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Upper Bound for the Heat Kernel on Higher-Rank NA Groups

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Abstract. Let S be a semi-direct product $S = N \rtimes A$ where N is a connected and simply connected, non-abelian, nilpotent meta-abelian Lie group and A is isomorphic with $\mathbb{R}^k$, $k > 1$. We consider a class of second order left-invariant differential operators $\mathcal{L}_\alpha$, $\alpha \in \mathbb{R}^k$, on S. We obtain an upper bound for the heat kernel for $\mathcal{L}_\alpha$.
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1. Introduction

Let S be a semi-direct product $S = N \rtimes A$ where N is a connected and simply connected, non-abelian, nilpotent Lie group and $A = \mathbb{R}^k$ with the ordinary additive structure (abelian group). In particular, the group operation on A is written additively. The dimension k of A is called the *rank* of S. For $g \in S$ we let $x(g) = x$ and $a(g) = a$ denote the components of g in this product so that $g = (x, a)$. Since $A = \mathbb{R}^k$, its Lie algebra $\mathfrak{a}$ is identified with A and the exponential mapping $\exp_A$ is the identity mapping.

We use the exponential map to identify N with its Lie algebra $\mathfrak{n}$. Thus N is $\mathfrak{n}$ endowed with the Campbell-Hausdorff product and $\exp_N$ is also the identity mapping. The action of A on N is then defined by

$$n^a = e^{\mathrm{ad}(a)}(n), \quad a \in A = \mathfrak{a}, \quad n \in N.$$

The multiplication in S then has the form $(x, a)(y, b) = (xy^a, a + b)$.

We assume that there is a basis $X_1, \ldots, X_{\dim \mathfrak{n}}$ of $\mathfrak{n}$ that diagonalizes the A-action, i.e., there are corresponding root functionals $\lambda_1, \ldots, \lambda_{\dim \mathfrak{n}}, \in \mathfrak{a}^*$ such that

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for every $a \in \mathfrak{a}$, $[a, X_j] = \lambda_j(a)X_j$, $j = 1, \dots, \dim \mathfrak{n}$. Thus the groups under study are solvable, non-unimodular, and hence have an exponential volume growth.

Let the usual scalar product on $\mathbb{R}^k$ be denoted by $\langle \cdot, \cdot \rangle$. We use this scalar product to identify $\mathfrak{a}$ with $\mathfrak{a}^*$, the space of linear forms on $\mathfrak{a}$. For $v \in \mathbb{R}^k$ we write $\|v\|^2 = v^2 = v \cdot v = \langle v, v \rangle = \sum_{i=1}^k v_i^2$.

In what follows we consider meta-abelian N. Specifically we assume that

$$N = M \rtimes V,$$

where M and V are abelian Lie groups with the corresponding Lie algebras $\mathfrak{m}$ and $\mathfrak{v}$. Let $\{Y_1, \dots, Y_d\}$ and $\{X_1, \dots, X_n\}$ be bases for $\mathfrak{m}$ and $\mathfrak{v}$ respectively, consisting of left-invariant vector fields, such that $\{Y_1, \dots, Y_d, X_1, \dots, X_n\}$ forms an ordered Jordan-Hölder basis for the Lie algebra $\mathfrak{n}$ of N, ordered so that the matrix of ad_X in this basis is strictly lower triangular for all $X \in \mathfrak{n}$. We use these bases to identify $M = \mathfrak{m}$ and $V = \mathfrak{v}$ with $\mathbb{R}^d$ and $\mathbb{R}^n$ respectively. For $x \in N$ we let $m(x) = m$ and $v(x) = v$ denote the components of x in this product so that $x = (m, v)$. By e we denote the neutral element of the group N. We assume also that the $\{Y_i\}$ and $\{X_j\}$ are eigenvectors for the ad_H, $H \in \mathfrak{a}$, action, i.e., there are $\xi_1, \dots, \xi_d, \vartheta_1, \dots, \vartheta_n \in \mathfrak{a}^*$ such that for every $H \in \mathfrak{a}$,

$$\mathrm{ad}_H Y_i = [H, Y_i] = \xi_i(H)Y_i, \; 1 \le i \le d,$$
$$\mathrm{ad}_H X_j = [H, X_j] = \vartheta_j(H)X_j, \; 1 \le j \le n.$$

Let $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$. We define a left-invariant differential operator on S,

$$\mathcal{L}_\alpha = \Delta_\alpha + \sum_{j=1}^d e^{2\xi_j(a)} Y_j^2 + \sum_{j=1}^n e^{2\vartheta_j(a)} X_j^2, \tag{1}$$

where

$$\Delta_\alpha = \sum_{i=1}^k (\partial_{a_i}^2 + 2\alpha_i \partial_{a_i}). \tag{2}$$

Let

$$\rho_o = \sum_{j=1}^d \xi_j + \sum_{j=1}^n \vartheta_j$$

and set

$$\chi(g) = \det(\mathrm{Ad}(g)) = e^{\rho_o(a)}, \;\; g = (x, a) \in S,$$

where

$$\mathrm{Ad}(g)s = gsg^{-1}, \;\; s \in S.$$

A left-invariant Haar measure on $S = N \rtimes A$ is then given by

$$d\mathcal{S}(y, b) = e^{-\rho_o(b)} db dy = \chi(b)^{-1} db dy, \tag{3}$$

Let

$$T_t f(x, a) = \int_N \int_A p_t(x, a; y, b) f(y, b) d\mathcal{S}(y, b) \tag{4}$$

be the semigroup of operators on $N \rtimes A$ generated by $\mathcal{L}_\alpha$. The kernel p_t is called the *heat kernel* for $\mathcal{L}_\alpha$.

The aim of this note is to give off-diagonal upper bounds for $p_t(x, a; y, b)$, particularly as (x, a) and (y, b) vary. There is a vast body of work related to producing on-diagonal and off-diagonal upper bounds for the heat kernel, both on groups and on Riemannian manifolds. See for example [3, 8, 14] and the references contained there in. Much is known about the growth in t. For example Varopoulos in [13] gives two sided on-diagonal estimates for the heat kernel p_t, $t \geq 1$, on Lie groups. However, in the context of general solvable Lie groups, very little explicit information seems to be available relating to the growth in the other variables. One exception to this statement is the recent work of Melzi [8] in the special case of the affine group of $\mathbb{R}$. Hence, in particular, A and N are assumed to be one dimensional. We remark that the assumption that A have dimension one is particularly restrictive. (The dimension of A is the rank of the corresponding manifold.) Melzi's result is t! he only one in the case of solvable Lie groups that we are aware of.

Before we state our estimate we need to introduce some notation. The neutral element of S is $(e, 0)$. Let $d^R(g_1, g_2)$ be the left-invariant Riemannian distance between two points $g_1, g_2 \in NA$. We write $\tau(g)$ instead of $d^R(g, (e, 0))$. We note that $\tau(g)$ is a sub-additive function on S. Our main result is the following estimate.

Theorem 1.1. *For every $q \geq 1$ and for every $t > 0$ there is a constant $C_{t,q} > 0$ such that for every $z \in N$ and every $a \in A$,*

$$p_t(e, 0; z, a) \leq C_{t,q} e^{-a^2/(32t)+\rho_o(a)} e^{-q\tau(z)}. \tag{5}$$

Remark 1.2. The proof of Theorem 1.1 as written is valid only for $q \geq \|\rho_o\|$. However, it is clear that if we have the above upper bound for some q_o then we also have it with all $1 \leq q \leq q_o$.

Remark 1.3. Since $\mathcal{L}_\alpha$ commutes with left translation, the same is true for T_t. Hence, from (4),

$$p_t(x, a; y, b) = p_t(e, 0; (x, a)^{-1}(y, b)).$$

Thus Theorem 1.1 immediately yields an estimate for p_t on $S \times S$.

Remark 1.4. Notice that since S is non-unimodular the heat kernel $p_t(x, a; y, b)$ is not symmetric with respect to the left-invariant Haar measure $d\mathcal{S}$. Specifically,

$$p_t(y, b; x, a) = e^{-\rho_o(a)} e^{\rho_o(b)} p_t(x, a; y, b).$$

Remark 1.5. An upper bound for the constant $C_{t,q}$ in (5), for $t > 1$, can be obtained easily by methods of this paper. However, we do not care about the precise value as the result is not optimal – what one can extract from our proof is the bound of the form $t^{-\gamma} e^{\omega t}$ for some $\omega = \omega_q > 1$ and $\gamma = \gamma_q > 0$ (see the proofs of Lemmas 4.3 and 4.4, and Remark 4.5). Hence, as a corollary we get that for every $q > 0$ there are positive constants C, γ, ω such that for $t > 1$,

$$p_t(e, 0; z, a) \leq Ct^{-\gamma} e^{\omega t} e^{-a^2/(32t)+\rho_o(a)} e^{-q\tau(z)}. \tag{6}$$

2. Homogeneous norms and the Riemannian metric

Let ξ_i, ϑ_j be the linear forms defined in §1. We set

$$A^+ = \mathrm{Int}\{a \in \mathbb{R}^k : \xi_i(a) \geq 0 \text{ for } 1 \leq i \leq d \text{ and } \vartheta_j(a) \geq 0 \text{ for } 1 \leq j \leq n\}.$$

For $t \in \mathbb{R}^+$ and $\rho \in A^+$, let

$$\delta_t^\rho = \mathrm{Ad}((\log t)\rho)|_N.$$

Then $t \mapsto \delta_t^\rho$ is a one parameter group of automorphisms (*dilations*) of N for which the corresponding eigenvalues on $\mathfrak{n}$ are all positive. It is known [2] that then N has δ_t^ρ-*homogeneous norm*: a continuous function $|\cdot|_\rho \geq 0$ on N such that $|x|_\rho = 0$ if and only if $x = e$, $|x|_\rho = |x^{-1}|_\rho$, and

$$|\delta_t^\rho x|_\rho = t|x|_\rho.$$

By $\|g\|$ we denote ℓ^2-norm of $g \in S$ considered as an element of $\mathbb{R}^{\dim \mathfrak{n}} \times \mathbb{R}^k$.

Lemma 2.1. *For every $\rho \in A^+$, there are positive constants C_1, C_2, p, q such that for every $x \in N$,*

$$C_1\|x\|^p \leq |x|_\rho \leq C_2(\|x\|^q + 1).$$

Proof. Let

$$\begin{aligned} p_i &= \rho(Y_i), \quad 1 \leq i \leq d, \\ q_i &= \rho(X_i), \quad 1 \leq i \leq n. \end{aligned}$$

For $x \in N$,

$$x = \sum_{i=1}^d m_i Y_i + \sum_{i=1}^n v_i X_i.$$

we define

$$\nu(x) = \sum_{i=1}^d |m_i|^{1/p_i} + \sum_{i=1}^n |v_i|^{1/q_i}.$$

Then

$$\delta_t^\rho x = \sum_{i=1}^d m_i e^{p_i \log t} Y_i + \sum_{i=1}^n v_i e^{q_i \log t} X_i.$$

Hence

$$\nu(\delta_t^\rho x) = t\nu(x).$$

Thus ν is a homogeneous norm on N. From homogeniety, there are positive constants C_1 and C_2 such that for all $x \in N$,

$$C_1\nu(x) \leq |x|_\rho \leq C_2\nu(x).$$

Our lemma follows. ∎

It is also known that S has a unique left-invariant Riemannian metric $d^R(\cdot,\cdot)$ for which the corresponding Riemannian metric agrees with the obvious scalar product on $\mathfrak{s}$ at the identity. Let

$$\tau(g) = d^R(g,(e,0)).$$

The following estimate is due to Guivarc'h [4]. It shows how $\tau(g) = \tau(x,a)$ behaves for each component $x \in N$ and $a \in A$. We will use this result in the proofs of Lemmas 4.3 and 4.4.

Lemma 2.2. *For every $\rho \in A^+$ there is a positive constant C such that for every $x \in N$ and $a \in A$ we have,*

$$C^{-1}(\ln(1+|x|_\rho) + \|a\|) \le \tau(x,a) + 1 \le C(\ln(1+|x|_\rho) + \|a\| + 1).$$

3. Skew-product formula

Let $\mathcal{L}_\alpha$ be defined by (1). The process σ_t in $\mathbb{R}^k$ generated by the operator Δ_α, i.e, the Brownian motion with drift 2α, is called a *vertical component* of the diffusion generated by $\mathcal{L}_\alpha$. The corresponding *horizontal component* is defined as follows.

Let $C_\infty(N)$ be the space of continuous functions f on N for which there exists the limit $\lim_{x\to\infty} f(x)$. For $\mathcal{X} \in \mathfrak{n}$, we let $\tilde{\mathcal{X}}$ denote the corresponding right-invariant vector field. For a multi-index $I = (i_1,\dots,i_m)$, $i_j \in \mathbb{Z}^+$ and a basis $\mathcal{X}_1,\dots,\mathcal{X}_m$ of the Lie algebra $\mathfrak{n}$ we write $\mathcal{X}^I = \mathcal{X}_1^{i_1}\dots\mathcal{X}_m^{i_m}$. For $k,l = 0,1,2,\dots,\infty$ we define

$$C^{(k,\ell)}(N) = \{f : \tilde{\mathcal{X}}^I\mathcal{X}^J f \in C_\infty(N) \text{ for every } |I| < k+1 \text{ and } |J| < \ell+1\}$$

and

$$\begin{aligned} \|f\|^0_{(k,\ell)} &= \sup_{|I|=k,|J|=\ell} \|\tilde{\mathcal{X}}^I\mathcal{X}^J f\|_\infty, \\ \|f\|_{(k,\ell)} &= \sup_{|I|\le k,|J|\le\ell} \|\tilde{\mathcal{X}}^I\mathcal{X}^J f\|_\infty. \end{aligned} \tag{7}$$

In particular $C^{(0,k)}(N)$ is a Banach space with the norm $\|f\|_{0,k}$.

For a continuous function $\sigma : [0,\infty) \to \mathbb{R}^k$, we consider the operator

$$L^{\sigma_t} = \sum_{j=1}^d e^{2\xi_j(\sigma_t)} Y_j^2 + \sum_{j=1}^n e^{2\vartheta_j(\sigma_t)} X_j^2.$$

Let $\{U^\sigma(s,t) : 0 \le s \le t\}$ be the (unique) family of bounded operators on $C_\infty(N)$ which satisfies

i) $U^\sigma(s,s) = \mathrm{Id}$, for all $s \ge 0$,

ii) $\lim_{h\to 0} U^\sigma(s,s+h)f = f$ in $C_\infty(N)$,

iii) $U^\sigma(s,r)U^\sigma(r,t) = U^\sigma(s,t)$, $0 \le s \le r \le t$,

iv) $\partial_s U^\sigma(s,t)f = -L^{\sigma_s}U^\sigma(s,t)f$ for every $f \in C^{(0,2)}(N)$,

v) $\partial_t U^\sigma(s,t)f = U^\sigma(s,t)L^{\sigma_t}f$ for every $f \in C^{(0,2)}(N)$,

vi) $U^\sigma(s,t) : C^{(0,2)}(N) \to C^{(0,2)}(N)$.

The operator $U^\sigma(s,t)$ is a convolution operator with a probability measure with a smooth density, i.e., $U^\sigma(s,t)f = f * P^\sigma_{t,s}$. In particular, $U^\sigma(s,t)$ is left-invariant. By iii), $P^\sigma_{t,r} * P^\sigma_{r,s} = P^\sigma_{t,s}$ for $t \geq r \geq s$. Existence of $U^\sigma(s,t)$ follows from [12].

A stochastic process (evolution) in N corresponding to transition probabilities $P^\sigma_{t,s}$ is called a *horizontal component* of the diffusion generated by $\mathcal{L}_\alpha$.

Let $U^\sigma(s,t)$ and $P^\sigma_{t,s}$ be as above. For $f \in C_c(N \times \mathbb{R}^k)$ and $t \geq 0$, we put

$$T_t f(x,a) = \mathbf{E}_a U^\sigma(0,t)f(x,\sigma_t) = \mathbf{E}_a(f *_N P^\sigma_{t,0})(x,\sigma_t), \tag{8}$$

where the expectation is taken with respect to the distribution of the process σ_t (Brownian motion with drift) in $\mathbb{R}^k$ with the generator Δ_α and starting from a, i.e., $\sigma_0 = a$. The operator $U^\sigma(0,t)$ acts on the first variable of the function f (as a convolution operator).

We have the following theorem which plays a crucial role in the proof of the upper bound for the heat kernel $p_t(\cdot,\cdot)$.

Theorem 3.1. *The family T_t defined in* (8) *is the semigroup of operators generated by $\mathcal{L}_\alpha$. That is, for $f \in C_\infty(S)$,*

$$\partial_t T_t f = \mathcal{L}_\alpha T_t f$$

and

$$\lim_{t\to 0} T_t f = f.$$

We refer to formula (8) as the *skew-product formula*. In the case of $A = \mathbb{R}$ it was proved in [1]. Higher rank case was considered in [9]. Recently a more general skew-product formula was proved in [10].

4. Proof of Theorem 1.1

4.1. Upper bound for $P^\sigma_{t,0}$. By the skew-product formula (8) we have,

$$\begin{aligned} T_t f(x,a) =& \mathbf{E}_a U^\sigma(0,t) f(x,\sigma_t), \\ =& \mathbf{E}_a \int_N f(xy^{-1},\sigma_t) P^\sigma_{t,0}(y)dy. \end{aligned} \tag{9}$$

We recall the result from [11]. In order to state this result let, for a continuous function $\sigma : [0,\infty) \to A = \mathbb{R}^k$,

$$\begin{aligned} A^\sigma_{M,i}(s,t) &= \int_s^t e^{2\xi_i(\sigma(u))}du, \quad i = 1,\dots,d, \\ A^\sigma_{V,j}(s,t) &= \int_s^t e^{2\vartheta_j(\sigma(u))}du, \quad j = 1,\dots,n, \end{aligned} \tag{10}$$

and

$$A^{\sigma}_{M,\Sigma}(s,t) = \sum_{i=1}^{d} A^{\sigma}_{M,i}(s,t), \qquad A^{\sigma}_{V,\Sigma}(s,t) = \sum_{j=1}^{n} A^{\sigma}_{V,j}(s,t),$$
$$A^{\sigma}_{M,\Pi}(s,t) = \prod_{i=1}^{d} A^{\sigma}_{M,i}(s,t), \qquad A^{\sigma}_{V,\Pi}(s,t) = \prod_{j=1}^{n} A^{\sigma}_{V,j}(s,t).$$

We also set

$$A^{\sigma}_{N,\Pi}(s,t) = A^{\sigma}_{M,\Pi}(s,t)A^{\sigma}_{V,\Pi}(s,t),$$
$$A^{\sigma}_{N,\Sigma}(s,t) = A^{\sigma}_{M,\Sigma}(s,t) + A^{\sigma}_{V,\Sigma}(s,t).$$

Finally, for $k \in \mathbb{N}$, we let

$$\phi_k(m) = \left(\frac{\|m\|^{1/k}}{\|m\|^{1/k}+1}\right)^k, \quad m \in M.$$

We also let k_o be the smallest non-negative integer such that

$$(\mathrm{ad}_X)^{k_o+1}\big|_{\mathfrak{m}} = 0, \, \forall\, X \in \mathfrak{v}.$$

Note that if $k_o = 0$, then $\mathfrak{v}$ centralizes $\mathfrak{m}$; hence N is abelian. Thus our hypotheses imply that $k_o > 0$.

Theorem 4.1 ([11, Theorem 1.5]). *There are positive constants C, D and $k_o \in \mathbb{N}$ such that for all $(m,v) \in N$,*

$$P^{\sigma}_{t,0}(m,v) \le C A^{\sigma}_{N,\Pi}(0,t)^{-1/2}(\|m\|^{1/(2k_o)} + 1 + A^{\sigma}_{V,\Sigma}(0,t)^{1/2}) \times \exp\left(-D\frac{\|v\|^2}{A^{\sigma}_{V,\Sigma}(0,t)} - D\frac{\|m\|^{1/k_0}}{A^{\sigma}_{N,\Sigma}(0,t)}\phi_{2k_o}(m)\right). \quad (11)$$

In case when N is the $2n+1$-dimensional Heisenberg group the above estimate is proved in [10].

To simplify notation we write

$$m_{k_o} = \|m\|^{\frac{1}{2k_o}} + 1,$$

and

$$\mathcal{E}^{\sigma}_1(v) = \exp\left(-\frac{D\|v\|^2}{A^{\sigma}_{V,\Sigma}(0,t)}\right),$$
$$\mathcal{E}^{\sigma}_2(m) = \exp\left(-D\frac{\|m\|^{\frac{1}{k_o}}}{A^{\sigma}_{N,\Sigma}(0,t)}\phi_{2k_o}(m)\right).$$

4.2. Apriori L^2-weighted estimates. For $t \ge 0$ and $y = (m,v) \in N$, define

$$\mathcal{C}(t,y) = \mathcal{C}(t,m,v) = m_{k_o}^2 \mathbf{E}_0 A^{\sigma}_{N,\Pi}(0,t)^{-1}\mathcal{E}^{\sigma}_1(v)^2\mathcal{E}^{\sigma}_2(m)^2 + \mathbf{E}_0 A^{\sigma}_{N,\Pi}(0,t)^{-1}A^{\sigma}_{V,\Sigma}(0,t)\mathcal{E}^{\sigma}_1(v)^2\mathcal{E}^{\sigma}_2(m)^2. \quad (12)$$

The following result will be applied in §4.5 to $\xi(y) = e^{q\tau(y)}$ with appropriate $q \in \mathbb{R}$.

Lemma 4.2. *There exists a constant $C > 0$, such that for any measurable function $\xi : N \to \mathbb{R}$,*

$$\int_N \int_{\mathbb{R}^k} \left(p_t(e,0;y,b)e^{b^2/(8t)-\rho_o(b)/2}\xi(y)\right)^2 e^{-\rho_o(b)}dbdy \leq Ct^{-k/2}\int_N \mathcal{C}(t,y)\xi(y)^2 dy. \quad (13)$$

Proof. By (4) and (9) for $f = \delta_y \otimes \phi$, where $y = (m,v) \in N$ and $\phi \in L^2(\mathbb{R}^k, e^{-b^2/4t}db)$ we have,

$$|T_t f(e,0)| = |\int_{\mathbb{R}^k} p_t(e,0;y,b)\phi(b)\chi(b)^{-1}db| = |\mathbf{E}_0 P^\sigma_{t,0}(y^{-1})\phi(\sigma_t)| \leq \mathbf{E}_0 P^\sigma_{t,0}(y^{-1})|\phi(\sigma_t)|.$$

Since the right hand side of (11) in Theorem 4.1 is symmetric, i.e., has the same value for $y = (m,v)$ and $y^{-1} = (-m,-v)$, we bound $P^\sigma_{t,0}(y^{-1})$ using (11), and get

$$|\int_{\mathbb{R}^k} p_t(e,0;y,b)\phi(b)\chi(b)^{-1}db| \leq Cm_{k_o}\mathbf{E}_0 A^\sigma_{N,\Pi}(0,t)^{-\frac{1}{2}}\mathcal{E}^\sigma_1(v)\mathcal{E}^\sigma_2(m)|\phi(\sigma_t)| + C\mathbf{E}_0 A^\sigma_{N,\Pi}(0,t)^{-\frac{1}{2}}A^\sigma_{V,\Sigma}(0,t)^{1/2}\mathcal{E}^\sigma_1(v)\mathcal{E}^\sigma_2(m)|\phi(\sigma_t)|.$$

Using the Cauchy-Schwarz inequality to the expectations on the right we bound the above integral by

$$Cm_{k_o}(\mathbf{E}_0 A^\sigma_{N,\Pi}(0,t)^{-1}\mathcal{E}^\sigma_1(v)^2\mathcal{E}^\sigma_2(m)^2)^{1/2}(\mathbf{E}_0\phi^2(\sigma_t))^{1/2} + C(\mathbf{E}_0 A^\sigma_{N,\Pi}(0,t)^{-1}A^\sigma_{V,\Sigma}(0,t)\mathcal{E}^\sigma_1(v)^2\mathcal{E}^\sigma_2(m)^2)^{1/2}(\mathbf{E}_0\phi^2(\sigma_t))^{1/2}.$$

Thus,

$$\left(\int_{\mathbb{R}^k} p_t(e,0;y,b)\phi(b)\chi(b)^{-1}db\right)^2 \leq C\mathcal{C}(t,m,v)\mathbf{E}_0\phi^2(\sigma_t).$$

Therefore, for all $\phi \in L^2(\mathbb{R}^k)$, and $y = (m,v) \in N$,

$$|\int_{\mathbb{R}^k} p_t(e,0;y,b)\phi(b)\chi(b)^{-1}db| \leq C\mathcal{C}(t,m,v)^{1/2}(\mathbf{E}_0\phi^2(\sigma_t))^{1/2}.$$

Since

$$\mathbf{E}_0\phi^2(\sigma_t) = Ct^{-k/2}\int_{\mathbb{R}^k}\phi^2(b)e^{-b^2/4t}db,$$

we get that for all $\phi \in L^2(\mathbb{R}^k, e^{-b^2/4t}db)$,

$$|\int_{\mathbb{R}^k} p_t(e,0;y,b)\phi(b)\chi(b)^{-1}db| \leq Ct^{-k/4}\mathcal{C}(t,m,v)^{1/2}\left(\int_{\mathbb{R}^k}\phi^2(b)e^{-b^2/4t}db\right)^{1/2}.$$

It follows that

$$\phi \mapsto \int_{\mathbb{R}^k} p_t(e,0;y,b)\phi(b)e^{b^2/8t}\chi(b)^{-1}db$$

defines a continuous linear functional on $L^2(\mathbb{R}^k, db)$ with norm bounded by

$$Ct^{-k/4}\mathcal{C}(t,y)^{1/2}.$$

Hence,

$$\int_{\mathbb{R}^k} \left(p_t(e,0;y,b)e^{b^2/8t}\right)^2 \chi(b)^{-2}db \le Ct^{-k/2}\mathcal{C}(t,y),$$

and consequently, (13) follows. ∎

4.3. A geometric ingredient.

Lemma 4.3. *Let $\gamma \in \mathbb{R}$ be given. There exist positive constants C, D such that for every $t > 0$ and $2q \ge |\gamma|\,\|\rho_o\|$ and every $b \in A$ and $y, z \in N$,*

$$e^{-q\tau(y)}e^{-q\tau(b^{-1}y^{-1}zb)}e^{-\frac{b^2}{16t}+\gamma\rho_o(b)} \le e^{64C^2q^2t+2qD}e^{-q\tau(z)}.$$

Proof. For simplicity, for $g \in S$, we write

$$|g| = e^{\tau(g)}.$$

Since $\tau(gh) \le \tau(g)+\tau(h)$ it follows that $|\cdot|$ is sub-multiplicative, i.e., $|gh| \le |g|\,|h|$ for $g, h \in S$.

By Lemma 2.2 there are positive constants C, D such that, for $b \in A$,

$$\tau(b) \le C\|b\| + D.$$

Obviously, the above inequality holds also for $b^{-1} = -b$. Thus,

$$|b|^q|b^{-1}|^q e^{\gamma\rho_o(b)} \le e^{2q(C\|b\|+D)\|+|\gamma|\,\|\rho_o\|\,\|b\|} = e^{(2qC+|\gamma|\|\rho_o\|)\|b\|+2qD}.$$

Consequently, with $2q \ge |\gamma|\,\|\rho_o\|$ and $u = (16t)^{-1/2}b$ we have

$$\begin{aligned}|b^{-1}|^q\,|b|^q e^{-\frac{b^2}{16t}+\gamma\rho_o(b)} \le & e^{(2qC+|\gamma|\|\rho_o\|)\|b\|-\frac{\|b\|^2}{16t}}e^{2qD}\\ \le & e^{4qC\|b\|-\frac{\|b\|^2}{16t}}e^{2qD}\\ = & e^{4qC(16t)^{1/2}\|u\|-\|u\|^2}e^{2qD}\\ = & e^{-(\|u\|-2qC(16t)^{1/2})^2+64C^2q^2t}e^{2qD}\\ \le & e^{64C^2q^2t+2qD}.\end{aligned} \tag{14}$$

Now, using sub-multiplicavity of $|\cdot|$, we get for $b \in A$,

$$|y^{-1}z| = |bb^{-1}y^{-1}zbb^{-1}| \le |b|\,|b^{-1}y^{-1}zb|\,|b^{-1}|.$$

In particular,

$$|b^{-1}y^{-1}zb|^{-q} \le |y^{-1}z|^{-q}\,|b^{-1}|^q|b|^q. \tag{15}$$

Of course, $|z| = |y(y^{-1}z)| \le |y||y^{-1}z|$, and so

$$|y|^{-q} \le |z|^{-q}|y^{-1}z|^q. \tag{16}$$

Hence, by (15) and (16),

$$|y|^{-q}|b^{-1}y^{-1}zb|^{-q}e^{-\frac{b^2}{16t}+\gamma\rho_o(b)} \le |b^{-1}|^q|b|^q e^{-\frac{b^2}{16t}+\gamma\rho_o(b)}|z|^{-q},$$

and so the lemma follows from (14). ∎

4.4. Moments estimates of $\mathcal{C}(t,y)$. Let $\mathcal{C}(t,y)$ be as defined in (12). We have the following result.

Lemma 4.4. *For every $q \geq 1$ and for every $t > 0$ there is a constant $C = C_{t,q}$ such that*

$$\int_{\mathbb{R}^{\dim \mathfrak{n}}} \mathcal{C}(t,y) e^{2q\tau(y)} dy \leq C_{t,q} < +\infty.$$

Proof. The function $\mathcal{C}(t,y)$ consists of three terms. Thus, we need to estimate three integrals. By the sub-multiplicativity of the Riemannian distance for $y = mv$, $e^{\tau(y)} \leq e^{\tau(m)} e^{\tau(v)}$. Thus, changing the order of integration, the first integral is

$$\begin{aligned}\mathbf{E}_0 A^{\sigma}_{N,\Pi}(0,t)^{-1} &\int_{\mathbb{R}^{\dim \mathfrak{n}}} \mathcal{E}^{\sigma}_1(v)^2 \mathcal{E}^{\sigma}_2(m)^2 e^{2q\tau(y)} m^2_{k_o} dy \\ &\leq \mathbf{E}_0 A^{\sigma}_{N,\Pi}(0,t)^{-1} \int_{\mathbb{R}^d} \mathcal{E}^{\sigma}_2(m)^2 e^{2q\tau(m)} m^2_{k_o} dm \int_{\mathbb{R}^n} \mathcal{E}^{\sigma}_1(v)^2 e^{2q\tau(v)} dv. \qquad (17)\end{aligned}$$

By Lemmas 2.2 and 2.1 it follows that for every $\rho \in A^+$ there are positive constants C, r_o, C_1, D_1 such that for all $y \in N$, with $\|y\| \geq 1$,

$$e^{\tau(y)} \leq C_1 |y|^{D_1}_{\rho} \leq C \|y\|^{r_o}. \qquad (18)$$

By (18), for every $q \geq 1$ and $y \in N$ we have $e^{q\tau(y)} \leq C \max(1, \|y\|^{r_o q})$. Let $r = r_o q$. We estimate the above integrals as follows

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{E}^{\sigma}_1(v)^2 e^{2q\tau(v)} dv &= \int_{\mathbb{R}^n} \exp\left(-\frac{2D\|v\|^2}{A^{\sigma}_{V,\Sigma}(0,t)}\right) e^{2q\tau(v)} dv \\ &\leq C \int_{\|v\|\leq 1} \exp\left(-\frac{D\|v\|^2}{A^{\sigma}_{V,\Sigma}(0,t)}\right) dv + C \int_{\|v\|>1} \exp\left(-\frac{D\|v\|^2}{A^{\sigma}_{V,\Sigma}(0,t)}\right) \|v\|^{2r} dv \\ &\leq C A^{\sigma}_{V,\Sigma}(0,t)^{n/2} + C_r A^{\sigma}_{V,\Sigma}(0,t)^{(n+2r)/2}.\end{aligned}$$

Similarly we have

$$\begin{aligned}\int_{\mathbb{R}^d} &\mathcal{E}^{\sigma}_2(m)^2 e^{2q\tau(m)} m^2_{k_o} dm \\ &\leq \int_{\mathbb{R}^d} \mathcal{E}^{\sigma}_2(m)^2 e^{2q\tau(m)} \|m\|^{1/k_o} dm + \int_{\mathbb{R}^d} \mathcal{E}^{\sigma}_2(m)^2 e^{2q\tau(m)} dm\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{E}^{\sigma}_2(m)^2 e^{2q\tau(m)} dm &= \int_{\mathbb{R}^d} \exp\left(-D\frac{\|m\|^{\frac{1}{k_o}}}{A^{\sigma}_{N,\Sigma}(0,t)} \phi_{2k_o}(m)\right) e^{2q\tau(m)} dm \\ &\leq C \int_{\|m\|\leq 1} \exp\left(-\frac{D\|m\|^{\frac{1}{k_o}+1}}{2^{2k_o} A^{\sigma}_{N,\Sigma}(0,t)}\right) dm \\ &\quad + C \int_{\|m\|>1} \exp\left(-\frac{D\|m\|^{\frac{1}{k_o}}}{2^{2k_o} A^{\sigma}_{N,\Sigma}(0,t)}\right) \|m\|^{2r} dm \\ &\leq C A^{\sigma}_{N,\Sigma}(0,t)^{dk_o/(k_o+1)} + C_r A^{\sigma}_{N,\Sigma}(0,t)^{(d+2r)k_o}.\end{aligned}$$

Exactly in the same way

$$\int_{\mathbb{R}^d} \mathcal{E}_2^\sigma(m)^2 e^{2q\tau(m)} \|m\|^{1/k_o} dm$$
$$\leq C A_{N,\Sigma}^\sigma(0,t)^{(dk_o/(k_o+2))+(1/(k_o+2))} + C_r A_{N,\Sigma}^\sigma(0,t)^{(d+2r+1/(k_o))k_o/2}.$$

Thus the expectation in (17) is bounded by a constant C_r depending on r (and therefore on q) times a sum of expectations of the form

$$\mathbf{E}_0 A_{N,\Pi}^\sigma(0,t)^{-1} A_{V,\Sigma}^\sigma(0,t)^\gamma A_{N,\Sigma}^\sigma(0,t)^\delta$$

with $\gamma, \delta > 0$. The above expectations are finite. This follows from the fact that the functionals $A_{M,i}^\sigma$ and $A_{V,j}^\sigma$ have finite moments (positive and negative) which follows from the fact that the exponential functionals $A_t^{(\mu)}$ defined in the Appendix in (19) have finite moments (see e.g. [6]).

The remaining integrals can be estimated in the same way. ■

Remark 4.5. Using asymptotic given in Theorem A.1 of the Appendix A one can bound, for $t > 1$, the constant $C_{t,q}$ in Lemma 4.4 and so get the estimate (6).

4.5. End of the proof of Theorem 1.1.

Proof. We now make use of the fact that $p_t = p_{t/2} * p_{t/2}$ to get an estimate for $p_t(e, 0; z, a)$. Specifically, let

$$\begin{aligned}
\psi(y,b) &= p_{t/2}(e,0;y,b),\\
\tilde{\psi}(y,b) &= p_{t/2}(e,0;y,b) e^{b^2/8t - \rho_o(b)/2} e^{q\tau(y)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
p_t(e,0;z,a) &= \int_S p_{t/2}(e,0;y,b) p_{t/2}(e,0;(y,b)^{-1}(z,a)) d\mathcal{S}(y,b)\\
&= \psi * \psi(z,a)\\
&= \int_S \psi(y,b)\psi((y,b)^{-1}(z,a)) d\mathcal{S}(y,b)\\
&= \int_S \psi(y,b)\psi(b^{-1}y^{-1}zb, a-b)) d\mathcal{S}(y,b)\\
&= \int_S \tilde{\psi}(y,b)\tilde{\psi}(b^{-1}y^{-1}zb, a-b)\\
&\quad \times e^{-\frac{b^2+(a-b)^2}{8t} + \frac{\rho_o(a)}{2}} e^{-q\tau(y)} e^{-q\tau(b^{-1}y^{-1}zb)} d\mathcal{S}(y,b),
\end{aligned}$$

where $d\mathcal{S}(y,b)$ is the left-invariant Haar measure on S (see (3)). Since

$$b^2 + (a-b)^2 \geq (\|b\| + \|a-b\|)^2/2 \geq a^2/2,$$

we get from Lemma 4.3 with $\gamma = -1/2$,

$$\begin{aligned}
&p_t(e,0;z,a)\\
&\quad\le e^{-\frac{a^2}{32t}+\frac{\rho_o(a)}{2}}\int_S \tilde\psi(y,b)\tilde\psi(b^{-1}y^{-1}zb,a-b)e^{-\frac{b^2}{16t}}e^{-q\tau(y)}e^{-q\tau(b^{-1}y^{-1}zb)}d\mathcal{S}(y,b)\\
&\quad\le c_{t,q}e^{-a^2/(32t)+\rho_o(a)/2}e^{-q\tau(z)}\int_S \tilde\psi(y,b)\tilde\psi(b^{-1}y^{-1}zb,a-b)e^{\rho_o(b)/2}d\mathcal{S}(y,b)\\
&\quad= c_{t,q}e^{-a^2/(32t)+\rho_o(a)/2}e^{-q\tau(z)}\int_S \tilde\psi(y,b)\tilde\psi((y,b)^{-1}(z,a))e^{-\rho_o(-b)/2}d\mathcal{S}(y,b),
\end{aligned}$$

where $c_{t,q} = e^{64C^2q^2t+2qD}$.

For any function f on S,

$$\int_S f(s^{-1})\,d\mathcal{S}(s) = \int_S f(s)\chi(s)d\mathcal{S}(s),$$

where $\chi(s) = e^{\rho_o(s)}$ is the modular function for $d\mathcal{S}(s)$. Thus

$$\begin{aligned}
\int_S \tilde\psi((y,b)^{-1}(z,a))^2e^{-\rho_o(-b)}d\mathcal{S}(y,b) &= \int_S \tilde\psi((y,b)(z,a))^2d\mathcal{S}(y,b)\\
&= e^{\rho_o(a)}\|\tilde\psi\|^2_{L^2(d\mathcal{S})}.
\end{aligned}$$

Hence from the Cauchy-Schwartz inequality

$$p_t(e,0;z,a) = \psi*\psi(z,a) \le c_{t,q}e^{-a^2/(32t)+\rho_o(a)}e^{-q\tau(z)}\|\tilde\psi\|^2_{L^2(d\mathcal{S})}.$$

By Lemma 4.2 applied to $\xi(y) = e^{q\tau(y)}$ and Lemma 4.4 with $q \ge \|\rho_o\|$,

$$\|\tilde\psi\|^2_{L^2(d\mathcal{S})} \le C_{t,q}.$$

Our theorem follows. ∎

A. Exponential functionals of Brownian motion

Let $b_s,\ s\ge 0$, be the Brownian motion on $\mathbb{R}$ normalized so that $\operatorname{Var} b_s = 2s$. For $\mu\in\mathbb{R}$ and $t>0$, define the following exponential functional

$$A_t^{(\mu)} = \int_0^t e^{2(b_s+\mu s)}ds. \tag{19}$$

Exponential functionals $A_t^{(\mu)}$ have been thoroughly studied and their properties are very well known. The reason for such an interest is that these functionals play an important role in financial mathematics (see e.g. [15, 6, 7] and the references therein).

The following theorem is a simplified version of the result proved in [5] (all the constants are explicit in [5]; notice also that we use a different normalization of Brownian motion than in [5]). Let, for $(\mu,m)\in\mathbb{R}^2$,

$$\Delta_t^{(\mu,m)} = \mathbf{E}_0[(A_t^{(\mu)})^{-m}].$$

Theorem A.1 ([5, Theorem 2.2]). *The following limits exists:*

$$\begin{aligned}
\lim_{t\to\infty} t^{3/2}e^{\mu^2 t}\Delta_t^{(\mu,m)} =& C_1 \ \textit{if}\ \ 2m>\mu,\ \mu>0,\\
\lim_{t\to\infty} t^{1/2}e^{\mu^2 t}\Delta_t^{(\mu,m)} =& C_2 \ \textit{if}\ \ 2m=\mu,\ \mu>0,\\
\lim_{t\to\infty} e^{4m(\mu-m)t}\Delta_t^{(\mu,m)} =& C_3 \ \textit{if}\ \ m<\mu,\ 2m<\mu,\\
\lim_{t\to\infty} t^{-1}\Delta_t^{(\mu,m)} =& C_4 \ \textit{if}\ \ m=\mu,\ \mu<0,\\
\lim_{t\to\infty} \Delta_t^{(\mu,m)} =& C_5 \ \textit{if}\ \ m>\mu,\ \mu<0,\\
\lim_{t\to\infty} t^{1/2}\Delta_t^{(\mu,m)} =& C_6 \ \textit{if}\ \ m>0,\ \mu=0.
\end{aligned}$$

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References

[1] Damek, E., A. Hulanicki and R. Urban, *Martin boundary for homogeneous Riemannian manifolds of negative curvature at the bottom of the spectrum*, Rev. Mat. Iberoamericana **17** (2001), 257–293.

[2] Folland, G. B., and E. Stein, “Hardy spaces on homogeneous groups,” Princeton Univ. Press, 1982.

[3] Grigor’yan, A., *Gaussian upper bounds for the heat kernel on arbitrary manifolds*, J. Differential Geom. **45** (1997), 33–52.

[4] Guivarc’h, Y., *Sur la loi des grands nombres et la rayon spectral d’une marche aléatoire*, in: Journées sur les marches aléatoires, Astérisque **74** (1980), 47–98.

[5] Hiraya, Y., and M. Yor, *Limiting distributions associated with moments of exponential Brownian functionals*, Studia Sci. Math. Hungar. **41** (2004), 193–242.

[6] Matsumoto, H., and M. Yor, *Exponential functionals of Brownian motion, I: Probability laws at fixed time*, Probab. Surv. **2** (2005), 312–347.

[7] —, *Exponential functionals of Brownian motion, II: some related diffusion processes*, Probab. Surv. **2** (2005), 348–384.

[8] Melzi, C., *Large time estimates for non-symmetric heat kernel on the affine group*, Ann. Math. Blaise Pascal **9** (2002), 63–78.

[9] Penney, R., and R. Urban, *Estimates for the Poisson kernel on higher rank NA groups*, Colloq. Math. **118** (2010), 259–281.

[10] —, *Estimates for the Poisson kernel and the evolution kernel on the Heisenberg group*, J. Evol. Equ. **12** (2012), 327–351.

[11] —, *Estimates for the Poisson kernel and the evolution kernel on nilpotent meta-abelian groups*, arXiv:math.FA/1108.2515, (2011).

[12] Tanabe, H., "Equations of evolutions," translated from the Japanese by N. Mugibayashi and H. Haneda. Monographs and Studies in Mathematics **6**, Pitman, London–San Francisco–Melbourne, 1979.

[13] Varopoulos, N. Th., *Analysis on Lie groups*, Rev. Mat. Iberoamericana **12** (1996), 791–917.

[14] Varopoulos, N. Th., L. Saloff-Coste, and T. Coulhon, "Analysis and geometry on groups," Cambridge Tracts in Math. **100**, Cambridge Univ. Press, 1992.

[15] Yor, M., "Exponential functionals of Brownian motion and related processes," with an introductory chapter by Hélyette Geman. Chapters 1, 3, 4, 8 translated from the French by Stephen S. Wilson, Springer Finance. Springer-Verlag, Berlin, 2001.

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