

Capelli Elements for the Algebra \mathfrak{g}_2

D. V. Artamonov and V. A. Golubeva*

Communicated by S. Gindikin

Abstract. Itoh and Umeda (see Itoh, M., and T. Umeda, *On Central Elements in the Universal Enveloping Algebras of the Orthogonal Lie Algebras*, *Compositio Mathematica* **127** (2001), 333–359) constructed central elements in the universal enveloping algebra $U(\mathfrak{o}_N)$, named Capelli elements, as sums of squares of noncommutative Pfaffians of some matrices, whose entries belong to \mathfrak{o}_N . However for exceptional algebras there are no construction of this type. In the present paper we construct central elements in $U(\mathfrak{g}_2)$ as sums of squares of Pfaffians of some matrices, whose elements belong to \mathfrak{g}_2 . For $U(\mathfrak{g}_2)$, as in the case $U(\mathfrak{o}_N)$, we give characterization of these central elements in terms of their vanishing properties. Also for $U(\mathfrak{g}_2)$ an explicit relations between constructed central elements and higher Casimir elements defined in Zhelobenko, D. P., “Compact Lie groups and their representations,” Amer. Math. Soc., Providence, R.I, 1973, are obtained.

Mathematics Subject Classification 2010: 17B25, 16S30.

Key Words and Phrases: Central elements, universal enveloping algebra, pfaffian.

1. Introduction

It is known that the for a simple Lie algebra \mathfrak{g} the center of the corresponding universal enveloping algebra $U(\mathfrak{g})$ is generated by Casimir elements, that are defined as follows. Let e_α be elements of \mathfrak{g} corresponding to roots α of the algebra \mathfrak{g} , and h_i base elements in the Cartan subalgebra. Suppose that h_i form an orthonormal base in the Cartan subalgebra with respect to the Killing form. Suppose also that e_α and $e_{-\alpha}$ are dual elements with respect to this form. Put

$$L = \sum_{\alpha} e_{\alpha} \otimes e_{-\alpha} + \sum_i h_i \otimes h_i \in U(\mathfrak{g}) \otimes U(\mathfrak{g}).$$

In other words

$$L = \frac{1}{2}(\Delta C_2 - C_2 \otimes 1 - 1 \otimes C_2),$$

where

$$C_2 = \sum_{\alpha} e_{\alpha} e_{-\alpha} + \sum_i h_i h_i \in U(\mathfrak{g})$$

*The work was supported by grants NSh-5998.2012.1, MK-4270.2011.1 and RFBR 12-01-31414

is the Casimir element of the second order. Let ρ be a standard representation of \mathfrak{g} and

$$\mathcal{L} = (1 \otimes \rho)(L) \in U(\mathfrak{g}) \otimes gl(V).$$

This element is a matrix whose entries belong to $U(\mathfrak{g})$. Then the Casimir elements Cas_k of order k are defined by the formula [15]

$$Cas_k = \text{Tr } \mathcal{L}^k.$$

Note that as in [4] we call "the higher Casimir elements" not all higher central elements, but some certain central elements defined in [15] and named there "the higher Casimir elements". In some other papers the same elements are called the Gelfand invariants.

The Casimir elements generate the center but they are not algebraically independent. The orders of primitive Casimir elements are known for all simple Lie algebras. It can be proved that the center of the universal enveloping algebra is isomorphic to the algebra of commutative polynomials of the primitive Casimir elements. However it is difficult to express nonprimitive Casimirs through the primitive ones. Such calculations are done only for certain cases [1],[6].

Also there are known other construction of central elements in $U(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}_N, \mathfrak{sp}_N, \mathfrak{o}_N$, in particular of Capelli elements. Remind their definitions for $\mathfrak{g} = \mathfrak{o}_N$.

Let \mathfrak{o}_N be a Lie algebra that preserves the quadratic form defined by the unit matrix. Then \mathfrak{o}_N is the algebra of skew symmetric matrices. It is generated by the elements

$$F_{ij} = E_{ij} - E_{ji}.$$

Define a matrix

$$F = (F_{ij}).$$

To define Capelli elements we need the notion of a Pfaffian of a matrix.

Let $\Phi = (\Phi_{ij})$ be a $k \times k$ matrix, where k is even, whose elements belong to some noncommutative ring. The noncommutative Pfaffian is defined by the formula

$$\text{Pfaff } \Phi = \frac{1}{2^{\frac{k}{2}} (\frac{k}{2})!} \sum_{\sigma \in S_k} (-1)^\sigma \Phi_{\sigma(1)\sigma(2)} \cdots \Phi_{\sigma(k-1)\sigma(k)}.$$

For a subset $I \subset \{1, \dots, n\}$ define a submatrix

$$F_I = (F_{ij})_{i,j \in I}.$$

Definition 1.1. [4] The Capelli elements C_k , $k = 2, 4, \dots, 2[\frac{N}{2}]$ are elements of $U(\mathfrak{o}_N)$ defined by the formula

$$C_k = \sum_{I \subset \{1, \dots, N\}, |I|=k} (\text{Pfaff } F_I)^2,$$

The term "Capelli elements" is explained by the fact that the same elements can be defined using the Capelli-type determinant. Let us give this definition.

The row determinant of a $N \times N$ matrix $\Phi = (\Phi_{ij})$ whose elements belong to some noncommutative ring is defined as follows

$$\det \Phi = \sum_{\sigma \in S_N} (-1)^\sigma \Phi_{\sigma(1)1} \dots \Phi_{\sigma(N)N}.$$

Definition 1.2. Consider the determinant in $U(\mathfrak{o}_N)$:

$$\det(F + \text{diag}(\frac{N}{2}, \frac{N}{2} - 1, \dots) + u).$$

Capelli elements C_k , $k = 2, 4, \dots, 2[\frac{N}{2}]$ are coefficients of the decomposition

$$\det(F + \text{diag}(\frac{N}{2}, \frac{N}{2} - 1, \dots) + u) = \sum_{r=0}^{[\frac{N}{2}]} C_{2r} (u + \frac{N}{2} - r)(u + \frac{N}{2} - r - 1) \dots (u - \frac{N}{2} + r + 1).$$

The definition 1.2 was given in [3]. The equivalence of these two definitions is proved in [4].

A relation between the Casimir and Capelli elements is given in [4].

The images of Capelli elements under the Harish-Chandra isomorphism were explicitly found in [9]. Using this calculation in [10] the following characterization Capelli elements for \mathfrak{o}_N was pointed out.

Theorem 1.3. *Up to multiplication by a constant C_k is a uniquely defined as a central element of order k which annihilates $V^{\otimes t}$ for $1 \leq t < \frac{k}{2}$, where V is the standard representation of \mathfrak{o}_N .*

Theorem 1.4. *[9] For odd N the elements C_k are algebraically independent and generate the center, for even N the same is true if one takes instead of the highest Capelli element $C_N = (PfF)^2$ the central element PfF .*

For exceptional Lie algebras these constructions are unknown, although the center of the universal enveloping algebras for exceptional Lie algebras was intensively studied. In particular, the expressions of some nonprimitive Casimir elements through the primitive ones were found [5],[7].

The problem of a definition of Capelli elements for the exceptional Lie algebras was raised in [2].

The aim of the present paper is to construct Capelli central elements for the algebras \mathfrak{g}_2 . The construction is based on Definition 1.1. In Sec. 2 for the algebra \mathfrak{g}_2 we construct matrix G which is an analog of the matrix $F = (F_{ij})$ for the algebra \mathfrak{o}_N . The matrix G is investigated using the relation between \mathfrak{g}_2 and \mathfrak{o}_8 .

In Sec. 3 Capelli elements G_k in $U(\mathfrak{g}_2)$ are defined as sums of squares of Pfaffians of submatrices in G . The centrality of these elements is proved.

In contrast with Capelli elements for \mathfrak{o}_N these elements are not algebraically independent. The relations between Capelli elements for \mathfrak{g}_2 are investigated in Sec. 4. It is proved that G_2, G_6 are algebraically independent and G_4 can be expressed as a polynomial of G_2 . Since it is well known that $Z(U(\mathfrak{g}_2))$ is generated by primitive elements of orders 2 and 6, we see that G_2, G_6 generate algebraically the center $Z(U(\mathfrak{g}_2))$.

In the rest part of the paper an analog of the Theorem 1.3 for Capelli elements for \mathfrak{g}_2 (Theorem 5.1) is proved.

In Sec. 5 we show that a Pfaffian of order k for the algebra \mathfrak{g}_2 annihilates $V^{\otimes t}$ for $1 \leq t < \frac{k}{2}$.

In Sec. 6 the final part of proof of Theorem 5.1 is given. To complete the proof we obtain relations between the Capelli and Casimir elements for the algebra \mathfrak{g}_2 .

1.1. Notation. In the paper we use two nonstandard pieces of notation. Consider the space \mathbb{C}^N with standart quadratic form. Identify the Lie algebra \mathfrak{o}_N with the space $\Lambda^2(\mathbb{C}^N)$ of skew-symmetric tensors. Denote as

$$\zeta : \Lambda^2(\mathbb{C}^N) \rightarrow \mathfrak{o}_N$$

this identification.

Note that if $x = e_i, y = e_j$, where e_i, e_j are elements of the standart base of \mathbb{C}^N , then

$$\zeta(e_i \wedge e_j) = F_{ij},$$

where F_{ij} was defined earlier. Below we shall use both notations for elements of \mathfrak{o}_N : the standart notation F_{ij} and the the nonstandart notation F_{xy} . Let $I = \{i_1, \dots, i_t\}, i_r \in \{1, \dots, N\}$ be a set of indices. For $g \in U(\mathfrak{g})$ let gI be a formal linear combination of set of indices that is defined as follows. Identify the indexing set $I = \{i_1, \dots, i_t\}$ with the tensor $e_{i_1} \wedge \dots \wedge e_{i_t} \in (\mathbb{C}^k)^{\wedge t}$. Let $g(e_{i_1} \wedge \dots \wedge e_{i_t}) = \sum_{j_1, \dots, j_s} c_{j_1, \dots, j_s} e_{j_1} \wedge \dots \wedge e_{j_t}$. Put

$$gI = \sum_{j_1, \dots, j_s} c_{j_1, \dots, j_s} \{j_1, \dots, j_t\}.$$

Let Φ be an $N \times N$ skew-symmetric matrix, whose columns and rows are indexed by $1, \dots, N$. As before, for a set of indices let $I = \{i_1, \dots, i_t\}$ let $\Psi_I = (\Phi_{ij})_{i, j \in I}$ be a submatrix of Φ .

Definition 1.5. Given a formal linear combination of indexing sets $\sum c_I I, c_I \in \mathbb{C}$ of indexing sets define pf $\Phi_{\sum c_I I}$ as follows

$$\text{pf } \Phi_{\sum c_I I} = \sum_I c_I \text{Pfaff } \Phi_I.$$

Then pf Φ_{gI} for $g \in U(\mathfrak{g})$ is also well-defined.

2. The algebra \mathfrak{g}_2

In this section we consider the algebra \mathfrak{g}_2 . A description of this algebra can be found in [14],[11],[13]. Below we remind this description and obtain special sets of generators for this algebra and derive commutation relations between these generators.

The group \mathfrak{G}_2 is the group of automorphisms of the algebra of octonions \mathbb{O} . As it is known, the group \mathfrak{G}_2 consists of orthogonal transformation. Thus $\mathfrak{G}_2 \subset O(8)$. Since every automorphism preserves the unit $1 \in \mathbb{O}$ one has $\mathfrak{G}_2 \subset O(7) \subset O(8)$. The Lie algebra \mathfrak{g}_2 is the Lie algebra of derivations of the algebra \mathbb{O} . One has $\mathfrak{g}_2 \subset \mathfrak{o}_7$.

Given octonions x, y , one can construct a derivation D_{xy} , defined by the formula

$$D_{xy}(z) = [[x, y], z] - 3[x, y, z], \quad (1)$$

where $[x, y, z] = (xy)z - x(yz)$ [14]. Obviously $D_{xy} = -D_{yx}$.

These derivations satisfy the following commutation relations

$$[D_{xy}, D_{ab}] = D_{D_{xy}a, b} + D_{a, D_{xy}b}. \quad (2)$$

It is known that every derivations is a linear combination of derivations D_{xy} [14]. Thus these elements generate the algebra \mathfrak{g}_2 . They are not linearly independent but relations between them are known.

The key observation is that the commutation relations between generators of the orthogonal algebra have the same form. Indeed the commutation relations between generators F_{ij} introduced in the introduction are the following

$$[F_{ij}, F_{kl}] = \delta_{kj}F_{il} - \delta_{il}F_{kj} - \delta_{ik}F_{jl} - \delta_{lj}F_{ki}. \quad (3)$$

Using notations introduced in 1 we can rewrite the commutation relations (3) in the following way

$$[\zeta(x \wedge y), \zeta(a \wedge b)] = \zeta((\zeta(x \wedge y)a) \wedge b) + \zeta(a \wedge (\zeta(x \wedge y)b)). \quad (4)$$

Take a standard base in the space of imaginary octonions. In the definitions below the indices x, y denote the elements of one of this base.

Definition 2.1. In the case of \mathfrak{g}_2 define a skew-symmetric matrix $G = (D_{xy})$ of size 7×7 , where D_{xy} are defined by the formula (1).

Below we denote the elements of the standart base in the space of imaginary octonions as e_1, \dots, e_7 and the elements $D_{e_i e_j}$ are denoted shortly as D_{ij} .

2.1. Triality.

Construct a 8×8 matrix G by adding elements $D_{1x} = D_{x1} = 0$, $D_{11} = 0$.

Let us find an interpretation of the 8×8 matrix G from the point of view of representation theory.

Take as x, y the unit and imaginary octonions by means of which the multiplication in the octonions is defined. They give an orthonormal base of the algebra of octonions. Then the algebra \mathfrak{o}_8 is generated by the elements $\zeta(x \wedge y)$.

As it is known, the algebra \mathfrak{o}_8 has a diagrammatic automorphism δ of order 3 and its set of fixed points is the subalgebra $\mathfrak{g}_2 \subset \mathfrak{o}_8$.

The mapping

$$\zeta(x \wedge y) \mapsto \frac{1}{3}(\zeta(x \wedge y) + \delta(\zeta(x \wedge y)) + \delta^2(\zeta(x \wedge y)))$$

is a linear (not a Lie) projection $\mathfrak{o}_8 \rightarrow \mathfrak{g}_2$.

Proposition 2.2. $\zeta(x \wedge y) + \delta(\zeta(x \wedge y)) + \delta^2(\zeta(x \wedge y)) = D_{xy}$.

Proof. This statement is very natural but we have not found its proof anywhere. Nevertheless it can be easily derived from statements from [14],[13],[8].

Take the following simple roots $\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_3 + \epsilon_4$ in the root system D_4 . Define diagrammatic automorphisms t and s of the algebra \mathfrak{o}_8 . The automorphism t corresponds to a permutation of simple roots $e_3 - e_4$ and $\epsilon_1 - \epsilon_2$, the automorphism s corresponds to a permutation of simple roots $\epsilon_3 + \epsilon_4$ and $\epsilon_1 - \epsilon_2$. One has

$$\delta = ts, \delta^2 = st.$$

Let V be the standard representation of \mathfrak{o}_8 and S_0, S_1 two spinor representations of \mathfrak{o}_8 . The spaces of all these representations can be identified with the algebra of octonions. Using explicit formulas for these representations (see [8]) one proves that $\zeta(x \wedge y) + \delta(\zeta(x \wedge y)) + \delta^2(\zeta(x \wedge y))$ acts on the standard representation in the same way as $\zeta(x \wedge y) + t(\zeta(x \wedge y)) + s(\zeta(x \wedge y))$

From explicit formulas for the automorphisms t and s in [14], [13], [8] it follows that $\zeta(x \wedge y) + t(\zeta(x \wedge y)) + s(\zeta(x \wedge y)) = D_{xy}$. ■

2.2. Split realization of \mathfrak{o}_N and \mathfrak{g}_2 .

Below we need also the split realization of the orthogonal algebra and the algebra \mathfrak{g}_2 . The split realization of \mathfrak{o}_N corresponds to another choice of the quadratic form that is preserved by the algebra, namely

$$G = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & & & & \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Let the rows and columns be indexed by $i, j = -n, -n + 1, \dots, n - 1, n$. The zero is skipped in the case $N = 2n$ and is included in the case $N = 2n + 1$. The algebra \mathfrak{o}_N is generated by matrices

$$\tilde{F}_{ij} = E_{ij} - E_{-j-i}.$$

The diagonal elements \tilde{F}_{ii} are the Cartan elements and the elements \tilde{F}_{ij} are root elements.

Now we describe a split realization of \mathfrak{g}_2 since it will be used below.

Remember that the split realization of the algebra \mathfrak{o}_8 corresponds to the quadratic form $G = (\delta_{i,-j})$, where the rows and columns of matrices from \mathfrak{o}_8 are indexed by $i, j = -4, -3, \dots, 3, 4$. The algebra \mathfrak{o}_8 is generated by the matrices $\tilde{F}_{ij} = E_{ij} - E_{-j-i}$. Consider the matrix $\tilde{F} = (\tilde{F}_{ij})$. The elements \tilde{F}_{ii} on the diagonal are Cartan elements and the elements \tilde{F}_{ij} below the diagonal are positive root elements, the elements above the diagonal are negative root elements.

Consider matrices $\delta(\tilde{F}) = (\delta(\tilde{F}_{ij}))$ and $\delta^2(\tilde{F}) = (\delta^2(\tilde{F}_{ij}))$.

Definition 2.3. Set $\tilde{G} = \tilde{F} + \delta(\tilde{F}) + \delta^2(\tilde{F})$.

It can be proved that $F = B^{-1}\tilde{F}B$, where B is a matrix of constants [9]. From the other hand one has $\delta(F) = B^{-1}\delta(\tilde{F})B$, $\delta^2(F) = B^{-1}\delta^2(\tilde{F})B$. We obtain a proposition.

Proposition 2.4. $G = B^{-1}\tilde{G}B$.

An important property of the matrix \tilde{G} is the following. Since the automorphism δ is generated by an automorphism of the Dynkin diagram, the matrix \tilde{G} has the same property as the matrix \tilde{F} : the diagonal elements are Cartan elements, the elements below the diagonal and above it are root elements.

Since matrix elements of the matrix G are denoted as D_{ij} we denote matrix elements of \tilde{G} as \tilde{D}_{ij} .

3. Capelli elements for \mathfrak{g}_2

In this section the definitions of Capelli elements for \mathfrak{g}_2 are given.

Definition 3.1. For the algebra \mathfrak{g}_2 the Capelli elements are defined as

$$G_k = \sum_{I \subset \{1, \dots, 7\}, |I|=k} (\text{Pfaff } G_I)^2, \quad k = 2, 4, 6.$$

Theorem 3.2. *The Capelli elements G_k are central, that is $G_k \in Z(U(\mathfrak{g}_2))$.*

Proof. The proof of the theorem consists of the following steps. At first, we find commutation relations between generators F_{ij} of the algebra \mathfrak{o}_N and Pfaffians $\text{Pfaff } F_I$ (Proposition 3.3). Using this calculation we easily obtain commutation relations between an arbitrary element $g \in \mathfrak{o}_N$ and Pfaffians $\text{Pfaff } F_I$ (Proposition 3.4) and between generators D_{xy} and Pfaffians $\text{Pfaff } G_I$ respectively (Proposition 3.5).

At last using Proposition 3.6 and previous calculation we complete the proof of Theorem 3.2.

The first step in the proof of this theorem consists of derivation of commutation relations between Pfaffians and elements of the algebra. In all cases $\mathfrak{g} = \mathfrak{o}_N, \mathfrak{g}_2$ this is done in a similar way.

Proposition 3.3. $[F_{ij}, \text{Pfaff } F_I] = \text{pf } F_{F_{ij}I}$.

Proof. It is more convenient to work with nonstandard notations for the elements of the orthogonal algebra. In these notations the proposition is written as follows

$$[\zeta(e_i \wedge e_j), \text{Pfaff } F_I] = \text{pf } F_{\zeta(e_i \wedge e_j)I}.$$

By the definition one has

$$\text{Pfaff } F_I = \frac{1}{2^{\frac{k}{2}} \binom{k}{2}!} \sum_{\sigma \in S_k} (-1)^\sigma \zeta(e_{\sigma(i_1)} \wedge e_{\sigma(i_2)}) \dots \zeta(e_{\sigma(i_{k-1})} \wedge e_{\sigma(i_k)}),$$

where $I = \{i_1, \dots, i_k\}$.

Denote the product $\zeta(e_{i_1} \wedge e_{i_2}) \dots \zeta(e_{i_{k-1}} \wedge e_{i_k})$ by F^I . Also for two sets of indices I, J and numbers α, β put by definition

$$F^{\alpha I + \beta J} = \alpha F^I + \beta F^J.$$

Due to relations (4), one has

$$[\zeta(x \wedge y), F^{\sigma(I)}] = F^{\zeta(x \wedge y)\sigma(I)}.$$

The Pfaffian can be rewritten as follows

$$\text{Pfaff } F_I = \frac{1}{2^{\frac{k}{2}} \binom{k}{2}!} \sum_{\sigma \in S_k} (-1)^\sigma F^{\sigma(I)}.$$

Thus one has

$$[\zeta(x \wedge y), \text{Pfaff } F_I] = \frac{1}{2^{\frac{k}{2}} \binom{k}{2}!} \sum_{\sigma \in S_k} (-1)^\sigma [\zeta(x \wedge y), F^{\sigma(I)}].$$

Note that $\zeta(e_i \wedge e_j)\sigma(I) = \sigma(\zeta(e_i \wedge e_j)I)$ modulo summands that contain coinciding indices. Thus one has

$$[\zeta(e_i \wedge e_j), \text{Pfaff } F_I] = \frac{1}{2^{\frac{k}{2}} \binom{k}{2}!} \sum_{\sigma \in S_n} (-1)^\sigma F^{\sigma(\zeta(e_i \wedge e_j)I)} = \text{pf } F_{\zeta(e_i \wedge e_j)I}. \quad \blacksquare$$

As a corollary of Proposition 3.3 one has the following statement:

Proposition 3.4. For an arbitrary element $g \in \mathfrak{o}_n$ one has $[g, \text{Pfaff } F_I] = \text{pf } F_{gI}$.

The proofs of the following two propositions are similar to the proof of the proposition 3.3.

Proposition 3.5. For the algebra \mathfrak{g}_2 we have $[D_{xy}, \text{Pfaff } G_I] = \text{pf } G_{D_{xy}I}$.

Let $e_I = e_{i_1} \wedge \dots \wedge e_{i_k}$, where I is an ordered set of indices $\{i_1, \dots, i_k\}$.

Corollary 3.6. *Let g be an arbitrary element of \mathfrak{o}_N . Then*

$$\sum_{|I|=k} g(e_I \otimes e_I) = 0.$$

The proof of this proposition is contained in the proof of Theorem 7.5.1 in [9]. ■

Let us finish the proof of Theorem 3.2. One has to prove that $[D_{xy}, G_k] = 0$. We have

$$[D_{xy}, \sum_{|I|=k} \text{Pfaff } G_I \text{ Pfaff } G_I] = \sum_{|I|=k} [D_{xy}, \text{Pfaff } G_I] \text{ Pfaff } G_I + \text{Pfaff } G_I [D_{xy}, \text{Pfaff } G_I] = \sum_{|I|=k} \text{pf } G_{D_{xy}I} \text{ Pfaff } G_I + \text{Pfaff } G_I \text{ pf } G_{D_{xy}I}.$$

Since $\mathfrak{g}_2 \subset \mathfrak{o}_7$, using Proposition 3.6 for $g = D_{xy}$ one obtains

$$\sum_{|I|=k} D_{xy}(e_I \otimes e_I) = 0.$$

Thus, one gets that

$$\sum_{|I|=k} \text{pf } G_{D_{xy}I} \text{ Pfaff } G_I + \text{Pfaff } G_I \text{ pf } G_{D_{xy}I} = 0.$$

The theorem is proved. ■

4. Independent Capelli elements in the case \mathfrak{g}_2

Consider the question of algebraic dependence of Capelli elements for \mathfrak{g}_2 . In this section for the algebra \mathfrak{g}_2 the following is proved.

Theorem 4.1. *The Capelli elements G_2, G_6 are algebraically independent, thus they algebraically generate the center $Z(U(\mathfrak{g}_2))$.*

The fact that G_4 can be expressed via G_2 follows from the general fact that in $Z(U(\mathfrak{g}_2))$ there are no primitive elements of the fourth order.

The proof of the theorem is based on an explicit calculation of the highest homogeneous components of images of Capelli elements for \mathfrak{g}_2 under the Harich-Chandra homomorphism (see [16]). In Subsection 4 we give the construction of Capelli elements in the split realization of \mathfrak{g}_2 . In this realization we easily find homogeneous components of images of Capelli elements for \mathfrak{g}_2 under the Harich-Chandra homomorphism. This is done in Subsection 4. Using these formulas in Subsection 4 we complete the proof of Theorem 4.1.

4.1. Capelli elements for \mathfrak{g}_2 in the split realization. Definitions 1.2-1.1 have analogs in the split realization of the orthogonal algebra. They give the same Capelli elements. Remind their construction following [9]. The construction of Capelli elements in the split realization of \mathfrak{o}_N is the following. For a subset

$I = \{i_1, \dots, i_k\} \subset \{-n, \dots, n\}$ let \tilde{F}_I be a submatrix $(\tilde{F}_{ij})_{-i,j \in I}$. Associate to this submatrix a Pfaffian

$$\text{Pfaff } \tilde{F}_I := \text{Pfaff}(\tilde{F}_{-ij})_{-i,j \in I}.$$

Define the Capelli elements by the formula

$$\tilde{C}_k = \sum_{I \subset \{-n, \dots, n\}, |I|=k} \text{Pfaff } \tilde{F}_I \cdot \text{Pfaff } \tilde{F}_{-I},$$

where $-I = \{-i_1, \dots, -i_k\}$. Since $F = B^{-1}\tilde{F}B$, where B is a matrix of constants, one has

$$\tilde{C}_k = C_k.$$

Similar construction can be applied to \mathfrak{g}_2 . So we have defined the matrix $\tilde{G} = (\tilde{D}_{ij})$. For a subset $I = \{i_1, \dots, i_k\} \subset \{-4, \dots, 4\}$ let \tilde{G}_I be a submatrix $(\tilde{D}_{ij})_{-i,j \in I}$. Associate to this submatrix a Pfaffian

$$\text{Pfaff } \tilde{G}_I := \text{Pfaff}(\tilde{G}_{-ij})_{-i,j \in I}.$$

Define Capelli elements by the formula

$$\tilde{G}_k = \sum_{I \subset \{-4, \dots, 4\}, |I|=k} \text{Pfaff } \tilde{G}_I \cdot \text{Pfaff } \tilde{G}_{-I},$$

where $-I = \{-i_1, \dots, -i_k\}$. In Proposition 2.4 it was proved that $G = B^{-1}\tilde{G}B$, where B is a matrix of constants, therefore

$$\tilde{G}_k = G_k.$$

So we can deal with the elements $\tilde{G}_k = \sum_{|I|=k} \text{Pfaff } \tilde{G}_I \text{Pf} \tilde{G}_{-I}$.

4.2. Highest coefficients of images of Capelli elements. Find the highest homogeneous components of images of Capelli elements $\tilde{G}_2, \tilde{G}_4, \tilde{G}_6$ under the Harish-Chandra homomorphism. One has

$$\begin{aligned} \tilde{G}_k = & \sum_{|I|=k} \sum_{\sigma, \sigma' \in \text{Aut}(I)} (-1)^\sigma (-1)^{\sigma'} \tilde{D}_{-\sigma(i_1)\sigma(i_2)} \dots \\ & \dots \tilde{D}_{-\sigma(i_{k-1})\sigma(i_k)} \tilde{D}_{\sigma(i_1)'\sigma'(i_2)} \dots \tilde{D}_{\sigma'(i_{k-1})-\sigma'(i_k)}. \end{aligned}$$

Using the commutation relations we transform this expression into a sum of products where at first (from the left to the right) positive root elements occur, then Cartan elements occur and then negative root elements occur. The application of the Harish-Chandra homomorphism means that we remove all summands except those that consist only of Cartan elements.

Some of the summands originally consist only of Cartan elements. If a product of Cartan elements appears after application of commutation relation then it consists of a less number of factors than an original product of Cartan elements. Thus the highest homogeneous component is a sum of products that

consist only of Cartan elements. The highest homogeneous component $[\tilde{G}_2]$ of \tilde{G}_2 equals

$$[\tilde{G}_2] = \tilde{D}_{11}^2 + \tilde{D}_{22}^2 + \tilde{D}_{33}^2 + \tilde{D}_{44}^2. \tag{5}$$

The highest homogeneous component $[\tilde{G}_4]$ of \tilde{G}_4 is equal

$$[\tilde{G}_4] = (\tilde{D}_{11}\tilde{D}_{22})^2 + (\tilde{D}_{11}\tilde{D}_{33})^2 + (\tilde{D}_{11}\tilde{D}_{44})^2 + (\tilde{D}_{22}\tilde{D}_{33})^2 + (\tilde{D}_{22}\tilde{D}_{44})^2 + (\tilde{D}_{33}\tilde{D}_{44})^2. \tag{6}$$

The highest homogeneous component $[\tilde{G}_6]$ of \tilde{G}_6 is equal

$$[\tilde{G}_6] = (\tilde{D}_{11}\tilde{D}_{22}\tilde{D}_{33})^2 + (\tilde{D}_{11}\tilde{D}_{22}\tilde{D}_{44})^2 + (\tilde{D}_{11}\tilde{D}_{33}\tilde{D}_{44})^2 + (\tilde{D}_{22}\tilde{D}_{33}\tilde{D}_{44})^2 + (\tilde{D}_{22}\tilde{D}_{33}\tilde{D}_{44})^2. \tag{7}$$

Using the matrix of the mapping $id + \delta + \delta^2$ on the Cartan subalgebra of \mathfrak{o}_8 we obtain the following expression of $\tilde{D}_{11}, \tilde{D}_{22}, \tilde{D}_{33}, \tilde{D}_{44}$ through the base of the Cartan subalgebra of \mathfrak{o}_8

$$\begin{aligned} \tilde{D}_{11} &= 2\tilde{F}_{11} - \tilde{F}_{22} + \tilde{F}_{33}, \\ \tilde{D}_{22} &= \tilde{F}_{11} + 2\tilde{F}_{22} - \tilde{F}_{33}, \\ \tilde{D}_{33} &= \tilde{F}_{11} - \tilde{F}_{22} + 2\tilde{F}_{33}, \\ \tilde{D}_{44} &= 0. \end{aligned}$$

Substituting these formulae into (5),(6),(7) one gets

1. $[\tilde{G}_4] = \frac{1}{4}[\tilde{G}_2]^2,$
2. $[\tilde{G}_6]$ is not a multiple of $[\tilde{G}_2].$

So the element \tilde{G}_4 can be expressed through \tilde{G}_2 . This follows from the general fact that there is no primitive element in $Z(U(\mathfrak{g}_2))$ of the order 4.

The element \tilde{G}_6 cannot be expressed through \tilde{G}_2 . Otherwise the highest homogeneous component $[\tilde{G}_6]$ would be equal to $[\tilde{G}_2]^3$, but this does not take place.

The theorem 4.1 is completely proved. ■

Below the formulae for highest homogeneous component will be used. If we take $\tilde{D}_{11}, \tilde{D}_{22}$ as a base in the Cartan subalgebra, we obtain $\tilde{D}_{33} = \tilde{D}_{11} - \tilde{D}_{22}$.

Thus the highest homogeneous component of images of Capelli elements under the Harish-Chandra homomorphism have the following form

$$[\tilde{G}_2] = 2\tilde{D}_{11}^2 + 2\tilde{D}_{22}^2 - 2\tilde{D}_{11}\tilde{D}_{22}, \tag{8}$$

$$[\tilde{G}_4] = \tilde{D}_{11}^4 - 2\tilde{D}_{11}^3\tilde{D}_{22} + 3\tilde{D}_{11}\tilde{D}_{22}^3 - 2\tilde{D}_{11}\tilde{D}_{22}^3 + \tilde{D}_{22}^4, \tag{9}$$

$$[\tilde{G}_6] = \tilde{D}_{11}^2\tilde{D}_{22}^2(\tilde{D}_{11} - \tilde{D}_{22})^2, \tag{10}$$

5. The action of Capelli elements on the tensor representations

In the next two sections we obtain a new characterization of Capelli elements for the algebra \mathfrak{g}_2 in terms of their eigenvalues.

Theorem 5.1. *The element G_k , $k = 2, 4, 6$, is the central element of order k , having the following properties*

1. *Let V be a standard representation of \mathfrak{g}_2 . Then G_k acts as a zero operator on $V^{\otimes t}$ if $1 \leq t < \frac{k}{2}$.*
2. *The highest homogeneous components of G_k under the Harich-Chandra homomorphism are given by formula (8), (9), (10).*

This and the next section are devoted to the proof of equivalence of Definitions 5.1 and 3.1. In the current section it is proved that a noncommutative Pfaffian of order k acts as zero on $V^{\otimes t}$ if $1 \leq t < \frac{k}{2}$ in the case \mathfrak{g}_2 .

5.1. Some formulae. Let us prove some formulae concerning with noncommutative Pfaffians. These formulae take place for both realizations of \mathfrak{o}_N and \mathfrak{g}_2 . Consider the case of standard realization of \mathfrak{g}_2

Lemma 5.2.

$$\text{Pfaff } G_I = \binom{k/2}{p/2}^{-1} \sum_{I=I' \sqcup I'', |I'|=p, |I''|=k-p} (-1)^{(I'I'')} \text{Pfaff } G_{I'} \text{Pfaff } G_{I''}.$$

Here $(-1)^{(I'I'')}$ is a sign of a permutation of the set $I = \{i_1, \dots, i_k\}$ that places first the subset $I' \subset I$ and then the subset $I'' \subset I$. The natural order on both subsets is preserved.

The number p is an arbitrary fixed integer, such that $p \leq k = |I|$.

Proof. By definition one has

$$\text{Pfaff } G_I = \frac{1}{2^{\frac{k}{2}} \left(\frac{k}{2}\right)!} \sum_{\sigma \in S_k} (-1)^\sigma D_{\sigma(i_1)\sigma(i_2)} \dots D_{\sigma(i_{k-1})\sigma(i_k)}.$$

Denote as q the number $k - p$. The summand

$$(-1)^\sigma D_{\sigma(i_1)\sigma(i_2)} \dots D_{\sigma(i_{k-1})\sigma(i_k)}$$

can be written in the form

$$(-1)^{(I'I'')} (-1)^{\sigma'} D_{\sigma'(i'_1)\sigma'(i'_2)} \dots D_{\sigma'(i'_{p-1})\sigma'(i'_p)} (-1)^{\sigma''} D_{\sigma''(i''_1)\sigma''(i''_2)} \dots D_{\sigma''(i''_{q-1})\sigma''(i''_q)}.$$

Here $I' = \{i'_1, \dots, i'_p\}$ is the set of indices $\{\sigma(i_1), \dots, \sigma(i_p)\}$ placed in a natural order, $I'' = \{i''_1, \dots, i''_q\}$ is the set of indices $\{\sigma(i_{p+1}), \dots, \sigma(i_k)\}$ placed in a natural order, σ' is a permutation $\{\sigma(i_1), \dots, \sigma(i_p)\}$ of the set I' and σ'' is a permutation of the set I'' defined in a similar way. Note, that

$$(-1)^{(I'I'')} (-1)^{\sigma'} (-1)^{\sigma''} = (-1)^\sigma.$$

The mapping $\sigma \mapsto I', I'', \sigma', \sigma''$ is bijective. One has

$$\frac{\left(\frac{p}{2}\right)!\left(\frac{q}{2}\right)!}{\left(\frac{k}{2}\right)!} = \binom{k/2}{p/2}^{-1},$$

thus the Pfaffian can be written in the form

$$\begin{aligned} & \binom{k/2}{p/2}^{-1} \sum_{I=I' \sqcup I'', |I'|=p, |I''|=q} (-1)^{(I' I'')} \frac{1}{2^{\frac{k}{2}} \left(\frac{p}{2}\right)! \left(\frac{q}{2}\right)!} \sum_{\sigma'} (-1)^{\sigma'} (-1)^{\sigma''} D_{\sigma'(i'_1)\sigma'(i'_2)} \cdots \\ & \cdots D_{\sigma'(i'_{p-1})\sigma'(i'_p)} D_{\sigma''(i''_1)\sigma''(i''_2)} \cdots D_{\sigma''(i''_{q-1})\sigma''(i''_q)} = \\ & = \binom{k/2}{p/2}^{-1} \sum_{I=I' \sqcup I'', |I'|=p, |I''|=q} (-1)^{(I' I'')} \text{Pfaff } G_{I'} \text{Pfaff } G_{I''} \quad \blacksquare \end{aligned}$$

Denote as Δ the standard comultiplication in $U(\mathfrak{o}_N)$.

Lemma 5.3. $\Delta \text{Pfaff } G_I = \sum_{I' \sqcup I'' = I} (-1)^{(I' I'')} \text{Pfaff } G_{I'} \otimes \text{Pfaff } G_{I''}$.

Here $(-1)^{(I' I'')}$ is a sign of a permutation of the set $I = \{i_1, \dots, i_k\}$ that places first the subset $I' \subset I$ and then places the subset $I'' \subset I$. The natural order on both subsets is preserved.

Proof. By definition one has

$$\text{Pfaff } G_I = \frac{1}{2^{\frac{k}{2}} \left(\frac{k}{2}\right)!} \sum_{\sigma \in S_k} (-1)^\sigma D_{\sigma(i_1)\sigma(i_2)} \cdots D_{\sigma(i_{k-1})\sigma(i_k)}.$$

Applying the comultiplication, one obtains

$$\begin{aligned} \Delta \text{Pfaff } G_I &= \frac{1}{2^{\frac{k}{2}} \left(\frac{k}{2}\right)!} \sum_{\sigma \in S_k} (-1)^\sigma (D_{\sigma(i_1)\sigma(i_2)} \otimes 1 + 1 \otimes D_{\sigma(i_1)\sigma(i_2)}) \cdots \\ & \cdots (D_{\sigma(i_{k-1})\sigma(i_k)} \otimes 1 + 1 \otimes D_{\sigma(i_{k-1})\sigma(i_k)}). \end{aligned}$$

The product

$$(D_{\sigma(i_1)\sigma(i_2)} \otimes 1 + 1 \otimes D_{\sigma(i_1)\sigma(i_2)}) \cdots (D_{\sigma(i_{k-1})\sigma(i_k)} \otimes 1 + 1 \otimes D_{\sigma(i_{k-1})\sigma(i_k)})$$

is equal to

$$\sum_{I=J' \sqcup J''} D_{\sigma(j'_1)\sigma(j'_2)} \cdots D_{\sigma(j'_{p-1})\sigma(j'_p)} \otimes D_{\sigma(j''_1)\sigma(j''_2)} \cdots D_{\sigma(j''_{q-1})\sigma(j''_q)}.$$

Here $J' = \{j'_1, j'_2, \dots, j'_p\}$, $J'' = \{j''_1, j''_2, \dots, j''_q\}$ are subset of I such that $I = J' \sqcup J''$ and also the following condition are satisfied. If $\sigma(i_{2t-1}) \in J'$ then $\sigma(i_{2t}) \in J'$, if $\sigma(i_{2t-1}) \in J''$ then $\sigma(i_{2t}) \in J''$. In other words, the partitions $I = J' \sqcup J''$ must induce a partition of $\frac{k}{2}$ pairs $(\sigma(i_{2t-1}), \sigma(i_{2t}))$. The summand

$$(-1)^\sigma D_{\sigma(j'_1)\sigma(j'_2)} \cdots D_{\sigma(j'_{p-1})\sigma(j'_p)} \otimes D_{\sigma(j''_1)\sigma(j''_2)} \cdots D_{\sigma(j''_{q-1})\sigma(j''_q)},$$

with $\sigma \in \text{Aut}(I)$, J', J'' can be written in the following form

$$(-1)^{(I' I'')} (-1)^{\sigma'} D_{\sigma'(i'_1)\sigma'(i'_2)} \cdots D_{\sigma'(i'_{p-1})\sigma'(i'_p)} \otimes (-1)^{\sigma''} D_{\sigma''(i''_1)\sigma''(i''_2)} \cdots D_{\sigma''(i''_{p-1})\sigma''(i''_p)}$$

with $\sigma' \in \text{Aut}(I')$, $\sigma'' \in \text{Aut}(I'')$, I', I'' . Here I' is the set $\{\sigma(j'_1), \dots, \sigma(j'_p)\}$ written in the natural order, I'' is the set $\{\sigma(j''_1), \dots, \sigma(j''_q)\}$ written in a natural

order. The permutation σ' is the permutation $\{\sigma(j'_1), \dots, \sigma(j'_p)\}$ of J' and σ'' is a permutation of J'' defined in a similar way.

But the mapping $\sigma, J', J'' \mapsto I', I'', \sigma', \sigma''$ is not injective. To get the triple σ, J', J'' with the prescribed image $I', I'', \sigma', \sigma''$, one must divide $\frac{k}{2}$ pairs $\{(i_1, i_2), \dots, (i_{k-1}, i_k)\}$ into two subsets J' and J'' with $\frac{|I'|}{2}$ and $\frac{|I''|}{2}$ elements respectively. Take a permutation σ , such that $\sigma(J') = \sigma'(I')$ (as order sets), and $\sigma(J'') = \sigma''(I'')$ (as order sets). The only freedom is the choice of two subsets J' and J'' . Thus the number of elements in the preimage equals to the number partitions of $\frac{k}{2}$ pairs into two subsets: one consists of $\frac{|I'|}{2}$ pairs and the other consists of $\frac{|I''|}{2}$ pairs. The number is $\frac{(\frac{k}{2})!}{(\frac{|I'|}{2})!(\frac{|I''|}{2})!} = \binom{k/2}{|I'|/2}$.

Note, that

$$(-1)^{|I'I''|}(-1)^{\sigma'}(-1)^{\sigma''} = (-1)^\sigma.$$

Thus $\Delta \text{Pfaff } G_I$ can be written as

$$\begin{aligned} & \frac{1}{2^{\frac{k}{2}}(\frac{k}{2})!} \sum_{I=I' \sqcup I''} \frac{(\frac{k}{2})!}{(\frac{|I'|}{2})!(\frac{|I''|}{2})!} (-1)^{|I'I''|} \left(\sum_{\sigma' \in \text{Aut}(I')} (-1)^{\sigma'} D_{\sigma'(i'_1)\sigma'(i'_2)} \dots \right. \\ & \left. \dots D_{\sigma'(i'_{p-1})\sigma'(i'_p)} \right) \otimes \left(\sum_{\sigma'' \in \text{Aut}(I'')} (-1)^{\sigma''} D_{\sigma''(i''_1)\sigma''(i''_2)} \dots D_{\sigma''(i''_{p-1})\sigma''(i''_p)} \right). \end{aligned}$$

This expression equals $\sum_{I' \sqcup I'' = I} (-1)^{|I'I''|} \text{Pfaff } G_{I'} \otimes \text{Pfaff } G_{I''}$. ■

5.2. The action of Pfaffians on tensor representations of \mathfrak{g}_2 . The aim of this section is the proof of the Theorem.

Theorem 5.4. *Pfaff G_I , $|I| = k$ acts as a zero operator on $V^{\otimes t}$ if $t < \frac{k}{2}$*

Firstly we prove the Lemma 5.5. Using it we prove the theorem in the case $k = 4$ and $t = 1$ (Proposition 5.6), then in the case of arbitrary k and $t = 1$ (Proposition 5.7) and at last in the general case (Proposition 5.8).

Lemma 5.5. $\zeta(\text{Pfaff } G_I(e_{j_1} \wedge e_{j_2})) = 0$.

Proof. For $I = \{i_1, i_2, i_3, i_4\}$ by definition one has

$$\begin{aligned} \text{Pfaff } G_I &= \frac{1}{2}(D_{i_1 i_2} D_{i_3 i_4} + D_{i_3 i_4} D_{i_1 i_2}) - \frac{1}{2}(D_{i_1 i_3} D_{i_2 i_4} + \\ &+ D_{i_2 i_4} D_{i_1 i_3}) + \frac{1}{2}(D_{i_1 i_4} D_{i_2 i_3} + D_{i_2 i_3} D_{i_1 i_4}) \end{aligned} \tag{11}$$

Using Proposition 3.3 let us calculate a commutator of this expression with $\text{Pfaff } F_J$.

$$\begin{aligned} [\text{Pfaff } G_I, \text{Pfaff } F_J] &= \frac{1}{2}(\text{pf } F_{D_{i_1 i_2}\{j_1, j_2\}} D_{i_3 i_4} + D_{i_1 i_2} \text{pf } F_{D_{i_3 i_4}\{j_1, j_2\}} + \\ &\text{pf } F_{D_{i_3 i_4}\{j_1, j_2\}} D_{i_1 i_2} + D_{i_3 i_4} \text{pf } F_{D_{i_1 i_2}\{j_1, j_2\}}) - \dots \end{aligned}$$

Let us move all $D_{i_k i_l}$ in the previous expression to the left, the result is the following.

$$[\text{Pfaff } G_I, \text{Pfaff } F_J] = D_{i_1 i_2} \text{pf } F_{D_{i_3 i_4} \{j_1, j_2\}} + D_{i_3 i_4} \text{pf } F_{D_{i_1 i_2} \{j_1, j_2\}} - D_{i_1 i_3} \text{pf } F_{D_{i_2 i_4} \{j_1, j_2\}} - D_{i_2 i_4} \text{pf } F_{D_{i_1 i_3} \{j_1, j_2\}} + D_{i_1 i_4} \text{pf } F_{D_{i_2 i_3} \{j_1, j_2\}} + D_{i_2 i_3} \text{pf } F_{D_{i_1 i_4} \{j_1, j_2\}} + \frac{1}{2} \text{pf } F_{\text{Pfaff } G_I \{j_1 j_2\}}$$

If we move all $D_{i_k i_l}$ to the right we get

$$[\text{Pfaff } G_I, \text{Pfaff } F_J] = \text{pf } F_{D_{i_3 i_4} \{j_1, j_2\}} D_{i_1 i_2} + \text{pf } F_{D_{i_1 i_2} \{j_1, j_2\}} D_{i_3 i_4} - \text{pf } F_{D_{i_2 i_4} \{j_1, j_2\}} D_{i_1 i_3} - \text{pf } F_{D_{i_1 i_3} \{j_1, j_2\}} D_{i_2 i_4} + \text{pf } F_{D_{i_2 i_3} \{j_1, j_2\}} D_{i_1 i_4} + \text{pf } F_{D_{i_1 i_4} \{j_1, j_2\}} D_{i_2 i_3} + \frac{1}{2} \text{pf } F_{\text{Pfaff } D_I \{j_1 j_2\}}.$$

Note that: $\text{Pfaff } F_{G_{i_3 i_4} \{j_1, j_2\}} = \zeta(D_{i_3 i_4} \{j_1, j_2\}) = [D_{i_3 i_4}, F_{j_1 j_2}]$.

Take a sum of two formulas for $[\text{Pfaff } F_I, \text{Pfaff } F_J]$. One obtains

$$2[\text{Pfaff } G_I, \text{Pfaff } F_J] = \text{pf } F_{D_{i_3 i_4} \{j_1, j_2\}} D_{i_1 i_2} + D_{i_1 i_2} \text{pf } F_{D_{i_3 i_4} \{j_1, j_2\}} + \text{pf } F_{D_{i_1 i_2} \{j_1, j_2\}} D_{i_3 i_4} + D_{i_3 i_4} \text{pf } F_{D_{i_1 i_2} \{j_1, j_2\}} - D_{i_1 i_3} \text{Pfaff } F_{D_{i_2 i_4} \{j_1, j_2\}} - \text{pf } F_{D_{i_2 i_4} \{j_1, j_2\}} D_{i_1 i_3} - D_{i_2 i_4} \text{pf } F_{D_{i_1 i_3} \{j_1, j_2\}} - \text{pf } F_{D_{i_1 i_3} \{j_1, j_2\}} D_{i_2 i_4} + D_{i_2 i_4} \text{pf } F_{D_{i_1 i_3} \{j_1, j_2\}} + \text{pf } F_{D_{i_2 i_3} \{j_1, j_2\}} D_{i_1 i_4} + \text{pf } F_{D_{i_1 i_4} \{j_1, j_2\}} D_{i_2 i_3} + D_{i_2 i_3} \text{pf } F_{D_{i_1 i_4} \{j_1, j_2\}} + \text{pf } F_{\text{Pfaff } D_I \{j_1 j_2\}}$$

Using (11) one sees that the sum of all summand except the last one equals to the $2[\text{Pfaff } G_I, \text{Pfaff } F_J]$.

Thus $\text{Pfaff } F_{\text{Pfaff } G_I \{j_1 j_2\}} = 0$, that is $\zeta(\text{Pfaff } G_I(e_{j_1} \wedge e_{j_2})) = 0$. ■

Proposition 5.6. *The Pfaffian $\text{Pfaff } G_I$, $|I| = 4$, acts as a zero operator on \mathfrak{g}_2 .*

Proof. Denote as V the standard representation of \mathfrak{g}_2 . Then $\Lambda^2 V$ is a sum of two irreducibles. One of them is isomorphic to the standard representation, the other one is an adjoint representation.

The standard subrepresentation in $\Lambda^2 V$ is defined by the condition

$$\xi_{k, j_1, j_2} e_{j_1} \wedge e_{j_2} = 0,$$

where ξ_{k, j_1, j_2} are structure constants of octonions. To an imaginary octonion x there corresponds $\sum_{e_{j_1} \cdot e_{j_2} = x} e_{j_1} \wedge e_{j_2}$, where \cdot is a multiplication of octonions.

From the relation $\zeta(\text{Pfaff } G_I(e_{j_1} \wedge e_{j_2})) = 0$ it follows that

$$\sum_{e_{j_1} \cdot e_{j_2} = x} \zeta(\text{Pfaff } G_I(e_{j_1} \wedge e_{j_2})) = 0.$$

From this it follows that $\sum_{e_{j_1} \cdot e_{j_2} = x} \text{Pfaff } G_I \{j_1 j_2\} = 0$. Hence $\text{Pfaff } G_I x = 0$. ■

Now we prove an analog of the previous statement in an arbitrary dimension:

Proposition 5.7. *On the base vectors e_1, \dots, e_7 of the standard representation of \mathfrak{g}_2 the Pfaffians $\text{Pfaff } F_I$ for $|I| > 2$ act as zero operators.*

Proof. By Lemma 5.2 for $q = 4, p = k - 4$ one has

$$\text{Pfaff } G_I e_j = \binom{k/2}{2}^{-1} \sum_{I' \sqcup I'' = I, |I'| = k-4} (-1)^{|I''|} \text{Pfaff } G_{I'} \text{Pfaff } G_{I''} e_j$$

If $j \notin I''$, then obviously $\text{Pfaff } G_{I''} e_j = 0$. If $j \in I''$, then using Proposition 5.6 one also obtains $\text{Pfaff } G_{I''} e_j = 0$. ■

Let us find an action of a Pfaffian of the order k on a tensor product of $< \frac{k}{2}$ vectors, that is on a tensor product $e_{r_1} \otimes e_{r_2} \otimes \dots \otimes e_{r_t}$, where $t < \frac{k}{2}$.

Proposition 5.8. $\text{Pfaff } G_I e_{r_1} \otimes \dots \otimes e_{r_t} = 0$ if $t < \frac{k}{2}$

Proof. By Lemma 5.3 the following formula holds

$$\Delta \text{Pfaff } G_I = \sum_{I' \sqcup I'' = I} (-1)^{|I''|} \text{Pfaff } G_{I'} \otimes \text{Pfaff } G_{I''}.$$

By definition one has $\text{Pfaff } G_I e_{r_1} \otimes e_{r_2} \otimes \dots \otimes e_{r_t} = (\Delta^t \text{Pfaff } F_I) e_{r_1} \otimes e_{r_2} \otimes \dots \otimes e_{r_t}$. Since $t < \frac{k}{2}$, the comultiplication $\Delta^t \text{Pfaff } F_I$ contains only summands in which on some place the Pfaffian stands whose indexing set I satisfies an equality $|I| \geq 4$ (Lemma 5.3). From Proposition 5.7 it follows that every such a summand acts as a zero operator.

Theorem 5.4 is proved. ■

6. The eigenvalues of the Capelli elements

It is proved in Theorem 5.4 that the Capelli elements for \mathfrak{g}_2 satisfy the first condition of Theorem 5.1. Note that the second condition of the definition is satisfied automatically.

In order to prove that only the Capelli elements satisfy the conditions of the Theorem 5.1 we do the following. Every central element is a polynomial in Casimir elements. We prove that there is only one polynomial of the Casimir elements that satisfy two conditions of Theorem 5.1. Thus we also find relation between Casimir and Capelli elements in the case \mathfrak{g}_2 .

Theorem 6.1. *The central elements that satisfy Definition 5.1 are unique and they are related with the Casimir elements by the formulae*

$$\begin{aligned} G_2 &= \text{const} \cdot Cas_2, \\ G_4 &= \text{const} \cdot (Cas_2^2 - 4Cas_2), \\ G_6 &= \text{const} \cdot (Cas_6 - \frac{11}{144} Cas_2^3 - \frac{149}{2304} Cas_2^2 - \frac{583}{55296} Cas_2). \end{aligned}$$

The constants in these three cases are different. They can be found by comparing highest homogeneous components of images under Harich-Chandra isomorphism of left and right sides of equalities.

Proof. First we give expressions for eigenvalues of Casimir elements of the second and the sixth orders. They are calculated using the method from [12]. Let

the highest weight of an irreducible representation be written as $m_1\omega_1 + m_2\omega_2$, where $m_1, m_2 \in \mathbb{Z}_{\geq 0}$, and $\omega_1 = \frac{1}{2\sqrt{3}}(0, 1)$, $\omega_2 = \frac{1}{2\sqrt{3}}(\frac{3}{2}, \frac{\sqrt{3}}{2})$ - are fundamental weights. The eigenvalues of the Casimir operators on an irrep with the highest weight $m_1\omega_1 + m_2\omega_2$ can be calculated by the formulae:

$$\begin{aligned}
 Cas_2 &= 1/48 m_1^2 + 1/16 m_2^2 + 1/16 m_1 m_2 + \frac{5}{48} m_1 + 3/16 m_2 \\
 Cas_6 &= \frac{5575}{5308416} m_1^2 + \frac{35}{442368} m_1^4 m_2 + \frac{35}{49152} m_2^4 + \frac{35}{98304} m_1 m_2^4 + \frac{35}{65536} m_1^3 m_2 + \\
 &\frac{1535}{1179648} m_1^2 m_2^2 + \frac{295}{196608} m_1 m_2^3 + \frac{145}{65536} m_2^3 + \frac{6925}{15925248} m_1^3 + \frac{115}{1327104} m_1^4 + \frac{55}{5308416} m_1^5 + \\
 &\frac{9}{65536} m_2^5 + \frac{55}{221184} m_1^3 m_2^2 + \frac{5}{12288} m_1^2 m_2^3 + \frac{11}{15925248} m_1^6 + \frac{1}{65536} m_2^6 + \frac{25}{393216} m_1^2 m_2^4 + \\
 &\frac{65536}{3} m_1 m_2^5 + \frac{85}{3538944} m_1^4 m_2^2 + \frac{5}{98304} m_1^3 m_2^3 + \frac{11}{1769472} m_1^5 m_2 + \frac{5705}{1769472} m_2^2 + \\
 &+ \frac{3175}{884736} m_1 m_2 + \frac{3145}{884736} m_1 m_2^2 + \frac{115}{55296} m_1^2 m_2 + \frac{2375}{2654208} m_1 + \frac{463}{294912} m_2
 \end{aligned}$$

Now express the Capelli elements through the Casimir elements using the first condition of Definition 5.1.

The Capelli and the Casimir elements of the second orders are propositional. The Capelli element of the fourth order is expressed through the Casimir elements as follows

$$G_4 = const \cdot (Cas_2^2 + \alpha Cas_2).$$

Find α such that G_4 is zero on the standard representation, that is when $(m_1, m_2) = (0, 1)$. Then one gets the formula

$$G_4 = const \cdot (Cas_2^2 - \frac{1}{4} Cas_2).$$

The Capelli element of the sixth order is expressed through the Casimir elements as follows:

$$G_6 = const \cdot (Cas_6 + \alpha Cas_2^3 + \beta Cas_2^2 + \gamma Cas_2).$$

Find coefficients such that G_6 is zero on all representation into which the tensor square of the standard representation splits. These are a trivial representation and representations with highest weights $(m_1, m_2) = (0, 1), (1, 0), (0, 2)$. Thus we have three unknowns α, β, γ and three restrictions. The solution is

$$G_6 = const \cdot (Cas_6 - \frac{11}{144} Cas_2^3 - \frac{149}{2304} Cas_2^2 - \frac{583}{55296} Cas_2). \quad \blacksquare$$

Theorem 5.1 is completely proved. \blacksquare

References

- [1] de Azcarraga, A., and A. J. Macfarlane, A. J. Mountain, and J. C.Parez Bueno, *Invariant tensors for simple groups*, Nuclear Physics B **510** (1998), 657–687.
- [2] Barton, C. H., and A. Sudbery, *Magic squares and matrix models of Lie algebras*, Adv. in Math. **180** (2003), 596–647.
- [3] Howe, R., and T. Umeda, *The Capelli identity, the double commutant theorem and multiplicity-free actions*, Math. Ann. **290** (1991), 565–619.

- [4] Itoh, M., and T. Umeda, *On Central Elements in the Universal Enveloping Algebras of the Orthogonal Lie Algebras*, *Compositio Mathematica* **127** (2001), 333–359.
- [5] Macfarlane, A. J., *Lie Algebra and Invariant Tensor Technology for \mathfrak{g}_2* , *Int. J. Mod. Phys. A* **16** (2001), 3067–3097.
- [6] Macfarlane, A. J., and H. Pfeiffer, *On characteristic equations, trace identities and Casimir operators of simple Lie algebras*, *J. Math. Phys.* **41** (2000), 3192–3225.
- [7] —, *Development of a unified tensor calculus for the exceptional Lie algebras*, *Int. J. Mod. Phys. A* **19** (2004), 287–316.
- [8] Manougue, C. , and J. Schray, *Octonionic representations of Clifford algebras and triality*, *Found. Phys.* **261** (1996), 17–70.
- [9] Molev, A. I., “Yangians and classical Lie algebras,” *Mathematical Surveys and Monographs* **143**, Amer. Math. Soc., Providence, RI, 2007.
- [10] Molev, A.I., and M. Nazarov, *Capelli identities for classical Lie algebras*, *Math. Ann.* **313** (1999), 315–357.
- [11] Onishchik, A. L., and E. B. Vinberg, “Lie groups and algebraic groups,” Springer, New York, 1990.
- [12] Perelomov, A. M., and V. S. Popov, *Casimir operators for semisimple Lie groups*, *Mathematics of the USSR-Izvestiya, Ser. Math.* **2** (1968), 1313–1335.
- [13] Postnikov, M. M., “Lectures in Geometry: Semester V, Lie Groups and Lie Algebras,” (Russian) Moscow, Nauka, 1982.
- [14] Shafer, R., “An introduction to nonassociative algebras,” Dover Publications, New York, 1995.
- [15] Zhelobenko, D. P., “Compact Lie groups and their representations,” Amer. Math. Soc., Providence, RI, 1973.
- [16] —, “Representations of reductive Lie algebras,” (Russian) Moscow, Nauka, 1994.

Dmitry Artamonov
Moscow State University
Leninskie gory 1
119421, Moscow, GSP-1, Russia
artamonov.dmitri@gmail.com

Valentina Golubeva
Moscow Aviation Institute
Volokolamskoe Shosse 4
125993, Moscow, GSP-3, A-80, Russia
golubeva@yahoo.com

Received March 21, 2012
and in final form October 23, 2012