A Characterization of the Unitary
Highest Weight Modules by Euclidean Jordan Algebras

Zhanqiang Bai

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Abstract.   Let $\mathfrak{co}(J)$ be the conformal algebra of a simple Euclidean Jordan algebra J. We show that a (non-trivial) unitary highest weight $\mathfrak{co}(J)$-module has the smallest positive Gelfand-Kirillov dimension if and only if a certain quadratic relation is satisfied in the universal enveloping algebra $U(\mathfrak{co}(J)_{\mathbb{C}})$. In particular, we find an quadratic element in $U(\mathfrak{co}(J)_{\mathbb{C}})$. A prime ideal in $U(\mathfrak{co}(J)_{\mathbb{C}})$ equals the Joseph ideal if and only if it contains this quadratic element.
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1.   Introduction

1.1. Unitary highest weight modules.   Let (G, K) be an irreducible Hermitian symmetric pair of non-compact type. If we denote their Lie algebras by $(\mathfrak{g}_0, \mathfrak{k}_0)$, denote their complexification by $\mathfrak{g} = \mathfrak{g}_0 \bigotimes \mathbb{C}$ and $\mathfrak{k} = \mathfrak{k}_0 \bigotimes \mathbb{C}$, then $\mathfrak{k} = \mathbb{C}H \bigoplus [\mathfrak{k}, \mathfrak{k}]$ with $\mathrm{ad}(H)$ having eigenvalues $0, 1, -1$ on $\mathfrak{g}$. Usually we use the real lie group G for the Hermitian symmetric pair (G, K) or merely the Lie algebra $\mathfrak{g}_0$. There are seven cases of Hermitian symmetric pairs which we refer to as follows:

$$\mathfrak{su}(p, q), \ \mathfrak{sp}(n, \mathbb{R}), \ \mathfrak{so}^*(2n), \ \mathfrak{so}(2, 2n-1), \ \mathfrak{so}(2, 2n-2), \ \mathfrak{e}_{6(-14)}, \ \mathfrak{e}_{7(-25)}.$$

Let M be an irreducible non-compact Hermitian symmetric space. Let G be the identity component of the automorphism group of M. Let K be the stabilizer of a fixed point of M, which is a maximal compact subgroup of G. Then M can be identified with G/K as smooth manifolds and (G, K) will be a Hermitian symmetric pair of non-compact type.

We call (G, K) a tube type Hermitian symmetric pair when G/K is a tube type Hermitian symmetric space. In our notation, the tube type Hermitian symmetric pairs are as follows:

$$\mathfrak{su}(p, p), \ \mathfrak{sp}(n, \mathbb{R}), \ \mathfrak{so}^*(4n), \ \mathfrak{so}(2, 2n-1), \ \mathfrak{so}(2, 2n-2), \ \mathfrak{e}_{7(-25)}.$$

Recall that the irreducible unitary representations of G are in one-to-one correspondence with the irreducible unitary $(\mathfrak{g}, K)$-modules. And a representation of G is called a highest weight representation if its underlying $(\mathfrak{g}, K)$-module is a highest weight $\mathfrak{g}$-module. Then we know a unitary highest weight module is irreducible. When (G, K) is a Hermitian symmetric pair, the unitary highest weight modules of G had been classified by Enright-Howe-Wallach [5] and Jakobsen [15, 16]. We sometime call such G-modules simply by $\mathfrak{g}$-modules or $\mathfrak{g}_0$-modules.

1.2. Euclidean Jordan algebras. The Euclidean Jordan algebras were initially introduced by P. Jordan [17] for the purpose of reformulating quantum mechanics in a minimal way. Then Jordan-von Neumann-Wigner [18] classified the simple finite dimensional Euclidean Jordan algebras: they consist of four infinity series (i.e. $\Gamma(n), \mathcal{H}_n(\mathbb{R}), \mathcal{H}_n(\mathbb{C})$ and $\mathcal{H}_n(\mathbb{H})$) and one exceptional (i.e. $\mathcal{H}_3(\mathbb{O})$).

Although Euclidean Jordan algebras are abandoned by physicists quickly, the Jordan methods have proved useful tools in a variety of settings by mathematicians since 1950's. Some applications can be found in McCrimmon [25, 26]. From Koufany [23] and Faraut-Korányi [9], we know the tube type Hermitian symmetric spaces, irreducible tube domains and the conformal algebras $\mathfrak{co}(J)$ of simple Euclidean Jordan algebras are in natural one-to-one correspondence. So tube type Hermitian symmetric pairs and the conformal algebras of simple Euclidean Jordan algebras are in one-to-one correspondence. In this paper, we will use the language of Euclidean Jordan algebras to study the unitary highest weight $\mathfrak{co}(J)$-modules.

1.3. Kepler problems and quadratic relations. The Kepler problem is a physics problem about two bodies with an attractive force obeying the inverse square law. Mathematically it is a mechanical system with configuration space $\mathbb{R}^3_* := \mathbb{R}^3 \backslash \{\mathbf{0}\}$ and Lagrangian

$$L = \frac{1}{2}\mathbf{r}'^2 + \frac{1}{r},$$

where $\mathbf{r}$ is a function of t taking value in $\mathbb{R}^3_*$, $\mathbf{r}'$ is the velocity vector and r is the length of $\mathbf{r}$. Therefore, quantum mechanically the Hamiltonian for the Kepler problem becomes

$$\hat{H} = -\frac{1}{2}\Delta - \frac{1}{r},$$

where Δ is the Laplace operator on $\mathbb{R}^3$.

The MICZ-Kepler problems are generalizations of the Kepler problems, and they were independently discovered by McIntosh-Cisneros [27] and Zwanziger [43] more than thirty years ago. The (classical) MICZ-Kepler problem with magnetic charge $\mu \in \mathbb{R}$ is a natural mathematical generalization of the Kepler problem, with the Lagrangian

$$L = \frac{1}{2}\mathbf{r}'^2 + \frac{1}{r} - \mathbf{A} \cdot \mathbf{r}' - \frac{\mu^2}{2r^2},$$

where $\mathbf{A}$ is the magnetic potential. Let $\mathbf{B} := \nabla \times \mathbf{A} = \mu\frac{\mathbf{r}}{r^3}$. Then the equation of motion is

$$\mathbf{r}'' = -\mathbf{r}' \times \mathbf{B} + \left(\frac{\mu^2}{r^4} - \frac{1}{r^3}\right)\mathbf{r}.$$

Recently Meng [28, 29, 35, 33] generalized the MICZ-Kepler problems and discovered a family of quadratic relations. Let m be an integer and $m \geq 2$. We define a $(3+m) \times (3+m)$ matrix $[\eta_{\mu\nu}] = \text{diag}(1,1,-1,...,-1)$. We denote its inverse by $[\eta^{\mu\nu}]$. Let $\text{Cl}_{2,m+1}$ be the Clifford algebra over $\mathbb{C}$ generated by $X'_\mu s$ subject to relations:

$$X_\mu X_\nu + X_\nu X_\mu = -2\eta_{\mu\nu}.$$

Let us denote $M_{\mu\nu} = \frac{i}{4}(X_\mu X_\nu - X_\nu X_\mu)$. Then we have

$$[M_{ab}, M_{cd}] = -i(\eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc}). \tag{1.1}$$

So $\{M_{\mu\nu}\}$ generate $\mathfrak{so}(2, m+1)$.

Let (π, V) denote the unitary highest $\mathfrak{so}(2, m+1)$-module, and $\pi(\mathcal{O}) := \tilde{\mathcal{O}}$ for any $\mathcal{O} \in \mathfrak{so}(2, m+1)$. We use $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ to denote the commutator and anticommutator in a Lie algebra throughout the paper.

Then Meng proved the following theorem.

Theorem 5.1 [30, *Theorem 2*] *A (non-trivial) unitary highest weight module of* $\mathfrak{so}(2, m+1) = (\mathfrak{co}(\Gamma(m)))$ *has the smallest positive Gelfand-Kirillov dimension if and only if it satisfies the following quadratic relations on the underlying module space:*

$$\{\tilde{M}_{\mu\lambda}, \tilde{M}_\nu^\lambda\} = c\eta_{\mu\nu} \text{ with } \mu, \nu = -1, 0, 1, ..., m+1, \tag{1.2}$$

where $\tilde{M}_\nu^\lambda = \eta^{\lambda\kappa}\tilde{M}_{\kappa\nu}$ and c is a representation-dependent real number.

We denote a unitary highest $\mathfrak{co}(J)$-module by (π, V), $\pi(\mathcal{O}) := \tilde{\mathcal{O}}$ for any $\mathcal{O} \in \mathfrak{co}(J)$. In this paper we will show the following main theorem.

Theorem 4.1 *A (non-trivial) unitary highest weight* $\mathfrak{co}(J)$*-module has the smallest positive Gelfand-Kirillov dimension if and only if the following primary quadratic relation is satisfied:*

(Q1) $\frac{2}{\rho} \sum_{1 \leq \alpha \leq D} \tilde{L}_{e_\alpha}^2 - \tilde{L}_e^2 - \frac{1}{2}\{\tilde{X}_e, \tilde{Y}_e\} = -aI_V$ *as an operator on* V.

Here $D = \dim(J)$. $a = a(J, k)$ *(given in Remark 7.2) is a nonzero constant and only depends on the highest weight* $\lambda = \lambda(k)$ *(given in Corollary 3.1).* $\{e_\alpha\}$ *is an orthonormal basis for* J. ρ *is the rank of* J. L_{e_α}, X_e *and* Y_e *are generators of the conformal algebra* $\mathfrak{co}(J)$.

We find that the two quadratic relations in the two theorems are equivalent when $J = \Gamma(m)$. So we call (Q1) a generalized quadratic relation.

This generalized quadratic relation is firstly introduced by Meng [36]. When Meng reconstructed the various Kepler-type problems [31, 32, 34] in the unified language of Euclidean Jordan algebras in Ref.[36], he constructed the minimal representation for the conformal algebra $\mathfrak{co}(J)$. Then he showed that this representation actually satisfied a quadratic relation, which corresponded to $k = 0$ in (Q1).

For a simple complex Lie algebra $\mathfrak{g}$ not of type A_n, Joseph [19] constructed a completely prime 2-sided ideal J_0 in the universal enveloping algebra $U(\mathfrak{g})$, whose associated variety is the closure of the minimal nilpotent orbit in $\mathfrak{g}^*$. In this paper,

we find that any prime ideal $I \subseteq U(\mathfrak{g})$ equals the Joseph ideal J_0 if and only if I contains $Q'_1 \triangleq \left(\frac{2}{\rho} \sum_{1 \leq \alpha \leq D} L^2_{e_\alpha} - L^2_e - \frac{1}{2}\{X_e, Y_e\} + a\right)$ with $a = a(J, 0) = \frac{\rho d}{4}(1 + \frac{(\rho-2)d}{4})$.

1.4. Outline of the paper. In section 2, we recall some basic definitions and properties about Jordan algebras. We will give their relation with tube type Hermitian symmetric pairs. In section 3, we recall the unitary highest weight modules and the classification theorem by Enright-Howe-Wallach [5]. In section 4, we state the main theorem and give some corollaries. Then in section 5, 6 and 7, we give a case-by-case proof for the main theorem.

2. Some basic facts in Euclidean Jordan algebras

The facts and theorems reviewed in this section can be found in Koecher [22] and Faraut-Korányi [9].

A linear *Jordan algebra* J is an(not necessarily associative) algebra over a field $\mathbb{F}$ whose multiplication satisfies the following axioms:

1. $xy = yx$ (*commutative law*).

2. $x(x^2 y) = x^2(xy)$ (*Jordan identity*).

If we define the left multiplication in the Jordan algebra J by $L_u(v) = uv$, then a Jordan algebra over a field $\mathbb{F}$ is just a commutative algebra over $\mathbb{F}$ such that $[L_u, L_{u^2}] = 0$.

Suppose V is an associative algebra over a field $\mathbb{F}(Char \neq 2)$, and A is a linear subspace of V, closed under square operations, i.e, $u \in A \Rightarrow u^2 \in A$. Any such A gives rise to a Jordan algebra A^+ under a new product:

$$x \circ y := ((x+y)^2 - x^2 - y^2)/2.$$

For example, if we take $V = \mathrm{Cl}(\mathbb{R}^n, Q)$ (i.e, the Clifford algebra of $\mathbb{R}^n$ equipped with a quadratic form Q) and $A = \mathbb{R} \oplus \mathbb{R}^n$, then we get a Jordan algebra A^+, denoted by $\Gamma(n)$.

A Jordan algebra J is called *special* if it can be realized as a Jordan subalgebra of some A^+. All other Jordan algebras are called *exceptional* Jordan algebras.

A Jordan algebra J is called *semi-simple* if its canonical symmetric bilinear form τ (where $\tau(u, v) =$ the trace of L_{uv}) is non-degenerate. J is called *simple* if it is *semi-simple* and has no nontrivial ideals. Every semi-simple Jordan algebra J is a direct sum of simple Jordan algebras.

A Jordan algebra J over $\mathbb{R}$ is called *Euclidean* if its canonical symmetric bilinear form τ is positive definite. So a *Euclidean* Jordan algebra is *semi-simple*.

Theorem 2.1. (Jordan, von Neumann and Wigner). *The complete list of simple Euclidean Jordan algebras are the following:*

1. *The algebra $\Gamma(n) = \mathbb{R} \oplus \mathbb{R}^n$ $(n \geq 2)$.*

2. *The algebra $\mathcal{H}_n(\mathbb{R})$ $(n \geq 3$ or $n = 1)$.*

3. *The algebra $\mathcal{H}_n(\mathbb{C})$ $(n \geq 3)$.*

4. *The algebra $\mathcal{H}_n$ $(\mathbb{H})$ $(n \geq 3)$.*

5. *The algebra $\mathcal{H}_3$ $(\mathbb{O})$.*

Remark 2.1. $\mathcal{H}_n(\mathbb{F})$ denotes Hermitian $n \times n$ matrices with entries in $\mathbb{F}$. In 1934, Albert [1] showed that $\mathcal{H}_3$ $(\mathbb{O})$ is not special.

We define the *trace* of an element in a simple Euclidean Jordan algebra J as: $\mathrm{tr}(\lambda, \vec{u}) = 2\lambda$ for $\Gamma(n)$, and $\mathrm{tr}(u) = \mathrm{trace}(u)$ for the other types.

We define the *inner product* on a Euclidean Jordan algebra J by

$$\langle u|v \rangle := \frac{1}{\dim(J)}\tau(u, v),$$

for all $u, v \in J$.

Every semi-simple Jordan algebra(over a field K with Char $= 0$) has a unit element e. We define the *rank* of a Euclidean Jordan algebra J as $\rho := \mathrm{tr}(e)$.

For a simple Euclidean Jordan algebra J, we have:

$$\mathrm{tr}(u) = \frac{\rho}{\dim(J)}\mathrm{Tr}(L_u) = \frac{\rho}{\dim(J)}\tau(u, e).$$

Then we see $\langle u|v \rangle = \frac{1}{\rho}\mathrm{tr}(uv)$.

Theorem 2.2. *Let J be a simple Euclidean Jordan algebra of rank ρ , $x_0 \in J$ is non-zero and $x_0^2 = tr(x_0)x_0$. Then there is an orthogonal basis for J: $\{e_{11}, \cdots, e_{\rho\rho}, e_{ij}^\mu\}$ with ($1 \leq i < j \leq \rho$, $1 \leq \mu \leq d$) such that*

1. *each basis vector has length $\frac{1}{\sqrt{\rho}}$;*

2. *$e_{ii}^2 = e_{ii}$, $e_{ii}e_{jj} = 0$ for $i \neq j$;*

3. *$\sum\limits_{i=1}^{\rho} e_{ii} = e$;*

4. *$(e_{jk}^\mu)^2 = \frac{1}{2}(e_{jj} + e_{kk})$, $e_{ii}e_{ij}^\mu = e_{jj}e_{ij}^\mu = \frac{1}{2}e_{ij}^\mu$, $e_{ii}e_{jk}^\mu = 0$ if $i \neq j, i \neq k$;*

5. *$tre_{ii} = 1$, $tre_{ij}^\mu = 0$;*

6. *$x_0 = (trx_0)e_{11}$.*

Remark 2.2. The parameter d in the above theorem is called the *degree* of J, and $dim(J) = \rho + \frac{d\rho(\rho-1)}{2}$. The set $\{e_{11}, \cdots, e_{\rho\rho}\}$ is called a *Jordan frame*.

We have the following table:

A *derivation* D of a Jordan algebra J is a linear transformation of J such that

$$D(xy) = Dx \cdot y + x \cdot Dy,$$

Table 2.1:

J	$\Gamma(n)$	$\mathcal{H}_n(\mathbb{R})$	$\mathcal{H}_n(\mathbb{C})$	$\mathcal{H}_n(\mathbb{H})$	$\mathcal{H}_3(\mathbb{O})$
ρ	2	n	n	n	3
d	$n-1$	1	2	4	8

for all $x, y \in J$. The set $\mathfrak{der}(J)$ of all derivations of J is a Lie algebra with respect to the usual bracket, $[D_1, D_2] = D_1 D_2 - D_2 D_1$. An *automorphism* W of J is an invertible linear transformation of J such that

$$W(xy) = W(x) \cdot W(y),$$

for all $x, y \in J$. The set $Aut(J)$ of all automorphisms of J is a a Lie group since it is a closed subgroup of $GL(J)$.

The Lie algebra of $Aut(J)$ is $\mathfrak{der}(J)$. If J is semi-simple(over $\mathbb{R}$ or $\mathbb{C}$), and L_x is the left multiplication of J, then every $D \in \mathfrak{der}(J)$ is a finite sum of $[L_x, L_y]$, with $x, y \in J$.

If J is a Jordan algebra, we define the *quadratic representation* of J:

$$P(x) = 2L_x^2 - L_{x^2}.$$

Then we can define the *structure group* by $Str(J) := \{W \in GL(J) \mid P(Wx) = WP(x)V$ for all $x \in J$ and some fixed $V \in GL(J)\}$.

If we denote $S_{uv} = [L_u, L_v] + L_{uv}, S_{uv}(z) = \{uvz\}$ and $S'_{uv} = S_{vu}$, then

$$[S_{uv}, S_{zw}] = S_{\{uvz\}w} - S_{z\{vuw\}}.$$

So $\mathfrak{str}(J) := span\{S_{uv} \mid u, v \in J\}$ becomes a real Lie algebra, called the *structure algebra* of J. When J is a semi-simple Jordan algebra (over $\mathbb{R}$ or $\mathbb{C}$), the Lie algebra of $Str(J)$ will be equal to $\mathfrak{str}(J)$.

Let J be a simple Euclidean Jordan algebra. We denote

$$\Omega := \{\text{the interior of } J^2\}, T_\Omega := J + i\Omega.$$

Let $Aut(T_\Omega)$ be the group of holomorphic automorphisms of the tube domain T_Ω. Then the Lie algebra of $Aut(T_\Omega)$ is $\mathfrak{co}(J)$, called the *conformal algebra* of J with Lie bracket defined in the next theorem. As a vector space,

$$\mathfrak{co}(J) = J \oplus \mathfrak{str}(J) \oplus J^*.$$

In the following we shall rewrite $u \in J$ as X_u and $v \in J^*$ as Y_v.

Theorem 2.3. *(Tits-Kantor-Koecher Construction). Let J be a simple Euclidean Jordan algebra, then $\mathfrak{co}(J) := J \oplus \mathfrak{str}(J) \oplus J^*$ becomes a simple real Lie algebra with the definitions as following:*

1. $[X_u, X_v] = 0$, $[Y_u, Y_v] = 0$;

2. $[X_u, Y_v] = -2S_{uv}$;

3. $[S_{uv}, X_z] = X_{\{uvz\}} = X_{S_{uv}(z)}$, $[S_{uv}, Y_z] = -Y_{\{vuz\}} = -Y_{S'_{uv}(z)}$;

4. $[S_{uv}, S_{zw}] = S_{\{uvz\}w} - S_{z\{vuw\}} = S_{S_{uv}(z)w} - S_{zS'_{uv}(w)}$,
   for $u, v, z, w \in J$.

Remark 2.3. Actually, we have the following Table 2.2:

Table 2.2:

J	$\mathfrak{der}$	$\mathfrak{str}$	$\mathfrak{co}$
$\Gamma(n)$	$\mathfrak{so}(n)$	$\mathfrak{so}(n,1) \oplus \mathbb{R}$	$\mathfrak{so}(2, n+1)$
$\mathcal{H}_n(\mathbb{R})$	$\mathfrak{so}(n)$	$\mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R}$	$\mathfrak{sp}(n, \mathbb{R})$
$\mathcal{H}_n(\mathbb{C})$	$\mathfrak{su}(n)$	$\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}$	$\mathfrak{su}(n,n)$
$\mathcal{H}_n(\mathbb{H})$	$\mathfrak{sp}(n)$	$\mathfrak{su}^*(2n) \oplus \mathbb{R}$	$\mathfrak{so}^*(4n)$
$\mathcal{H}_3(\mathbb{O})$	$\mathfrak{f}_4$	$\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$	$\mathfrak{e}_{7(-25)}$

3.   The classification of unitary highest weight modules

In this section we review some well-known facts and notations about the classification of unitary highest weight moduless. The details can be found in the paper Enright, Howe and Wallach [5], hereafter referred to as EHW[5].

Let (G, K) be an irreducible Hermitian symmetric pair with real Lie algebras $\mathfrak{g}_0$ and $\mathfrak{k}_0$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be their complexification. Then $\mathfrak{k} = \mathbb{C}H \bigoplus [\mathfrak{k}, \mathfrak{k}]$ with $\text{ad}(H)$ having eigenvalues $0, 1, -1$ on $\mathfrak{g}$. If we denote $\mathfrak{p}^{\pm} = \{X \in \mathfrak{g} | [H, X] = \pm X\}$, then $\mathfrak{g} = \mathfrak{p}^- \bigoplus \mathfrak{k} \bigoplus \mathfrak{p}^+$.

Let $\mathfrak{h}$ be a Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{k}$. Let Δ denote the roots of $(\mathfrak{g}, \mathfrak{h})$ and Δ_c the roots of $(\mathfrak{k}, \mathfrak{h})$. Let Δ_n denote the complement so that $\Delta = \Delta_c \cup \Delta_n$. We call the roots in these two sets the *compact roots* and *non-compact roots* respectively. Let Δ^+ denote a fixed positive root system for which $\Delta^+ = \Delta_c^+ \cup \Delta_n^+$ and $\Delta_n^+ = \{\alpha \in \Delta^+ | \mathfrak{g}_\alpha \subset \mathfrak{p}^+\}$. Let β denote the unique maximal non-compact root of Δ^+ and denote $\beta^\vee = \frac{2\beta}{(\beta, \beta)}$. Now choose $\zeta \in \mathfrak{h}^*$ so that ζ is orthogonal to Δ_c and $(\zeta, \beta^\vee) = 1$. Let $\lambda \in \mathfrak{h}^*$ be Δ_c^+-dominant integral and $F(\lambda)$ be the irreducible $\mathfrak{k}$-module with highest weight λ. By letting $\mathfrak{p}^+$ act by zero, we may consider $F(\lambda)$ as a module of $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$. Then we define:

$$N(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} F(\lambda).$$

Let $L(\lambda)$ denote the irreducible quotient of $N(\lambda)$. If $L(\lambda)$ is unitarizable(i.e it is equivalent to the $\mathfrak{g}$-module of K-finite vectors in a unitary representation of G, or it is a unitary $(\mathfrak{g}, K)$-module), then $\lambda = \lambda_0 + z\zeta$, with $\lambda_0 \in \mathfrak{h}^*$ such that $(\lambda_0 + \rho, \beta) = 0$, and $z \in \mathbb{R}$. For a fixed λ_0, the set of all $z \in \mathbb{R}$ with $L(\lambda_0 + z\zeta)$ a unitary $\mathfrak{g}$-module takes the following form:

$$\bullet \quad \bullet \quad \bullet \cdots \bullet \quad \bullet$$
$$A(\lambda_0) \hspace{5cm} B(\lambda_0)$$

Let $Z(\lambda_0) = \{z \in \mathbb{R} \mid L(\lambda_0 + z\zeta) \text{ is unitarizable }\}$. Let $Z(\lambda_0)_r$ denote the subset of $Z(\lambda_0)$ for which $N(\lambda)$ is reducible. We call $Z(\lambda_0)_r$ the unitary reduction points on the line. These points correspond to the value of z on the above line which are equally spaced from $A(\lambda_0)$ to $B(\lambda_0)$. $A(\lambda_0)$ is called the first reduction point, and $B(\lambda_0)$ is called the last reduction point. The *reduction level* $r(\lambda_0)$ of λ_0 is the number of points in $Z(\lambda_0)_r$. We list these reduction points by

$$A(\lambda_0) = z_{r(\lambda_0)}^{\lambda_0} < z_{r(\lambda_0)-1}^{\lambda_0} < ... < z_1^{\lambda_0} = B(\lambda_0).$$

From now on we simply call $L(\lambda)$ a unitary highest weight $\mathfrak{g}_0$-module since it is a module of the hermitian symmetric pair $\mathfrak{g}_0$.

We define $C(\lambda_0) = \frac{B(\lambda_0)-A(\lambda_0)}{r(\lambda_0)-1}$. Let r be the split rank of $\mathfrak{g}_0$. Then r equals the reduction level of the weight $\lambda_0 = -(\rho, \beta^\vee)\zeta$. And we have the following table from EHW[5]:

Table 3.1:

$\mathfrak{g}_0$	$\mathfrak{su}(p,q)$	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{so}^*(2n)$	$\mathfrak{so}(2,2n-2)$	$\mathfrak{so}(2,2n-1)$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{7(-25)}$
$C(\lambda_0)$	1	1/2	2	$n-2$	$n-3/2$	3	4
r	$min\{p,q\}$	n	$[n/2]$	2	2	2	3
$(\rho,\beta^\vee)$	$p+q-1$	n	$2n-3$	$2n-3$	$2n-2$	11	17

So we can denote $C(\lambda_0)$ by C since it is independent of λ_0.

Lemma 3.1.   *Suppose J is one of the 5 simple Euclidean Jordan algebras, then*

1. *$r = \rho$, i.e., the split rank of $co(J)$ equals the rank of J;*

2. *$2C = d$, here d is the degree of J.*

Let $\mathfrak{g}$ be a complex Lie algebra and $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. We denote by $U_n(\mathfrak{g})$ the finite dimensional subspace of $U(\mathfrak{g})$, which are spanned by products of at most n elements in $\mathfrak{g}$. Then $\{U_n(\mathfrak{g})\}_{n=0}^\infty$ is the natural filtration of $U(\mathfrak{g})$. By the Poincaré-Birkhoff-Witt theorem, we can identify the associated graded ring $\mathrm{gr}U(\mathfrak{g}) = \oplus_{n=0}^\infty U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ with the symmetric algebra $S(\mathfrak{g})$.

Let M be a finitely generated $U(\mathfrak{g})$-module. We can take a finite dimensional subspace M_0 of M such that $M = U(\mathfrak{g})M_0$. Define $M_n = U_n(\mathfrak{g})M_0 (n = 1, 2, ...)$. Then $\{M_n\}$ gives an increasing filtration of M. We get a finitely generated graded $S(\mathfrak{g})$-module $\mathrm{gr}M = \oplus_{n=0}^\infty \mathrm{gr}_n M$, $\mathrm{gr}_n M = M_n/M_{n-1}$.

Let $\mathrm{Ann}(L(\lambda)) \subseteq U(\mathfrak{g})$ be the annihilator of the unitary highest weight module $L(\lambda)$. Then its graded ideal is:

$$\mathrm{gr}(\mathrm{Ann}(L(\lambda))) = \mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{gr}L(\lambda)) = \{D \in S(\mathfrak{g}) \mid Dw = 0 \text{ for all } w \in \mathrm{gr}(L(\lambda))\}.$$

The *associated variety* of the $(\mathfrak{g}, K)$-module $L(\lambda)$ is defined to be:

$$\mathcal{V}(L(\lambda)) = \mathcal{V}(\mathrm{gr}(\mathrm{Ann}(L(\lambda)))) = \{X \in \mathfrak{g}^* \mid D(X) = 0 \text{ for all } D \in \mathrm{gr}(\mathrm{Ann}(L(\lambda)))\}.$$

Here $S(\mathfrak{g})$ is viewed as the polynomial ring over $\mathfrak{g}^*$ through the Killing form of $\mathfrak{g}$. In the language of algebraic geometry, $\mathcal{V}(L(\lambda))$ is just the set of all prime ideals containing $\mathrm{Ann}_{S(\mathfrak{g})}\mathrm{gr}(L(\lambda))$, a closed subvariety of the affine algebraic variety $\mathrm{Spec}\,S(\mathfrak{g})$. It is also a K-invariant subvariety of $\mathfrak{g}^*$ (or of $(\mathfrak{g}/\mathfrak{k})^*$) since the compatibility of the K action and the module structure on $L(\lambda)$. In fact $\mathcal{V}(L(\lambda))$ is equal to the closure of some nilpotent $K_{\mathbb{C}}$-orbit, also equal to the union of a finite number of nilpotent $K_{\mathbb{C}}$-orbits by Vogan [42].

We define the *Gelfand-Kirillov dimension* of $L(\lambda)$ by:

$$\mathrm{GK}\dim(L(\lambda)) = \dim\mathcal{V}(L(\lambda)).$$

For a simple complex Lie algebra $\mathfrak{g}$ not of type A_n, Joseph [19] constructed a unique completely prime 2-sided ideal J_0 in the universal enveloping algebra $U(\mathfrak{g})$, with the property that its associated variety $\mathcal{V}(\mathrm{gr}(J_0))$ is equal to the closure of the minimal nilpotent orbit in $\mathfrak{g}^*$. This ideal is calllled the *Joseph Ideal*. However, it was noticed by Savin [38] that there is a gap in the proof of Lemma 8.8 in Joseph[19]. Gan and Savin[10] proved the uniqueness property of the Joseph ideal.

A unitary highest weight $\mathfrak{g}_0$-module $L(\lambda)$ is called *minimal* if its annihilator is the Joseph ideal. So it must have the smallest positive Gelfand-Kirillov dimension. From Refs. [6, 8, 20], we have the following lemma.

Lemma 3.2. *A unitary highest weight $\mathfrak{g}_0$-module $L(\lambda) = L(\lambda_0 + z\zeta)$ has the smallest positive Gelfand-Kirillov dimension if and only if $z = (\rho, \beta^\vee) - C$, and $L(\lambda_0 + z\zeta)$ has the zero Gelfand-Kirillov dimension if and only if $z = (\rho, \beta^\vee)$.*

Following the notations in Bourbaki [3], we can write the highest weight λ as a tuple of real numbers. Then we have the following corollary.

Corollary 3.1. *The unitary highest weight $\mathfrak{co}(J)$-module $L(\lambda)=L(\lambda_0+z\zeta)$ has the smallest positive Gelfand-Kirillov dimension if and only if:*

$$\lambda = \begin{cases} (-(n+k-\frac{1}{2}), k, ..., k), k = 0 \ or \ 1/2, & if \ J = \Gamma(2n) \\ (-(n+|k|-1), |k|, ..., |k|, k), k \ is \ a \ half \ integer, & if \quad J = \Gamma(2n-1) \\ -(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, \frac{1}{2}+k), k = 0, 1. & if \ J = \mathcal{H}_n(\mathbb{R}) \\ (\underbrace{-\frac{n+k}{2n}, ..., -\frac{n+k}{2n}}_{n}, \frac{n-k}{2n}+k, \frac{n-k}{2n}, ..., \frac{n-k}{2n}), k \in \mathbb{Z}_{\geq 0}, & \\ or \ (-\frac{n-k}{2n}, ..., -\frac{n-k}{2n}, -\frac{n-k}{2n}-k, \underbrace{\frac{n+k}{2n}, ..., \frac{n+k}{2n}}_{n}), k \in \mathbb{Z}_{\geq 0}, & if \ J = \mathcal{H}_n(\mathbb{C}) \\ -(1, ..., 1, k+1), k \in \mathbb{Z}_{\geq 0}, & if \ J = \mathcal{H}_n(\mathbb{H}) \\ (0, 0, 0, 0, 0, -4, 2, -2), & if \ J = \mathcal{H}_3(\mathbb{O}) \end{cases}$$

$L(\lambda_0 + z\zeta)$ has the zero GK dimension if and only if λ equals zero weight.

Proof. From the above lemma, a unitary highest weight $\mathfrak{g}_0$-module $L(\lambda) = L(\lambda_0+z\zeta)$ has the smallest positive Gelfand-Kirillov dimension means $z = (\rho, \beta^\vee)-C$. Then case by case from EHW[5], we can compute all possible λ_0 for this z. ∎

From Hilgert-Kobayashi-Möllers [13] and Torasso [39], we have the following lemma about minimal representations.

Lemma 3.3. *A unitary highest weight $\mathfrak{co}(J)$-module $L(\lambda) = L(\lambda_0 + z\zeta)$ is minimal if and only if:*

$$\lambda = \begin{cases} (-(n-\frac{1}{2}), 0, ..., 0), & \text{if } J = \Gamma(2n) \\ (-(n-1), 0, ..., 0), & \text{if } J = \Gamma(2n-1) \\ -(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, \frac{1}{2} + k), k = 0, 1, & \text{if } J = \mathcal{H}_n(\mathbb{R}) \\ (-1, ..., -1), & \text{if } J = \mathcal{H}_n(\mathbb{H}) \\ (0, 0, 0, 0, 0, -4, 2, -2), & \text{if } J = \mathcal{H}_3(\mathbb{O}) \end{cases}$$

In other words, $\lambda = -C\zeta$ except for the odd part of the metaplectic representation.

The unitary highest $\mathfrak{g}_0$-module $L(\lambda) = L(-C\zeta)$ is called the *first Wallach representation.*

4. Main theorem

We denote a unitary highest $\mathfrak{co}(J)$-module $L(\lambda) = L(\lambda_0 + z\zeta)$ by (π, V), $\pi(\mathcal{O}) := \tilde{\mathcal{O}}$ for any $\mathcal{O} \in \mathfrak{co}(J)$.

Theorem 4.1. *A (non-trivial) unitary highest weight $\mathfrak{co}(J)$-module $L(\lambda) = L(\lambda_0 + z\zeta)$ has the smallest positive Gelfand-Kirillov dimension if and only if the following primary quadratic relation is satisfied:*

(Q1) $\frac{2}{\rho} \sum\limits_{1 \leq \alpha \leq D} \tilde{L}_{e_\alpha}^2 - \tilde{L}_e^2 - \frac{1}{2}\{\tilde{X}_e, \tilde{Y}_e\} = -aI_V$ *as an operator on V.*

    Here $D = \dim(J)$. $a = a(J, k)$ (the explicit values will be given in Remark 7.2) is a nonzero constant and only depends on the highest weight $\lambda = \lambda(k)$ (given in Corollary 3.1). $\{e_\alpha\}$ is an orthonormal basis for J.

Corollary 4.1. *From the quadratic relation (Q1), we can get the following secondary quadratic relations:*

(Q2) $\sum\limits_{1 \leq \alpha \leq D} \tilde{X}_{e_\alpha}^2 = \rho \tilde{X}_e^2$.

(Q3) $\sum\limits_{1 \leq \alpha \leq D} \{\tilde{X}_{e_\alpha}, \tilde{X}_{e_\alpha u}\} = \rho\{\tilde{X}_e, \tilde{X}_u\}$, *for any $u \in J$.*

(Q4) $\frac{2}{\rho} \sum\limits_{1 \leq \alpha \leq D} \{\tilde{L}_{e_\alpha}, \tilde{L}_{e_\alpha u^2 - u(ue_\alpha)}\} + \frac{4}{\rho} \sum\limits_{1 \leq \alpha \leq D} [\tilde{L}_u, \tilde{L}_{e_\alpha}]^2 + \{\tilde{X}_u, \tilde{Y}_u\} - \frac{1}{2}\{\tilde{X}_{u^2}, \tilde{Y}_e\} - \frac{1}{2}\{\tilde{X}_e, \tilde{Y}_{u^2}\} = 0$, *for any $u \in J$.*

Proof. Meng [36] had proved that the quadratic relation (Q1) implies (Q2) for all J.

We compute $[(Q1), \tilde{X}_e]$ and get:

$$\sum\limits_{1 \leq \alpha \leq D} \{\tilde{X}_{e_\alpha}, \tilde{L}_{e_\alpha}\} = \rho\{\tilde{X}_e, \tilde{L}_e\}. \tag{4.1}$$

Then $[(4.1), \tilde{X}_u]$ implies (Q3).

We compute $[(Q1), \tilde{L}_u]$ and get:

$$\frac{2}{\rho} \sum_{1 \leq \alpha \leq D} \{\tilde{L}_{e_\alpha}, [\tilde{L}_{e_\alpha}, \tilde{L}_u]\} + \frac{1}{2}(\{\tilde{X}_u, \tilde{Y}_e\} - \{\tilde{X}_e, \tilde{Y}_u\}) = 0. \qquad (4.2)$$

Then $[(4.2), \tilde{L}_u]$ implies (Q4).

$\blacksquare$

Corollary 4.2. *Suppose $\mathfrak{g}_0 = \mathfrak{co}(J) \neq \mathfrak{su}(n,n)$ (i.e. not of type A) and (π, V) is a (non-trivial) unitary highest weight $\mathfrak{co}(J)$-module. Then (π, V) is minimal if and only if the following quadratic relation is satisfied:*

$$\frac{2}{\rho} \sum_{1 \leq \alpha \leq D} \tilde{L}_{e_\alpha}^2 - \tilde{L}_e^2 - \frac{1}{2}\{\tilde{X}_e, \tilde{Y}_e\} = -aI_V \text{ as an operator on } V. \qquad (4.3)$$

Here $D = \dim(J)$, $a = \frac{\rho d}{4}(1 + \frac{(\rho-2)d}{4})$ is a nonzero constant and equal to $a(J, 0)$ in our main theorem, and $\{e_\alpha\}$ is an orthonormal basis for J.

Remark 4.1. The quadratic relation in the above corollary is a special case of (Q1). Actually all highest weights in a minimal $\mathfrak{co}(J)$-module correspond to $a = a(J, 0)$ in Corollary 7.2. From Meng[36], we know $a(J, 0) = \frac{\rho d}{4}(1 + \frac{(\rho-2)d}{4})$.

For a simple complex Lie algebra $\mathfrak{g} = \mathfrak{co}(J)_{\mathbb{C}}$ not of type A_n, we know that the Joseph ideal J_0 in $U(\mathfrak{g})$ is the unique completely prime ideal whose associated variety is the closure of the minimal nilpotent orbit in $\mathfrak{g}^*$. From the definition of minimality and our Corollary 4.2, we must have:

the annihilator ideal $\text{Ann}(L(\lambda)) \subseteq U(\mathfrak{co}(J))_{\mathbb{C}}$ is equal to the Joseph ideal J_0 if and only if $\text{Ann}(L(\lambda))$ contains

$$Q_1' \triangleq \left(\frac{2}{\rho} \sum_{1 \leq \alpha \leq D} L_{e_\alpha}^2 - L_e^2 - \frac{1}{2}\{X_e, Y_e\} + \frac{\rho d}{4}(1 + \frac{(\rho-2)d}{4})\right). \qquad (4.4)$$

Then any prime ideal $I \subseteq U(\mathfrak{g})$ equals the Joseph ideal J_0 if and only if I contains $Q_1' \triangleq \left(\frac{2}{\rho} \sum_{1 \leq \alpha \leq D} L_{e_\alpha}^2 - L_e^2 - \frac{1}{2}\{X_e, Y_e\} + a\right)$ with $a = a(J, 0) = \frac{\rho d}{4}(1 + \frac{(\rho-2)d}{4})$.

In Ref. [36], Meng constructed the models of some generalized Kepler problems. He found that: a minimal unitary highest weight $\mathfrak{co}(J)$-module can be realized by a L^2-model, i.e. the Hilbert space of bound states of a model from a generalized Kepler problem he defined in his paper. Then Meng showed: the quadratic relation (Q1) with $a = a(J, 0)$ is in fact in the annihilator of the minimal $\mathfrak{co}(J)$-module he defined.

Before we give a proof for our main theorem, we define some notations which will be used throughout the rest of this paper. We take the same notation with Meng[36]. Denote

$$E_u^{\pm} = iL_u \mp \frac{1}{2}(X_u - Y_u), \quad h_u = -i(X_u + Y_u),$$

for any $u \in J$.

Then we have the following property according to the TKK commutation relations.

Lemma 4.2.   *1.* $[h_u, E_v^\pm] = \pm 2 E_{uv}^\pm$;

   2. $[E_u^+, E_v^-] = -h_{uv} - 2[L_u, L_v]$;

   3. $[E_u^+, E_v^+] = [E_u^-, E_v^-] = 0$;

   4. $[h_u, h_v] = 4[L_u, L_v]$.

5.   Proof of the main theorem-the case when $J = \Gamma(m)$ $(m \geq 2)$

Actually this case had been proved by Meng [30].

Let m be an integer and $m \geq 2$. We define a $(3+m) \times (3+m)$ matrix $[\eta_{\mu\nu}] = \mathrm{diag}(1, 1, -1, ..., -1)$. We denote its inverse by $[\eta^{\mu\nu}]$. Let $\mathrm{Cl}_{2,m+1}$ be the Clifford algebra over $\mathbb{C}$ generated by $X_\mu's$ subject to relations:

$$X_\mu X_\nu + X_\nu X_\mu = -2\eta_{\mu\nu}.$$

We denote $M_{\mu\nu} = \frac{i}{4}(X_\mu X_\nu - X_\nu X_\mu)$, then we have

$$[M_{ab}, M_{cd}] = -i(\eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc}). \tag{5.1}$$

Then $\{M_{\mu\nu}\}$ generate $\mathfrak{so}(2, m+1)$.

Let (π, V) denote the unitary highest $\mathfrak{so}(2, m+1)$-module, and $\pi(\mathcal{O}) := \tilde{\mathcal{O}}$ for any $\mathcal{O} \in \mathfrak{so}(2, m+1)$. In Ref. [30], Meng proved the following theorem.

Theorem 5.1.   (Meng). *A (non-trivial) unitary highest weight module of* $\mathfrak{co}(\Gamma(m)) = \mathfrak{so}(2, m+1)$ *has the smallest positive Gelfand-Kirillov dimension if and only if it satisfies the following quadratic relations on the underlying module space:*

$$\{\tilde{M}_{\mu\lambda}, \tilde{M}_\nu^\lambda\} = c\eta_{\mu\nu} \text{ with } \mu, \nu = -1, 0, 1, ..., m+1, \tag{5.2}$$

where $\tilde{M}_\nu^\lambda = \eta^{\lambda\kappa} \tilde{M}_{\kappa\nu}$ *and* c *is a representation-dependent real number.*

From (5.1) we find that:

Lemma 5.2.   *The quadratic relation in Meng's theorem is equivalent to the following relation:*

$$\{M_{0\lambda}, M^\lambda{}_0\} = c\eta_{00} = c, \tag{5.3}$$

plus the commutation relations between the M's in (5.1).

The Jordan algebra $\Gamma(m)$ has an orthogonal basis:

$$\begin{cases} e_{11} = (\frac{1}{2}, \frac{1}{2}, 0, ..., 0), \\ e_{22} = (\frac{1}{2}, -\frac{1}{2}, 0, ..., 0), \\ e_{12}^1 = (0, 0, \frac{1}{\sqrt{2}}, 0, ..., 0) = \frac{1}{\sqrt{2}} e_3, \\ e_{12}^\mu = (0, 0, ..., 0, \frac{1}{\sqrt{2}}, 0, ..., 0) = \frac{1}{\sqrt{2}} e_{\mu+2}, 1 \leq \mu \leq d = m - 1. \end{cases}$$

We define the following linear transformation:

$$
\begin{aligned}
\varphi: \ co(\Gamma(m)) &\to so(2, m+1) \\
X_e &\mapsto -i(M_{-1,0} + M_{0,m+1}), \\
Y_e &\mapsto -i(M_{-1,0} - M_{0,m+1}), \\
L_{(\lambda, \vec{u}_0)} &\mapsto -i(-\lambda M_{-1,m+1} + \sum_{1 \leq i \leq m} u_i M_{0,i}), \\
&\text{here } \vec{u}_0 = (u_1, u_2, ..., u_m).
\end{aligned}
\tag{5.4}
$$

By using the TKK commutation relations, we can get

$$\varphi(X_{e_{11}}) = -\frac{i}{2}(M_{-1,0} + M_{-1,1} + M_{0,m+1} + M_{1,m+1}),$$

$$\varphi(X_{e_{22}}) = -\frac{i}{2}(M_{-1,0} - M_{-1,1} + M_{0,m+1} - M_{1,m+1}),$$

$$\varphi(Y_{e_{11}}) = -\frac{i}{2}(M_{-1,0} - M_{-1,1} - M_{0,m+1} + M_{1,m+1}),$$

$$\varphi(Y_{e_{22}}) = -\frac{i}{2}(M_{-1,0} + M_{-1,1} - M_{0,m+1} - M_{1,m+1}),$$

$$\varphi(X_{e_{12}^\mu}) = -\frac{i}{\sqrt{2}}(M_{-1,\mu+1} + M_{\mu+1,m+1}),$$

$$\varphi(Y_{e_{12}^\mu}) = -\frac{i}{\sqrt{2}}(-M_{-1,\mu+1} + M_{\mu+1,m+1}).$$

Then we can easily check that φ is an isomorphism by computing its inverse φ^{-1} and φ^{-1} satisfies the commutators.

From this linear transformation we find that:

$$\{M_{0\lambda}, M^{\lambda}{}_0\} = c\eta_{00} = c := -a$$

is equivalent to

$$\frac{2}{\rho} \sum_{1 \leq \alpha \leq D} L_{e_\alpha}^2 - L_e^2 - \frac{1}{2}\{X_e, Y_e\} = -a.$$

And the commutation relations in (5.1) is equivalent to the TKK commutation relations.

The TKK commutation relations is naturally satisfied by such unitary highest weight $\mathfrak{co}(J)$-modules from the definition of Lie algebra modules. So Meng's theorem is equivalent to:

Theorem 5.3. *A (non-trivial) unitary highest weight module of $\mathfrak{so}(2, m+1) = (\mathfrak{co}(\Gamma(m)))$ has the smallest positive Gelfand-Kirillov dimension if and only if the following quadratic relation is satisfied as operators on the underlining module space V:*

$$\frac{2}{\rho} \sum_{1 \leq \alpha \leq D} \tilde{L}_{e_\alpha}^2 - \tilde{L}_e^2 - \frac{1}{2}\{\tilde{X}_e, \tilde{Y}_e\} = -a.$$

6.   Proof of the main theorem-the hermitian cases (Part I)

For the remaining Hermitian cases, firstly we suppose a unitary highest weight $\mathfrak{co}(J)$-module satisfies a quadratic relation (Q1) in the main theorem. We will show that it will have the smallest positive Gelfand-Kirillov dimension by computing its highest weight in a case-by-case way. Then in the next section, we will show the other direction of the main theorem.

The idea is very simple. Actually, if we denote $Q_1 \triangleq \left(\frac{2}{\rho} \sum\limits_{1 \leq \alpha \leq D} \tilde{L}_{e_\alpha}^2 - \tilde{L}_e^2 - \frac{1}{2}\{\tilde{X}_e, \tilde{Y}_e\} + a\right)$, then the quadratic relation (Q1) is equivalent to $(Q_1)v_\lambda = 0$ and $[(Q_1), \pi(E_{\beta_1}^{n_1} \cdots E_{\beta_k}^{n_k})]v_\lambda = 0$, where $\beta_i \in \Delta$ is a root and v_λ is any highest weight vector in V. From theses equations, we can work out the highest weight λ. Mainly we will use the equations (Q2), (Q3) and (Q4) to compute the highest weight λ.

6.1. The case when $J = \mathcal{H}_n(\mathbb{R})(n \geq 3)$.

We will follow the approach in Meng [30]. The idea is to construct a convenient Cartan basis and then rewrite the quadratic relations in terms of the Cartan basis elements.

In this case, $\mathfrak{co}(J) = \mathfrak{sp}(n, \mathbb{R})$. The positive roots are $e_i - e_j$ (compact roots), $e_i + e_j$ (non-compact roots) with $i < j$ and $2e_i$ (non-compact roots) with $1 \leq i \leq n$. We choose the following Cartan basis for $\mathfrak{sp}(n, \mathbb{R})$:

$$H_{2e_i} = h_{e_{ii}} = -i(X_{e_{ii}} + Y_{e_{ii}}) \qquad\qquad H_{e_i \pm e_j} = h_{e_{ii}} \pm h_{e_{jj}}$$

$$E_{2e_i} = E_{e_{ii}}^+ = iL_{e_{ii}} - \frac{1}{2}(X_{e_{ii}} - Y_{e_{ii}}) \qquad E_{-2e_i} = E_{e_{ii}}^- = iL_{e_{ii}} + \frac{1}{2}(X_{e_{ii}} - Y_{e_{ii}})$$

$$E_{e_i + e_j} = \sqrt{2}E_{e_{ij}}^+ \qquad\qquad\qquad E_{-e_i - e_j} = \sqrt{2}E_{e_{ij}}^-$$

$$E_{\pm(e_i - e_j)} = \frac{1}{\sqrt{2}}(h_{e_{ij}} \pm 4[L_{e_{ij}}, L_{e_{jj}}])$$

Then we get:

$$L_{e_{ii}} = -\frac{i}{2}(E_{2e_i} + E_{-2e_i})$$

$$L_{e_{ij}} = -\frac{i}{2\sqrt{2}}(E_{e_i + e_j} + E_{-e_i - e_j})$$

$$X_{e_{ii}} = \frac{1}{2}(E_{-2e_i} - E_{2e_i} + iH_{2e_i})$$

$$Y_{e_{ii}} = -\frac{1}{2}(E_{-2e_i} - E_{2e_i} - iH_{2e_i})$$

$$X_{e_{ij}} = \frac{1}{2\sqrt{2}}(iE_{e_i - e_j} + iE_{-e_i + e_j} + E_{-e_i - e_j} - E_{e_i + e_j})$$

$$Y_{e_{ij}} = \frac{1}{2\sqrt{2}}(iE_{e_i - e_j} + iE_{-e_i + e_j} - E_{-e_i - e_j} + E_{e_i + e_j})$$

Let (π, V) be the unitary highest weight module of $\mathfrak{co}(J) = \mathfrak{sp}(n, \mathbb{R})$ with highest weight $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$. Then for any highest weight vector $v \in V$, we have $\pi(H_\alpha)v = \tilde{H}_\alpha v = \lambda(H_\alpha)v = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}v$ and $\pi(E_\alpha)v = 0$ if α is a positive root.

We let the quadratic relations (Q1) and (Q2) act on any highest weight vector, then we get:

$$(\sum_{1\leq i\leq n} \lambda_i)^2 + (n+1)(\sum_{1\leq i\leq n} \lambda_i) + 4a = 0, \tag{6.1}$$

and

$$(\sum_{1\leq i\leq n} \lambda_i)^2 - \sum_{1\leq i\leq n} \lambda_i^2 + \sum_{1\leq i<j\leq n} \lambda_j = 0. \tag{6.2}$$

By the unitarity, we can get:

$$\lambda_n \leq \lambda_{n-1} \leq ... \leq \lambda_1 \leq 0. \tag{6.3}$$

From the quadratic relation (Q3), we can get

$$\begin{cases} \sum_{1<j\leq n} \lambda_j + 2\lambda_1(\sum_{1<j\leq n} \lambda_j) = 0, \text{ here we take } u = \sqrt{\rho}e_{11}; \\ \sum_{i<j\leq n} \lambda_j + \sum_{1\leq k<i} \lambda_i + 2\lambda_i(\sum_{p\neq i} \lambda_p) = 0, \\ \quad \text{here we take } u = \sqrt{\rho}e_{ii}, 2 \leq i \leq n-1; \\ \sum_{1\leq k<n} \lambda_n + 2\lambda_n(\sum_{1\leq j<n} \lambda_j) = 0, \text{ here we take } u = \sqrt{\rho}e_{nn}. \end{cases} \tag{6.4}$$

Since λ is not a zero weight, then from (6.4) and (6.3), we must have

$$\lambda_1 = \lambda_2 = ... = \lambda_{n-1} = -\frac{1}{2} \geq \lambda_n.$$

From the quadratic relation (Q4), we can get

$$(2\lambda_1 + 2\lambda_n + 1)\sum_{1\leq i\leq n} \lambda_i + (3+n)\lambda_n - 3\lambda_1 = 0, \text{ here we take } u = \sqrt{\rho}e_{1n}. \tag{6.5}$$

Then we can get $\lambda_n = -\frac{1}{2} - k$, with $k = 0, 1$.

So $\lambda = \begin{cases} -(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}), & \text{if } k = 0, \\ -(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, \frac{1}{2} + 1), & \text{if } k = 1. \end{cases}$

6.2. The case when $J = \mathcal{H}_n(\mathbb{C})(n \geq 3)$.

In this case, $\mathfrak{co}(J) = \mathfrak{su}(n, n)$. The compact positive roots are $e_i - e_j$ with $1 \leq i < j \leq n$ or $n+1 \leq i < j \leq 2n$, and the non-compact roots are $e_i - e_j$ with $1 \leq i \leq n$ and $2n \geq j \geq n+1$. Let $\{\sqrt{\rho}e_{ij} = \sqrt{\rho}\frac{E_{ij}+E_{ji}}{\sqrt{2}}, \sqrt{\rho}e_{ij}^{\alpha} = \sqrt{\rho}\sqrt{-1}\frac{E_{ij}-E_{ji}}{\sqrt{2}}\}$ be the orthonormal basis for J_{ij}. We choose the following Cartan basis for $\mathfrak{su}(n, n)$:

$$H_{e_{i+\bar{n}_1}-e_{j+\bar{n}_2}} = \frac{1}{2}(h_{\eta_{\bar{n}_1}e_{ii}+\eta_{\bar{n}_2}e_{jj}} - 4i[L_{e_{ij}}, L_{e_{ij}^{\alpha}}])$$
$$\text{where } \bar{n}_1 = 0, n; \bar{n}_2 = 0, n; \eta_0 = 1, \eta_n = -1$$
$$H_{e_i-e_{n+i}} = h_{e_{ii}}$$

$$E_{e_i-e_{n+i}} = E^+_{e_{ii}} = iL_{e_{ii}} - \frac{1}{2}(X_{e_{ii}} - Y_{e_{ii}})$$

$$E_{-e_i+e_{n+i}} = E^-_{e_{ii}} = iL_{e_{ii}} + \frac{1}{2}(X_{e_{ii}} - Y_{e_{ii}})$$

$$E_{e_i-e_j} = \frac{1}{2\sqrt{2}}(h_{e_{ij}+ie^\alpha_{ij}} + 4[L_{e_{ij}+ie^\alpha_{ij}}, L_{e_{jj}}])$$

$$E_{-e_i+e_j} = \frac{1}{2\sqrt{2}}(h_{e_{ij}-ie^\alpha_{ij}} - 4[L_{e_{ij}-ie^\alpha_{ij}}, L_{e_{jj}}])$$

$$E_{e_{n+i}-e_{n+j}} = \frac{1}{2\sqrt{2}}(h_{e_{ij}+ie^\alpha_{ij}} - 4[L_{e_{ij}+ie^\alpha_{ij}}, L_{e_{jj}}])$$

$$E_{-e_{n+i}+e_{n+j}} = \frac{1}{2\sqrt{2}}(h_{e_{ij}-ie^\alpha_{ij}} + 4[L_{e_{ij}-ie^\alpha_{ij}}, L_{e_{jj}}])$$

$$E_{e_i-e_{n+j}} = \frac{1}{\sqrt{2}}E^+_{e_{ij}+ie^\alpha_{ij}} \qquad\qquad E_{-e_i+e_{n+j}} = \frac{1}{\sqrt{2}}E^-_{e_{ij}-ie^\alpha_{ij}}$$

$$E_{e_j-e_{n+i}} = \frac{1}{\sqrt{2}}E^+_{e_{ij}-ie^\alpha_{ij}} \qquad\qquad E_{-e_j+e_{n+i}} = \frac{1}{\sqrt{2}}E^-_{e_{ij}+ie^\alpha_{ij}}$$

Then we get:

$$L_{e_{ii}} = -\frac{i}{2}(E_{e_i-e_{n+i}} + E_{-e_i+e_{n+i}})$$

$$L_{e_{ij}} = -\frac{i}{2\sqrt{2}}(E_{e_i-e_{n+j}} + E_{-e_i+e_{n+j}} + E_{e_j-e_{n+i}} + E_{-e_j+e_{n+i}})$$

$$L_{e^\alpha_{ij}} = -\frac{1}{2\sqrt{2}}(E_{e_i-e_{n+j}} - E_{-e_i+e_{n+j}} - E_{e_j-e_{n+i}} + E_{-e_j+e_{n+i}})$$

$$X_{e_{ii}} = \frac{1}{2}(E_{-e_i+e_{n+i}} - E_{e_i-e_{n+i}} + iH_{e_i-e_{n+i}})$$

$$Y_{e_{ii}} = -\frac{1}{2}(E_{-e_i+e_{n+i}} - E_{e_i-e_{n+i}} - iH_{e_i-e_{n+i}})$$

$$X_{e_{ij}} = \frac{1}{2\sqrt{2}}(iE_{e_i-e_j} + iE_{-e_i+e_j} + iE_{e_{n+i}-e_{n+j}} + iE_{-e_{n+i}+e_{n+j}}$$
$$+ E_{-e_i+e_{n+j}} + E_{-e_j+e_{n+i}} - E_{e_i-e_{n+j}} - E_{e_j-e_{n+i}})$$

$$Y_{e_{ij}} = \frac{1}{2\sqrt{2}}(iE_{e_i-e_j} + iE_{-e_i+e_j} + iE_{e_{n+i}-e_{n+j}} + iE_{-e_{n+i}+e_{n+j}}$$
$$- E_{-e_i+e_{n+j}} - E_{-e_j+e_{n+i}} + E_{e_i-e_{n+j}} + E_{e_j-e_{n+i}})$$

$$X_{e^\alpha_{ij}} = \frac{1}{2\sqrt{2}}(E_{e_i-e_j} + E_{-e_i+e_j} - E_{e_{n+i}-e_{n+j}} - E_{-e_{n+i}+e_{n+j}}$$
$$+ iE_{-e_i+e_{n+j}} - iE_{-e_j+e_{n+i}} + iE_{e_i-e_{n+j}} - iE_{e_j-e_{n+i}})$$

$$Y_{e^\alpha_{ij}} = \frac{1}{2\sqrt{2}}(E_{e_i-e_j} + E_{-e_i+e_j} - E_{e_{n+i}-e_{n+j}} - E_{-e_{n+i}+e_{n+j}}$$
$$- iE_{-e_i+e_{n+j}} + iE_{-e_j+e_{n+i}} - iE_{e_i-e_{n+j}} + iE_{e_j-e_{n+i}})$$

where $1 \leq i < j \leq n$.

Let (π, V) be the unitary highest weight module of $\mathfrak{co}(J) = \mathfrak{su}(n,n)$ with highest weight $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$. Then for any highest weight vector $v \in V$, we have $\pi(H_\alpha)v = \tilde{H}_\alpha v = \lambda(H_\alpha)v = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}v$ and $\pi(E_\alpha)v = 0$ if α is a positive root.

We let the quadratic relations (Q1) and (Q2) act on any highest weight vector, then we get:

$$(\sum_{1 \leq i \leq n} \lambda_i - \lambda_{n+i})^2 + 2n(\sum_{1 \leq i \leq n} \lambda_i - \lambda_{n+i}) = -4a, \tag{6.6}$$

$$\sum_{1 \leq i \leq n} (\lambda_i - \lambda_{n+i})^2 - (\sum_{1 \leq i \leq n} \lambda_i - \lambda_{n+i})^2 - 2 \sum_{1 \leq i < j \leq n} (\lambda_j - \lambda_{n+i}) = 0. \tag{6.7}$$

From the unitarity we have

$$\lambda_n \leq \lambda_{n-1} \leq ... \leq \lambda_1 \leq \lambda_{2n} \leq ... \leq \lambda_{n+2} \leq \lambda_{n+1}. \tag{6.8}$$

For A-type root system, we have

$$\sum_{1 \leq i \leq 2n} \lambda_i = 0. \tag{6.9}$$

From the quadratic relations (Q3) and (Q4), we can get:

$$\begin{cases} (\lambda_i - \lambda_{n+i})^2 + (-\sum_{1 \leq i \leq n} (\lambda_i - \lambda_{n+i}) + 2 - i)(\lambda_i - \lambda_{n+i}) \\ +(n+2-2i)\lambda_{n+i} - \sum_{1 \leq j \leq n} \lambda_j + \sum_{1 \leq k \leq i-1} (\lambda_k + \lambda_{n+k}) = 0, \\ \text{here we take } u = \sqrt{\rho}e_{ii} \text{ for } 1 \leq i \leq n, \end{cases} \tag{6.10}$$

and

$$\begin{cases} (\lambda_i - \lambda_{n+i})^2 + (-\sum_{1 \leq i \leq n} (\lambda_i - \lambda_{n+i}) + 2 - i)(\lambda_i - \lambda_{n+i}) \\ +(n+2-2i)\lambda_{n+i} - \sum_{1 \leq j \leq n} \lambda_j + \sum_{1 \leq k \leq i-1} (\lambda_k + \lambda_{n+k}) \\ +(\lambda_j - \lambda_{n+j})^2 + (-\sum_{1 \leq i \leq n} (\lambda_i - \lambda_{n+i}) + 2 - j)(\lambda_j - \lambda_{n+j}) \\ +(n+2-2j)\lambda_{n+j} - \sum_{1 \leq j \leq n} \lambda_j + \sum_{1 \leq k \leq j-1} (\lambda_k + \lambda_{n+k}) \\ +2(\lambda_i - \lambda_{n+i})(\lambda_j - \lambda_{n+j}) + 2(\lambda_j - \lambda_{n+i}) = 0, \\ \text{here we take } u = e_{ii} + e_{jj} \text{ for } 1 \leq i < j \leq n. \end{cases} \tag{6.11}$$

From (6.10) and (6.11) we get:

$$(\lambda_i - \lambda_{n+i})(\lambda_j - \lambda_{n+j}) = -(\lambda_j - \lambda_{n+i}). \tag{6.12}$$

If $\lambda_{i_0} - \lambda_{n+i_0} = 0$ for some i_0, then we have $\lambda_j - \lambda_{n+i_0} = 0$ for $i_0 < j \leq n$ and $\lambda_{i_0} - \lambda_{n+k} = 0$ for $1 \leq k < i_0$ since (6.12). This result and (6.8) imply $\lambda_n = ... = \lambda_1 = \lambda_{2n} = ... = \lambda_{n+1}$. Since the constant a is not zero, we get a contradiction with (6.6). So we must have

$$\lambda_i - \lambda_{n+i} \leq -1, \text{ for } 1 \leq i \leq n.$$

Lemma 6.1. $\lambda_n - \lambda_{2n} = -1$ or $\lambda_1 - \lambda_{n+1} = -1$. And $\lambda_j - \lambda_{n+j} = -1$ for all $1 < j < n$.

Proof.   Suppose $\lambda_1 - \lambda_{n+1} \leq -2$ and $\lambda_n - \lambda_{2n} \leq -2$. Then from (6.12) we have

$$\lambda_n - \lambda_{n+1} \leq 2\lambda_1 - 2\lambda_{n+1}, \lambda_n - \lambda_{n+1} \leq 2\lambda_n - 2\lambda_{2n}$$
$$\Rightarrow 2\lambda_{2n} \leq 2\lambda_1$$
$$\Rightarrow \lambda_{2n} = \lambda_1 \text{ by } (6.8)$$
$$\Rightarrow (\lambda_1 - \lambda_{n+1})(\lambda_n - \lambda_1) = -(\lambda_n - \lambda_{n+1}) = -(\lambda_1 - \lambda_{n+1}) - (\lambda_n - \lambda_1)$$
$$\Rightarrow \lambda_1 - \lambda_{n+1} = -2, \lambda_n - \lambda_{2n} = -2, \lambda_n - \lambda_{n+1} = -4 \text{ by } (6.12)$$
$$\Rightarrow -2(\lambda_i - \lambda_{n+i}) = (\lambda_i - \lambda_{n+i})(\lambda_n - \lambda_{2n}) = -\lambda_n + \lambda_{n+i},$$
$$\quad - 2(\lambda_i - \lambda_{n+i}) = (\lambda_1 - \lambda_{n+1})(\lambda_i - \lambda_{n+i}) = -\lambda_i + \lambda_{n+1}$$
$$\Rightarrow -4(\lambda_i - \lambda_{n+i}) = -(\lambda_n - \lambda_{n+i} + \lambda_i - \lambda_{n+1}) = -(-4 + \lambda_i - \lambda_{n+i})$$
$$\Rightarrow \lambda_i - \lambda_{n+i} = -\frac{4}{3}.$$

We get a contradiction since all $\lambda_i - \lambda_{n+i}$ are integers. So we may suppose

$$\lambda_n - \lambda_{2n} = -1.$$

If we also have $\lambda_1 - \lambda_{n+1} = -1$, then

$$- 1 = -(\lambda_1 - \lambda_{n+1})(\lambda_n - \lambda_{2n}) = \lambda_n - \lambda_{n+1}$$
$$\Rightarrow -1 = \lambda_n - \lambda_{n+1} \leq \lambda_i - \lambda_{n+1} = -(\lambda_1 - \lambda_{n+1})(\lambda_i - \lambda_{n+i})$$
$$\Rightarrow -1 \leq \lambda_i - \lambda_{n+i} \leq -1$$
$$\Rightarrow \lambda_i - \lambda_{n+i} = -1 \text{ for } 1 \leq i \leq n.$$

Now we suppose $\lambda_1 - \lambda_{n+1} = -1 - k \leq -2$. Then

$$\lambda_n - \lambda_{2n} = -1$$
$$\Rightarrow \lambda_i - \lambda_{n+i} = -(\lambda_i - \lambda_{n+i})(\lambda_n - \lambda_{2n}) = \lambda_n - \lambda_{n+i}$$
$$\Rightarrow \lambda_i = \lambda_n \text{ for } 1 \leq i \leq n - 1$$
$$\Rightarrow (\lambda_1 - \lambda_{n+1})(\lambda_j - \lambda_{n+j}) = -\lambda_j + \lambda_{n+1} = -\lambda_1 + \lambda_{n+1} > 0$$
$$\Rightarrow \lambda_j - \lambda_{n+j} = -1 \text{ for } 1 < j \leq n.$$

The result is similar if we suppose $\lambda_1 - \lambda_{n+1} = -1$.   ∎

So we can get:

$$\lambda = (\underbrace{b, \ldots, b}_{n}, b + 1 + k, b + 1, \ldots b + 1), k \in \mathbb{Z}_{\geq 0},$$

or

$$\lambda = (b, \ldots b, b - k, \underbrace{b + 1, \ldots, b + 1}_{n}), k \in \mathbb{Z}_{\geq 0}.$$

Then from (6.9), we have

$$\lambda = (\underbrace{-\frac{n + k}{2n}, \ldots, -\frac{n + k}{2n}}_{n}, \frac{n - k}{2n} + k, \frac{n - k}{2n}, \ldots, \frac{n - k}{2n}), k \in \mathbb{Z}_{\geq 0},$$

or

$$\lambda = (-\frac{n-k}{2n}, ..., -\frac{n-k}{2n}, -\frac{n-k}{2n} - k, \underbrace{\frac{n+k}{2n}, ..., \frac{n+k}{2n}}_{n}), k \in \mathbb{Z}_{\geq 0}.$$

6.3. The case when $J = \mathcal{H}_n(\mathbb{H})(n \geq 3)$.

In this case, $\mathfrak{co}(J) = \mathfrak{so}^*(4n)$. The compact positive roots are $e_i - e_j$ with $1 \leq i < j \leq 2n$, and the non-compact roots are $e_i + e_j$ with $1 \leq i < j \leq 2n$. Let $\{\sqrt{\rho}e_{ij} = \sqrt{\rho}\frac{E_{ij}+E_{ji}}{\sqrt{2}}, \sqrt{\rho}e_{ij}^2 = \sqrt{\rho}\sqrt{-1}\frac{E_{ij}-E_{ji}}{\sqrt{2}}, \sqrt{\rho}e_{ij}^3 = \sqrt{\rho}j\frac{E_{ij}-E_{ji}}{\sqrt{2}}, \sqrt{\rho}e_{ij}^4 = \sqrt{\rho}k\frac{E_{ij}-E_{ji}}{\sqrt{2}}\}$ be the orthonormal basis for J_{ij}. We choose the following Cartan basis for $\mathfrak{so}^*(4n)$:

$$H_{e_i+e_{n+i}} = h_{e_{ii}}$$

$$H_{e_i-e_{n+i}} = 2i([L_{e_{ij}}, L_{e_{ij}^4}] - [L_{e_{ij}^2}, L_{e_{ij}^3}])$$

$$E_{e_i+e_{n+i}} = E_{e_{ii}}^+ = iL_{e_{ii}} - \frac{1}{2}(X_{e_{ii}} - Y_{e_{ii}})$$

$$E_{-e_i-e_{n+i}} = E_{e_{ii}}^- = iL_{e_{ii}} + \frac{1}{2}(X_{e_{ii}} - Y_{e_{ii}})$$

$$E_{e_i-e_{n+i}} = [L_{e_{ij}^2} + iL_{e_{ij}^3}, L_{e_{ij}} - iL_{e_{ij}^4}]$$

$$E_{-e_i+e_{n+i}} = [L_{e_{ij}} + iL_{e_{ij}^4}, L_{e_{ij}^2} - iL_{e_{ij}^3}]$$

$$H_{e_{\bar{n}+i}-e_{\bar{n}+j}} = \frac{1}{2}(h_{e_{ii}-e_{jj}} + 4i\eta_{\bar{n}}[L_{e_{ij}}, L_{e_{ij}^4}])$$

$$E_{e_{\bar{n}+i}-e_{\bar{n}+j}} = \frac{1}{2\sqrt{2}}(h_{e_{ij}-i\eta_{\bar{n}}e_{ij}^4} + 4[L_{e_{ij}-i\eta_{\bar{n}}e_{ij}^4}, L_{e_{jj}}])$$

$$E_{-e_{\bar{n}+i}+e_{\bar{n}+j}} = \frac{1}{2\sqrt{2}}(h_{e_{ij}+i\eta_{\bar{n}}e_{ij}^4} - 4[L_{e_{ij}+i\eta_{\bar{n}}e_{ij}^4}, L_{e_{jj}}])$$

$$H_{e_{\bar{n}+i}-e_{n+j-\bar{n}}} = \frac{1}{2}(h_{e_{ii}-e_{jj}} - 4i\eta_{\bar{n}}[L_{e_{ij}^2}, L_{e_{ij}^3}])$$

$$E_{e_{\bar{n}+i}-e_{n+j-\bar{n}}} = \frac{1}{2\sqrt{2}}(h_{e_{ij}^2+i\eta_{\bar{n}}e_{ij}^3} + 4[L_{e_{ij}^2+i\eta_{\bar{n}}e_{ij}^3}, L_{e_{jj}}])$$

$$E_{-e_{\bar{n}+i}+e_{n+j-\bar{n}}} = \frac{1}{2\sqrt{2}}(h_{e_{ij}^2-i\eta_{\bar{n}}e_{ij}^3} - 4[L_{e_{ij}^2-i\eta_{\bar{n}}e_{ij}^3}, L_{e_{jj}}])$$

$$H_{e_{i+\bar{n}}+e_{\bar{n}+j}} = \frac{1}{2}(h_{e_{ii}+e_{jj}} - 4i\eta_{\bar{n}}[L_{e_{ij}^2}, L_{e_{ij}^3}])$$

$$E_{e_{i+\bar{n}}+e_{\bar{n}+j}} = \frac{1}{\sqrt{2}}E_{e_{ij}^2+i\eta_{\bar{n}}e_{ij}^3}^+$$

$$E_{-e_{i+\bar{n}}-e_{\bar{n}+j}} = \frac{1}{\sqrt{2}}E_{e_{ij}^2-i\eta_{\bar{n}}e_{ij}^3}^-$$

$$E_{e_{\bar{n}+i}+e_{n+j-\bar{n}}} = \frac{1}{\sqrt{2}}E_{e_{ij}-i\eta_{\bar{n}}e_{ij}^4}^+$$

$$E_{-e_{\bar{n}+i}-e_{n+j-\bar{n}}} = \frac{1}{\sqrt{2}}E_{e_{ij}+i\eta_{\bar{n}}e_{ij}^4}^-$$

$$H_{e_{\bar{n}+i}+e_{n+j-\bar{n}}} = \frac{1}{2}(h_{e_{ii}+e_{jj}} + 4i\eta_{\bar{n}}[L_{e_{ij}}, L_{e_{ij}^4}])$$

$$\text{where } \bar{n} = 0, n; \eta_0 = 1, \eta_n = -1.$$

Let (π, V) be the unitary highest weight module of $\mathfrak{co}(J) = \mathfrak{so}^*(4n)$ with highest weight $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$. Then for any highest weight vector $v \in V$, we have $\pi(H_\alpha)v = \tilde{H}_\alpha v = \lambda(H_\alpha)v = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}v$ and $\pi(E_\alpha)v = 0$ if α is a positive root.

Similar to the previous case, we can write the generators of $\mathfrak{so}^*(4n)$ as linear combinations of these Cartan basis and then from the quadratic relations (Q1) and (Q2), we can get:

$$(\sum_{1 \leq i \leq n} \lambda_i + \lambda_{n+i})^2 + (4n - 2)(\sum_{1 \leq i \leq n} \lambda_i + \lambda_{n+i}) = -4a, \tag{6.13}$$

and

$$\sum_{1 \leq i \leq n} (\lambda_i + \lambda_{n+i})^2 - (\sum_{1 \leq i \leq n} \lambda_i + \lambda_{n+i})^2 - 2 \sum_{1 \leq i < j \leq n} (\lambda_j + \lambda_{n+i} + 2\lambda_{n+j}) = 0. \tag{6.14}$$

From the unitarity we have:

$$\lambda_{2n} \leq \lambda_{2n-1} \leq ... \leq \lambda_{n+1} \leq \lambda_n \leq ... \leq \lambda_2 \leq -|\lambda_1| \leq 0. \tag{6.15}$$

From the quadratic relations (Q3) and (Q4), we can get:

$$\begin{cases} (\lambda_i + \lambda_{n+i})^2 + (-\sum_{1 \leq l \leq n} (\lambda_l + \lambda_{n+l}) + 2 - i)(\lambda_i + \lambda_{n+i}) \\ -(n - 2)\lambda_{n+i} - \sum_{1 \leq j \leq n} \lambda_{n+j} - \sum_{1 \leq l \leq n} (\lambda_l + \lambda_{n+l}) \\ + \sum_{1 \leq k \leq i-1} (\lambda_k + \lambda_{n+k}) = 0, \\ \text{here we take } u = \sqrt{\rho}e_{ii} \text{ for } 1 \leq i \leq n, \end{cases} \tag{6.16}$$

and

$$\begin{cases} (\lambda_i + \lambda_{n+i})^2 + (-\sum_{1 \leq l \leq n} (\lambda_l + \lambda_{n+l}) + 2 - i)(\lambda_i + \lambda_{n+i}) \\ -(n - 2)\lambda_{n+i} - \sum_{1 \leq l \leq n} \lambda_{n+l} - \sum_{1 \leq l \leq n} (\lambda_l + \lambda_{n+l}) \\ + \sum_{1 \leq k \leq i-1} (\lambda_k + \lambda_{n+k}) \\ + (\lambda_j + \lambda_{n+j})^2 + (-\sum_{1 \leq l \leq n} (\lambda_l + \lambda_{n+l}) + 2 - j)(\lambda_j + \lambda_{n+j}) \\ -(n - 2)\lambda_{n+j} - \sum_{1 \leq l \leq n} \lambda_{n+l} - \sum_{1 \leq l \leq n} (\lambda_l + \lambda_{n+l}) \\ + \sum_{1 \leq k \leq j-1} (\lambda_k + \lambda_{n+k}) \\ + 2(\lambda_i + \lambda_{n+i})(\lambda_j + \lambda_{n+j}) + 2(\lambda_j + \lambda_{n+i} + 2\lambda_{n+j}) = 0, \\ \text{here we take } u = e_{ii} + e_{jj} \text{ for } 1 \leq i < j \leq n. \end{cases} \tag{6.17}$$

From (6.16) and (6.17) we get:

$$(\lambda_i + \lambda_{n+i})(\lambda_j + \lambda_{n+j}) = -(\lambda_j + \lambda_{n+i} + 2\lambda_{n+j}). \tag{6.18}$$

From (6.15) we know $\lambda_i + \lambda_{n+i} \leq 0$. If $\lambda_{i_0} + \lambda_{n+i_0} = 0$ for some $i_0 < n$, then from (6.18) we have

$$0 = \lambda_n + \lambda_{n+i_0} + 2\lambda_{2n} \leq 0$$
$$\Rightarrow \lambda_n = 0, \lambda_{n+i_0} = 0, \lambda_{i_0} = 0, \lambda_{2n} = 0.$$

Then we get all $\lambda_i = 0, \lambda_{n+i} = 0$.

If $\lambda_n + \lambda_{2n} = 0$, then from (6.15), we have all $\lambda_i = 0, \lambda_{n+i} = 0$. But this is a contradiction since the given highest weight module is nontrivial. So we must have:

$$2\lambda_{n+i} \leq \lambda_i + \lambda_{n+i} \leq -1 \text{ for } 1 \leq i \leq n.$$

Lemma 6.2. $\lambda_i + \lambda_{n+i} = -2$ *for* $1 \leq i < n$.

Proof.   If $\lambda_{i_0} + \lambda_{n+i_0} = -1$ for some $i_0 < n$, then from (6.18) we can get

$$\lambda_n + \lambda_{2n} = \lambda_n + \lambda_{n+i_0} + 2\lambda_{2n}$$
$$\Rightarrow 0 = \lambda_{n+i_0} + \lambda_{2n} \leq 2\lambda_{n+i_0} \leq -1.$$

This contradiction implies

$$\lambda_i + \lambda_{n+i} \leq -2 \text{ for } 1 \leq i < n.$$

If $\lambda_{i_0} + \lambda_{n+i_0} \leq -4$ for some $i_0 < n$, then from (6.18) we can get

$$\lambda_n + \lambda_{n+i_0} + 2\lambda_{2n} \leq 4\lambda_n + 4\lambda_{2n}$$
$$\Rightarrow \lambda_{n+i_0} \leq 3\lambda_n + 2\lambda_{2n} \leq 2\lambda_{2n} \leq 2\lambda_{n+i_0}$$
$$\Rightarrow \lambda_{n+i_0} \leq 2\lambda_{n+i_0} \leq 0$$
$$\Rightarrow \lambda_{n+i_0} = 0$$
$$\Rightarrow \lambda_{i_0} = 0 \text{ by (6.15)}.$$

This contradiction implies

$$\lambda_i + \lambda_{n+i} \geq -3 \text{ for } 1 \leq i < n.$$

If $\lambda_{i_0} + \lambda_{n+i_0} = -3$ for some $i_0 < n$, then from (6.18) we can get

$$\lambda_n + \lambda_{n+i_0} + 2\lambda_{2n} = 3\lambda_n + 3\lambda_{2n}$$
$$\Rightarrow \lambda_{n+i_0} = 2\lambda_n + \lambda_{2n} \leq \lambda_{2n} \leq \lambda_{n+i_0}$$
$$\Rightarrow \lambda_{2n} = ... = \lambda_{n+i_0}, \lambda_n = 0$$
$$\Rightarrow \lambda_i = 0 \text{ for } 1 \leq i \leq n, \lambda_{2n} = ... = \lambda_{n+i_0} = -3.$$

When $i_0 > 1$, we take $k < i_0$, then from (6.18) we have

$$3(\lambda_k + \lambda_{n+k}) = \lambda_{i_0} + \lambda_{n+k} + 2\lambda_{n+i_0}$$
$$\Rightarrow \lambda_{n+k} = -3.$$

So we always have $\lambda_{2n} = ... = \lambda_{n+1} = -3, \lambda_n = ... = \lambda_1 = 0$. But from the quadratic relation (Q4), we have

$$
\begin{cases}
(\lambda_1 + \lambda_2 + \lambda_{n+1} + \lambda_{n+2})(n + 1 + \sum_{1 \le i \le n} (\lambda_i + \lambda_{n+i})) \\
+2 \sum_{1 \le i \le n} (\lambda_i + 2\lambda_{n+i}) - (\lambda_1 - \lambda_2 - \lambda_{n+1} + \lambda_{n+2})^2 \\
-(n+1)\lambda_1 - (n-3)\lambda_2 - 5\lambda_{n+1} - \lambda_{n+2} = 0, \\
\text{here we take } u = \sqrt{\rho}e_{12}.
\end{cases}
\tag{6.19}
$$

Then we get: $-6(n+1-3n)+2(-6n)+15+3 = 0$, i.e. $12=0$, a contradiction! So we must have

$\lambda_i + \lambda_{n+i} = -2$ for all $1 \le i < n$.   ∎

From the above lemma and (6.18), we have:

$$2(\lambda_i + \lambda_{n+i}) = (\lambda_i + \lambda_{n+1} + 2\lambda_{n+i}),$$

which implies $\lambda_i = \lambda_{n+1}$ for $1 < i \le n$.

From the above lemma and (6.15) we also have $\lambda_1 = ... = \lambda_{n-1}, \lambda_{n+1} = ... = \lambda_{2n-1}$. Then we can get $\lambda_1 = ... = \lambda_n = \lambda_{n+1} = ... = \lambda_{2n-1} = -1$.

So we have $\lambda = (-1, ..., -1, -k-1), k \in \mathbb{Z}_{\ge 0}$.

6.4. The case when $J = \mathcal{H}_3(\mathbb{O})$.

In this case, $\mathbf{co}(J) = \mathfrak{e}_{7(-25)}$. The compact positive roots are $\{\pm e_i + e_j$ with $1 \le i < j \le 5\} \bigcup \{\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{1 \le i \le 5} (-1)^{n(i)}e_i)| \sum_{1 \le i \le 5} n(i) \text{ is even}\}$, and the non-compact positive roots are $\{\pm e_i + e_6$ with $1 \le i \le 5\} \bigcup \{e_8 - e_7\} \bigcup \{\frac{1}{2}(e_8 - e_7 + e_6 + \sum_{1 \le i \le 5} (-1)^{n(i)}e_i)| \sum_{1 \le i \le 5} n(i) \text{ is odd}\}$. Let $\{\sqrt{\rho}e_{pq}^\alpha = \sqrt{\rho}\alpha \frac{E_{pq} - E_{qp}}{\sqrt{2}}|\alpha = 1, i, j, k, l, il, jl, kl\}$ be the orthonormal basis for J_{pq}. We choose the following Cartan basis for $\mathfrak{e}_{7(-25)}$:

$$E_{e_6+e_3} = E_{e_{22}}^+ \qquad\qquad E_{e_6-e_3} = E_{e_{11}}^+$$

$$E_{e_6+e_1} = E_{z_1}^+ = \frac{1}{\sqrt{2}}E_{e_{12}+\sqrt{-1}e_{12}^i}^+ \qquad E_{e_6+e_2} = E_{z_2}^+ = \frac{1}{\sqrt{2}}E_{e_{12}^j+\sqrt{-1}e_{12}^k}^+$$

$$E_{e_6+e_4} = E_{z_4}^+ = \frac{1}{\sqrt{2}}E_{e_{12}^l+\sqrt{-1}e_{12}^{il}}^+ \qquad E_{e_6+e_5} = E_{z_5}^+ = \frac{1}{\sqrt{2}}E_{e_{12}^{jl}-\sqrt{-1}e_{12}^{kl}}^+$$

$$E_{e_6-e_i} = E_{\bar{z}_i}^+ \text{ for } i = 1, 2, 4, 5$$

$$E_{e_i+e_j} = [L_{z_i}, L_{z_j}] \text{ for } 1 \le i < j \le 5 \text{ and } i \ne 3, j \ne 3$$

$$E_{-e_i+e_j} = [L_{\bar{z}_i}, L_{z_j}] \qquad\qquad E_{-e_i-e_j} = [L_{\bar{z}_i}, L_{\bar{z}_j}]$$

$$E_{e_i-e_j} = [L_{z_i}, L_{\bar{z}_j}] \qquad\qquad E_{e_i\pm e_3} = \frac{1}{2}(h_{z_i} \pm 4[L_{z_i}, L_{e_{11}}])$$

$$E_{-e_i\pm e_3} = \frac{1}{2}(h_{\bar{z}_i} \pm 4[L_{\bar{z}_i}, L_{e_{11}}])$$

$$E_{-\alpha} = E_{\bar{z}}^- \quad \text{if } E_\alpha = E_z^+ \text{ for non-compact positive root } \alpha$$

$$E_{\beta(-5)} = E_{\frac{1}{2}(e_8 - e_7 + e_6 + e_3 - e_5 + e_4 + e_2 + e_1)} = \frac{1}{\sqrt{2}} E^+_{e_{23}^{jl} + \sqrt{-1}e_{23}^{kl}}$$

$$E_{\beta(+5)} = E_{\frac{1}{2}(e_8 - e_7 + e_6 + e_3 + e_5 - e_4 - e_2 - e_1)} = \frac{1}{\sqrt{2}} E^+_{e_{23}^{jl} - \sqrt{-1}e_{23}^{kl}}$$

$$E_{\beta(-4)} = E_{\frac{1}{2}(e_8 - e_7 + e_6 + e_3 + e_5 - e_4 + e_2 + e_1)} = \frac{1}{\sqrt{2}} E^+_{e_{23}^l - \sqrt{-1}e_{23}^{il}}$$

$$E_{\beta(+4)} = E_{\frac{1}{2}(e_8 - e_7 + e_6 + e_3 - e_5 + e_4 - e_2 - e_1)} = \frac{1}{\sqrt{2}} E^+_{e_{23}^l + \sqrt{-1}e_{23}^{il}}$$

$$E_{\beta(-2)} = \frac{1}{\sqrt{2}} E^+_{e_{23}^j - \sqrt{-1}e_{23}^k} \qquad\qquad E_{\beta(+2)} = \frac{1}{\sqrt{2}} E^+_{e_{23}^j + \sqrt{-1}e_{23}^k}$$

$$E_{\beta(-1)} = \frac{1}{\sqrt{2}} E^+_{e_{23} + \sqrt{-1}e_{23}^i} \qquad\qquad E_{\beta(+1)} = \frac{1}{\sqrt{2}} E^+_{e_{23} - \sqrt{-1}e_{23}^i}$$

$$E_{\mu(4-)} = \frac{1}{\sqrt{2}} E_{\frac{1}{2}(e_8 - e_7 + e_6 - e_3 - e_5 - e_4 - e_2 - e_1)} = \frac{1}{\sqrt{2}} E^+_{e_{13} - \sqrt{-1}e_{13}^i}$$

$$E_{\mu(4+)} = \frac{1}{\sqrt{2}} E_{\frac{1}{2}(e_8 - e_7 + e_6 - e_3 + e_5 + e_4 + e_2 + e_1)} = \frac{1}{\sqrt{2}} E^+_{e_{13} + \sqrt{-1}e_{13}^i}$$

$$E_{\mu(+1+2)} = \frac{1}{\sqrt{2}} E_{\frac{1}{2}(e_8 - e_7 + e_6 - e_3 - e_5 - e_4 + e_2 + e_1)} = \frac{1}{\sqrt{2}} E^+_{e_{13}^j + \sqrt{-1}e_{13}^k}$$

$$E_{\mu(-1-2)} = \frac{1}{\sqrt{2}} E_{\frac{1}{2}(e_8 - e_7 + e_6 - e_3 + e_5 + e_4 - e_2 - e_1)} = \frac{1}{\sqrt{2}} E^+_{e_{13}^j - \sqrt{-1}e_{13}^k}$$

$$E_{\mu(+1+4)} = \frac{1}{\sqrt{2}} E_{\frac{1}{2}(e_8 - e_7 + e_6 - e_3 - e_5 + e_4 - e_2 + e_1)} = \frac{1}{\sqrt{2}} E^+_{e_{13}^l + \sqrt{-1}e_{13}^{il}}$$

$$E_{\mu(-1-4)} = \frac{1}{\sqrt{2}} E_{\frac{1}{2}(e_8 - e_7 + e_6 - e_3 + e_5 - e_4 + e_2 - e_1)} = \frac{1}{\sqrt{2}} E^+_{e_{13}^l - \sqrt{-1}e_{13}^{il}}$$

$$E_{\mu(+2+4)} = \frac{1}{\sqrt{2}} E_{\frac{1}{2}(e_8 - e_7 + e_6 - e_3 - e_5 + e_4 + e_2 - e_1)} = \frac{1}{\sqrt{2}} E^+_{e_{13}^{jl} + \sqrt{-1}e_{13}^{kl}}$$

$$E_{\mu(-2-4)} = \frac{1}{\sqrt{2}} E_{\frac{1}{2}(e_8 - e_7 + e_6 - e_3 + e_5 - e_4 - e_2 + e_1)} = \frac{1}{\sqrt{2}} E^+_{e_{13}^{jl} - \sqrt{-1}e_{13}^{kl}}$$

$$H_\alpha = [E_\alpha, E_{-\alpha}] \text{ if } \alpha \text{ is compact}$$

$$H_\alpha = -[E_\alpha, E_{-\alpha}] \text{ if } \alpha \text{ is non-compact}$$

Let (π, V) be the unitary highest weight module of $\mathfrak{co}(J) = \mathfrak{e}_{7(-25)}$ with highest weight $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$. Then for any highest weight vector $v \in V$, we have $\pi(H_\alpha)v = \tilde{H}_\alpha v = \lambda(H_\alpha)v = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}v$ and $\pi(E_\alpha)v = 0$ if α is a positive root.

Similar to the previous case, we can write the generators of $\mathfrak{e}_{7(-25)}$ as linear combinations of these Cartan basis and then from the quadratic relations (Q1) and (Q2), we can get:

$$(2\lambda_6 + \lambda_8 - \lambda_7)^2 + 18(2\lambda_6 + \lambda_8 - \lambda_7) + 4a = 0, \tag{6.20}$$

and

$$\begin{cases} (2\lambda_6 + \lambda_8 - \lambda_7)^2 - (\lambda_6 - \lambda_3)^2 - (\lambda_6 + \lambda_3)^2 \\ -(\lambda_8 - \lambda_7)^2 + (24\lambda_6 - 4\lambda_3 - 2\lambda_4 - 2\lambda_5) = 0. \end{cases} \tag{6.21}$$

By unitarity we conclude that

$$-\lambda_6 \geq \lambda_5 \geq \lambda_4 \geq \lambda_3 \geq \lambda_2 \geq |\lambda_1| \geq 0, \lambda_8 \leq 0 \leq \lambda_7. \tag{6.22}$$

By unitarity, we also have
$$\lambda_8 - \lambda_7 - \sum_{1 \leq j \leq 5, j \neq i} \lambda_j \geq (\lambda_6 - \lambda_i).$$
Take sum on i, then we have:

$$10\lambda_7 + 5\lambda_6 + 3 \sum_{1 \leq j \leq 5} \lambda_j \leq 0,$$

$$\Rightarrow 10\lambda_7 + 5\lambda_6 \leq -3 \sum_{1 \leq j \leq 5} \lambda_j \leq 0,$$

$$\Rightarrow 2\lambda_7 + \lambda_6 \leq 0.$$

If $\lambda_6 = 0$, then $0 \leq -\lambda_8 = \lambda_7 \leq -\frac{\lambda_6}{2} = 0$. From (6.22), we can get:

$$0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5.$$

This implies: $\lambda = (0, 0, 0, 0, 0, 0, 0, 0)$.
So we can assume $\lambda_6 < 0$.
From the quadratic relation (Q3), we have

$$\lambda_6(4 - 2\lambda_7) = 0, \text{ here we take } u = \sqrt{\rho}e_{33}. \tag{6.23}$$

This implies $\lambda_7 = 2, \lambda_8 = -2$.
From the quadratic relation (Q4), we have

$$\lambda_7(4 + \lambda_6) = 0, \text{ here we take } u = \sqrt{\rho}e_{33}. \tag{6.24}$$

This implies $\lambda_6 = -4$.
Then from (6.21) and (6.22), we have $0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5$.
So $\lambda = (0, 0, 0, 0, 0, -4, 2, -2)$.

7. Proof of the main theorem-the Hermitian cases (Part II)

In this section, we will show that for any unitary highest weight $\mathfrak{co}(J)$-module $L(\lambda)$, if it has the smallest positive Gelfand-Kirillov dimension, then it must satisfy the quadratic relation (Q1) for some value a only depending on $\lambda = \lambda(k)$ in Corollary 3.1.

We have the following lemma.

Lemma 7.1. *Our quadratic relation* $(Q1)$ *is equivalent to the following relation:*

$$\sum_{1 \leq \alpha \leq D} \{\tilde{X}_{e_\alpha}, \tilde{Y}_{e_\alpha}\} = 2\rho(\tilde{L}_e^2 + a).$$

We denote it by $(Q1)'$.

Proof. We suppose the relation $(Q1)$ holds, then we compute $[(Q1), \tilde{X}_e]$ and get (4.1):

$$\sum_{1 \leq \alpha \leq D} \{\tilde{X}_{e_\alpha}, \tilde{L}_{e_\alpha}\} = \rho\{\tilde{X}_e, \tilde{L}_e\}.$$

Then from $[(4.1), \tilde{Y}_e]$ we can get:

$$4 \sum_{1 \leq \alpha \leq D} \tilde{L}_{e_\alpha}^2 + \sum_{1 \leq \alpha \leq D} \{\tilde{X}_{e_\alpha}, \tilde{Y}_{e_\alpha}\} = 4\rho\tilde{L}_e^2 + \rho\{X_e, Y_e\},$$

which implies the relation $(Q1)'$.

Conversely, if the relation $(Q1)'$ holds, the we compute $[(Q1)', \tilde{X}_e]$ and get (4.1):

$$\sum_{1 \leq \alpha \leq D} \{\tilde{X}_{e_\alpha}, \tilde{L}_{e_\alpha}\} = \rho\{\tilde{X}_e, \tilde{L}_e\}.$$

Then from $[(4.1), \tilde{Y}_e]$, we find that the relation $(Q1)$ holds. ∎

If a (non-trivial) unitary highest weight $\mathfrak{co}(J)$-module has the smallest positive Gelfand-Kirillov dimension, then from the previous lemma we need to show that $\left(\sum_{1 \leq \alpha \leq D} \{\tilde{X}_{e_\alpha}, \tilde{Y}_{e_\alpha}\} - 2\rho(\tilde{L}_e^2 + a)\right)v = 0$ for any element $v \in V$.

If we denote $Q_1' \triangleq \left(\sum_{1 \leq \alpha \leq D} \{\tilde{X}_{e_\alpha}, \tilde{Y}_{e_\alpha}\} - 2\rho(\tilde{L}_e^2 + a)\right)$, then from the above case-by-case computation we know: $Q_1' v_\lambda = 0$ for any highest weight vector $v_\lambda \in V$.

From the TKK commutation relations and the property of highest weight module, we know any element $v \in V$ is a linear combination of the form $(\tilde{L}_{e_{\alpha_1}} \cdots \tilde{L}_{e_{\alpha_k}} \tilde{X}_{e_{\beta_1}} \cdots \tilde{X}_{e_{\beta_p}} \tilde{Y}_{e_{\gamma_1}} \cdots \tilde{Y}_{e_{\gamma_q}})v_\lambda$, with $k, p, q \geq 0$. So we need to show

$$Q_1'(\tilde{L}_{e_{\alpha_1}} \cdots \tilde{L}_{e_{\alpha_k}} \tilde{X}_{e_{\beta_1}} \cdots \tilde{X}_{e_{\beta_p}} \tilde{Y}_{e_{\gamma_1}} \cdots \tilde{Y}_{e_{\gamma_q}})v_\lambda = 0, \tag{7.1}$$

or equivalently,

$$[[[[[[\cdots[Q_1', \tilde{L}_{e_{\alpha_1}}], \cdots], \tilde{L}_{e_{\alpha_k}}], \tilde{X}_{e_{\beta_1}}], \cdots], \tilde{X}_{e_{\beta_p}}], \tilde{Y}_{e_{\gamma_1}}], \cdots], \tilde{Y}_{e_{\gamma_q}}]v_\lambda = 0. \tag{7.2}$$

Let J be a Hermitian type simple Euclidean Jordan algebra $\mathcal{H}_n(K)$, then we can take the following orthonormal basis:
$\{\sqrt{\rho}e_{pp}, \sqrt{\rho}e_{ab}^\mu | 1 \leq p \leq \rho, 1 \leq a < b \leq \rho, e_{pp} = E_{pp}, e_{ab} = \frac{1}{\sqrt{2}}(E_{ab} + E_{ba}), e_{ab}^\mu = \frac{a_\mu}{\sqrt{2}}(E_{ab} - E_{ba})$ *for* $2 \leq \mu \leq d$ *and* $a_2 = i, a_3 = j, a_4 = k, a_5 = l, a_6 = il, a_7 = jl, a_8 = kl\}$, where $E_{ab} = (x_{pq})_{n \times n}, x_{pq} = \delta_{ap}\delta_{bq}$.

We denote this basis by O_J.

Then we have the following lemma.

Lemma 7.2. *For any $e_\beta, e_{\beta_1}, e_\gamma, \ldots \in O_J$, we simply write $\tilde{L}_{e_\beta}$ by $\tilde{L}_\beta$, $\tilde{X}_{e_\beta}$ by $\tilde{X}_\beta$, $\tilde{Y}_{e_\beta}$ by $\tilde{Y}_\beta$, then*

1. $[Q_1', \tilde{L}_\beta] = 0;$

2. $A_1 := [Q_1', \tilde{X}_\beta] = 2 \sum_{1 \leq \alpha \leq D} \{\tilde{X}_\alpha, \tilde{S}_{\beta\alpha}\} - 2\rho\{\tilde{L}_e, \tilde{X}_\beta\};$

3. $A_2 := [Q_1', \tilde{Y}_\gamma] = -2 \sum_{1 \leq \alpha \leq D} \{\tilde{Y}_\alpha, \tilde{S}_{\alpha\gamma}\} + 2\rho\{\tilde{L}_e, \tilde{Y}_\gamma\};$

4. $A_{11} := [[Q_1', \tilde{X}_\beta], \tilde{X}_{\beta_1}] = 4 \sum_{1 \leq \alpha \leq D} \tilde{X}_\alpha \tilde{X}_{S_{\beta\alpha}(\beta_1)} - 4\rho\tilde{X}_\beta \tilde{X}_{\beta_1}, \quad [A_{11}, \tilde{X}_{\beta_2}] = 0;$

5. $A_{22} := [[Q_1', \tilde{Y}_\gamma], \tilde{Y}_{\gamma_1}] = 4 \sum_{1 \leq \alpha \leq D} \tilde{Y}_\alpha \tilde{Y}_{S_{\gamma\alpha}(\gamma_1)} - 4\rho\tilde{Y}_\gamma \tilde{Y}_{e_{\gamma_1}}, \quad [A_{22}, \tilde{Y}_{\gamma_2}] = 0;$

6.
$$
\begin{aligned}
A_{12} :=& [[Q_1', \tilde{X}_\beta], \tilde{Y}_\gamma] \\
=& -2 \sum_{1 \leq \alpha \leq D} \{\tilde{X}_\alpha, \tilde{Y}_{S_{\beta\alpha}(\gamma)}\} - 4 \sum_{1 \leq \alpha \leq D} \{\tilde{S}_{\alpha\gamma}, \tilde{S}_{\beta\alpha}\} \\
& + 2\rho\{\tilde{X}_\beta, \tilde{Y}_\gamma\} + 4\rho\{\tilde{S}_{\beta\gamma}, \tilde{L}_e\};
\end{aligned}
$$

7.
$$
\begin{aligned}
A_{122} :=& [[[Q_1', \tilde{X}_\beta], \tilde{Y}_\gamma], \tilde{Y}_{\gamma_1}] \\
=& 4 \sum_{1 \leq \alpha \leq D} \{\tilde{S}_{\alpha\gamma}, \tilde{Y}_{S_{\alpha\beta}(\gamma_1)}\} + 4 \sum_{1 \leq \alpha \leq D} \{\tilde{S}_{\alpha\gamma_1}, \tilde{Y}_{S_{\alpha\beta}(\gamma)}\} \\
& + 4 \sum_{1 \leq \alpha \leq D} \{\tilde{S}_{\beta\alpha}, \tilde{Y}_{S_{\gamma\alpha}(\gamma_1)}\} - 4\rho\{\tilde{S}_{\beta\gamma_1}, \tilde{Y}_\gamma\} \\
& - 4\rho\{\tilde{S}_{\beta\gamma}, \tilde{Y}_{\gamma_1}\} - 4\rho\{\tilde{L}_e, \tilde{Y}_{S_{\gamma\beta}(\gamma_1)}\};
\end{aligned}
$$

8.
$$
\begin{aligned}
A_{1222} :=& [[[[Q_1', \tilde{X}_\beta], \tilde{Y}_\gamma], \tilde{Y}_{\gamma_1}], \tilde{Y}_{\gamma_2}] \\
=& -8 \sum_{1 \leq \alpha \leq D} \tilde{Y}_{S_{\gamma\alpha}(\gamma_2)} \tilde{Y}_{S_{\alpha\beta}(\gamma_1)} - 8 \sum_{1 \leq \alpha \leq D} \tilde{Y}_{S_{\gamma_1\alpha}(\gamma_2)} \tilde{Y}_{S_{\alpha\beta}(\gamma)} \\
& - 8 \sum_{1 \leq \alpha \leq D} \tilde{Y}_{S_{\alpha\beta}(\gamma_2)} \tilde{Y}_{S_{\gamma\alpha}(\gamma_1)} + 8\rho\tilde{Y}_{S_{\gamma_1\beta}(\gamma_2)} \tilde{Y}_\gamma \\
& + 8\rho\tilde{Y}_{S_{\gamma\beta}(\gamma_2)} \tilde{Y}_{\gamma_1} + 8\rho\tilde{Y}_{\gamma_2} \tilde{Y}_{S_{\gamma\beta}(\gamma_1)}
\end{aligned}
$$
$$[A_{1222}, \tilde{Y}_{\gamma_3}] = 0;$$

9.
$$A_{112} := [A_{11}, \tilde{Y}_\gamma]$$

$$= -4 \sum_{1 \leq \alpha \leq D} \{\tilde{X}_\alpha, \tilde{S}_{S_{\beta\alpha}(\beta_1)\cdot\gamma}\} - 4 \sum_{1 \leq \alpha \leq D} \{\tilde{X}_{S_{\beta\alpha}(\beta_1)}, \tilde{S}_{\alpha\gamma}\}$$
$$+ 4\rho\{\tilde{X}_\beta, \tilde{S}_{\beta_1\gamma}\} + 4\rho\{\tilde{X}_{\beta_1}, \tilde{S}_{\beta\gamma}\};$$

10. *We simply write $S_{uv}(z)$ by $\{uvz\}$, then*

$$A_{1122} := [A_{112}, \tilde{Y}_{\gamma_1}]$$
$$= 4 \sum_{1 \leq \alpha \leq D} \{\tilde{X}_\alpha, \tilde{Y}_{\{\gamma\{\beta\alpha\beta_1\}\gamma_1\}}\} + 8 \sum_{1 \leq \alpha \leq D} \{\tilde{S}_{\alpha\gamma_1}, \tilde{S}_{\{\beta\alpha\beta_1\}\gamma}\}$$
$$+ 4 \sum_{1 \leq \alpha \leq D} \{\tilde{X}_{S_{\beta\alpha}(\beta_1)}, \tilde{Y}_{S_{\gamma\alpha}(\gamma_1)}\} + 8 \sum_{1 \leq \alpha \leq D} \{\tilde{S}_{\{\beta\alpha\beta_1\}\gamma_1}, \tilde{S}_{\alpha\gamma}\}$$
$$- 4\rho\{\tilde{X}_\beta, \tilde{Y}_{\{\gamma\beta_1\gamma_1\}}\} - 8\rho\{\tilde{S}_{\beta\gamma_1}, \tilde{S}_{\beta_1\gamma}\}$$
$$- 4\rho\{\tilde{X}_{\beta_1}, \tilde{Y}_{\{\gamma\beta\gamma_1\}}\} - 8\rho\{\tilde{S}_{\beta_1\gamma_1}, \tilde{S}_{\beta\gamma}\};$$

11.

$$A_{11222} := [A_{1122}, \tilde{Y}_{\gamma_2}]$$
$$= -8 \sum_{1 \leq \alpha \leq D} \{\tilde{S}_{\alpha\gamma_2}, \tilde{Y}_{\{\gamma\{\beta\alpha\beta_1\}\gamma_1\}}\} - 8 \sum_{1 \leq \alpha \leq D} \{\tilde{S}_{\alpha\gamma_1}, \tilde{Y}_{\{\gamma\{\beta\alpha\beta_1\}\gamma_2\}}\}$$
$$- 8 \sum_{1 \leq \alpha \leq D} \{\tilde{S}_{\{\beta\alpha\beta_1\}\gamma}, \tilde{Y}_{\{\gamma_1\alpha\gamma_2\}}\} - 8 \sum_{1 \leq \alpha \leq D} \{\tilde{S}_{\{\beta\alpha\beta_1\}\gamma_2}, \tilde{Y}_{S_{\gamma\alpha}(\gamma_1)}\}$$
$$- 8 \sum_{1 \leq \alpha \leq D} \{\tilde{S}_{\{\beta\alpha\beta_1\}\gamma_1}, \tilde{Y}_{\{\gamma\alpha\gamma_2\}}\} - 8 \sum_{1 \leq \alpha \leq D} \{\tilde{S}_{\alpha\gamma}, \tilde{Y}_{\{\gamma_1\{\beta\alpha\beta_1\}\gamma_2\}}\}$$
$$+ 8\rho\{\tilde{S}_{\beta\gamma_2}, \tilde{Y}_{\{\gamma\beta_1\gamma_1\}}\} + 8\rho\{\tilde{S}_{\beta\gamma_1}, \tilde{Y}_{\{\gamma\beta_1\gamma_2\}}\}$$
$$+ 8\rho\{\tilde{S}_{\beta_1\gamma}, \tilde{Y}_{\{\gamma_1\beta\gamma_2\}}\} + 8\rho\{\tilde{S}_{\beta_1\gamma_2}, \tilde{Y}_{\{\gamma\beta\gamma_1\}}\}$$
$$+ 8\rho\{\tilde{S}_{\beta_1\gamma_1}, \tilde{Y}_{\{\gamma\beta\gamma_2\}}\} + 8\rho\{\tilde{S}_{\beta\gamma}, \tilde{Y}_{\{\gamma_1\beta_1\gamma_2\}}\};$$

12.

$$A_{112222} := [A_{11222}, \tilde{Y}_{\gamma_3}]$$
$$= 16 \sum_{1 \leq \alpha \leq D} \tilde{Y}_{\{\gamma_2\alpha\gamma_3\}} \tilde{Y}_{\{\gamma\{\beta\alpha\beta_1\}\gamma_1\}} + 16 \sum_{1 \leq \alpha \leq D} \tilde{Y}_{\{\gamma_1\alpha\gamma_3\}} \tilde{Y}_{\{\gamma\{\beta\alpha\beta_1\}\gamma_2\}}$$
$$+ 16 \sum_{1 \leq \alpha \leq D} \tilde{Y}_{\{\gamma\{\beta\alpha\beta_1\}\gamma_3\}} \tilde{Y}_{\{\gamma_1\alpha\gamma_2\}} + 16 \sum_{1 \leq \alpha \leq D} \{\tilde{Y}_{\{\gamma_2\{\beta\alpha\beta_1\}\gamma_3\}} \tilde{Y}_{S_{\gamma\alpha}(\gamma_1)}$$
$$+ 16 \sum_{1 \leq \alpha \leq D} \{\tilde{Y}_{\{\gamma_1\{\beta\alpha\beta_1\}\gamma_3\}} \tilde{Y}_{\{\gamma\alpha\gamma_2\}} + 16 \sum_{1 \leq \alpha \leq D} \tilde{Y}_{\{\gamma\alpha\gamma_3\}} \tilde{Y}_{\{\gamma_1\{\beta\alpha\beta_1\}\gamma_2\}}$$
$$- 16\rho\tilde{Y}_{\{\gamma_2\beta\gamma_3\}} \tilde{Y}_{\{\gamma\beta_1\gamma_1\}} - 16\rho\tilde{Y}_{\{\gamma_1\beta\gamma_3\}} \tilde{Y}_{\{\gamma\beta_1\gamma_2\}}$$
$$- 16\rho\tilde{Y}_{\{\gamma\beta_1\gamma_3\}} \tilde{Y}_{\{\gamma_1\beta\gamma_2\}} - 16\rho\tilde{Y}_{\{\gamma_2\beta_1\gamma_3\}} \tilde{Y}_{\{\gamma\beta\gamma_1\}}$$
$$- 16\rho\tilde{Y}_{\{\gamma_1\beta_1\gamma_3\}} \tilde{Y}_{\{\gamma\beta\gamma_2\}} - 16\rho\tilde{Y}_{\{\gamma\beta\gamma_3\}} \tilde{Y}_{\{\gamma_1\beta_1\gamma_2\}};$$
$$[A_{112222}, \tilde{Y}_{\gamma_4}] = 0;$$

13.

$$A_1 v_\lambda = 0, \qquad\qquad\qquad A_2 v_\lambda = 0;$$
$$A_{11} v_\lambda = 0, \qquad\qquad\qquad A_{22} v_\lambda = 0;$$
$$A_{12} v_\lambda = 0, \qquad\qquad\qquad A_{122} v_\lambda = 0;$$
$$A_{1222} v_\lambda = 0, \qquad\qquad\qquad A_{112} v_\lambda = 0;$$
$$A_{1122} v_\lambda = 0, \qquad\qquad\qquad A_{11222} v_\lambda = 0;$$
$$A_{112222} v_\lambda = 0.$$

Remark 7.1. For each Hermitian type simple Euclidean Jordan algebra $\mathcal{H}_n(\mathbb{K})$, its basis O_J is a finite set. So the number of the operators in the form A_α in the above lemma must be finite.

The proof for this lemma is a straightforward computation, so we skip it.

Then we have the following:

Proof. (Proof of equation (7.2)). We use the same notation A_α with some operator in the lemma to mean they are in the same form. From the above lemma, we only need to show

$$[[[[\cdots[Q_1', \tilde{X}_{\beta_1}], \cdots], \tilde{X}_{\beta_p}], \tilde{Y}_{\gamma_1}], \cdots], \tilde{Y}_{\gamma_q}] v_\lambda = 0, \qquad (7.3)$$

since $[Q_1', \tilde{L}_\beta] = 0$.

Also we know $[[[[Q_1', \tilde{X}_{\beta_1}], \tilde{X}_{\beta_2}], \tilde{X}_{\beta_3}] = [A_{11}, \tilde{X}_{\beta_3}] = 0$.

So we only need to show the cases $p = 0$, $p = 1$ and $p = 2$.

For $p = 0$, we have

$$[[[Q_1', \tilde{Y}_{\gamma_1}], \cdots], \tilde{Y}_{\gamma_q}] v_\lambda = \left\{ \begin{array}{ll} A_2 v_\lambda, & \text{if } q = 1; \\ A_{22} v_\lambda, & \text{if } q = 2; \\ 0, & \text{if } q \geq 3. \end{array} \right\} = 0. \qquad (7.4)$$

For $p = 1$, we have

$$[[[[Q_1', \tilde{X}_{\beta_1}], \tilde{Y}_{\gamma_1}], \cdots], \tilde{Y}_{\gamma_q}] v_\lambda$$
$$= [[[A_1, \tilde{Y}_{\gamma_1}], \cdots], \tilde{Y}_{\gamma_q}] v_\lambda$$
$$= \left\{ \begin{array}{ll} A_1 v_\lambda, & \text{if } q = 0; \\ A_{12} v_\lambda, & \text{if } q = 1; \\ A_{122} v_\lambda, & \text{if } q = 2; \\ A_{1222} v_\lambda, & \text{if } q = 3; \\ 0, & \text{if } q \geq 4. \end{array} \right\}$$
$$= 0.$$

For $p = 2$, we have

$$[[[[[Q_1', \tilde{X}_{\beta_1}], \tilde{X}_{\beta_2}], \tilde{Y}_{\gamma_1}], \cdots], \tilde{Y}_{\gamma_q}] v_\lambda$$
$$= [[[A_{11}, \tilde{Y}_{\gamma_1}], \cdots], \tilde{Y}_{\gamma_q}] v_\lambda$$

$$= \begin{cases} A_{11}v_\lambda, & \text{if } q = 0; \\ A_{112}v_\lambda, & \text{if } q = 1; \\ A_{1122}v_\lambda, & \text{if } q = 2; \\ A_{11222}v_\lambda, & \text{if } q = 3; \\ A_{112222}v_\lambda, & \text{if } q = 4; \\ 0, & \text{if } q \geq 5. \end{cases}$$

$$= 0.$$

So we have proved the equation (7.3). Then the equation (7.2) is proved. ∎

Combined with the previous argument, we have shown that: the quadratic relation $(Q1)$ is satisfied by any unitary highest weight $\mathfrak{co}(J)$-module which has the smallest positive Gelfand-Kirillov dimension.

Remark 7.2. From the computations in the proof of our main theorem, we have

$$a = a(J, k)$$

$$= \begin{cases} -(n-1)k + n - \frac{1}{2}, \ (k = 0, \frac{1}{2}), & \text{if } J = \Gamma(2n) \\ -k^2 - (n-2)|k| + n - 1, \ (k \text{ is a half integer}), & \text{if } J = \Gamma(2n-1) \\ \frac{n(n+2)}{16}, \ (k = 0, 1), & \text{if } J = \mathcal{H}_n(\mathbb{R}) \\ \frac{n^2 - k^2}{4}, \ (k = 0, 1, ...), & \text{if } J = \mathcal{H}_n(\mathbb{C}) \\ n^2 - n - \frac{k}{2} - \frac{k^2}{4}, \ (k = 0, 1, ...), & \text{if } J = \mathcal{H}_n(\mathbb{H}) \\ 18, \ (k = 0), & \text{if } J = \mathcal{H}_3(\mathbb{O}) \end{cases}$$

i.e., $a = a(J, k)$ only depends on the highest weight $\lambda = \lambda(k)$ given in Corollary 3.1.

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Zhanqiang Bai
Department of Mathematics
Hong Kong Univ. of Sci. and Tech.
Clear Water Bay
Kowloon, Hong Kong
mazqbai@ust.hk
zhanqiang.bai@gmail.com