

Cohomology of $\mathfrak{osp}(n|2)$ Acting on the Spaces of Linear Differential Operators on the Superspace $\mathbb{R}^{1|n}$

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Abstract. We compute the first differential cohomology of the orthosymplectic Lie superalgebra $\mathfrak{osp}(n|2)$ with coefficients in the superspace of linear differential operators acting on the space of weighted densities on the $(1, n)$ -dimensional real superspace. Likewise we compute its $\mathfrak{osp}(n-1|2)$ -relative cohomology. We explicitly give 1-cocycles spanning these cohomology groups. This work is the simplest superization of a result by Lecomte (see: On the cohomology of $\mathfrak{sl}(n+1, \mathbb{R})$ acting on differential operators and $\mathfrak{sl}(n+1, \mathbb{R})$ -equivariant symbols, *Indag. Math. NS.* **11**(1), 95–114 (2000).)

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1. Introduction

This work is a direct continuation of the work contained in [3, 8, 13], which listed, among other things, binary differential operators invariant with respect to a supergroup of diffeomorphisms and computed the cohomology of polynomial versions of various infinite dimensional Lie superalgebras. Our main motivation is the study of deformations of the natural action of the orthosymplectic Lie superalgebra $\mathfrak{osp}(n|2)$ on the superspace of symbols of differential operators on densities.

The space of weighted densities with weight λ (or λ -densities) on \mathbb{R} , denoted by:

$$\mathcal{F}_\lambda = \{f(dx)^\lambda \mid f \in C^\infty(\mathbb{R})\}, \quad \lambda \in \mathbb{R},$$

is the space of sections of the line bundle $(T^*\mathbb{R})^{\otimes \lambda}$ for positive integer λ . Let $\text{Vect}(\mathbb{R})$ be the Lie algebra of all vector fields $X_F = F \frac{d}{dx}$ on \mathbb{R} , where $F \in C^\infty(\mathbb{R})$. The *Lie derivative* L_{X_F} along the vector field X_F makes \mathcal{F}_λ a $\text{Vect}(\mathbb{R})$ -module for any $\lambda \in \mathbb{R}$:

$$L_{X_F}(f(dx)^\lambda) = L_{X_F}^\lambda(f)(dx)^\lambda \quad \text{with} \quad L_{X_F}^\lambda(f) = Ff' + \lambda fF', \quad (1)$$

where f', F' are $\frac{df}{dx}, \frac{dF}{dx}$. Denote by $D_{\lambda,\mu} := \text{Hom}_{\text{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$ the $\text{Vect}(\mathbb{R})$ -module of linear differential operators with the natural $\text{Vect}(\mathbb{R})$ -action. Lecomte [13] computed $H_{\text{diff}}^1(\mathfrak{sl}(2); D_{\lambda,\mu})$ and $H_{\text{diff}}^2(\mathfrak{sl}(2); D_{\lambda,\mu})$, where $\mathfrak{sl}(2)$ is realized as the Lie subalgebra of $\text{Vect}(\mathbb{R})$ spanned by $\{X_1, X_x, X_{x^2}\}$ and where H_{diff}^* denotes the differential cohomology; that is, only cochains given by differential operators are considered. These spaces appear naturally in the problem of describing the deformations of the $\mathfrak{sl}(2)$ -module $\mathcal{S}_{\mu-\lambda} = \bigoplus_{k=0}^\infty \mathcal{F}_{\mu-\lambda-k}$, the space of symbols of differential operators of $D_{\lambda,\mu}$. More precisely, the elements of $H^1(\mathfrak{sl}(2); V)$ classify the infinitesimal deformations of a $\mathfrak{sl}(2)$ -module V and the obstructions to integrability of a given infinitesimal deformation of V are elements of $H^2(\mathfrak{sl}(2); V)$ (for example, see [1, 2, 4, 5, 14]).

It is natural to study the simplest super analog of the problems solved in [13], namely, we consider the superspace $\mathbb{R}^{1|n}$ endowed with its standard contact structure defined by the 1-form α_n , and the Lie superalgebra $\mathcal{K}(n)$ of contact vector fields on $\mathbb{R}^{1|n}$. We introduce the $\mathcal{K}(n)$ -module \mathfrak{F}_λ^n of λ -densities on $\mathbb{R}^{1|n}$ and the $\mathcal{K}(n)$ -module of linear differential operators, $\mathfrak{D}_{\lambda,\mu}^n := \text{Hom}_{\text{diff}}(\mathfrak{F}_\lambda^n, \mathfrak{F}_\mu^n)$, which are super analogs of the spaces \mathcal{F}_λ and $D_{\lambda,\mu}$, respectively. The Lie superalgebra $\mathfrak{osp}(n|2)$, a super analog of $\mathfrak{sl}(2)$, can be realized as a subalgebra of $\mathcal{K}(n)$. The space $H_{\text{diff}}^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu}^1)$ has been computed by Basdouri and Ben Ammar [3], and the space $H_{\text{diff}}^1(\mathfrak{osp}(2|2); \mathfrak{D}_{\lambda,\mu}^2)$ has been computed by the first author and Boujelben [8].

In this paper, we compute $H_{\text{diff}}^1(\mathfrak{osp}(n|2), \mathfrak{osp}(n-1|2); \mathfrak{D}_{\lambda,\mu}^n)$ and $H_{\text{diff}}^1(\mathfrak{osp}(n|2); \mathfrak{D}_{\lambda,\mu}^n)$ for $n > 2$. Moreover, we give explicit formulae for all the nontrivial 1-cocycles. These spaces arise in the classification of infinitesimal deformations of the $\mathfrak{osp}(n|2)$ -module $\mathfrak{S}_{\mu-\lambda}^n = \bigoplus_{k \geq 0} \mathfrak{F}_{\mu-\lambda-\frac{k}{2}}^n$, a super analogue of $\mathcal{S}_{\mu-\lambda}$, see [12]. We mention that the same problem was considered in [6] for the case of the Lie superalgebra $\mathcal{K}(n)$ of contact vector fields on the $(1, n)$ -dimensional real superspace instead of the orthosymplectic Lie superalgebra $\mathfrak{osp}(n|2)$. We also mention that Duval and Michel studied a similar problem related to the link between discrete projective invariants of the supercircle $S^{1|n}$ and the cohomology of the group of its contactomorphisms [10].

2. Definitions and Notation

Let $\mathbb{R}^{1|n}$ be the superspace with coordinates (x, θ) where $\theta = (\theta_1, \dots, \theta_n)$. The coordinates $\theta_1, \dots, \theta_n$ are the odd variables: $\theta_i \theta_j = -\theta_j \theta_i$. Consider the standard contact structure given by the following 1-form:

$$\alpha_n = dx + \sum_{i=1}^n \theta_i d\theta_i. \tag{2}$$

On the space $C^\infty(\mathbb{R}^{1|n})$, we consider the contact bracket

$$\{F, G\} = FG' - F'G - \frac{1}{2}(-1)^{|F|} \sum_{i=1}^n \eta_i(F) \cdot \eta_i(G), \tag{3}$$

where $\eta_i = \frac{\partial}{\partial \theta_i} - \theta_i \frac{\partial}{\partial x}$ and $|F|$ is the parity of F . Note that the derivations η_i are the generators of n -extended supersymmetry and generate the kernel of the form (2) as a module over the ring of functions. Let $\text{Vect}(\mathbb{R}^{1|n})$ be the superspace of vector fields on $\mathbb{R}^{1|n}$:

$$\text{Vect}(\mathbb{R}^{1|n}) = \left\{ F_0 \partial_x + \sum_{i=1}^n F_i \partial_i \mid F_i \in C^\infty(\mathbb{R}^{1|n}) \text{ for all } i \right\},$$

where $\partial_i = \frac{\partial}{\partial \theta_i}$ and $\partial_x = \frac{\partial}{\partial x}$, and consider the superspace $\mathcal{K}(n)$ of contact vector fields on $C^\infty(\mathbb{R}^{1|n})$. That is, $\mathcal{K}(n)$ is the superspace of vector fields on $\mathbb{R}^{1|n}$ preserving the distribution singled out by the 1-form α_n :

$$\mathcal{K}(n) = \{X \in \text{Vect}(\mathbb{R}^{1|n}) \mid \text{there exists } F \in C^\infty(\mathbb{R}^{1|n}) \text{ such that } L_X(\alpha_n) = F\alpha_n\}.$$

The Lie superalgebra $\mathcal{K}(n)$ is spanned by the fields of the form:

$$X_F = F\partial_x - \frac{1}{2} \sum_{i=1}^n (-1)^{|F|} \eta_i(F)\eta_i, \text{ where } F \in C^\infty(\mathbb{R}^{1|n}).$$

In particular, we have $\mathcal{K}(0) = \text{Vect}(\mathbb{R})$. Observe that $L_{X_F}(\alpha_n) = X_1(F)\alpha_n$. The bracket in $\mathcal{K}(n)$ can be written as:

$$[X_F, X_G] = X_{\{F,G\}}. \tag{4}$$

The orthosymplectic Lie superalgebra $\mathfrak{osp}(n|2)$ can be realized as a subalgebra of $\mathcal{K}(n)$:

$$\mathfrak{osp}(n|2) = \text{Span}(X_1, X_x, X_{x^2}, X_{x\theta_i}, X_{\theta_i}, X_{\theta_i\theta_j}), \quad 1 \leq i, j \leq n.$$

We easily see that $\mathfrak{osp}(n-1|2)$ is a subalgebra of $\mathfrak{osp}(n|2)$:

$$\mathfrak{osp}(n-1|2) = \{X_F \in \mathfrak{osp}(n|2) \mid \partial_n F = 0\}$$

Note also that, for any i in $\{1, 2, \dots, n-1\}$, $\mathfrak{osp}(n-1|2)$ is isomorphic to

$$\mathfrak{osp}(n-1|2)^i = \{X_F \in \mathfrak{osp}(n|2) \mid \partial_i F = 0\}.$$

We define the space of λ -densities as

$$\mathfrak{F}_\lambda^n = \{F\alpha_n^\lambda \mid F \in C^\infty(\mathbb{R}^{1|n})\}. \tag{5}$$

As a vector space, \mathfrak{F}_λ^n is isomorphic to $C^\infty(\mathbb{R}^{1|n})$, but the Lie derivative of the density $G\alpha_n^\lambda$ along the vector field X_F in $\mathcal{K}(n)$ is now:

$$\mathfrak{L}_{X_F}(G\alpha_n^\lambda) = \mathfrak{L}_{X_F}^\lambda(G)\alpha_n^\lambda, \quad \text{with } \mathfrak{L}_{X_F}^\lambda = X_F + \lambda F', \quad \lambda \in \mathbb{R}. \tag{6}$$

Here, we restrict ourselves to the subalgebra $\mathfrak{osp}(n|2)$, thus we obtain a one-parameter family of $\mathfrak{osp}(n|2)$ -modules on $C^\infty(\mathbb{R}^{1|n})$ still denoted by \mathfrak{F}_λ^n .

A differential operator on $\mathbb{R}^{1|n}$ is an operator on $C^\infty(\mathbb{R}^{1|n})$ of the form:

$$A = \sum_{k=0}^M \sum_{\varepsilon=(\varepsilon_1, \dots, \varepsilon_n)} a_{k,\varepsilon}(x, \theta) \partial_x^k \partial_1^{\varepsilon_1} \dots \partial_n^{\varepsilon_n}; \quad \varepsilon_i = 0, 1; \quad M \in \mathbb{N}. \quad (7)$$

Of course any differential operator defines a linear mapping $F\alpha_n^\lambda \mapsto (AF)\alpha_n^\mu$ from \mathfrak{F}_λ^n to \mathfrak{F}_μ^n for any $\lambda, \mu \in \mathbb{R}$, thus the space of differential operators becomes a family of $\mathcal{K}(n)$ -modules $\mathfrak{D}_{\lambda,\mu}^n$ for the natural action:

$$X_F \cdot A = \mathfrak{L}_{X_F}^\mu \circ A - (-1)^{|A||F|} A \circ \mathfrak{L}_{X_F}^\lambda. \quad (8)$$

Since $-\eta_i^2 = \partial_x$, and $\partial_i = \eta_i - \theta_i \eta_i^2$, every differential operator $A \in \mathfrak{D}_{\lambda,\mu}^n$ can be expressed in the form

$$A(F\alpha_n^\lambda) = \sum_{\ell=(\ell_1, \dots, \ell_n)} a_\ell(x, \theta) \eta_1^{\ell_1} \dots \eta_n^{\ell_n} (F)\alpha_n^\mu, \quad (9)$$

where the coefficients $a_\ell(x, \theta)$ are arbitrary functions.

Proposition 2.1. *As an $\mathfrak{osp}(n-1|2)$ -module, we have*

$$\mathfrak{D}_{\lambda,\mu}^n \simeq \mathfrak{D}_{\lambda,\mu}^{n-1} \oplus \mathfrak{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}^{n-1} \oplus \Pi \left(\mathfrak{D}_{\lambda,\mu+\frac{1}{2}}^{n-1} \oplus \mathfrak{D}_{\lambda+\frac{1}{2},\mu}^{n-1} \right), \quad (10)$$

where Π is the change of parity operator.

Proof. Any element $F \in C^\infty(\mathbb{R}^{1|n})$ can be uniquely written as follows: $F = F_1 + F_2\theta_n$, where $\partial_n F_1 = \partial_n F_2 = 0$. Therefore, for any $X_H \in \mathfrak{osp}(n-1|2)$, we easily check that

$$\mathfrak{L}_{X_H}^\lambda (F) = \mathfrak{L}_{X_H}^\lambda (F_1) + \mathfrak{L}_{X_H}^{\lambda+\frac{1}{2}} (F_2)\theta_n.$$

Thus, the following map is an $\mathfrak{osp}(n-1|2)$ -isomorphism:

$$\begin{aligned} \Phi_\lambda : \mathfrak{F}_\lambda^n &\rightarrow \mathfrak{F}_\lambda^{n-1} \oplus \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^{n-1}) \\ F\alpha_n^\lambda &\mapsto \left(F_1\alpha_{n-1}^\lambda, \Pi(F_2\alpha_{n-1}^{\lambda+\frac{1}{2}}) \right) \end{aligned} \quad (11)$$

So, we deduce an $\mathfrak{osp}(n-1|2)$ -isomorphism:

$$\begin{aligned} \Psi_{\lambda,\mu} : \mathfrak{D}_{\lambda,\mu}^{n-1} \oplus \mathfrak{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}^{n-1} \oplus \Pi \left(\mathfrak{D}_{\lambda,\mu+\frac{1}{2}}^{n-1} \oplus \mathfrak{D}_{\lambda+\frac{1}{2},\mu}^{n-1} \right) &\rightarrow \mathfrak{D}_{\lambda,\mu}^n \\ A &\mapsto \Phi_\mu^{-1} \circ A \circ \Phi_\lambda. \end{aligned} \quad (12)$$

Here, we identify the $\mathfrak{osp}(n-1|2)$ -modules via the following isomorphisms:

$$\begin{aligned} \Pi \left(\mathfrak{D}_{\lambda,\mu+\frac{1}{2}}^{n-1} \right) &\rightarrow \text{Hom}_{\text{diff}} \left(\mathfrak{F}_\lambda^{n-1}, \Pi(\mathfrak{F}_{\mu+\frac{1}{2}}^{n-1}) \right) & \Pi(A) &\mapsto \Pi \circ A, \\ \Pi \left(\mathfrak{D}_{\lambda+\frac{1}{2},\mu}^{n-1} \right) &\rightarrow \text{Hom}_{\text{diff}} \left(\mathfrak{F}_{\lambda+\frac{1}{2}}^{n-1}, \Pi(\mathfrak{F}_\mu^{n-1}) \right) & \Pi(A) &\mapsto A \circ \Pi, \\ \mathfrak{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}^{n-1} &\rightarrow \text{Hom}_{\text{diff}} \left(\Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^{n-1}), \Pi(\mathfrak{F}_{\mu+\frac{1}{2}}^{n-1}) \right) & \Pi(A) &\mapsto \Pi \circ A \circ \Pi. \quad \blacksquare \end{aligned}$$

Note that this Proposition is a simplest generalization of a result by the first author and Boujelben for $n = 2$ (see [8]).

3. $\mathfrak{osp}(n|2)$ -invariant binary differential operators

Observe that, as an $\mathfrak{osp}(n - 1|2)$ -module, we have

$$\mathfrak{osp}(n|2) \simeq \mathfrak{osp}(n - 1|2) \oplus \Pi(\mathfrak{h}^{n-1}), \quad \text{where}$$

\mathfrak{h}^{n-1} is the subspace of $\mathfrak{F}_{-\frac{1}{2}}^{n-1}$ spanned by $\{\alpha_{n-1}^{-\frac{1}{2}}, x\alpha_{n-1}^{-\frac{1}{2}}, \theta_1\alpha_{n-1}^{-\frac{1}{2}}, \dots, \theta_{n-1}\alpha_{n-1}^{-\frac{1}{2}}\}$. In fact, by (4) and (6), the Lie superalgebra $\mathcal{K}(n)$ is isomorphic to \mathfrak{F}_{-1}^n which is isomorphic, as $\mathfrak{osp}(n - 1|2)$ -module, to $\mathfrak{F}_{-1}^{n-1} \oplus \Pi(\mathfrak{F}_{-\frac{1}{2}}^{n-1})$. So, the space $\mathfrak{osp}(n|2)$ is isomorphic, as an $\mathfrak{osp}(n - 1|2)$ -module, to $\Phi_\lambda(\mathfrak{osp}(n|2))$, where Φ_λ is given by (11). More precisely, any element X_F is decomposed into $X_F = X_{F_1} + X_{F_2\theta_n}$ where $\partial_n F_1 = \partial_n F_2 = 0$, and then $X_{F_1} \in \mathfrak{osp}(n - 1|2)$ and $X_{F_2\theta_n}$ is identified to $\Pi(F_2\alpha_{n-1}^{-\frac{1}{2}}) \in \Pi(\mathfrak{h}^{n-1})$.

To compute the $\mathfrak{osp}(n - 1|2)$ -relative cohomology of $\mathfrak{osp}(n|2)$ for $n \geq 3$, we need the description of $\mathfrak{osp}(n - 1|2)$ -invariant mappings from $\mathfrak{h}^{n-1} \otimes \mathfrak{F}_\lambda^{n-1}$ to \mathfrak{F}_μ^{n-1} . As a first step towards these descriptions, we shall need the list of binary $\mathfrak{osp}(1|2)$ -invariant differential operators $\mathfrak{h}^1 \otimes \mathfrak{F}_\lambda^1 \rightarrow \mathfrak{F}_\mu^1$.

3.1. $\mathfrak{osp}(1|2)$ -invariant binary differential operators. The nontrivial binary $\mathfrak{osp}(1|2)$ -invariant differential operators

$$\mathfrak{h}^1 \otimes \mathfrak{F}_\lambda^1 \rightarrow \mathfrak{F}_\mu^1, \quad (H\alpha_1^{-\frac{1}{2}}, F\alpha_1^\lambda) \mapsto \mathcal{A}(H, F)\alpha_1^\mu$$

were described in [8]. Recall that their list consists of:

- i) If $\mu = \lambda + k - \frac{1}{2}$ where k is a non-negative integer satisfying $k(k - 1)(2\lambda + k - 1) = 0$, then, up to a scalar factor, the map \mathcal{A} is given by:

$$\mathcal{A}(H, F) = HF^{(k)} + k(2\lambda + k - 1)H'F^{(k-1)} - (-1)^{|H|}k\eta_1(H)\eta_1(F^{(k-1)}). \quad (13)$$

- ii) If $\mu = \lambda + k$, where k is a non-negative integer satisfying $k(2\lambda + k)(2\lambda + k - 1) = 0$, then, up to a scalar factor, the map \mathcal{A} is given by:

$$\mathcal{A}(H, F) = (-1)^{|H|}H\eta_1(F^{(k)}) + (2\lambda + k)(\eta_1(H)F^{(k)} + kH'\eta_1(F^{(k-1)})). \quad (14)$$

3.2. $\mathfrak{osp}(n|2)$ -invariant binary differential operators for $n \geq 2$. Now, we describe the spaces of $\mathfrak{osp}(n|2)$ -invariant binary differential operators $\mathfrak{h}^n \otimes \mathfrak{F}_\lambda^n \rightarrow \mathfrak{F}_\mu^n$ for $n \geq 2$. We prove that these spaces are nontrivial only if $\mu - \lambda = \pm \frac{1}{2}$

$$\text{or } (\lambda, \mu) = \left(-\frac{k+n-1}{2}, \frac{k}{2}\right) \quad \text{or } (\lambda, \mu) = \left(-\frac{k+n-2}{2}, \frac{k+1}{2}\right), \quad k \in \mathbb{N}.$$

More precisely, we have: Hereafter all ϵ 's are constants and we will use the superscript i when we consider the superalgebra $\mathfrak{osp}(n|2)^i$ instead of $\mathfrak{osp}(n|2)$.

Theorem 3.1. *Let $\mathfrak{T}_{\lambda,\mu}^n : \mathfrak{h}^n \times \mathfrak{F}_{\lambda}^n \rightarrow \mathfrak{F}_{\mu}^n, (H\alpha_n^{-\frac{1}{2}}, F\alpha_n^{\lambda}) \mapsto \mathfrak{T}_{\lambda,\mu}^n(H, F)\alpha_n^{\mu}$ be a non-zero $\mathfrak{osp}(n|2)$ -invariant bilinear differential operator. Then, up to a scalar factor, the map $\mathfrak{T}_{\lambda,\mu}^n$ is given by:*

$$\begin{aligned} \mathfrak{T}_{\lambda,\lambda-\frac{1}{2}}^n(H, F) &= HF, \\ \mathfrak{T}_{\lambda,\lambda+\frac{1}{2}}^n(H, F) &= \lambda H'F + \frac{1}{2}M_n(H, F) \text{ for } n > 2 \text{ or } n = 2 \text{ but } \lambda \neq 0, -\frac{1}{2}, \\ \mathfrak{T}_{0,\frac{1}{2}}^2(H, F) &= \epsilon_1 M_2(H, F) + \\ &\quad \epsilon_2 \left((-1)^{|H|} (\eta_1(H)\eta_2(F) - \eta_2(H)\eta_1(F)) - H\eta_2\eta_1(F) \right), \\ \mathfrak{T}_{-\frac{1}{2},0}^2(H, F) &= \epsilon_1 (H'F - M_2(H, F)) + \epsilon_2 H\eta_1\eta_2(F), \\ \mathfrak{T}_{-\frac{k+n-1}{2},\frac{k}{2}}^n(H, F) &= (-1)^{n|H|} H\eta_1 \cdots \eta_n(F^{(k)}) \text{ with } k \in \mathbb{N} \text{ if } n > 2 \\ &\quad \text{and } k \in \mathbb{N} \setminus \{0\} \text{ if } n = 2, \\ \mathfrak{T}_{-\frac{k+n-2}{2},\frac{k+1}{2}}^n(H, F) &= (-1)^{n|H|} \left(H\eta_1 \cdots \eta_n(F^{(k)}) + kH'\eta_1 \cdots \eta_n(F^{(k-1)}) \right) + \\ &\quad (-1)^{(n-1)|H|} \sum_{\sigma \in S_n} \frac{\text{sgn}(\sigma)}{(n-1)!} \eta_{\sigma(1)}(H)\eta_{\sigma(2)} \cdots \eta_{\sigma(n)}(F^{(k)}) \\ &\quad \text{with } k \in \mathbb{N} \text{ if } n > 2 \text{ and } k \in \mathbb{N} \setminus \{0\} \text{ if } n = 2, \end{aligned} \tag{15}$$

where $M_n(H, F) = HF' - (-1)^{|H|} \sum_{i=1}^n \eta_i(H)\eta_i(F)$ and S_n is the symmetric group.

Proof. By induction on n . First note that the $\mathfrak{osp}(n|2)$ -invariance of any binary differential operators $\mathfrak{T}_{\lambda,\mu}^n : \mathfrak{h}^n \otimes \mathfrak{F}_{\lambda}^n \rightarrow \mathfrak{F}_{\mu}^n$ is equivalent to invariance with respect to the subalgebras $\mathfrak{osp}(n-1|2)$ and $\mathfrak{osp}(n-1|2)^i, i = 1, \dots, n-1$. Using, the fact that, as $\mathfrak{osp}(n-1|2)$ -module,

$$\mathfrak{h}^n \otimes \mathfrak{F}_{\lambda}^n \simeq \mathfrak{h}_0^{n-1} \otimes \mathfrak{F}_{\lambda}^{n-1} \oplus \Pi(\mathfrak{h}_1^{n-1}) \otimes \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^{n-1}) \oplus \mathfrak{h}_0^{n-1} \otimes \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^{n-1}) \oplus \Pi(\mathfrak{h}_1^{n-1}) \otimes \mathfrak{F}_{\lambda}^{n-1}$$

and

$$\mathfrak{F}_{\mu}^n \simeq \mathfrak{F}_{\mu}^{n-1} \oplus \Pi(\mathfrak{F}_{\mu+\frac{1}{2}}^{n-1}),$$

where \mathfrak{h}_1^{n-1} is the subspace of \mathfrak{F}_0^{n-1} spanned by 1 and \mathfrak{h}_0^{n-1} is the subspace of $\mathfrak{F}_{-\frac{1}{2}}^{n-1}$ spanned by

$$\left\{ \alpha_{n-1}^{-\frac{1}{2}}, x\alpha_{n-1}^{-\frac{1}{2}}, \theta_1\alpha_{n-1}^{-\frac{1}{2}}, \dots, \theta_{n-1}\alpha_{n-1}^{-\frac{1}{2}} \right\},$$

we can see that, $\mathfrak{T}_{\lambda,\mu}^n$ is decomposed into eight $\mathfrak{osp}(n-1|2)$ -invariant maps:

$$\mathfrak{h}_0^{n-1} \otimes \Pi^i(\mathfrak{F}_{\lambda+\frac{1}{2}}^{n-1}) \longrightarrow \Pi^j(\mathfrak{F}_{\mu+\frac{1}{2}}^{n-1}), \quad \Pi(\mathfrak{h}_1^{n-1}) \otimes \Pi^i(\mathfrak{F}_{\lambda+\frac{1}{2}}^{n-1}) \longrightarrow \Pi^j(\mathfrak{F}_{\mu+\frac{1}{2}}^{n-1}), \quad i, j = 0, 1. \tag{16}$$

We will need the following classical fact:

Lemma 3.2. [7] *Let \mathfrak{g} be a Lie superalgebra and $A : U \otimes V \rightarrow W$ a bilinear map, where U, V and W are \mathfrak{g} -modules. We consider the following associated maps*

$$\begin{aligned} A_1 : \Pi(U) \otimes V &\rightarrow W, & A_2 : \Pi(U) \otimes \Pi(V) &\rightarrow \Pi(W), \\ A_3 : \Pi(U) \otimes V &\rightarrow \Pi(W), & A_4 : \Pi(U) \otimes \Pi(V) &\rightarrow W, \\ A_{i,j} : U \otimes \Pi^i(V) &\rightarrow \Pi^j(W), \quad i, j = 0, 1, \end{aligned}$$

defined by

$$\begin{aligned} A_1(\Pi(u) \otimes v) &= (-1)^{|u|}A(u \otimes v), \quad A_2(\Pi(u) \otimes \Pi(v)) = (-1)^{|u|}\Pi(A(u \otimes v)), \\ A_3(\Pi(u) \otimes v) &= (-1)^{|v|}\Pi(A(u \otimes v)), \quad A_4(\Pi(u) \otimes \Pi(v)) = (-1)^{|v|}A(u \otimes v), \\ A_{i,j}(u \otimes \Pi^j(v)) &= (-1)^{(i+j)(|u|+|v|)}\Pi^j(A(u \otimes v)). \end{aligned}$$

The maps A_1, A_2, A_3, A_4 and $A_{i,j}$ are \mathfrak{g} -invariant if and only if A is \mathfrak{g} -invariant.

We need also the following

Proposition 3.3. *Let $\mathcal{B}_n : \mathfrak{h}_1^n \otimes \mathfrak{F}_\lambda^n \rightarrow \mathfrak{F}_\mu^n, (h, F\alpha_n^\lambda) \mapsto \mathcal{B}_n(h, F)\alpha_n^\mu$ be a non-trivial $\mathfrak{osp}(n|2)$ -invariant bilinear differential operator, then one of the following holds:*

i) *If $\mu = \lambda$ then, up to a scalar factor, the map \mathcal{B}_n is given by:*

$$\mathcal{B}_n(h, F) = hF. \tag{17}$$

ii) *If $(\lambda, \mu) = (-\frac{k+n-1}{2}, \frac{k+1}{2}), k \in \mathbb{N}$, then, up to a scalar factor, the map \mathcal{B}_n is given by:*

$$\mathcal{B}_n(h, F) = h\eta_1 \cdots \eta_n(F^{(k)}). \tag{18}$$

Proof. By induction on n . For $n = 0$, the result holds from [8], (Lemma 4.2). For $n = 1$, let $\mathcal{B}_1 : \mathfrak{h}_1^1 \otimes \mathfrak{F}_\lambda^1 \rightarrow \mathfrak{F}_\mu^1$ be an $\mathfrak{osp}(1|2)$ -invariant binary differential operator. The $\mathfrak{osp}(1|2)$ -invariance of \mathcal{B}_1 is equivalent to invariance with respect just to the subalgebra $\mathfrak{sl}(2)$ and the vector fields X_{θ_1} and $X_{x\theta_1}$. Using the fact that, as $\mathfrak{sl}(2)$ -modules,

$$\mathfrak{F}_\mu^1 \simeq \mathcal{F}_\mu \oplus \Pi(\mathcal{F}_{\mu+\frac{1}{2}}) \quad \text{and} \quad \mathfrak{h}_1^1 \otimes \mathfrak{F}_\lambda^1 \simeq \mathfrak{h}_1^0 \otimes \mathcal{F}_\lambda \oplus \mathfrak{h}_1^0 \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}), \tag{19}$$

we can deduce, by induction hypothesis, the restriction of \mathcal{B}_1 to each component of the right-hand side of (19). Of course, \mathcal{B}_1 is identically zero if $2(\mu - \lambda) \neq -1, 0, 1$ and if

$$(\lambda, \mu) \notin \left\{ \left(\frac{1-k}{2}, \frac{1+k}{2}\right), \left(\frac{-k}{2}, \frac{k}{2}\right), \left(\frac{1-k}{2}, \frac{k}{2}\right) \right\}, \quad k \in \mathbb{N} \setminus \{0\}.$$

The invariance of \mathcal{B}_1 with respect X_{θ_1} and $X_{x\theta_1}$ determines thus completely the space of $\mathfrak{osp}(1|2)$ -invariant binary differential operators $\mathfrak{h}_1^1 \otimes \mathfrak{F}_\lambda^1 \rightarrow \mathfrak{F}_\mu^1$. Now, assume that the result holds for $n > 1$. Observe that the $\mathfrak{osp}(n|2)$ -invariance of any binary differential operators $\mathcal{B}_n : \mathfrak{h}_1^n \otimes \mathfrak{F}_\lambda^n \rightarrow \mathfrak{F}_\mu^n$ is equivalent to invariance with respect just to the subalgebras $\mathfrak{osp}(n-1|2)$ and $\mathfrak{osp}(n-1|2)^i, i = 1, \dots, n-1$, and that \mathcal{B}_n is decomposed into four $\mathfrak{osp}(n-1|2)$ -invariant maps:

$$\mathfrak{h}_1^{n-1} \otimes \Pi^i(\mathfrak{F}_{\lambda+\frac{1}{2}}^{n-1}) \longrightarrow \Pi^j(\mathfrak{F}_{\mu+\frac{1}{2}}^{n-1}), \quad i, j = 0, 1. \tag{20}$$

Thus, by Lemma 3.2 and induction assumption, we exhibit the $\mathfrak{osp}(n-1|2)$ -invariant binary differential operators $\mathfrak{h}_1^n \otimes \mathfrak{F}_\lambda^n \rightarrow \mathfrak{F}_\mu^n$. Of course, these operators are identically zero if $2(\mu - \lambda) \neq -1, 0, 1$ and if

$$(\lambda, \mu) \notin \left\{ \left(-\frac{k+n-1}{2}, \frac{k+1}{2}\right), \left(-\frac{k+n-2}{2}, \frac{k+1}{2}\right), \left(-\frac{k+n-1}{2}, \frac{k}{2}\right), \left(\frac{2-n}{2}, 0\right) \right\}, \quad k \in \mathbb{N}.$$

More precisely, any $\mathfrak{osp}(n-1|2)$ -invariant binary differential operators $\mathcal{B}_n : \mathfrak{h}_1^n \otimes \mathfrak{F}_\lambda^n \rightarrow \mathfrak{F}_\mu^n$ can be expressed as:

For $(\lambda, \mu) = (-\frac{k+n-1}{2}, \frac{k+1}{2})$, $k \in \mathbb{N}$,

$$\mathcal{B}_n(h, F) = \Omega_{\lambda, \mu} h \eta_{n-1} \cdots \eta_1 \partial_n(F^{(k)}) + \Gamma_{\lambda, \mu}(-1)^{|F|} h \eta_{n-1} \cdots \eta_1 (1 - \theta_n \partial_n)(F^{(k+1)}) \theta_n.$$

For $\lambda = \mu$, we have $\mathcal{B}_n(h, F) =$

$$\begin{cases} \Xi_{\lambda, n} h (1 - \theta_n \partial_n)(F) - \Theta_{\lambda, n}(-1)^{|F|} h \partial_n(F) \theta_n & \text{if } n > 2 \text{ or } n = 2 \text{ and } \lambda \neq 0, \\ \Xi_{\lambda, n} h (1 - \theta_2 \partial_2)(F) - \Theta_{\lambda, n}(-1)^{|F|} h \partial_2(F) \theta_2 + \\ + \Delta_{\lambda, n}(-1)^{|F|} h \eta_1 (1 - \theta_2 \partial_2)(F) \theta_2 & \text{if } n = 2 \text{ and } \lambda = 0, \end{cases}$$

where the coefficients $\Omega_{\lambda, \mu}, \Gamma_{\lambda, \mu}, \Xi_{\lambda, n}, \Theta_{\lambda, n}$ and $\Delta_{\lambda, n}$ are, a priori, arbitrary constants, but the invariance of \mathcal{B}_n with respect $\mathfrak{osp}(n-1|2)^i$, $i = 1, \dots, n-1$, shows that

$$\Gamma_{-\frac{k+n-1}{2}, \frac{k+1}{2}} = -\Omega_{-\frac{k+n-1}{2}, \frac{k+1}{2}}, \quad \Xi_{\lambda, n} = \Theta_{\lambda, n}, \quad \Delta_{0, 2} = 0.$$

Therefore, we easily check that \mathcal{B}_n is expressed as in Proposition 3.3. For the other cases, the same arguments as before, show that \mathcal{B}_n is identically zero. This completes the proof of Proposition 3.3. ■

Now, assume that $n = 2$. By Proposition 3.3 and formulas (13)–(14) together with decomposition (16) with the help of Lemma 3.2, we exhibit the $\mathfrak{osp}(1|2)$ -invariant binary differential operators $\mathfrak{h}^2 \otimes \mathfrak{F}_\lambda^2 \rightarrow \mathfrak{F}_\mu^2$. Of course, these elements are identically zero if

$2(\mu - \lambda) \neq -2, -1, 0, 1, 2$ and if

$$(\lambda, \mu) \notin \left\{ \left(-\frac{k}{2}, \frac{k+1}{2}\right), \left(-\frac{k+1}{2}, \frac{k}{2}\right), \left(-\frac{k+1}{2}, \frac{k+1}{2}\right), \left(-\frac{k+2}{2}, \frac{k}{2}\right), \left(-\frac{k}{2}, \frac{k+2}{2}\right) \right\}, \quad k \in \mathbb{N}.$$

The invariance of these elements with respect $\mathfrak{osp}(1|2)^1$ determines thus completely the space of $\mathfrak{osp}(2|2)$ -invariant binary differential operators $\mathfrak{h}^2 \otimes \mathfrak{F}_\lambda^2 \rightarrow \mathfrak{F}_\mu^2$. Thus, we can check that Theorem 3.1 is proved for $n = 2$. Now, assume that it holds for $n > 2$. Again, the same arguments as in the proof of the case $n = 2$ together with induction assumption show that any nontrivial $\mathfrak{osp}(n|2)$ -invariant binary differential operators $\mathfrak{T}_{\lambda, \mu}^{n+1} : \mathfrak{h}^{n+1} \otimes \mathfrak{F}_\lambda^{n+1} \rightarrow \mathfrak{F}_\mu^{n+1}$ only can appear if $2(\mu - \lambda) \neq -2, -1, 0, 1, 2$ and if (λ, μ) not belongs to (here $k \in \mathbb{N}$)

$$\left\{ \left(-\frac{k+n-1}{2}, \frac{k+\ell}{2}\right), \left(-\frac{k+n-2}{2}, \frac{k+i}{2}\right), \left(-\frac{k+n}{2}, \frac{k+j}{2}\right) \right\}, \quad j, \ell = -1, 0, 1, \quad i = 0, 1. \quad (21)$$

So, we consider only the cases where $2(\mu - \lambda) = -2, -1, 0, 1, 2$ and (λ, μ) as in (21). As before, the invariance of $\mathfrak{T}_{\lambda, \mu}^{n+1}$ with respect $\mathfrak{osp}(n|2)^i$, $i = 1, \dots, n$, shows that $\mathfrak{T}_{\lambda, \mu}^{n+1}$ is expressed as in Theorem 3.1. ■

4. Cohomology

Let us first recall some fundamental concepts from cohomology theory (see, e.g., [9, 11]). Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra acting on a superspace $V = V_0 \oplus V_1$

and let \mathfrak{h} be a subalgebra of \mathfrak{g} . (If \mathfrak{h} is omitted it assumed to be $\{0\}$.) The space of \mathfrak{h} -relative n -cochains of \mathfrak{g} with values in V is the \mathfrak{g} -module

$$C^n(\mathfrak{g}, \mathfrak{h}; V) := \text{Hom}_{\mathfrak{h}}(\Lambda^n(\mathfrak{g}/\mathfrak{h}); V).$$

The *coboundary operator* $\delta_n : C^n(\mathfrak{g}, \mathfrak{h}; V) \rightarrow C^{n+1}(\mathfrak{g}, \mathfrak{h}; V)$ is a \mathfrak{g} -map satisfying $\delta_n \circ \delta_{n-1} = 0$. The kernel of δ_n , denoted $Z^n(\mathfrak{g}, \mathfrak{h}; V)$, is the space of \mathfrak{h} -relative n -cocycles, among them, the elements in the range of δ_{n-1} are called \mathfrak{h} -relative n -coboundaries. We denote $B^n(\mathfrak{g}, \mathfrak{h}; V)$ the space of n -coboundaries.

By definition, the n^{th} \mathfrak{h} -relative cohomology space is the quotient space

$$H^n(\mathfrak{g}, \mathfrak{h}; V) = Z^n(\mathfrak{g}, \mathfrak{h}; V) / B^n(\mathfrak{g}, \mathfrak{h}; V).$$

We will only need the formula of δ_n (which will be simply denoted δ) in degrees 0 and 1: for $v \in C^0(\mathfrak{g}, \mathfrak{h}; V) = V^{\mathfrak{h}}$, $\delta v(g) := (-1)^{|g||v|} g \cdot v$, where

$$V^{\mathfrak{h}} = \{v \in V \mid h \cdot v = 0 \text{ for all } h \in \mathfrak{h}\},$$

and for $\Upsilon \in C^1(\mathfrak{g}, \mathfrak{h}; V)$,

$$\delta(\Upsilon)(g, h) := (-1)^{|g||\Upsilon|} g \cdot \Upsilon(h) - (-1)^{|h|(|g|+|\Upsilon|)} h \cdot \Upsilon(g) - \Upsilon([g, h]) \text{ for any } g, h \in \mathfrak{g}.$$

4.1. The space $H_{\text{diff}}^1(\mathfrak{osp}(n|2); \mathfrak{D}_{\lambda, \mu}^n)$.

In this subsection, we will compute the first differential cohomology spaces $H_{\text{diff}}^1(\mathfrak{osp}(n|2); \mathfrak{D}_{\lambda, \mu}^n)$ for $n \geq 3$. Our main result is the following:

Theorem 4.1. *The space $H_{\text{diff}}^1(\mathfrak{osp}(n|2); \mathfrak{D}_{\lambda, \mu}^n)$ has the following structure:*

$$H_{\text{diff}}^1(\mathfrak{osp}(n|2); \mathfrak{D}_{\lambda, \mu}^n) \simeq \begin{cases} \mathbb{R} & \text{if } \mu - \lambda = 0, \\ \mathbb{R}^2 & \text{if } (\lambda, \mu) = (-\frac{k+n-2}{2}, \frac{k}{2}) \text{ and } k \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases} \quad (22)$$

The following 1-cocycles span the corresponding cohomology spaces:

$$\begin{aligned} \Upsilon_{\lambda, \lambda}(X_G) &= G' \\ \Upsilon_{-\frac{k+n-2}{2}, \frac{k}{2}}(X_G) &= (-1)^{n|G|} G' \eta_1 \cdots \eta_n \partial_x^{k-1} \\ \bar{\Upsilon}_{-\frac{k+n-2}{2}, \frac{k}{2}}(X_G) &= (k-1) G'' \eta_1 \cdots \eta_n \partial_x^{k-2} + \\ &\quad + (-1)^{(n-1)|H|} \sum_{\sigma \in S_n} \frac{\text{sgn}(\sigma)}{(n-1)!} \eta_{\sigma(1)}(G') \eta_{\sigma(2)} \cdots \eta_{\sigma(n)} \partial_x^{k-1} \end{aligned} \quad (23)$$

The proof of Theorem 4.1 will be the subject of section 5. In fact, we need first the description of $H_{\text{diff}}^1(\mathfrak{osp}(n-1|2); \mathfrak{D}_{\lambda, \mu}^n)$ and $H_{\text{diff}}^1(\mathfrak{osp}(n|2), \mathfrak{osp}(n-1|2)^i; \mathfrak{D}_{\lambda, \mu}^n)$.

4.2. The space $H_{\text{diff}}^1(\mathfrak{osp}(n-1|2); \mathfrak{D}_{\lambda, \mu}^n)$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Lie superalgebra, where \mathfrak{k} is a subalgebra and \mathfrak{p} is a \mathfrak{k} -module such that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Consider a 1-cocycle $\Upsilon \in Z^1(\mathfrak{g}, V)$, where V is a \mathfrak{g} -module. The cocycle relation reads

$$\Upsilon([g, h]) - (-1)^{|g||\Upsilon|} g \cdot \Upsilon(h) + (-1)^{|h|(|g|+|\Upsilon|)} h \cdot \Upsilon(g) = 0, \quad g, h \in \mathfrak{g}.$$

Denote $\Upsilon_{\mathfrak{k}} = \Upsilon|_{\mathfrak{k}}$ and $\Upsilon_{\mathfrak{p}} = \Upsilon|_{\mathfrak{p}}$. Obviously, $\Upsilon_{\mathfrak{k}}$ is a 1-cocycle over \mathfrak{k} and if $\Upsilon_{\mathfrak{k}} = 0$ then $\Upsilon_{\mathfrak{p}}$ is \mathfrak{k} -invariant. Thus, the space $H^1(\mathfrak{g}; V)$ is closely related to the space $H^1(\mathfrak{k}; V)$. Furthermore, $\Upsilon_{\mathfrak{k}}$ and $\Upsilon_{\mathfrak{p}}$ subject to the following equations:

$$\Upsilon_{\mathfrak{p}}([h, p]) - (-1)^{|h||\Upsilon|} h \cdot \Upsilon_{\mathfrak{p}}(p) + (-1)^{|p|(|h|+|\Upsilon|)} p \cdot \Upsilon_{\mathfrak{k}}(h) = 0, \quad h \in \mathfrak{k}, p \in \mathfrak{p}, \quad (24)$$

$$\Upsilon_{\mathfrak{k}}([p, p']) - (-1)^{|p||\Upsilon|} p \cdot \Upsilon_{\mathfrak{p}}(p') + (-1)^{|p'|(|p|+|\Upsilon|)} p' \cdot \Upsilon_{\mathfrak{p}}(p) = 0, \quad p, p' \in \mathfrak{p}. \quad (25)$$

In our situation, $\mathfrak{g} = \mathfrak{osp}(n|2)$, $\mathfrak{k} = \mathfrak{osp}(n-1|2)$, $\mathfrak{p} = \Pi(\mathfrak{h}^{n-1})$ and $V = \mathfrak{D}_{\lambda, \mu}^n$. According to isomorphism (10), we can see that the knowledge of $H_{\text{diff}}^1(\mathfrak{k}; \mathfrak{D}_{\lambda, \mu}^{n-1})$ allows us to compute $H_{\text{diff}}^1(\mathfrak{k}; \mathfrak{D}_{\lambda, \mu}^n)$:

$$\begin{aligned} H_{\text{diff}}^1(\mathfrak{k}; \mathfrak{D}_{\lambda, \mu}^n) \simeq & H_{\text{diff}}^1(\mathfrak{k}; \mathfrak{D}_{\lambda, \mu}^{n-1}) \oplus H_{\text{diff}}^1(\mathfrak{k}; \mathfrak{D}_{\lambda+\frac{1}{2}, \mu+\frac{1}{2}}^{n-1}) \oplus \\ & H_{\text{diff}}^1(\mathfrak{k}; \Pi(\mathfrak{D}_{\lambda, \mu+\frac{1}{2}}^{n-1})) \oplus H_{\text{diff}}^1(\mathfrak{k}; \Pi(\mathfrak{D}_{\lambda+\frac{1}{2}, \mu}^{n-1})). \end{aligned} \quad (26)$$

Of course, we can deduce the structure of the cohomology space $H_{\text{diff}}^1(\mathfrak{k}; \Pi(\mathfrak{D}_{\lambda, \mu}^{n-1}))$ from $H_{\text{diff}}^1(\mathfrak{k}; \mathfrak{D}_{\lambda, \mu}^{n-1})$. Indeed, to any $\Upsilon \in Z_{\text{diff}}^1(\mathfrak{k}; \mathfrak{D}_{\lambda, \mu}^{n-1})$ there corresponds an element $\widehat{\Upsilon} \in Z_{\text{diff}}^1(\mathfrak{k}; \Pi(\mathfrak{D}_{\lambda, \mu}^{n-1}))$ where $\widehat{\Upsilon}(X_G) = \Pi(\pi \circ \Upsilon(X_G))$ with $\pi(F) = (-1)^{|F|} F$. Obviously, Υ is a coboundary if and only if $\widehat{\Upsilon}$ is a coboundary. Thus, we recall the space $H_{\text{diff}}^1(\mathfrak{osp}(2|2); \mathfrak{D}_{\lambda, \mu}^2)$ which was computed in [8]:

$$H_{\text{diff}}^1(\mathfrak{osp}(2|2); \mathfrak{D}_{\lambda, \mu}^2) \simeq \begin{cases} \mathbb{R}^2 & \text{if } \mu - \lambda = 0, \\ \mathbb{R}^3 & \text{if } (\lambda, \mu) = (-\frac{k}{2}, \frac{k}{2}) \text{ and } k \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases} \quad (27)$$

A basis for the space $H_{\text{diff}}^1(\mathfrak{osp}(2|2); \mathfrak{D}_{\lambda, \mu}^2)$ is given by the cohomology classes of the 1-cocycles $\Upsilon_{\lambda, \mu}^2$ and $\widetilde{\Upsilon}_{\lambda, \mu}^2$ defined by:

$$\begin{aligned} \Upsilon_{\lambda, \lambda}^2(X_G) &= G' \\ \widetilde{\Upsilon}_{\lambda, \lambda}^2(X_G) &= \begin{cases} \eta_1 \eta_2(G) & \text{if } \lambda = 0 \\ 2\lambda \eta_1(\partial_2(G)) - (-1)^{|G|} (\partial_2(G) \eta_1 + \theta_2 \eta_2 \eta_1(G) \eta_2) & \text{if } \lambda \neq 0 \end{cases} \\ \Upsilon_{-\frac{k}{2}, \frac{k}{2}}^2(X_G) &= G' \eta_1 \eta_2^{2k-1} \\ \widetilde{\Upsilon}_{-\frac{k}{2}, \frac{k}{2}}^2(X_G) &= k \eta_1(\partial_2(G)) \eta_1 \eta_2^{2k-1} - (-1)^{|G|} (\partial_2(G) \eta_2^{2k+1} - \eta_1(\theta_2 \partial_2(G)) \eta_1^{2k+1}) \\ \overline{\Upsilon}_{-\frac{k}{2}, \frac{k}{2}}^2(X_G) &= (k-1) G'' \eta_1 \eta_2^{2k-3} + (-1)^{|G|} (\eta_2(G') \eta_1^{2k-1} - \eta_1(G') \eta_2^{2k-1}) \end{aligned} \quad (28)$$

4.3. The space $H_{\text{diff}}^1(\mathfrak{osp}(n|2), \mathfrak{osp}(n-1|2)^i; \mathfrak{D}_{\lambda, \mu}^n)$.

As a first step towards the proof of Theorem 4.1, we shall need to study the $\mathfrak{osp}(n-1|2)^i$ -relative cohomology $H_{\text{diff}}^1(\mathfrak{osp}(n|2), \mathfrak{osp}(n-1|2)^i; \mathfrak{D}_{\lambda, \mu}^n)$.

Theorem 4.2. *The spaces $H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{osp}(1|2)^i; \mathfrak{D}_{\lambda, \mu}^2)$ are one dimensional if $\lambda = \mu \neq 0$ or $(\lambda, \mu) = (-\frac{k}{2}, \frac{k}{2})$ where $k \in \mathbb{N} \setminus \{0\}$. Moreover, these spaces are spanned by classes of some $\mathfrak{osp}(1|2)^i$ -relative cocycles which are cohomologous as $\mathfrak{osp}(2|2)$ -cocycles.*

Otherwise, the spaces $H_{\text{diff}}^1(\mathfrak{osp}(n|2), \mathfrak{osp}(n-1|2)^i; \mathfrak{D}_{\lambda, \mu}^n)$ are trivial.

Proof. For $n = 1$, the result holds from [3], (Lemma 3.3). For $n = 2$, we deduce the result from [8] (Theorem 4.3). Now, we deduce the result for $n \geq 3$ from the following Proposition. ■

Proposition 4.3. For $n \geq 3$, any element of $Z_{\text{diff}}^1(\mathfrak{osp}(n|2); \mathfrak{D}_{\lambda, \mu}^n)$ is a coboundary over $\mathfrak{osp}(n|2)$ if and only if at least one of its restrictions to the subalgebras $\mathfrak{osp}(n-1|2)^i$ is a coboundary.

Proof. Let $\Upsilon \in Z_{\text{diff}}^1(\mathfrak{osp}(n|2); \mathfrak{D}_{\lambda, \mu}^n)$ and assume that the restriction of Υ to $\mathfrak{osp}(n-1|2)^i$, for some i , is a coboundary, that is, there exists $b \in \mathfrak{D}_{\lambda, \mu}^n$ such that

$$\Upsilon(X_F) = \delta(b)(X_F) = (-1)^{|F||b|} X_F \cdot b \quad \text{for all } X_F \in \mathfrak{osp}(n-1|2)^i.$$

By replacing Υ by $\Upsilon - \delta b$, we can suppose that $\Upsilon|_{\mathfrak{osp}(n-1|2)^i} = 0$. Thus, the map Υ is $\mathfrak{osp}(n-1|2)^i$ -invariant and therefore the equation (25) becomes (here $|\tilde{g}| = |g| + 1$):

$$(-1)^{|\tilde{g}||\Upsilon|} X_{g\theta_i} \cdot \Upsilon(X_{h\theta_i}) - (-1)^{|\tilde{h}|(|\tilde{g}|+|\Upsilon|)} X_{h\theta_i} \cdot \Upsilon(X_{g\theta_i}) = 0. \tag{29}$$

According to the isomorphism (11), the map Υ is decomposed into four components

$$\begin{aligned} \Pi(\mathfrak{h}^{n-1,i}) \otimes \mathfrak{F}_{\lambda}^{n-1,i} &\rightarrow \mathfrak{F}_{\mu}^{n-1,i}, & \Pi(\mathfrak{h}^{n-1,i}) \otimes \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^{n-1,i}) &\rightarrow \Pi(\mathfrak{F}_{\mu+\frac{1}{2}}^{n-1,i}), \\ \Pi(\mathfrak{h}^{n-1,i}) \otimes \mathfrak{F}_{\lambda}^{n-1,i} &\rightarrow \Pi(\mathfrak{F}_{\mu+\frac{1}{2}}^{n-1,i}), & \Pi(\mathfrak{h}^{n-1,i}) \otimes \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^{n-1,i}) &\rightarrow \mathfrak{F}_{\mu}^{n-1,i}. \end{aligned} \tag{30}$$

So, each of these bilinear maps is $\mathfrak{osp}(n-1|2)^i$ -invariant. Therefore, their expressions are given by Theorem 3.1 with the help of Lemma (3.2) and equation (29). More precisely, we get up to a scalar factor:

$$\Upsilon = \begin{cases} \delta(\theta_i) & \text{if } \mu = \lambda - \frac{1}{2}, \\ \delta(1 - 2\theta_i \partial_i) & \text{if } \mu = \lambda, \\ \delta(\epsilon_1 \chi_{n,\lambda} \theta_i \eta_1 \cdots \tilde{\eta}_i \cdots \eta_n + \epsilon_2 \partial_i) & \text{if } \mu = \lambda + \frac{1}{2}, \\ \delta(\theta_i \eta_1 \cdots \eta_n \partial_x^{k-1}) & \text{if } (\lambda, \mu) = \left(-\frac{k+n-2}{2}, \frac{k-1}{2}\right), \\ \delta((\eta_1 \cdots \tilde{\eta}_i \cdots \eta_n + (-1)^i \theta_i \eta_1 \cdots \eta_n) \partial_x^{k-1}) & \text{if } (\lambda, \mu) = \left(-\frac{k+n-3}{2}, \frac{k}{2}\right), \\ \delta(\eta_1 \cdots \tilde{\eta}_i \cdots \eta_n \partial_i \partial_x^{k-1}) & \text{if } (\lambda, \mu) = \left(-\frac{k+n-2}{2}, \frac{k}{2}\right), \\ 0 & \text{otherwise,} \end{cases} \tag{31}$$

where $\chi_{3,-\frac{1}{2}} = 1$ and $\chi_{n,\lambda} = 0$ if $(n, \lambda) \neq (3, -\frac{1}{2})$. ■

5. Proof of Theorem 4.1

(i) According to Proposition 4.3, the restriction of any nontrivial differential 1-cocycle Υ of $\mathfrak{osp}(3|2)$ with coefficients in $\mathfrak{D}_{\lambda, \mu}^3$ to $\mathfrak{osp}(2|2)^i$, for $i=1, 2, 3$, is a nontrivial 1-cocycle. Furthermore, using isomorphisms (26) and (27) we deduce that if $2(\mu - \lambda) \neq -1, 0, 1$, and if

$$(\lambda, \mu) \notin \left\{ \left(-\frac{k}{2}, \frac{k}{2}\right), \left(-\frac{k+1}{2}, \frac{k-1}{2}\right), \left(-\frac{k+1}{2}, \frac{k}{2}\right) \right\}, \quad k \in \mathbb{N} \setminus \{0\} \tag{32}$$

then $H_{\text{diff}}^1(\mathfrak{osp}(2|2)^i; \mathfrak{D}_{\lambda,\mu}^3) = 0$ and therefore, $H_{\text{diff}}^1(\mathfrak{osp}(3|2); \mathfrak{D}_{\lambda,\mu}^3) = 0$. Now, for (λ, μ) as in (32) or $2(\mu - \lambda) = -1, 0, 1$, let Υ be a 1-cocycle from $\mathfrak{osp}(3|2)$ to $\mathfrak{D}_{\lambda,\mu}^3$. Note that the knowledge of Υ is equivalent to the knowledge of its restriction to $\mathfrak{osp}(2|2)^i$, for $i=1, 2, 3$. The map $\Upsilon|_{\mathfrak{osp}(2|2)^i}$ is a 1-cocycle of $\mathfrak{osp}(2|2)^i$. Therefore, using isomorphism (26) together with formulas (28), we deduce that, up to a coboundary, Υ is expressed as in Theorem 4.1.

(ii) We proceed by induction on n . First note that by Proposition 4.3, the restriction of any nontrivial differential 1-cocycle Υ of $\mathfrak{osp}(n|2)$ with coefficients in $\mathfrak{D}_{\lambda,\mu}^n$ to $\mathfrak{osp}(n|2)^i$, for $i = 1, \dots, n$, is a nontrivial 1-cocycle. Now, we assume that the result holds for some $n \geq 4$. Again, the same arguments as before together with induction assumption show that $H_{\text{diff}}^1(\mathfrak{osp}(n|2)^i; \mathfrak{D}_{\lambda,\mu}^{n+1}) = 0$ if $2(\mu - \lambda) \neq -1, 0, 1$, and if (λ, μ) not belongs to (with $k \in \mathbb{N} \setminus \{0\}$)

$$\left\{ \left(-\frac{k+n-2}{2}, \frac{k}{2} \right), \left(-\frac{k+n-1}{2}, \frac{k-1}{2} \right), \left(-\frac{k+n-1}{2}, \frac{k}{2} \right), \left(-\frac{n-1}{2}, 0 \right) \right\}. \quad (33)$$

Therefore, the corresponding cohomology spaces $H_{\text{diff}}^1(\mathfrak{osp}(n+1|2); \mathfrak{D}_{\lambda,\mu}^{n+1})$ vanish. So, we consider only the cases where $2(\mu - \lambda) = -1, 0, 1$ and (λ, μ) as in (33), we proceed as in (i) and we get the result for $n+1$. \square

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