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Infinite Loop Spaces Associated to Affine Kac-Moody Groups

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Abstract. The main purpose of this paper is to construct infinite loop spaces from affine Kac-Moody groups, It is well known that to each infinite class of classical groups over a commutative ring R, we can associate an infinite loop space $G(R)$ by Quillen's plus construction. In this paper we generalize this fact to the cases of affine Kac-Moody groups. Roughly speaking, for each commutative ring R there are seven infinite classes of affine Kac-Moody groups over R, and to each infinite class we can associate an analogous infinite loop space.
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1. Introduction

The loop space ΩX of a pointed space X is the space of pointed maps from the unit circle S^1 to X together with the compact-open topology. We say that a pointed space X is an *infinite loop space* if there is a sequence of (pointed) spaces $X_0, X_1, \cdots$ with $X_0 = X$ and weak homotopy equivalences $X_n \simeq \Omega X_{n+1}$ for each $n \geq 0$.

Example 1.1. Let $GL(n)$ be the general linear group over $\mathbb{C}$ and let BGL be the limit of classifying space $\varinjlim BGL(n)$. By the Bott peoriodicity theorem [1, 2] we have a weak homotopy equivalence

$$\mathbb{Z} \times BGL \simeq \Omega^2(\mathbb{Z} \times BGL);$$

thus BGL is an infinite loop space. Similar results hold for BO and BSp, where O (resp. Sp) is the infinite orthogonal (resp. symplectic) group over $\mathbb{C}$.

Now we introduce a theorem about construction of infinite loop spaces. First, we need some preliminaries.

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Let Σ_n be the symmetric group on the set $\{1, 2, \cdots, n\}$. For any $\sigma \in \Sigma_m$ and $\tau \in \Sigma_n$, $c(m, n) \in \Sigma_{m+n}$ is defined by

$$c(m,n)(i) = \begin{cases} n+i, & 1 \le i \le m, \quad (1a) \\ i-m, & m < i \le m+n. \quad (1b) \end{cases}$$

The definition above implies $c(m, n) = c(n, m)^{-1}$.

Theorem 1.2. *Given a sequence of topological groups*

$$G(0), G(1), G(2), \cdots, G(n), \cdots$$

together with homomorphisms $\phi_m : \Sigma_m \to G(m)$, $f_m : G(m) \to G(m+1)$, $m > 0$, *satisfying,*
0) ϕ_0 *is an isomorphism;*
1) $f_m \phi_m(\alpha) = \phi_{m+1}(\alpha)$ *for each* $m > 0$;
2)set $f_{m,n} := f_{m+n-1} \cdots f_{m+1} f_m$, *then* $\phi_n(c(n, m))(f_{n,m}(G(n)))\phi_n(c(m, n))$ *and* $f_{m,n}(G(m))$ *are commutative in* $G(m + n)$;
3)let $G = \lim_{n\to\infty} G(n)$ *and let* $\pi' = [\pi, \pi]$ *be the commutator subgroup of* $\pi = \pi_0(G)$, *we have* $\pi' = [\pi', \pi']$.
Then BG^+ *(where* $+$ *means the Quillen's plus construction for* BG *and* $\pi' \subseteq \pi_1(BG)$ *) is an infinite loop space.*

Proof. Define a topological category ξ as follows. The objects of ξ are nonnegative integers, $hom_\xi(m, n)$ is empty if $m \neq n$ and $hom_\xi(m, m) = G(m)$. One checks that $(\xi, \oplus, 0, c)$ has a structure of permutative category, $\coprod_{n\ge 0} BG(n)$ is the corresponding classifying space. Then the rest of the proof carries over as in [3]p.62. ∎

Remark 1.3. This theorem must be well known, but we can not find suitable reference.

Corollary 1.4. *Let* R *be a commutative ring and set*

$$GL(\infty, R) = \lim_{n\to\infty} GL(n, R),$$

then $BGL(\infty, R)^+$ *is an infinite loop space.*

Proof. We can easily find natural homomorphisms $\phi_n : \Sigma_n \to GL(n, R)$, $n > 0$ that satisfy conditions of Theorem 1.2. ∎

Similarly, we can show that $BSL(\infty, R)^+$, $BO(\infty, R)^+$, $BSO(\infty, R)^+$ and $BSp(\infty, R)^+$ are all infinite loop spaces. The main purpose of this paper is to construct infinite loop spaces from affine Kac-Moody groups, which are infinite dimensional generalization of algebraic groups. Roughly speaking, for each commutative ring R there are seven infinite classes of affine Kac-Moody groups over R, and to each infinite class we can associate an analogous infinite loop space.

This paper is structured as follows. Section 2 is a short review of Kac-Moody algebras and Kac-Moody groups. In Section 3 we construct the infinite loop spaces corresponding to affine Kac-Moody groups of type $A_{2l-1}^{(2)}$. In the final section we consider several variations and the other cases. Throughout this paper R will be an arbitrary commutative ring (not necessarily with unit).

2. Kac-Moody Algebras and Kac-Moody Groups

In this section, we give a brief review of the theory of Kac-Moody algebras and Kac-Moody groups, details can be found in [4, 8, 9].

Definition 2.1. A generalized Cartan matrix is a matrix $A = (a_{i,j})_{i,j=1}^n$ satisfying, $a_{i,i} = 2$, $a_{i,j}$ are non-positive integers for $i \neq j$, and $a_{i,j} \neq 0$ implies $a_{j,i} \neq 0$.

Definition 2.2. The Kac-Moody algebra $\mathfrak{g}(A)$ associated to a generalized Cartan matrix $A = (a_{ij})_{i,j=1}^n$ is the Lie algebra (over $\mathbb{C}$) generated by $3n$ elements e_i, f_i, h_i, $(i = 1, \cdots, n)$ with the following defining relations:

$$[h_i, h_j] = 0;\ [h_i, e_j] = a_{ij}e_j;\ [h_i, f_j] = -a_{ij}f_j; [e_i, f_j] = \delta_{i,j}h_i;$$

$$(ad\ e_i)^{1-a_{ij}}e_j = 0,\ (ad\ f_i)^{1-a_{ij}}f_j = 0,\ if\ i \neq j.$$

Let $A = (a_{i,j})_1^n$ be a generalized Cartan matrix. For a pair of indices i, j such that ij, set $m_{i,j} = 2, 3, 4$ *or* 6 if $a_{i,j}a_{j,i} = 0, 1, 2$ *or* 3 respectively and set $m_{i,j} = 0$ otherwise; set $m_{i,i} = 1$. We associate to A a discrete group $W(A)$ (the *Weyl group*) on n generators $s_1, \cdots, s_n$ with relations $\{(s_is_j)^{m_{i,j}} = 1\}_{0<i,j\leq n}$.

We also need another group $W'(A)$ which is defined by n generators $s'_1, \cdots, s'_n$ and the following relations:

$$s'_j s'^2_i s'^{-1}_j = s'^2_i s'^{-2a_{i,j}}_j$$

$$s'_i s'_j s'_i \cdots = s'_j s'_i s'_j \cdots (m_{i,j}\ facors\ on\ each\ side).$$

By the definitions above the map $s'_i \to s_i$ extends to a group homomorphism $\phi : W'(A) \to W(A)$. As $ad\ e_i$ and $ad\ f_i$ are locally nilpotent endomorphisms of $\mathfrak{g}(A)$ (cf.[4, p.33]), the expressions $exp(e_i) = \sum_{n\geq 0} \frac{(ad\ e_i)^n}{n!}$ and $exp(f_i) = \sum_{n\geq 0} \frac{(ad\ f_i)^n}{n!}$ make sense. The map $s'_i \to exp(e_i)exp(-f_i)exp(e_i) \in Aut(\mathfrak{g}(A))$ can be extended to a homomorphism $\psi : W'(A) \to Aut(\mathfrak{g}(A))$ (cf.[5, 188]); we also denote by s'_i the image of s'_i in $Aut(\mathfrak{g}(A))$.

Let V be the vector space over $\mathbb{Q}$, with basis $\{a_i\}_{i=1,\cdots,n}$ and let $W(A)$ act on V by $s_i(a_j) = a_j - a_{i,j}a_i$. *Real roots* of $A = (a_{i,j})_1^n$ are defined to be elements of V of the form $w(a_i)$, with $w \in W(A)$ and $0 < i \leq n$. Each real root a is an integral linear combination of $\{a_i\}$, the coefficients of which of all positive or negative; the real root a is said to be *positive* or *negative* accordingly. Denote by $\triangle$, $\triangle_+$, $\triangle_-$ the sets of all real roots, positive and negative real roots respectively. We say that a set of real roots θ is *pre-nilpotent* if there exist $w, w' \in W(A)$ such that all elements of $w(\theta)$ are positive and all elements of $w'(\theta)$ are negative;

if, moreover, $a, b \in \theta$ and $a + b \in \triangle$ imply $a + b \in \theta$, then we said that θ is *nilpotent*.

For $0 < i \leq n$ and $w' \in W'(A)$, the pair of opposite elements $w'\{e_i, -e_i\} \subset \mathfrak{g}(A)$ depends only on the real root $a = \phi(w')(a_i)$ (cf.[9, p.547]); set $E_a = w'\{e_i, -e_i\}$ and denote by L_a the $\mathbb{C}$-subalgebra of $\mathfrak{g}(A)$ generated by E_a.

For each real root a, we denote by $\mathfrak{U}_a$ the group scheme over $\mathbb{Z}$ isomorphic to *Spec* $\mathbb{Z}$ and whose Lie algebra is the $\mathbb{Z}$-subalgebra of $\mathfrak{g}(A)$ generated by E_a.

Let θ be a nilpotent set of real roots, then $L_\theta = \bigoplus_{a \in \theta} L_a$ is a nilpotent Lie algebra. Let U_θ be the unipotent complex algebraic group whose Lie algebra is L_θ. The following proposition was proved in [9].

Proposition 2.3. *There exist a uniquely defined group scheme $\mathfrak{U}_\theta$ over $\mathbb{Z}$ containing all $\mathfrak{U}_a$ for $a \in \theta$, whose fibre over $\mathbb{C}$ is the group U_θ and such that for any order on θ, the product morphism $\prod_{a \in \theta} \to \mathfrak{U}_\theta$ is an isomorphism of the underlying schemes.*

Now we present Tits' definition of Kac-Moody group associated to a generalized Cartan matrix $A = (a_{i,j})_{i,j=1}^n$ and a commutative ring R.

Let $\wedge$ be a free abelian group with basis $h_1, \cdots, h_n$, and $\wedge'$ its dual, then there are n elements $\alpha_1, \cdots, \alpha_n \in \wedge'$ satisfying $\langle h_i, \alpha_j \rangle = a_{i,j}$. Set $\mathfrak{T}(R) = Hom(\wedge', R^*)$, where R^* is the multiplicative group of invertible elements of R. The group $W(A)$ also acts on $\wedge'$ by $s_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i$. The automorphism of $\mathfrak{T}(R)$ induced by s_i will also be denoted by s_i.

For a real root a, and a nilpotent set of real roots θ, set $\mathfrak{U}_a(R)$, $\mathfrak{U}_\theta(R)$ to be the groups of R points of $\mathfrak{U}_a \times Spec\ R$ and $\mathfrak{U}_\theta \times Spec\ R$ respectively. For each pair of roots $\{a, b\}$, set $\vartheta(a, b) = (\mathbb{N}a + \mathbb{N}b) \cap \triangle$.

The *Steinberg group* $\mathfrak{S}(R)$ over R is defined to be the inductive limit of the groups $\mathfrak{U}_a(R)$ and $\mathfrak{U}_{\vartheta(a,b)}(R)$, where $a \in \triangle$ and $\{a, b\}$ runs over all pre-nilpotent pairs of real roots, relative to all the canonical injections $\mathfrak{U}_c(R) \to \mathfrak{U}_{\vartheta(a,b)}(R)$ for $c \in \vartheta(a, b)$. For each $0 < i \leq n$, $s'_i = exp(e_i)exp(-f_i)exp(e_i)$ is an automorphism of $\mathfrak{g}(A)$ which permutes the L_a and the E_a; therefore, it induces an automorphism of $\mathfrak{S}(R)$ which we also denote by s'_i.

Remark 2.4. For any a, b in a nilpotent set θ of real roots and any $r, r' \in R$, the following commutation relation holds inside $\mathfrak{S}(R)$:

$$[x_a(r), x_b(r')] = \prod_{c=ma+nb} x_c(k(a, b; c) r^m r'^n),$$

where $c = ma + nb$ runs over $\vartheta(a, b) - \{a, b\}$, $k(a, b; c) \in \mathbb{Z}$ and $x_a : R \to \mathfrak{S}(R)$, $x_b : R \to \mathfrak{S}(R)$ denote respectively the homomorphisms associated to a and b.

Definition 2.5. The Kac-Moody group $G_A(R)$ associated to A over R is defined to be the quotient of the free product of $\mathfrak{S}(R)$ and $\mathfrak{T}(R)$ by the following relations.

$$t x_i(r) t^{-1} = x_i(t(\alpha_i) r); \quad \widetilde{s}_i t \widetilde{s}_i^{-1} = s'_i(t);$$

$$\widetilde{s}_i(r^{-1}) = \widetilde{s}_i r^{h_i};\ \ \widetilde{s}_i u \widetilde{s}_i^{-1} = s_i'(u),$$

where t is an element from $\mathfrak{T}(R)$, r is an invertible element of R, u is an element from $\mathfrak{S}(R)$, $x_i : R \to \mathfrak{S}(R)$ and $x_{-i} : R \to \mathfrak{S}(R)$ are the homomorphisms associated to e_i and f_i respectively, $\widetilde{s}_i(r)$ is the canonical image of $x_i(r)x_{-i}(r^{-1})x_i(r)$ in $\mathfrak{S}(R)$, $\widetilde{s}_i = \widetilde{s}_i(1)$, and $r^{h_i} \in \mathfrak{T}(R)$ is defined by $r^{h_i}(\lambda) = r^{\langle \lambda, h_i \rangle}$ for $\lambda \in \wedge'$.

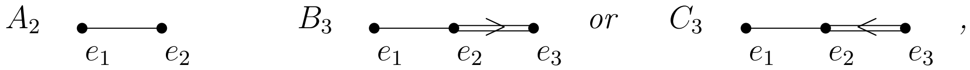
It is easy to see $G_A(R)$ is functorial in R, we call G_A the *Tits functor* associated to $A = (a_{ij})_{i,j=1}^n$. Set $r = 1$ in $\widetilde{s}_i(r^{-1}) = \widetilde{s}_i r^{h_i}$, we have $\widetilde{s}_i^2 = (-1)^{h_i}$; this formula will be used in the next section.

Remark 2.6. The above defining relations were given in [8, 196], and are slightly different from that of [9]; in fact the formula $\widetilde{s}_i^2 = (-1)^{h_i}$ cannot be derived from the defining relations in [9].

Remark 2.7. From the defining relations we see that $G_A(R)$ (as a group) is generated by the images of $\mathfrak{U}_{a_i}(R)$ $(0 < i \le n)$ in $G_A(R)$.

In Section 3 we need the following lemma.

Lemma 2.8. *Let A be a Cartan matrix of type*



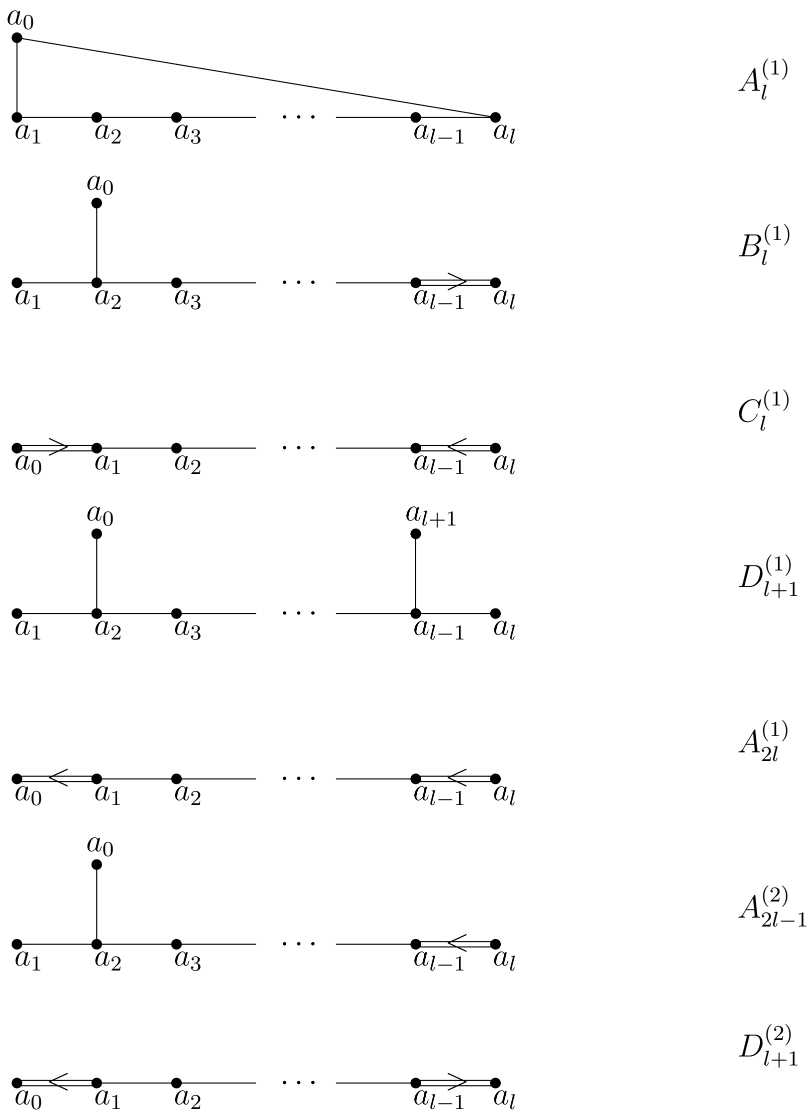
then the corresponding Kac-Moody group satisfies $G_A(R) = [G_A(R), G_A(R)]$.

Proof. In the case of A_2, we have the commutation relation $[x_{e_1}(1), x_{e_2}(r)] = x_{e_1+e_2}(r)$, hence the image of $\mathfrak{U}_{e_1+e_2}(R)$ is contained in $[G_A(R), G_A(R)]$. But the Weyl group acts transitively on the set of real roots, hence the images of $\mathfrak{U}_{e_1}(R)$ and $\mathfrak{U}_{e_2}(R)$ are contained in $[G_A(R), G_A(R)]$ too. Thus by Remark 2.7, we have $G_A(R) = [G_A(R), G_A(R)]$.

In the case of C_3, the above proof shows that the image of $\mathfrak{U}_{e_1}(R)$ and $\mathfrak{U}_{e_2}(R)$ is contained in $[G_A(R), G_A(R)]$. A direct computation shows that in $\mathfrak{U}_{\vartheta(e_2,e_3)}(R)$ we have $[x_{e_3}(r), x_{e_2}(1)] = x_{e_2+e_3}(-r)x_{e_2+2e_3}(-r)$, As the Weyl group acts transitively on the set of loot roots, we see that the image of $\mathfrak{U}_{e_2+2e_3}(R)$ is contained in $[G_A(R), G_A(R)]$ and so is $\mathfrak{U}_{e_2+e_3}(R)$. But the Weyl group acts transitively on the set of short roots too, hence the image of $\mathfrak{U}_{e_3}(R)$ is also contained in $[G_A(R), G_A(R)]$. By Remark 2.7 again, we have $G_A(R) = [G_A(R), G_A(R)]$. The proof for the case of B_3 is similar. ∎

3. Construction of infinite loop spaces associated to $A^{(2)}_{2l-1}$

It is well known that there are seven infinite classes of generalized Cartan matrices of affine type (cf.[4, p.51]), whose Dynkin diagrams are listed below.



To each infinite class and each commutative ring R we want to associate a sequence of Kac-Moody groups $G(n)$ that satisfies the conditions of Theorem 1.2. First consider the case of $A_{2l-1}^{(2)}$, let $\mathfrak{g}_l$ (resp. $G_l(R)$) be the corresponding Kac-Moody algebra (resp. group). In the following we use the notations of Section 2 freely, sometimes the subscript l will be added to indicate that the notations are associated to $A_{2l-1}^{(2)}$. For example, V_l will be the vector space over $\mathbb{Q}$, with basis $\{a_i\}_{i=0,\cdots,l}$. The group $W_l(A)$ acts on V_l and $\triangle_l$ denotes the set of real roots of $A_{2l-1}^{(2)}$

In $\mathfrak{g}_{l+1}$ set $e'_l = s'_l(e_{l+1})$, $f'_l = s'_l(f_{l+1})$, $h'_l = s'_l(h_{l+1}) = h_{l+1}+h_l$ respectively and for $i < l$ set $e'_i = e_i$, $f'_i = f_i$, $h'_i = h_i$ respectively.

Lemma 3.1. *In $\mathfrak{g}_{l+1}$ we have, for $i, j \leq l$,*

$$[h'_i, h'_j] = 0;\ [h'_i, e'_j] = a_{ij}e'_j;\ [h'_i, f'_j] = -a_{ij}f'_j; [e'_i, f'_j] = \delta_{i,j}h'_i;$$

$$(ad\ e_{l-1})^3 e'_l = 0;\ (ad\ f_{l-1})^3 f'_l = 0.$$

Proof. The first four relations follow from direct computations. Now set $\mathfrak{g}_0 = \mathbb{C}e_{l-1} \oplus \mathbb{C}f_{l-1} \oplus \mathbb{C}h_{l-1}$ and consider $\mathfrak{g}_{l+1}$ as a $\mathfrak{g}_0$-module by restricting of the adjoint representation. Since $[h_{l-1}, e'_l] = -2e'_l$ and $[f_{l-1}, e'_l] = 0$ (this follows from the fact that every root is either positive or negative), the representation theory of $\mathfrak{g}_0 \cong sl_2(\mathbb{C})$ implies $(ad\ e_{l-1})^3 e'_l = 0$. The proof for the last relation is exactly the same. ■

By the defining relations of $\mathfrak{g}_l$, the map $e_i \to e'_i$, $f_i \to f'_i$ extends to an injective Lie algebra homomorphism $\varphi_l : \mathfrak{g}_l \to \mathfrak{g}_{l+1}$.

Lemma 3.2. *Define a linear map $\tau_l : V_l \to V_{l+1}$ by $\tau_l(a_i) = a_i$ for $i < l$ and $\tau_l(a_l) = 2a_l + a_{l+1}$, then $\tau_l(\triangle_l^{\pm}) \subset \triangle_{l+1}^{\pm}$ and $\varphi_l(E_a) = E_{\tau_l(a)}$ for any $a \in \triangle_l$.*

Proof. It is easy to see that the map $s_i \to s_i$ for $i < l$ and $s_l \to s_l s_{l+1} s_l$ extends to a group homomorphism $w_l : W_l(A) \to W_{l+1}(A)$ and for any $v \in V_l$ and $W \in W_l(A)$ we have $\tau_l \cdot W(v) = w_l(W) \cdot \tau_l(v)$. Thus the first assertion follows readily. Similarly, the map $s'_i \to s'_i$ for $i < l$ and $s'_l \to s'_l s'_{l+1} (s'_l)^{-1}$ extends to a group homomorphism $w'_l : W'_l(A) \to W'_{l+1}(A)$. One checks that w_l and w'_l are compatible with the homomorphisms $\phi_l : W'_l(A) \to W_l(A)$ and $\phi_{l+1} : W'_{l+1}(A) \to W_{l+1}(A)$. We also have for any $\omega \in W'_l(A)$, $\varphi_l \cdot \psi_l(\omega) = (\psi_{l+1} w'_l(\omega)) \cdot \varphi_l$; recall the homomorphisms $\psi_l : W'_l(A) \to Aut(\mathfrak{g}(A)_l)$ and $\psi_{l+1} : W'_{l+1}(A) \to Aut(\mathfrak{g}(A)_{l+1})$ define in Section 2. Now we are ready to prove the second assertion. First, it is true for $a = a_i$, $i \le l$ by the definition of φ_l. Let $a = \phi_l(\omega)(a_i)$ be an element of $\triangle_l$, with $\omega \in W'_l(A)$, then $\varphi_l(E_a) = \varphi_l \cdot \psi_l(\omega)(E_{a_i}) = (\psi_{l+1} w'_l(\omega)) \cdot \varphi_l(E_{a_i}) = (\psi_{l+1} w'_l(\omega))(E_{\tau_l(a_i)}) = E_{\phi_{l+1} w'_l(\omega)(\tau_l(a_i))} = E_{w_l \phi_l(\omega)(\tau_l(a_i))} = E_{\tau_l(\phi_l(\omega)(a_i))} = E_{\tau_l(a)}$. This finishes the proof. ■

For any $a \in \triangle_l$, let $\mathfrak{U}_a$ be the corresponding group scheme defined in §2, then we can define a homomorphism $\psi_a : \mathfrak{U}_a \to \mathfrak{U}_{\tau_l(a)}$ that is compatible with the map $E_a \to E_{\tau_l(a)}$.

Lemma 3.3. *Let $\theta \subset \triangle_l$ be a nilpotent set of real roots, then $\tau_l(\theta) \subset \triangle_{l+1}$ is pre-nilpotent. Let θ' be the least nilpotent set containing $\tau_l(\theta)$, let $\mathfrak{U}_\theta$ and $\mathfrak{U}_{\theta'}$ be the group schemes in Proposition 2.3, then the homomorphisms $\psi_a : \mathfrak{U}_a(R) \to \mathfrak{U}_{\tau_l(a)}(R)$, $a \in \theta$ extend uniquely to a homomorphism $\psi_\theta : \mathfrak{U}_\theta(R) \to \mathfrak{U}_{\theta'}(R)$.*

Proof. By lemma 3.2 the homomorphism $L_\theta \to L_{\theta'}$ induced by φ_l is injective. Thus for $a, b \in \theta$, the commutation relation of $\mathfrak{U}_a(R)$ and $\mathfrak{U}_b(R)$ in $\mathfrak{U}_\theta(R)$ is exactly the same as that of $\mathfrak{U}_{\tau_l(a)}(R)$ and $\mathfrak{U}_{\tau_l(b)}(R)$ in $\mathfrak{U}_{\theta'}(R)$. Now the lemma follows readily. ■

By Lemma 3.2 and Lemma 3.3 the group homomorphisms $\psi_a(R) : \mathfrak{U}_a(R) \to \mathfrak{U}_{\tau_l(a)}(R)$, $a \in \triangle_l$, extend to a group homomorphism $\psi(R) : \mathfrak{G}_l(R) \to \mathfrak{G}_{l+1}(R)$.

Let $\wedge_l$ be a free abelian groups with basis $h_0, \cdots, h_l$ and $\wedge'_l$ its dual. Define linear map $\omega_l : \wedge_l \to \wedge_{l+1}$ by $\omega_l(h_i) = h_i$ for $i < l$ and $\omega_l(h_l) = h_l + 2h_{l+1}$. Denote by ω'_l the dual map of ω_l, then ω'_l induces a group homomorphism $\omega_l(R) : \mathfrak{T}_l(R) \to \mathfrak{T}_{l+1}(R)$.

From the defining relations of Kac-Moody groups and the constructions of $\psi(R)$ and $\omega_l(R)$ we see that the homomorphism of free products $\psi * \omega_l(R) : \mathfrak{S}_l(R) * \mathfrak{T}_l(R) \to \mathfrak{S}_{l+1}(R) * \mathfrak{T}_{l+1}(R)$ reduces to a homomorphism $g_l : G_l(R) \to G_{l+1}(R)$. Set $G(n) := G_{2n}(R)$ and $f_n := g_{2n+1} \cdot g_{2n}$. In order to apply Theorem 1.2, we have to define group homomorphism $\varsigma_n : \Sigma_n \to G(n)$ for each $n > 0$.

First we need some preliminaries. Let $\overline{W}_l$ be the signed permutation group, i.e., the group of linear transformations of $\mathbb{R}^l$ leaving invariant the set $\{\pm e_i\}$ of standard basis vectors and their negatives. It has $l-1$ generators $\overline{r}_1, \cdots, \overline{r}_{l-1}$ and the following defining relations:

$$\overline{r}_j \overline{r}_i^2 \overline{r}_j^{-1} = \overline{r}_i^2 \overline{r}_j^{-2a_{i,j}}$$

$$\overline{r}_i \overline{r}_j \overline{r}_i \cdots = \overline{r}_j \overline{r}_i \overline{r}_j \cdots (m_{i,j} \ factors \ on \ each \ side),$$

where $\overline{r}_i$ is defined by sending $\{e_i, e_{i+1}\}$ to $\{-e_{i+1}, e_i\}$ and leaves the other basis vectors invariant.

Lemma 3.4. *The $\widetilde{s}_i$, $0 < i < l$ in $G_l(R)$ satisfy the following two relations,*

$$\widetilde{s}_j \widetilde{s}_i^2 \widetilde{s}_j^{-1} = \widetilde{s}_i^2 \widetilde{s}_j^{-2a_{i,j}},$$

$$\widetilde{s}_i \widetilde{s}_j \widetilde{s}_i \cdots = \widetilde{s}_j \widetilde{s}_i \widetilde{s}_j \cdots (m_{i,j} \ factors \ on \ each \ side).$$

Let $\widetilde{W}_l$ be the subgroup of $G_l(R)$ generated by $\{\widetilde{s}_i\}_{0<i<l}$, then the maps $s_i' \to \widetilde{s}_i$ extend to a group homomorphism $h_l : W_l' \to \widetilde{W}_l$.

Proof. We prove the first assertion and the second assertion will follow directly. As $\widetilde{s}_i^2 = (-1)^{h_i}$ the first relation is equivalent to

$$\widetilde{s}_j (-1)^{h_i} \widetilde{s}_j^{-1} = (-1)^{h_i - 2a_{i,j} h_i},$$

which is one of the defining relations of $G_l(R)$. The second relation was proved in Remark 3.7 of [9]. ∎

For each $0 < i < n$ set

$$r_i = s_{2i+1}'^3 s_{2i}' s_{2i-1}' s_{2i+1}' s_{2i}' s_{2i-1}'$$

in $G_{2n}(R)$ and set $w_i = h_{2n}(r_i)$. Let $\sigma(i) \in \Sigma_n$ be the permutation that swaps the i-th element with the $(i+1)$-th one, then the map $\sigma(i) \to r_i$ extends to a group homomorphism $\varsigma_n' : \Sigma_n \to W_{2n}'$. Set $\varsigma_n = h_{2n} \varsigma_n'$.

Remark 3.5. In fact we can identify W_{2n}' with the signed permutation group, i.e., the group of linear transformations of $\mathbb{R}^{2n}$ leaving invariant the set $\{\pm e_i\}$ of standard basis vectors and their negatives. Then r_i is the linear isomorphism of $\mathbb{R}^{2n}$ that sends $\{e_{2i-1}, e_{2i}\}$ to $\{e_{2i+1}, e_{2i+2}\}$ and leaves the other basis vectors invariant.

Theorem 3.6. *let $G(R) = \lim_{n\to\infty} G_n(R)$, then $\pi = \pi_0(G)$ satisfies $\pi = [\pi, \pi]$. Applying Quillen's plus construction to $BG(R)$ and $\pi_1(BG) \cong \pi$, we get an infinite loop space $BG^+(R)$.*

Proof. Condition 1) of Theorem 1.2 follows directly from Lemma 2.8. Thus we only need to verify condition 2) of Theorem 1.2. Set $f_{m,n} = f_{m+n-1}\cdots f_{m+1}f_m$, we want to show that $f_{m,n}(G(m))$ and $c(n,m)(f_{n,m}(G(n)))c(m,n)$ are commutative in $G(m+n)$. Set $s_{nm} := \phi_{2m+2n}\varsigma'_{m+n}(c(n,m))$ in the following, recall that ϕ_{2m+2n} is the natural homomorphism $W'(A^{(2)}_{4m+4n-1}) \to W(A^{(2)}_{4m+4n-1})$.

By remark 2.7, $f_{m,n}(G(m))$ is generated by the subgroups $\{\mathfrak{U}_a(R)\}_{a\in\Theta}$ and $c(n,m)(f_{n,m}(G(n)))c(m,n)$ is generated by the subgroups $\{\mathfrak{U}_a(R)\}_{a\in\Theta'}$, where

$$\Theta = \{\pm a_0, \cdots, \pm a_{2m-1}, (s_{2m-1}\cdot s_{2m}\cdots s_{2m+2n-1})(\pm a_{2m+2n})\}$$

$$= \{\pm a_0, \cdots, \pm a_{2m-1}, \pm(2a_{2m-1} + \cdots + 2a_{2m+2n-1} + a_{2m+2n})\}$$

and

$$\Theta' = s_{nm}\{\pm a_0, \cdots, \pm a_{2n-1}, (s_{2n-1}\cdot s_{2m}\cdots s_{2m+2n-1})(\pm a_{2m+2n})\}.$$

Thus in order to verify condition 2) it suffices to show that for any $\alpha \in \Theta$ and $\beta \in \Theta'$, $\mathfrak{U}_\alpha(R)$ and $\mathfrak{U}_\beta(R)$ are commutative, but this can be deduced from the fact that the subalgebras $L_{\pm\alpha}$ and $L_{\pm\beta}$ of $\mathfrak{g}_{2m+2n}$ are commutative. Indeed, when $L_{\pm\alpha}$ and $L_{\pm\beta}$ are commutative, one checks that $\{\alpha,\beta\}$ is a prenilpotent pair and $\vartheta(a,b) = \{\alpha,\beta\}$, hence by Remark 2.4 the group $\mathfrak{U}_{\vartheta(a,b)}(R)$ is commutative. Thus in order to finish the proof it suffices to show that for any $\alpha \in \Theta$ and $\beta \in \Theta'$, $L_{\pm\alpha}$ and $L_{\pm\beta}$ are commutative.

Direct computation shows that

$$(s_{2m-1}\cdot s_{2m}\cdots s_{2m+2n-1})(\pm a_{2m+2n}) = s_{nm}(\pm a_{2m+2n});$$

$$(s_{2n-1}\cdot s_{2m}\cdots s_{2m+2n-1})(\pm a_{2m+2n}) = s_{mn}(\pm a_{2m+2n});$$

$$s_{mn}(\pm a_0) = \pm(a_0 + a_1 + 2(a_2 + \cdots + a_{2m}) + a_{2m+1});$$

$$s_{nm}\{\pm a_1, \cdots, \pm a_{2n-1}\} = \{\pm a_{2m+1}, \cdots, \pm a_{2m+2n-1}\};$$

$$s_{nm}\{\pm a_{2n+1}, \cdots, \pm a_{2m+2n-1}\} = \{\pm a_1, \cdots, \pm a_{2m-1}\}.$$

Thus we only need to show that $L_{\pm(a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1})}$ is commutative with $L_{\pm a_0}$, and $L_{\pm a_{2m+2n}}$ is commutative with $L_{\pm(2a_{2m-1}+\cdots+2a_{2m+2n-1}+a_{2m+2n})}$. We prove the first assertion, the proof for the second one is similar.

First, we have $[L_{-a_0}, L_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}] \in L_{a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}$, but it is well known that the highest root in $\mathbb{Z}a_1 + \mathbb{Z}a_2 + \cdots + \mathbb{Z}a_{2m+1} \cap \triangle_{2m+2n}$ is $a_1 + \cdots + a_{2m+1}$. Hence $[L_{-a_0}, L_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}] = 0$. We also have $[h_0, L_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}] = 0$. Set $\mathfrak{g}_0 = L_{a_0} \oplus L_{-a_0} \oplus \mathbb{C}h_0$ and consider $\mathfrak{g}_{2m+2n}$ as a $\mathfrak{g}_0$-module by restricting of the adjoint representation. By the representation theory of $\mathfrak{g}_0 \cong sl_2(\mathbb{C})$, it follows that

$$[L_{a_0}, e_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}] = 0.$$

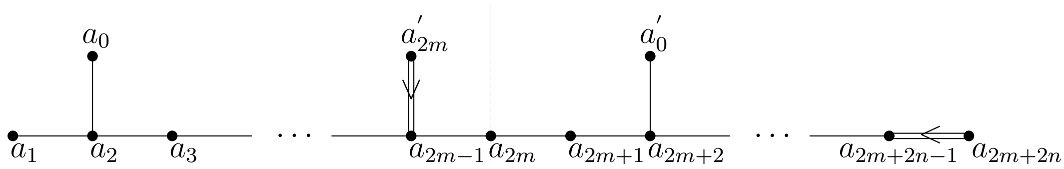
Similarly, we have

$$[L_{a_0}, f_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}] = 0$$

and

$$[L_{-a_0}, f_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}] = 0.$$

This finishes the proof of the theorem. The following Dynkin diagram would illustrate our proof, where $a_0^{'}$ (resp. $a_{2m}^{'}$) denotes $2a_{2m-1}+\cdots+2a_{2m+2n-1}+a_{2m+2n}$ (resp. $s_{n,m}(a_0)$).



■

Remark 3.7. It is easy to see, from the construction above, that $BG^+(R)$ as an infinite loop space is functorial in R (see [6, 7] for a delicate exposition of infinite loop spaces and its relation with E_∞ spaces). Thus we can, as in the classical cases, define a K-theory of rings by setting

$$K_i^G(R) := \pi_i(BG^+(R)).$$

4. The constructions in the other cases

The constructions in the other cases are similar. For example, in the case of $A_l^{(1)}$, let $\mathfrak{g}_l$ be the Kac-Moody algebra associated to $A_l^{(1)}$, and in $\mathfrak{g}_{l+1}$ set $e_l' = s_l'(e_{l+1})$, $f_l' = s_l'(f_{l+1})$, $h_l' = s_l'(h_{l+1}) = h_{l+1} + h_l$ respectively and for $i < l$ set $e_i' = e_i$, $f_i' = f_i$, $h_i' = h_i$ respectively. In the case of $D_{l+1}^{(1)}$, set $e_l' = s_l' \cdot s_{l-1}'(e_{l+1})$, $f_l' = s_l' \cdot s_{l-1}'(f_{l+1})$, $h_l' = s_l' \cdot s_{l-1}'(h_{l+1}) = h_{l+1} + h_l + h_{l-1}$ respectively. For the rest constructions we just repeat the arguments of the previous section.

Remark 4.1. In Section 3 we require that $\wedge_l$ is freely generated by $\{h_0, \cdots, h_l\}$, in fact this assumption is not necessary. For example, in the case of $A_l^{(1)}$ we can set $\wedge_l$ to be freely generated by $\{h_1, \cdots, h_l\}$ and add an $h_0 := -h_1 - \cdots - h_l$. When R is a field K, the corresponding Kac-Moody group $G_l(K)$ is isomorphic to $SL_{l+1}(K[t, t^{-1}])$, then $G(\infty, K)^+$ is of course an infinite loop space. However, we don't know any explicit realization of $G_l(R)$ in the general cases.

We can also treat the (topological) affine Kac-Moody groups over $\mathbb{C}$ (see [5] for the definition), and applying the method of Section 3 we have the following result.

Theorem 4.2. *Let $\{A_l\}_{l>2}$ be one of the seven (infinite) classes of affine generalized Cartan matrices and let $\{G_l\}_{l>2}$ be the associated simply-connected Kac-Moody groups over $\mathbb{C}$, then we can define for each $l > 2$ a natural homomorphism $f_l : G_l \to G_{l+1}$ such that $BG = \lim_{l\to\infty} BG_l$ is an infinite loop space.*

Remark 4.3. In fact there exists a (infinite) classes of classical Lie groups $\{G(l)\}_{l>2}$ such that G_l is isomorphic to a central extension of the group of polynomial loops or twisted polynomial loops on $G(l)$ (cf.[5, §2.8]).

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