Left Invariant Metrics on Lie Groups Associated with G-Associative Algebras

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Abstract. A left invariant connection associated with a left invariant metric on a Lie group defines a Lie-admissible algebra which provides a Lie-admissible algebraic approach to the study given by Milnor. In this paper, using such an approach, we study left invariant metrics on Lie groups associated with certain subclasses of Lie-admissible Lie algebras, namely, G-associative algebras explicitly. In particular, their classifications in low dimensions are given.
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1. Introduction

Milnor in [10] commenced to study the curvatures of left invariant metrics on Lie groups which outline "what is the Riemannian geometry of a Lie group which has been provided with a Riemannian metric invariant under the left translation". In fact, let H be a Lie group, and let $\mathfrak{h}$ be its Lie algebra consisting of all smooth vector fields on H which are invariant under left translations. Suppose that H admits a Riemannian metric g invariant under the left translation, then the left invariant Riemannian connection ∇ associated with g satisfies the "symmetry" condition (torsion free)

$$\nabla_x y - \nabla_y x = [x, y], \ \ \forall\, x, y \in \mathfrak{h}, \tag{1.1}$$

and the compatibility condition (the parallel translation preserves the metric g ([13]))

$$g(\nabla_x y, z) + g(y, \nabla_x z) = 0, \ \ \forall x, y \in \mathfrak{h}. \tag{1.2}$$

Milnor studied the sectional, Ricci and scalar curvatures in details in [10] by considering the Riemann curvature tensor R which associates with each pair of smooth vector fields $x \in \mathfrak{h}$ and $y \in \mathfrak{h}$

$$R_{xy} = \nabla_{[x,y]} - \nabla_x \nabla_y + \nabla_y \nabla_x \tag{1.3}$$

from smooth vector fields to smooth vector fields. Basically his study is pure Lie-algebraic and from "metrics" to "connections" by the following formula

$$g(\nabla_x y, z) = \frac{1}{2}(g([x, y], z]) - g([y, z], x) + g([z, x], y)), \quad \forall x, y, z \in \mathfrak{h}, \qquad (1.4)$$

which combines Eqs. (1.1) and (1.2) with the various identities obtained by permuting the variables.

Conversely, one may consider another approach from "connections" to "metrics" in the above study. That is, one can try to find a left invariant Riemannian metric g associated with a left invariant connection ∇ satisfying Eq. (1.1) such that the compatibility condition (1.2) holds. So it is natural to define a bilinear operation on $\mathfrak{h}$ by

$$xy = \nabla_x y, \quad \forall x, y \in \mathfrak{h}. \qquad (1.5)$$

Then Eq. (1.1) is equivalent to the fact that the commutator of the above operation defines a Lie algebra. In fact, an algebra whose commutator defines a Lie algebra is called a Lie-admissible algebra which connects with many fields in mathematics and physics ([16], [14], [11], etc.). Therefore, the Milnor's study can be carried on by considering the Lie-admissible algebra structures on the Lie algebra $\mathfrak{h}$ of a Lie group H with a compatible Riemannian metric g satisfying Eq. (1.2). As an extreme case, the classifications of left invariant flat metrics or connections whose Riemann curvature tensors given by Eq. (1.3) are zero were given by Milnor in [10] and the corresponding Lie-admissible algebraic (the Lie-admissible algebras in such a case are called left-symmetric algebras ([15], [7], [6], [2], etc.)) approach with a more explicit formula was given in [4].

On the other hand, obviously it is a little general to consider all Lie-admissible algebras since the identity of a Lie-admissible algebra given by Eq. (1.1) (or equivalently, Eq. (2.3) in Section 2) seems a little weak ([3]). Like the left-symmetric algebras mentioned as above, it is natural and interesting to consider some subclasses of the Lie-admissible algebras satisfying certain additional conditions. In [9], the Lie-admissible algebras are divided into several subclasses, namely, the G-associative algebras including left-symmetric algebras, according to certain "symmetry" of the associators under the transformation subgroup G in Σ_3 which is the symmetry group of degree 3. In this paper, except the left-symmetric algebras which have been given in [4] and the most general Lie-admissible algebras only satisfying Eq. (1.1), we mainly study the other subclasses of Lie-admissible algebras with positive definite and symmetric bilinear forms satisfying Eq. (1.2). We would like to point out that although our study may not provide more new results on the study of curvatures given by Milnor, the new algebraic approach would provide certain more explicit and practicable description and on the other hand, it provides plentiful of geometric background and motivation to the study of the (pure) algebra systems like the Lie-admissible algebras themselves.

The paper is organized as follows. In Section 2, we will briefly recall the definitions of G-associative algebras. In Section 3, we give the structure theories on G-associative algebras with positive definite and symmetric bilinear forms satisfying Eq. (1.2). In Section 4, we give the classification up to dimension 3.

Throughout this paper, all algebras are over the real number field $\mathbb{R}$, unless otherwise stated.

2.   *G*-associative algebras

In this section, we use the notations and terminologies in [9]. Let $A = (A, \mu)$ be a finite dimensional algebra over a field $\mathbb{F}$, where $\mu : A \times A \to A$ is a linear mapping (the law of A). Denote by $a_\mu : A^{\otimes 3} \to A$ the associator of the law μ:

$$a_\mu(X_1, X_2, X_3) = \mu(\mu(X_1, X_2), X_3) - \mu(X_1, \mu(X_2, X_3)), \quad \forall X_1, X_2, X_3 \in A. \quad (2.1)$$

Let Σ_n be the symmetric group of degree n. For every $\sigma \in \Sigma_3$, let

$$\sigma(X_1, X_2, X_3) = (X_{\sigma^{-1}(1)}, X_{\sigma^{-1}(2)}, X_{\sigma^{-1}(3)}), \quad \forall X_1, X_2, X_3 \in A. \quad (2.2)$$

Definition 2.1.   The algebra $A = (A, \mu)$ is called *Lie-admissible* if the law μ satisfies

$$\sum_{\sigma \in \Sigma_3} (-1)^{\varepsilon(\sigma)} a_\mu \circ \sigma = 0. \quad (2.3)$$

This definition is equivalent to that the mapping $[\,,\,]_\mu : A \times A \to A$ defined by $[X, Y] = \mu(X, Y) - \mu(Y, X)$ is a Lie bracket ([1]).

Definition 2.2.   Let G be a subgroup of Σ_3. We say that the algebra law is *G-associative* if

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} a_\mu \circ \sigma = 0. \quad (2.4)$$

It is known that there are a total of 6 subgroups of Σ_3 given as follows.

$$G_1 = \{Id\}, \quad G_2 = \{Id, \tau_{12}\}, \quad G_3 = \{Id, \tau_{23}\}, \quad G_4 = \{Id, \tau_{13}\},$$

$$G_5 = \{Id, (231), (312)\} = A_3, \quad G_6 = \Sigma_3, \quad (2.5)$$

where τ_{ij} is the transposition between i and j, (231) a 3-cycle, and A_3 is the alternating group. Let $xy = \mu(x, y)$ and $(x, y, z) = a_\mu(x, y, z)$ for any $x, y, z \in A$. Then

$$(x, y, z) = (xy)z - x(yz), \quad \forall x, y, z \in A. \quad (2.6)$$

There are the following types of Lie-admissible algebras:

1. If μ is G_1-associative, then μ is an *associative* law, that is, the identity

$$(x, y, z) = 0 \quad (2.7)$$

holds for any $x, y, z \in A$. In this case, A is an *associative algebra.*

2. If μ is G_2-associative, then μ is a law of *Vinberg algebra* ([7], [12]), that is, the identity

$$(x, y, z) = (y, x, z) \quad (2.8)$$

holds for any $x, y, z \in A$. In this case A is also called a *left-symmetric algebra.*

3. If μ is G_3-associative, then μ is a law of *pre-Lie algebra* ([6]), that is, the identity

$$(x, y, z) = (x, z, y) \tag{2.9}$$

holds for any $x, y, z \in A$. In this case A is also called a *right-symmetric algebra*.

4. If μ is G_4-associative, then A is called a G_4-*associative algebra*, that is, the identity

$$(x, y, z) = (z, y, x) \tag{2.10}$$

holds for any $x, y, z \in A$.

5. If μ is G_5-associative, then A is called a G_5-*associative algebra*, that is, the identity

$$(x, y, z) + (y, z, x) + (z, x, y) = 0 \tag{2.11}$$

holds for any $x, y, z \in A$. The identity (2.11) is called the *generalized Jacobi condition*.

6. If μ is G_6-associative, then μ is a Lie-admissible law.

3. Left invariant metrics on Lie groups associated with G-associative algebras

Let A be a G-associative algebra over the real number field $\mathbb{R}$ and let f be a bilinear form. We say that f is *left invariant* if f satisfies

$$f(xy, z) + f(y, xz) = 0, \quad \forall x, y, z \in A. \tag{3.1}$$

Suppose that f is positive definite, symmetric and left invariant. For any subspace V in A, set

$$V^{\perp} = \{x \in A \mid f(x, y) = 0, \ \forall y \in V\}. \tag{3.2}$$

For any $x, y \in A$, let $L(x)$ and $R(x)$ denote the left and right multiplication operators respectively, that is, $L(x)(y) = xy$, $R(x)(y) = yx$. The *adjoint* F^* of a linear transformation F on A associated with f is defined by the formula

$$f(F(x), y) = f(x, F^*(y)), \quad \forall x, y \in A. \tag{3.3}$$

The transformation F is called *skew-adjoint* if $F^* = -F$. So Eq. (3.1) is equivalent to the fact that $L(x)$ is skew-adjoint for any $x \in A$. Let

$$\mathrm{Ann}(A) = \{x \in A \mid xy = yx = 0, \ \forall y \in A\} \tag{3.4}$$

be the *annihilator* of A. Obviously, $\mathrm{Ann}(A)$ is an ideal of A, and an algebra A is called *trivial* if $A = \mathrm{Ann}(A)$. For any subalgebra V in A, set

$$C_R(V) = \{x \in V \mid R(x)|_V = 0\}. \tag{3.5}$$

Lemma 3.1 ([5]). *Let A be a Lie-admissible algebra with a positive definite, symmetric and left invariant bilinear form f. Then $C_R(A) = (AA)^{\perp}$. Therefore A can be decomposed into an orthogonal direct sum of vector spaces as*

$$A = C_R(A) \dotplus AA. \tag{3.6}$$

Theorem 3.2 ([10]). *A Lie group G with a left invariant metric is flat if and only if the associated Lie algebra $\mathfrak{g}$ splits as an orthogonal direct sum of vector spaces $\mathfrak{b}\dot{+}\mathfrak{u}$, where $\mathfrak{b}$ is a commutative subalgebra, $\mathfrak{u}$ is a commutative ideal, and the linear transformation $\mathrm{ad}b$ is skew-adjoint for any $b \in \mathfrak{b}$. Furthermore, if these conditions are satisfied, then*

$$\nabla_u = 0, \nabla_b = \mathrm{ad}(b), \forall u \in \mathfrak{u}, b \in \mathfrak{b}. \tag{3.7}$$

Theorem 3.3 ([4]). *A left-symmetric algebra A has a positive definite, symmetric and left invariant bilinear form if and only if A splits as an orthogonal direct sum of vector spaces $\mathfrak{b}\dot{+}\mathrm{Ann}(A)\dot{+}[A, A]$, where $\mathfrak{b}$ is a trivial subalgebra, $[A, A] = AA$ is a trivial ideal with even dimension, $\dim \mathfrak{b} \leq \dim[A, A]/2$, and the linear transformation $R(x) = 0$ and $L(x) \neq 0$ is skew-adjoint for any non-zero $x \in \mathfrak{b}$.*

Next we study the other cases of G-associative algebras, where $G = G_1, G_3, G_4$ and G_5 respectively.

Theorem 3.4. *An associative algebra with a positive definite, symmetric and left invariant bilinear form is trivial.*

Proof. Let A be an associative algebra and let f be a positive definite, symmetric and left invariant bilinear form. Obviously, A is a left-symmetric algebra. By theorem 3.3, A splits as an orthogonal direct sum of vector spaces

$$\mathfrak{b}\dot{+}\mathrm{Ann}(A)\dot{+}[A, A],$$

where $\mathfrak{b}$ is a trivial subalgebra, $[A, A] = AA$ is a trivial ideal, $\dim[A, A] = 2k$, and the linear transformation $R(x) = 0$ and $L(x) \neq 0$ is skew-adjoint for any non-zero $x \in \mathfrak{b}$.

Assume that A is not trivial. So $k \neq 0$. For any $x, y \in \mathfrak{b}$, we know that

$$x(yz) = (xy)z + y(xz) - (yx)z = (yx)z, \quad \forall z \in [A, A].$$

Hence $L(x)_{[A,A]}L(y)_{[A,A]} = L(y)_{[A,A]}L(x)|_{[A,A]}$, that is, $\{L(x)|_{[A,A]}\}_{x \in \mathfrak{b}}$ is a family of commutative linear transformations on $[A, A]$. Therefore there exists a basis $\{e_1, \cdots, e_{2k}\}$ of $[A, A]$ such that $f(e_i, e_j) = \delta_{ij}$ and for any $x \in \mathfrak{b}$,

$$xe_{2i-1} = -\alpha_i(x)e_{2i}, \quad xe_{2i} = \alpha_i(x)e_{2i-1}, \quad i = 1, \cdots, k,$$

and for any i ($1 \leq i \leq k$), there exists an element $y \in \mathfrak{b}$ such that $\alpha_i(y) \neq 0$ (due to the fact that $\mathfrak{b} \cap \mathrm{Ann}(A) = \{0\}$). Let $x \in \mathfrak{b}$ satisfy $\alpha_1(x) \neq 0$. Then

$$x(xe_1) = -\alpha_1(x)^2e_1 \neq 0 = (xx)e_1,$$

which is a contradiction. Hence A is trivial. ∎

Theorem 3.5. *Let A be a right-symmetric algebra with a positive definite, symmetric and left invariant bilinear form. Then $A = \mathrm{Ann}(A) \oplus AA$ as a direct sum of ideals, where $\mathrm{Ann}(A) = C_R(A)$ and AA is an ideal of A satisfying $AA = (AA)(AA)$.*

Proof. If A is a right-symmetric algebra, then $C_R(A)$ is an ideal of A ([8]). Hence for any $x \in C_R(A)$ and $y \in A$, we know that $xy \in C_R(A) \cap AA$. Therefore $xy = 0$ by Lemma 3.1. So $x \in \mathrm{Ann}(A)$ and then $C_R(A) \subseteq \mathrm{Ann}(A)$. Thus $\mathrm{Ann}(A) = C_R(A)$. Obviously AA is an ideal of A and $A = \mathrm{Ann}(A) \oplus AA$ as a direct sum of ideals, which implies that $AA = (AA)(AA)$.   ∎

Theorem 3.6. *Let A be a G_4-associative algebra with a positive definite, symmetric and left invariant bilinear form. Then $A = \mathrm{Ann}(A) \oplus AA$ as a direct sum of ideals, where $\mathrm{Ann}(A) = C_R(A)$ and AA is an ideal of A satisfying $AA = (AA)(AA)$.*

Proof. Let $x \in C_R(A)$ and $y \in A$. Since

$$(yx)x - y(xx) = (xx)y - x(xy),$$

we know that $x(xy) = 0$. Therefore, we obtain $0 = f(x(xy), y) = -f(xy, xy)$, which implies $xy = 0$. Hence $\mathrm{Ann}(A) = C_R(A)$. Therefore we know that AA is an ideal of A and $A = \mathrm{Ann}(A) \oplus AA$ as a direct sum of ideals. Moreover, $AA = (AA)(AA)$.   ∎

Lemma 3.7. *Let A be a Lie-admissible algebra. Then*
$$[A, A]^\perp = \{z \in A \mid R(z) = R(z)^*\}.$$

Proof. For any $x, y, z \in A$, $z \in [A, A]^\perp$ if and only if $f(z, [x, y]) = 0$, if and only if $f(z, xy) - f(z, yx) = 0$, if and only if $f(R(z)x, y) = f(R(z)y, x)$, if and only if $R(z) = R(z)^*$.   ∎

Lemma 3.8. *Let A be a G_5-associative algebra. Then*
$$[A, A]^\perp = \{u \in A \mid R(u) = 0\} = C_R(A).$$

Proof. For any $u \in [A, A]^\perp$ and $v \in A$, by Lemma 3.7, we know that

$$f(uu, v) = f(R(u)u, v) = f(u, R(u)v) = f(u, vu) = 0.$$

Hence $uu = 0$ due to the positive definiteness of f. For any $u \in [A, A]^\perp$, $R(u)$ is diagonalizable over the real number field $\mathbb{R}$ since it is self-adjoint. Let $\lambda \in \mathbb{R}$ be an arbitrary eigenvalue of $R(u)$ and $v \in A$ be a non-zero eigenvector associated with λ. Then
$$-(uv)u + \lambda uv + u(uv) = \lambda^2 v.$$
due to the equation $(vu)u + (uu)v + (uv)u = v(uu) + u(uv) + u(vu)$. Thus

$$\begin{aligned}
\lambda^2 f(v, v) &= f(u(uv) - (uv)u + \lambda uv, v) = f(u(uv) - (uv)u, v) \\
&= -f(uv, uv) + f(uv, vu) = \lambda f(uv, v) - f(uv, uv) \le 0.
\end{aligned}$$

Therefore $\lambda = 0$ and then $R(u) = 0$.   ∎

Theorem 3.9. *Let A be a G_5-associative algebra with a positive definite, symmetric and left invariant bilinear form. Then $A = \mathrm{Ann}(A) \oplus [A, A]$ as a direct sum of ideals.*

Proof.   By Lemma 3.1 and 3.8, we know that $[A, A] = AA$ is an ideal of A and $A = C_R(A) \dotplus [A, A]$ as an orthogonal direct sum of vector spaces. For any $x \in C_R(A)$ and $y \in A$, we know that $x(xy) = 0$ since

$$(yx)x + (xx)y + (xy)x = y(xx) + x(xy) + x(yx).$$

Thus $0 = f(x(xy), y) = -f(xy, xy)$. Hence $xy = 0$ by the positive definiteness of f. So $C_R(A) \subseteq \text{Ann}(A)$ and then $C_R(A) = \text{Ann}(A)$. Therefore $A = \text{Ann} \oplus [A, A]$ as a direct sum of ideals.   ∎

Corollary 3.10.   *A G_5-associative algebra on a solvable Lie algebra with a positive definite, symmetric and left invariant bilinear form is trivial.*

Proof.   Let A be a G_5-associative algebra with a positive definite, symmetric and left invariant bilinear form. By Theorem 3.9, $A = \text{Ann}(A) \oplus [A, A]$ as a direct sum of ideals. Then $[A, A] = [[A, A], [A, A]]$. Since the commutator Lie algebra of A is solvable, we know that $[A, A] = 0$. Therefore $A = \text{Ann}(A)$ is trivial.   ∎

4.   The classification in dimension ≤ 3

Lemma 4.1.   *Let A be a Lie-admissible algebra with a positive definite, symmetric and left invariant bilinear form f. If A is 1-dimensional, then A is trivial.*

Proof.   Let $\{e\}$ be a basis of A such that $f(e, e) = 1$. Since $f(ee, e) = -f(e, ee) = 0$, we know that $ee = 0$. So A is trivial.   ∎

Theorem 4.2 ([4]).   *Let A be a left-symmetric algebra in dimension 2 with a positive definite, symmetric and left invariant bilinear form. Then A is trivial.*

Theorem 4.3.   *Let A be a right-symmetric algebra or G_4-associative algebra in dimension 2 with a positive definite, symmetric and left invariant bilinear form. Then A is trivial.*

Proof.   Let f be the positive definite, symmetric and left invariant bilinear form, and let $\{e_1, e_2\}$ be a basis of A such that $f(e_i, e_j) = \delta_{ij}$. Set

$$e_1 e_1 = a_{11} e_1 + b_{11} e_2, \quad e_1 e_2 = a_{12} e_1 + b_{12} e_2,$$

$$e_2 e_1 = a_{21} e_1 + b_{21} e_2, \quad e_2 e_2 = a_{22} e_1 + b_{22} e_2.$$

By the left invariance of f, we know that

$$a_{11} = b_{12} = a_{21} = b_{22} = 0, \quad b_{11} = -a_{12}, \ a_{22} = -b_{21}.$$

Let $a = b_{11}$ and $b = a_{22}$. Then we have

$$e_1 e_1 = a e_2, \quad e_2 e_2 = b e_1, \quad e_1 e_2 = -a e_1, \quad e_2 e_1 = -b e_2. \tag{4.1}$$

If A is a right-symmetric algebra, then $a = 0$ by Eq. (4.1) and

$$(e_1 e_1) e_2 - e_1 (e_1 e_2) = (e_1 e_2) e_1 - e_1 (e_2 e_1),$$

and $b = 0$ by Eq. (4.1) and

$$(e_2e_1)e_2 - e_2(e_1e_2) = (e_2e_2)e_1 - e_2(e_2e_1).$$

If A is a G_4-associative algebra, then $a = b = 0$ by Eq. (4.1) and

$$(e_1e_1)e_2 - e_1(e_1e_2) = (e_2e_1)e_1 - e_2(e_1e_1).$$

So A is trivial in both cases.   ∎

Corollary 4.4. *Let A be a G_i-associative algebra in dimension 2 with a positive definite, symmetric and left invariant bilinear form for $1 \leq i \leq 5$. Then A is trivial.*

Proof.   It is clear due to Theorems 4.2 and 4.3 and Corollary 3.10 and the fact that any 2-dimensional Lie algebra is solvable.   ∎

Theorem 4.5 ([4]).   *Let A be a left-symmetric algebra in dimension 3 with a positive definite, symmetric and left invariant bilinear form. Then A is trivial or A is 3-dimensional with a basis $\{e_1, e_2, e_3\}$ such that the non-zero products are given by*

$$e_1e_2 = -e_3, \quad e_1e_3 = e_2. \tag{4.2}$$

Lemma 4.6.   *Let A be a right-symmetric or G_4-associative algebra in dimension 3 with a positive definite, symmetric and left invariant bilinear form. Then A is trivial or $AA = A$.*

Proof.   By Theorem 3.5 and 3.6, for both cases, $A = \text{Ann}(A) \oplus AA$, where $AA = (AA)(AA)$. By Lemma 4.1 and Theorem 4.3, we know that $AA = 0$ or $AA = A$. Then the result follows.   ∎

Theorem 4.7.   *Every right-symmetric algebra in dimension 3 with a positive definite, symmetric and left invariant bilinear form is trivial.*

Proof.   Let A be a 3-dimensional right-symmetric algebra with a positive definite, symmetric and left invariant bilinear form f. Assume that A is not trivial. By Lemma 4.6, we know that $AA = A$. Then there exists an element $e \in A$ such that $L(e) \neq 0$. Since $f(ex, y) + f(x, ey) = 0$, there exists a basis $\{e_1, e_2, e_3\}$ of A such that

$$f(e_i, e_j) = \delta_{ij}, \quad \text{and} \quad ee_1 = -ae_2, \quad ee_2 = ae_1, \quad ee_3 = 0, \quad a \neq 0.$$

Replacing e by $\frac{e}{a}$, which is still denoted by e, we know that

$$ee_1 = -e_2, \quad ee_2 = e_1, \quad ee_3 = 0. \tag{4.3}$$

Set

$$e_ie_j = \sum_{k=1}^{3} \alpha_{ij}^k e_k, \quad 1 \leq i, j \leq 3.$$

Then by the left invariance of f, we know that

$$\alpha_{ij}^k + \alpha_{ik}^j = 0.$$

Due to the identity

$$(ee_3)e_1 - e(e_3 e_1) = (ee_1)e_3 - e(e_1 e_3), \tag{4.4}$$

we know that

$$\alpha_{12}^3 + \alpha_{21}^3 + \alpha_{31}^2 = 0, \quad \alpha_{11}^3 - \alpha_{22}^3 = 0.$$

Replacing e_1 by e_2 in the identity (4.4), we know that

$$\alpha_{12}^3 + \alpha_{21}^3 - \alpha_{31}^2 = 0, \quad \alpha_{11}^3 - \alpha_{22}^3 = 0.$$

Replacing e_3 by e_2 in the identity (4.4), we know that

$$\alpha_{11}^2 = \alpha_{21}^2 = 0, \quad \alpha_{11}^3 + \alpha_{22}^3 = 0.$$

Therefore

$$\alpha_{11}^2 = \alpha_{21}^2 = \alpha_{11}^3 = \alpha_{22}^3 = \alpha_{31}^2 = 0, \quad \alpha_{12}^3 + \alpha_{21}^3 = 0. \tag{4.5}$$

Furthermore, due to the above Eq. (4.5) and the identity

$$(e_1 e_2)e_1 - e_1(e_2 e_1) = (e_1 e_1)e_2 - e_1(e_1 e_2),$$

we know that

$$\alpha_{12}^3(e_3 e_1 + 2e_1 e_3) = 0.$$

Then $\alpha_{12}^3 = 0$ or $e_3 e_1 + 2e_1 e_3 = 0$. Moreover, if $e_3 e_1 + 2e_1 e_3 = 0$ it is easy to know that $\alpha_{12}^3 = 0$. Hence

$$\alpha_{11}^2 = \alpha_{21}^2 = \alpha_{11}^3 = \alpha_{22}^3 = \alpha_{31}^2 = \alpha_{12}^3 = \alpha_{21}^3 = 0.$$

It follows that $\dim AA \leq 2$, which is a contradiction. Therefore A is trivial. ∎

Let $sl(2, \mathbb{C})$ be the 3-dimensional complex simple Lie algebra consisting of all 2×2 matrices with trace zero.

Theorem 4.8. *Let A be a complex G_5-associative algebra on $sl(2, \mathbb{C})$ with a nondegenerate, symmetric and left invariant bilinear form. Then there exists a basis $\{h, x, y\}$ of A such that the non-zero products are given by*

$$hx = x, \quad xh = -x, \quad hy = -y, \quad yh = y, \quad xy = \frac{1}{2}h, \quad yx = -\frac{1}{2}h.$$

Proof. Let f be the nondegenerate, symmetric and left invariant bilinear form, and let $\{h, x, y\}$ be a basis of $sl(2, \mathbb{C})$ such that

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

It is easy to know that the identity (2.11) is equivalent to

$$[xy, z] + [yz, x] + [zx, y] = 0, \quad \forall\, x, y, z \in A. \tag{4.6}$$

We prove this theorem by the following several lemmas.

Lemma 4.9.   *There exists $a \in \mathbb{C}$ such that $hh = ah$.*

Proof.   Let $hh = ah + bx + cy$. Since $[hh, x] + [hx, h] + [xh, h] = 0$, we know that $c = 0$. Replacing x by y, we know that $b = 0$. ∎

Lemma 4.10. $hx = \frac{a+2}{2}x$, $xh = \frac{a-2}{2}x$, $xx = 0$.

Proof.   By $[hh, x] + [hx, h] + [xh, h] = 0$ and

$$[hh, x] + [hx, h] + [xh, h] = 2ax + [hx, h] + [hx - 2x, h] = (2a + 4)x + [2hx, h],$$

we know that there exists $m \in \mathbb{C}$ such that $hx = \frac{a+2}{2}x + mh$. Therefore $xh = \frac{a-2}{2}x + mh$. Since $[hx, x] + [xx, h] + [xh, x] = 0$, we know that there exists $n \in \mathbb{C}$ such that $xx = 2mx + nh$. By the identity $[xx, x] = 0$, we know that $n = 0$. Thus $xx = 2mx$. Since

$$f(hx, x) = f(xh, h) = f(xx, x) = 0,$$

we have that

$$\frac{a+2}{2}f(x, x) + mf(h, x) = 0, \quad \frac{a-2}{2}f(x, h) + mf(h, h) = 0, \quad 2mf(x, x) = 0.$$

If $m \neq 0$, then $f(x, x) = f(h, x) = f(h, h) = 0$. It follows that f is degenerate which is a contradiction. So $m = 0$. ∎

Lemma 4.11. $hh = 0$, $hx = x$, $xh = -x$, $xx = 0$.

Proof.   By Lemma 4.9, $hh = ah$. Assume that $a \neq 0$. Since $f(hh, h) = af(h, h) = 0$, we know that $f(h, h) = 0$. Furthermore we know that

$$
\begin{aligned}
f(hh, x) &= af(h, x) = -f(h, hx) = -\frac{a+2}{2}f(h, x) \\
&= \frac{1}{2}(f([h, h], x) - f([h, x], h) + f([x, h], h)) = -2f(x, h).
\end{aligned}
$$

It follows that $f(h, x) = 0$. Then $f(hx, x) = \frac{a+2}{2}f(x, x) = 0$ and

$$f(hh, y) = af(h, y) = \frac{1}{2}(f([h, h], y) - f([h, y], h) + f([y, h], h)) = 2f(y, h).$$

Then $f(x, x) = 0$ or $f(h, y) = 0$. For any case, f is degenerate which is a contradiction. Thus $a = 0$. ∎

Similarly, we obtain the following conclusion.

Lemma 4.12. $hy = -y$, $yh = y$, $yy = 0$.

Lemma 4.13. $xy = \frac{1}{2}h$, $yx = -\frac{1}{2}h$.

Proof.   Since $[xx, y] + [xy, x] + [yx, x] = 0 = [h, x] + 2[yx, x]$, we know that $yx = -\frac{1}{2}h + bx$. Due to the identity $[yy, x] + [xy, y] + [yx, y] = 0 = [h, y] + 2[yx, y]$, we know that $b = 0$. Therefore $xy = \frac{1}{2}h$. ∎

The theorem follows from the above lemmas. ∎

Theorem 4.14.   *Let A be a G_5-associative algebra in dimension 3 with a positive definite, symmetric and left invariant bilinear form f. Then A is trivial or there exists a basis $\{e_1, e_2, e_3\}$ of A such that non-zero products are given by*

$$e_1 e_2 = -e_2 e_1 = e_3, \quad e_1 e_3 = -e_3 e_1 = -e_2, \quad e_2 e_3 = -e_3 e_2 = e_1. \qquad (4.7)$$

Proof.   Assume that A is not trivial. By Corollary 3.10, the associated Lie algebra of the complexification $A_{\mathbb{C}}$ of A must be the 3-dimensional complex simple Lie algebra $sl(2, \mathbb{C})$. Obviously $A_{\mathbb{C}}$ is a complex G_5-associative algebra. By Theorem 4.8, $A_{\mathbb{C}}$ is anti-commutative, i.e., $xy = -yx$ for any $x, y \in A_{\mathbb{C}}$. It is obvious that A is anti-commutative, too. Let $\{e_1', e_2', e_3'\}$ be a basis of A such that $f(e_i', e_j') = \delta_{ij}$. Since A is anti-commutative, we know that

$$e_1' e_1' = e_2' e_2' = e_3' e_3' = 0.$$

Let $e_1' e_2' = \alpha_1 e_1' + \alpha_2 e_2' + \alpha e_3'$. Then $\alpha_1 = \alpha_2 = 0$ by $f(e_1' e_2', e_2') = f(e_1' e_2', e_1') = 0$. Hence by letting $\alpha = \alpha_3$, we have

$$e_1' e_2' = -e_2' e_1' = \alpha e_3'.$$

Similarly, there exist $\beta, \gamma \in \mathbb{C}$ such that

$$e_1' e_3' = -e_3' e_1' = \beta e_2', \quad e_2' e_3' = -e_3' e_2' = \gamma e_1'.$$

Since A is anti-commutative and f is left invariant, we know that $f(e_1' e_2', e_3') = -f(e_2', e_1' e_3') = f(e_1', e_2' e_3')$. Since A is not trivial, we know that

$$\alpha = -\beta = \gamma \neq 0.$$

Therefore we have

$$e_1 e_2 = -e_2 e_1 = e_3, \quad e_1 e_3 = -e_3 e_1 = -e_2, \quad e_2 e_3 = -e_3 e_2 = e_1$$

if we replace the basis $\{\frac{e_1'}{\alpha}, \frac{e_2'}{\alpha}, \frac{e_3'}{\alpha}\}$ by $\{e_1, e_2, e_3\}$. ∎

Recall that a Lie group H is called *unimodular* if its left invariant Haar measure is also right invariant and a Lie algebra $\mathfrak{h}$ is called *unimodular* if trace $\mathrm{ad}(x) = 0$ for any $x \in \mathfrak{h}$, where $\mathrm{ad}(x)$ is the adjoint operator satisfying $\mathrm{ad}(x)y = [x, y]$ for any $y \in \mathfrak{h}$.

Lemma 4.15 ([10]).   *Let H be a connected 3-dimensional Lie group with a left invariant metric g and let $\mathfrak{h}$ be the associated Lie algebra. Then we have the following results.*

1) H is unimodular if and only if $\mathfrak{h}$ is a unimodular Lie algebra. Furthermore in this case, there exists a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{h}$ such that

$$g(e_i, e_j) = \delta_{ij}, \quad \text{and} \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3. \qquad (4.8)$$

2) If H is non-unimodular, then there exists a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{h}$ such that

$$g(e_i, e_j) = \delta_{ij}, \quad \text{and} \quad [e_1, e_2] = ae_2 + be_3, \quad [e_1, e_3] = ce_2 + de_3, \qquad (4.9)$$

where $ac + bd = 0$ and $a + d \neq 0$.

Theorem 4.16.   *Let A be a G_4-associative algebra in dimension 3 with a positive definite, symmetric and left invariant bilinear form f. Then A is trivial.*

Proof.   First we prove that the associated Lie algebra of A is a unimodular Lie algebra. Otherwise we assume that the associated Lie algebra of A is a non-unimodular Lie algebra. By Lemma 4.15, there exists a basis $\{e_1, e_2, e_3\}$ of A such that Eq. (4.9) holds, that is,

$$f(e_i, e_j) = \delta_{ij}, \quad [e_1, e_2] = ae_2 + be_3, \quad [e_1, e_3] = ce_2 + de_3,$$

where $ac + bd = 0$ and $a + d \neq 0$. By the above equation and Eq. (1.4), we have

$$e_1 e_1 = 0, \quad e_2 e_2 = ae_1, \quad e_3 e_3 = de_1;$$

$$e_1 e_2 = \frac{b-c}{2} e_3, \quad e_2 e_1 = -ae_2 - \frac{b+c}{2} e_3;$$

$$e_1 e_3 = \frac{c-b}{2} e_2, \quad e_3 e_1 = -\frac{b+c}{2} e_2 - de_3;$$

$$e_2 e_3 = e_3 e_2 = \frac{b+c}{2} e_1.$$

Since $(e_2, e_3, e_1) = (e_1, e_3, e_2)$ and $(e_3, e_2, e_1) = (e_1, e_2, e_3)$, we know that

$$bd + cd + 2ab = 0, \quad ab + ac + 2cd = 0. \qquad (4.10)$$

Therefore with the equation $ac + bd = 0$, we know that

$$ab + cd = 0.$$

Then by the above equation and Eq. (4.10) and $a + d \neq 0$, we know that $b = 0$. Since $(e_1, e_2, e_2) = (e_2, e_2, e_1)$, we know that

$$c^2 + bc + 2a^2 = 0,$$

which implies $a = c = 0$. It follows that $AA \neq A$. Then A is trivial by Lemma 4.6, which is a contradiction.

Then the associated Lie algebra of A is a unimodular Lie algebra. Thus by Lemma 4.15, there exists a basis $\{e_1, e_2, e_3\}$ of A such that Eq. (4.8) holds, i.e.

$$f(e_i, e_j) = \delta_{ij}, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3.$$

Then by the above equation and Eq. (1.4), we know that

$$e_1 e_1 = e_2 e_2 = e_3 e_3 = 0;$$

$$e_1 e_2 = \frac{-\lambda_1 + \lambda_2 + \lambda_3}{2} e_3, \quad e_2 e_1 = \frac{-\lambda_1 + \lambda_2 - \lambda_3}{2} e_3;$$

$$e_1 e_3 = \frac{\lambda_1 - \lambda_2 - \lambda_3}{2} e_2, \quad e_3 e_1 = \frac{\lambda_1 + \lambda_2 - \lambda_3}{2} e_2;$$

$$e_2 e_3 = \frac{\lambda_1 - \lambda_2 + \lambda_3}{2} e_1, \quad e_3 e_2 = \frac{-\lambda_1 - \lambda_2 + \lambda_3}{2} e_1.$$

Since $(e_1, e_3, e_3) = (e_3, e_3, e_1)$, we know that

$$(\lambda_1 - \lambda_2)^2 - \lambda_3^2 = (\lambda_1 + \lambda_2 - \lambda_3)^2.$$

Similarly, by the equations $(e_3, e_1, e_1) = (e_1, e_1, e_3)$ and $(e_1, e_2, e_2) = (e_2, e_2, e_1)$ respectively, we know that

$$(\lambda_2 - \lambda_3)^2 - \lambda_1^2 = (\lambda_1 - \lambda_2 - \lambda_3)^2,$$

$$(\lambda_1 - \lambda_3)^2 - \lambda_2^2 = (\lambda_1 - \lambda_2 + \lambda_3)^2.$$

Therefore we know that

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0.$$

It follows that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence A is trivial. ∎

Remark 4.17. Theorem 4.16 also can be obtained by a more complicated computation by considering directly the 3-dimensional G_4-associative algebras with a positive definite and symmetric bilinear form as we have done for the other G-associative algebras. On the other hand, by a similar study as in the proof of Theorem 4.16 which follows from Lemma 4.15 given by Milnor in [10], the corresponding results for the other G-associative algebras in this paper can be obtained, too. Obviously, for these cases, our approach is less complicated and certain part of the conclusions are more explicit and may extend to higher dimensions.

Remark 4.18. We have shown that besides two examples, most of the G-associative algebras ($G = G_i, i = 1, \cdots, 5$) in dimension ≤ 3 with a positive definite, symmetric and left invariant bilinear form are trivial. So it seems that these G-associative algebras are not very "compatible" with the positive definite, symmetric and left invariant bilinear forms. However, the situation will be different in higher dimensions, although the study in higher dimensions is very complicated (Milnor only gave a complete classification in dimension 3 in [10] and it is quite difficult to extend his study to higher dimensions). For example, there are more non-trivial cases of left-symmetric algebras in dimension ≥ 4 ([4]). Moreover, our approach may be helpful to the study in higher dimensions and the study in this paper can be a guide for further development.

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