Lie algebra with a free presentation $L = F/R$ and for some suitable ideal S of F, $I = S/R$. Then $(R \cap [F, S])/[F, R]$ is a homomorphic image of $\mathcal{M}(L, I)$.*

By the consequence of former arguments we can refine conditions of [6, Theorem 2.3] to give the following bounds. One can check that if N is complemented in L then equality will happen in the best conditions.

Theorem 4.2. *Let (L, N) be a pair of finite dimensional Lie algebras and I be an ideal contained in N such that commutes the diagram (2) with the notations and assumptions of Lemma 3.1, the following statements hold:*

(i) $\dim(\mathcal{M}(L/I, N/I)) + \dim(\frac{[F,S]}{[R,F]}) \leq \dim(I \cap [N, L]) + \dim(\mathcal{M}(L, N))$.

(ii) $\dim(I \cap [N, L]) + \dim(\mathcal{M}(L, N)) \leq \dim(\mathcal{M}(L/I, N/I)) + \frac{1}{2}\dim(I)(2\dim(\frac{L}{I}) + \dim(I) - 1)$.

In particular, if $I \subseteq Z(L)$ then

$$\dim(I \cap [N, L]) + \dim(\mathcal{M}(L, N)) \leq \dim(\mathcal{M}(L/I, N/I)) + \dim(\frac{L}{L^2} \otimes I) + \dim(\mathcal{M}(I)).$$

Proof. Using sequence (3),

$$\dim(\mathcal{M}(L/I, N/I)) + \dim(\mathcal{M}(L, I)) + \dim([I, L]) = \dim(\mathcal{M}(L, N)) + \dim(I \cap [N, L]).$$

$[I, L] = [S, F]/(R \cap [S, F])$ and by Lemma 4.1 $\dim(\mathcal{M}(L, I)) \geq \dim(\frac{R \cap [F,S]}{[F,R]})$, so we have

$$\dim(\mathcal{M}(L, I)) + \dim([I, L]) \geq \dim(\frac{R \cap [T, F]}{[R, F]}) + \dim(\frac{[S, F]}{R \cap [S, F]}) = \dim(\frac{[F, S]}{[R, F]}),$$

which completes the first part. For the second part, we remind that

$$\dim(\mathcal{M}(L, I)) + \dim([I, L]) = \dim(L \wedge I),$$

as the non-abelian exterior product $L \wedge I$ is a homomorphic image of usual exterior product of Λ-modules L and I, we have

$$\dim(L \wedge I) \leq \dim(L \wedge_\Lambda I) = \frac{1}{2}\dim(I)(2\dim(\frac{L}{I}) + \dim(I) - 1).$$

For the rest of the proof, we remind that since I is an abelian ideal

$$\dim(\mathcal{M}(I)) = \dim(I \wedge I) = \frac{1}{2}\dim(I)(\dim(I) - 1),$$

which completes the proof. ■

Corollary 4.3. *Let (L,N) be an arbitrary pair of finite dimensional Lie algebras then*

(i) $\dim([L,N])+\dim(\mathcal{M}(L,N))\geq\frac{1}{2}\dim\frac{N}{[L,N]}\dim\frac{L}{[L,N]}+\dim\frac{L}{N}-1)$

(ii) $\dim([L,N])+\dim(\mathcal{M}(L,N))\leq\frac{1}{2}(\dim N)(2\dim\frac{L}{N}+\dim N-1)$.

Proof. (i) Put $I=[L,N]$ in Theorem 4.2 (i), we have

$$\dim([L,N])+\dim(\mathcal{M}(L,N))\geq\dim\mathcal{M}(\frac{L}{[L,N]},\frac{N}{[L,N]})=\dim(\frac{L}{[L,N]}\wedge\frac{N}{[L,N]}).$$

Since $N/[L,N]$ centralizes $L/[L,N]$, $L/[L,N]\wedge N/[L,N]$ is the usual exterior product of vector spaces which has dimension $\frac{1}{2}\dim(N/[L,N])(\dim(L/[L,N])+\dim(L/N)-1)$. Part (ii) holds if we put $I=N$ in Theorem 4.2 (ii). ■

Finally, we gives a condition for equality to hold in first result of Theorem 4.2

Theorem 4.4. *Let (L,N) be a pair of finite dimensional Lie algebras and I be an ideal of L which is contained in $Z^\wedge(L,N)$ then*

$$\dim(\mathcal{M}(L/I,N/I))=\dim(I\cap[N,L])+\dim(\mathcal{M}(L,N)).$$

Proof. Since $I\subseteq Z^\wedge(L,N)$, I is a central ideal of L, hence $[I,L]=0$ and using sequence (3), the following exact sequence exists:

$$0\to\mathcal{M}(L,N)\to\mathcal{M}(L/I,N/I)\to[L,N]\cap I\to 0,$$

which gives the desired result. ■

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