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Clifford-Wolf Homogeneous Randers Spaces*

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Abstract. A Clifford–Wolf translation of a connected Finsler space is an isometry which moves all points the same distance. A Finsler space (M,F) is called Clifford-Wolf homogeneous if for any two points $x_1, x_2 \in M$ there is a Clifford-Wolf translation ρ such that $\rho(x_1) = x_2$. In this paper, we give a complete classification of connected simply connected Clifford-Wolf homogeneous Randers spaces.
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1. Introduction

Let (M,F) be a connected Finsler space and d its distance function. An isometry ρ of (M,F) is called a Clifford-Wolf translation (CW-translation for short) if the function $d(x,\rho(x))$ is constant on M. The Finsler space (M,F) is called Clifford-Wolf homogeneous (CW-homogeneous) if for any two points $x_1, x_2 \in M$, there is a CW-translation ρ such that $\rho(x_1) = x_2$.

These definitions are natural generalizations of the related notions in Riemannian geometry. CW-translations have been studied extensively in Riemannian geometry, due to its relevance in the classification of space forms. Let (N, Q_1) be a connected complete Riemannian manifold with constant sectional curvature k. Then (N, Q_1) is isometric to a quotient manifold $(M,Q)/\Gamma$, where (M,Q) is a connected simply connected complete Riemannian manifold with constant sectional curvature k, and Γ is a discrete discontinuous subgroup of the full group of isometries of (M,Q) which acts freely on M. It is well-known that a connected simply connected complete Riemannian manifold of constant curvature is homogeneous. However, the quotient manifold $(M,Q)/\Gamma$ is no longer homogeneous in general. J. A. Wolf proved in [16] that it is homogeneous if and only of Γ consists of CW-translations. Thus the classification of homogeneous space form is reduced to the study of CW-subgroups of the full group of isometries. This formulation was generalized to symmetric Riemannian spaces by J. A. Wolf in [17]; see [11, 12] for other proofs.

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Recently, connected simply connected Riemannian CW-homogeneous manifolds were classified by Berestovskii and Nikonorov in [2, 3, 4]. Their list consists of the euclidean spaces, the odd-dimensional spheres with constant curvature, the connected simply connected compact simple Lie groups with bi-invariant Riemannian metrics, and the direct products of the above Riemannian manifolds. The purpose of this paper is to give a classification of connected simply connected CW-homogeneous Randers spaces. Our main result is the following

Theorem 1.1. *A connected simply connected Randers space (M, F) is CW-homogeneous if and only if M is a product of odd dimensional spheres, connected simply connected compact simple Lie groups and a euclidian space, and the Finsler metric F has the navigation data (h, W) such that (M, h) is CW-homogeneous, i.e., h is a Riemannian product of Riemannian metrics on spheres of constant curvature, bi-invariant metrics on Lie groups and a flat metric on the euclidian space with respect to the product decomposition of M, and W is a Killing vector field of constant length with $||W||_h < 1$.*

This result presents many examples of non-reversible CW-homogeneous Finsler spaces. It follows easily from the main results of Bao-Robles-Shen [6] that, on an odd dimensional sphere, a CW-homogeneous Randers space must be of constant flag curvature. However, the converse statement is not true. In fact, the Randers space (S^{2n-1}, F) of constant flag curvature is CW-homogeneous if and only if in its navigation data (h, W), h is the standard Riemannian metric and W is a Killing field of constant length with $||W||_h < 1$ ([9]). When W is not zero and of constant length, it defines a complex structure on $\mathbb{R}^{2n}$, and correspondingly the sphere can be denoted as $U(2n)/U(2n-1)$. It is easily seen that W is the Killing vector field generating the center of $U(2n)$ ([15]).

In view of the above result, it would be an interesting problem to classify CW-homo-geneous Finsler spaces in general.

In Section 2, we recall the preliminary knowledge about Finsler geometry, CW-translation and CW-homogeneity. In Section 3, we prove an interrelation theorem between CW-translations and Killing vector fields of constant length on a homogeneous Finsler space. In Section 4, we prove the main theorem.

2. Preliminaries

Let M be a connected manifold. A Finsler metric is a continuous function $F : TM \to \mathbb{R}^+$, which is smooth on the slit tangent bundle $TM\backslash 0$. In a standard local coordinates (x^i, y^j) for TM, where $x = (x^i)$ is the local coordinates for M, and $y = y^j\partial_{x^j}$ is the linear coordinates for $y \in T_x(M)$, the Finsler metric F is required to satisfy the following properties:

(1) $F(x, y) > 0$ for any $y \neq 0$.

(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $y \in T_x(M)$ and $\lambda > 0$.

(3) The Hessian matrix defined by $g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}$ is positive definite.

A Randers metric on M is a Finsler metrics of the form $F=\alpha+\beta$, where α is a Riemannian metric and β is a smooth 1-form on M whose length with respect to α is everywhere less than 1. This kind of metric was introduced by G. Randers in 1941 ([13]) in the context of general relativity.

The above expression of a Randers metric is called the defining form in the literature. Using standard local coordinates (x^i, y^j) for TM, a Randers metric can be presented as

$$F=\alpha+\beta=\sqrt{a_{ij}(x)y^iy^j}+b_i(x)y^i. \tag{1}$$

There is another method introduced by Shen [14] to express a Randers metric. The main idea is that the Randers metric F can also be uniquely written as

$$F(x,y)=\frac{\sqrt{h(y,W)^2+\lambda h(y,y)}}{\lambda}-\frac{h(y,W)}{\lambda},$$

where h is a Riemannian metric, W is a vector field on M with $h(W,W)<1$ and $\lambda=1-h(W,W)$. The pair (h,W) is called the navigation data of the Randers metric F. The navigation data is convenient when handling some problems concerning the flag curvatures and Ricci scalar of a Randers space. For example, using navigation data, Bao, Robles and Shen presented a very explicit description of Einstein-Randers metrics and Randers spaces of constant flag curvature; see [5, 6].

In a local coordinate system, the transformation laws between the defining form and navigation data can be described as the following. If

$$F=\alpha+\beta=\sqrt{a_{ij}y^iy^j}+b_iy^i,$$

then the navigation data has the form

$$h_{ij}=(1-||\beta||^2)(a_{ij}-b_ib_j),\quad W^i=-\frac{a^{ij}b_j}{1-||\beta||_\alpha^2}.$$

Conversely, the defining form can also be expressed by the navigation data through the formula:

$$a_{ij}=\frac{h_{ij}}{\lambda}+\frac{W_i}{\lambda}\frac{W_j}{\lambda},\quad b_i=\frac{-W_i}{\lambda}, \tag{1.2}$$

here $W_i=h_{ij}W^j$ and $\lambda=1-W^iW_i=1-h(W,W)$. All these formulas can be found in [5].

A Finsler metric F defines the length of a tangent vector, and the integration along a piece-wise smooth path defines the arc length of curves. Taking the infimum of arc lengths for all piece-wise smooth paths, one defines a distance function $d(\cdot,\cdot)$ on (M,F), which satisfies all the conditions of a distance in the general sense except the reversibility. Based on the distance function, we have generalized the concept of a Clifford-Wolf translation (CW-translation) to Finsler geometry.

Definition 2.1 Let (M,F) be a connected Finsler space. An isometry ρ of (M,F) is called a Clifford-Wolf translation if the function $d(x,\rho(x))$ is a constant on M.

One can also define the notions of Clifford-Wolf homogeneous (CW-homogenous) spaces and of restrictively Clifford-Wolf homogenous (restrictively CW-homogeneous) spaces.

Definition 2.2 A Finsler manifold (M,F) is called Clifford-Wolf homogeneous if for any two points $x, x' \in M$, there is a CW-translation ρ such that $\rho(x) = x'$. It is called restrictively Clifford-Wolf homogeneous if for any point $x \in M$ there is a neighborhood V of x, such that for any two points $x_1, x_2 \in V$ there is a CW-translation ρ such that $\rho(x_1) = x_2$.

¿From the above definitions, it is obvious that a Finsler manifold (M,F) is restrictively CW-homogeneous if it is CW-homogeneous. Moreover, a connected restrictively CW-homogeneous Finsler space (M,F) must be a homogeneous Finsler space, i.e., its isometry group $G = I(M,F)$ acts transitively on M. In fact, if it is connected and restrictively CW-homogeneous, then each G-orbit in M is an open as well as a closed subset.

3. Killing vector fields of constant length

CW-translations of a connected Finsler manifold (M,F) have a natural interrelation with the Killing vector fields of constant length, which has been proved by Berestovskii and Nikonorov ([4]) in the Riemannian case, and by the authors in Finsler geometry ([8]).

Theorem 3.1. *([8]) Let (M,F) be a complete Finsler manifold with a positive injectivity radius. If X is a Killing vector field of constant length, then the flow φ_t generated by X is a CW-translation for all sufficiently small $t > 0$.*

Theorem 3.2. *([8]) Let (M,F) be a compact Finsler manifold. Then there is a $\delta > 0$, such that any CW-translation ρ with $d(x, \rho(x)) < \delta$ is generated by a Killing vector field of constant length.*

Theorem 3.2 is not convenient when dealing with non-compact Finsler spaces. We now prove the following theorem which will be useful in proving the main theorem of this paper.

Theorem 3.3. *Let (M,F) be a connected homogeneous Finsler space. Then there is a $\delta > 0$, such that any CW-translation ρ with $d(x, \rho(x)) \leq \delta$ is generated by a Killing vector field of constant length.*

Proof. The connected isometry group $G = I_0(M,F)$ is Lie group ([7]) which acts transitively on M. Let $\mathfrak{g}$ be its Lie algebra. The exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism when restricted to a small round open disk neighborhood B of $0 \in \mathfrak{g}$. When the positive number δ is small enough, all the CW-translations ρ with $d(x, \rho(x)) \leq \delta$ is contained in $\exp(B)$. If we denote the set of all CW-translations ρ with $d(x, \rho(x)) \leq \delta$ as $\mathcal{C}_\delta$, then both $\mathcal{C}_\delta$ and $\exp^{-1}(\mathcal{C}_\delta) \cap B$ are compact. As the homogeneous space (M,F) has a positive injectivity radius $r > 0$, we can choose B to be small enough such that for any $X \in B$, $F(X(x)) < r$.

Now any CW-translation $\rho \in \mathcal{C}_\delta$ is generated by a unique Killing vector field $X \in B$, such that $\rho = \varphi_{1;X} = \exp X$, where $\varphi_{t;X}$ is the local one-parameter subgroup of diffeomorphisms generated by X. We need to prove that $F(X)$ is a constant function on M. Let g_n be a sequence in G such that $F(X(g_n^{-1}x))$ converge to $\sup F(X)$. The supremum of $F(X)$ is not required to be finite, but we can see from the later argument that $\sup F(X) \leq r$. We have a sequence of CW-translations $\rho_n = \mathrm{Ad}(g_n)\rho$ in $\mathcal{C}_\delta$. Each ρ_n is generated by the Killing vector field $X_n = g_{n*}X$, i.e., $\rho_n = \varphi_{1;X_n} = \exp X_n$, and we have $F(X(g_n^{-1}x)) = F(X_n(x))$. By the connectedness of G one easily sees that all the Killing vector fields X_n is contained $\exp^{-1}(\mathcal{C}_\delta)\cap B$. Choosing subsequence if necessary, one can assume that the sequence $\{\rho_n\}$ converges to $\rho' \in \mathcal{C}_\delta$, with $d(\cdot,\rho'(\cdot)) \equiv d(\cdot,\rho(\cdot))$, that $\{X_n\}$ converges to a Killing vector field $X' \in \exp^{-1}\mathcal{C}_\delta\cap B$ and that $\rho' = \varphi_{1;X'} = \exp X'$. At each point, we have

$$X'(\cdot) = \lim_{n\to\infty} X_n(\cdot).$$

In particular, at the point x, the function

$$F(X'(\cdot)) = \lim_{n\to\infty} F(X_n(\cdot))$$

reaches its maximum. If $F(X'(x)) = 0$, then $X = 0$ and ρ is the identity map. Otherwise $\nabla_X^X X = -\frac{1}{2}\tilde{\nabla}^{(X)}F(X)^2$ vanishes along the flow curve $\varphi_{t;X'}(x)$, $t \geq 0$. Then by Corollary 3.2 of [8], the flow curve $\varphi_{t;X'}(x)$, $t \geq 0$ is a geodesic. Because $F(X'(\varphi_{t;X'}(x))) = F(X'(x)) \leq r$ for all $t \geq 0$, the geodesic $\varphi_{t;X'}(x)$, $t \in [0,1]$ is minimizing. Thus

$$F(X'(x)) = \sup(F(X)) = d(x,\rho'(x)) = d(x,\rho(x)).$$

A similar argument for $\inf(F(X))$ (which must be positive when the CW-translation is not the identity map) gives $\inf(F(X)) = d(x,\rho(x))$. Therefore X is a Killing vector field of constant length. ■

As a corollary, we can give a description of restrictive CW-homogeneity by Killing vector fields of constant length, which will be used in the proof of the main theorem.

Theorem 3.4. *Let (M,F) be a connected homogeneous Finsler space. Then it is restrictively CW-homogeneous if and only if the Killing fields of constant length can exhaust all tangent directions, or equivalently, any geodesic starting from any point is the flow curve of a Killing field of constant length.*

Proof. If (M,F) is homogeneous, it has a positive injectivity radius $r > 0$. At each point, there is a small neighborhood, in which for any points x and x', $d(x,x') < r$. Then the proof of Theorem 3.1 indicates the existence of a CW-translation ρ which maps x to x'. So (M,F) is restrictive CW-homogeneous.

If (M,F) is connected and restrictively homogeneous, then it is homogenous. Then Theorem 3.3 indicates that all tangent vectors are exhausted by Killing vector fields of constant length. ■

In the Riemannian case, the above arguments give an alternative proof for Theorem 7 in [4].

4. Proof of Theorem 1.1

We first prove that, if (M,F) is a connected simply connected restrictively CW-homogeneous Randers space, then M has a product form with a metric F as described in the theorem.

Let (h,W) be the navigation data of F and $\varphi_{t;W}$ be the flow generated by the vector field W. By Theorem 3.4, for any $y \in TM_x$, there is a Killing vector field X of constant length for F, such that $X(x) = y$. Let $x' \in M$ and $t \geq 0$, such that $\varphi_{t;W}(x) = x'$. Since X is a Killing vector field for F, we have $L_X W = [X,W] = 0$. Moreover, $y = X(x)$ has the same F-length as $y' = X(x') = \varphi_{t;W*}(y)$. It follows that all $\varphi_{t;W}$'s are isometries and W is a Killing vector field of F. For any Killing vector field X of constant length 1 of F, $X+W$ is a Killing vector field for h. It is easily seen that $X+W$ has constant length 1 with respect to h. Since (M,F) is restrictively CW-homogeneous, the set of Killing vector fields X's of constant length 1 of F exhaust all tangent directions. Because the length of W with respect to F is less than 1, the set of all the vector fields $X+W$ exhaust all tangent directions too. So (M,h) is restrictively CW-homogeneous. By the classification theorem in [4], (M,h) is a Riemannian product of odd dimensional spheres with constant curvature metrics, compact connected simply connected Lie groups with bi-invariant metrics, and a flat Euclidian space. The vector field W is of constant length with respect to F, so it is of constant length with respect to h.

Next we prove that, if M is the product manifold with a Randers metric F as described in the theorem, then (M,F) is restrictively CW-homogeneous.

If (M,F) is the product manifold as described in the theorem, then it is homogeneous. Let $h^2 = \sum_{i=1}^n h_i^2$ be the decomposition for the symmetric metric, with respect to the decomposition $M = M_1 \times \cdots \times M_n$, and denote $W = \sum_{i=1}^n W_i$. Then each W_i is a Killing field of constant length for h_i. A diffeomorphism $\psi = \psi_1 \times \cdots \times \psi_n \in I(M,h)$ in which each $\psi_i \in I(M_i,h_i)$ is an isometry for (M,F) if and only if ψ_i keeps W_i invariant for each i. We only need to check that it acts transitively for each factor. If $W_i = 0$, then the argument for the corresponding factor is trivial. Thus we assume $W_i \neq 0$. If (M_i,h_i) is a compact connected simply connected simple Lie group and h_i is bi-invariant, then W_i must belong to the Lie algebra for $L(M_i)$ or $R(M_i)$. Without losing generality, we assume W_i belongs to the Lie algebra for $L(M_i)$, then all right translations acts transitively on M_i. If (M_i,h_i) is an odd dimensional sphere S^{2k-1} with constant curvature metric, then W_i defines a complex structure J such that $W_i = cJ$ with $c \neq 0$, when both are regarded as matrices in $\mathfrak{so}(2k)$. Then the isometries keeping W_i invariant are just the group $\mathrm{U}(k)$ with respect to the complex structure J, and it is obvious that $\mathrm{U}(k)$ act transitively on M_i. If (M_i,h_i) is a flat euclidian space, then W_i is a constant vector field, and ψ_i can be any parallel translation, which also acts transitively on M_i.

We now prove that the set of Killing fields of constant length for h which commute with W exhaust all tangent directions at each point. Any Killing fields X of constant length for h can be decomposed as $X = \sum_{i=1}^n X_i$, in which each X_i is a Killing field of constant length for h_i. The condition $[X,W] = 0$ is equivalent to $[X_i,W_i] = 0$ for each i. So the discussion breaks down to a case by case discussion for each factor (M_i, h_i). We only consider the case that $W_i \neq 0$. When (M_i, h_i) is a compact connected simply connected simple Lie group with bi-invariant metric or a flat euclidian space, the assertion is obvious because the group of right translations or the parallel translations are CW-translations for h_i, and its Lie algebra provides Killing fields of constant length for h_i which commutes with W_i. If (M_i, h_i) is an odd dimensional sphere with constant curvature, the assertion follows immediately from the fact ([9]) that the Randers space with navigation data $(h_i, \lambda W_i)$ is CW-homogeneous, where λ is any constant such that $||\lambda W_i||_{h_i} < 1$.

When $||W||_h < 1$, the set of the vector fields $X - W$, where X is a Killing vector field of constant length 1 with respect to h commuting with W, exhaust all tangent directions. It is easily seen that any vector field $X - W$ as above is a Killing vector field of constant length 1 for F. By Theorem 3.4, (M, F) is restrictively CW-homogeneous.

Finally, we prove the CW-homogeneity for a Randers space (M, F) as described in the theorem. If (M, F) is not CW-homogeneous, then we can find a Killing vector field of the form $X - W$ of constant length 1 with respect to F, where X is a Killing vector field of constant length 1 for h and $[X, W] = 0$. Moreover, there is a positive constant t_0 and two pairs of points x_i and x'_i, $i = 0, 1$, such that the diffeomorphisms $\varphi_{X-W;t}$ generated by $X - W$ satisfy the condition $\varphi_{X-W;t_0}(x_i) = x'_i$, $i = 0, 1$, and the geodesic flow curve of $X - W$ is minimizing from x_0 to x'_0 but not minimizing from x_1 to x'_1. Since $[X, W] = 0$, the diffeomorphisms $\varphi_{X;t}$ commute with $\varphi_{W;t'}$. Furthermore, we have

$$\varphi_{X-W;t} = \varphi_{X;t}\varphi_{W;-t} = \varphi_{W;-t}\varphi_{X;t}. \tag{2}$$

Since the flow curve from x_1 to x'_1 is not minimizing, there is a Killing vector field $X' - W$ of constant length 1 for F, where X' is a Killing vector field of constant length 1 for h with $[X', W] = 0$, and a constant $t' \in [0, t_0)$ such that

$$\varphi_{W;-t_0}\varphi_{X;t_0}(x_1) = \varphi_{W;-t'}\varphi_{X';t'}(x_1), \tag{3}$$

i.e., $\varphi_{X;t_0}(x_1) = \varphi_{W;t_0-t'}\varphi_{X',t'}(x_1)$. Since h is a symmetric Riemannian metric, and since X is a Killing field of constant length 1 with respect to h, the centralizer of X in the full group of isometries of h acts transitively on M (see [17, 12]). Thus there is an isometry g of h, which sends x_1 to x_0, such that $\mathrm{Ad}(g)X = X$. Then we have

$$\varphi_{\mathrm{Ad}(g)X;t_0}(x_0) = \varphi_{X;t_0}(x_0) = \varphi_{\mathrm{Ad}(g)W;t_0-t'}\varphi_{\mathrm{Ad}(g)X';t'}(x_0). \tag{4}$$

Since the right side of the above equality gives a path with arc length $t' + (t_0 - t')|W|_h < t_0$, the geodesic $\varphi_{X;t}(x_0)$ from $t = 0$ to $t = t_0$ is not minimizing with respect to h. Now the next lemma asserts that the above equality still holds with g changed to e, for some other X' and t' satisfying the same properties as above. This implies that the geodesic $\varphi_{X-W;t}(x_0)$ for F is not minimizing between $t = 0$ to $t = t_0$, which is a contradiction. Therefore (M, F) must be CW-homogeneous.

Lemma 4.1. *Let (M, h) be connected simply connected Riemannian CW-homogeneous space, X be a Killing vector field of constant length* 1 *for h, and W be* *a Killing field of constant length less than* 1. *Assume the geodesic* $\varphi_{X;t}(x_0)$ *of* h *from* $t=0$ *to* $t=t_0$ *is not minimizing. Then there is a* $t' \in [0,t_0)$ *and a Killing vector field* X' *of constant length* 1 *for* h *with* $[X',W]=0$, *such that*

$$\varphi_{X;t_0}(x_0) = \varphi_{W;t_0-t'}\varphi_{X';t'}(x_0). \tag{5}$$

Proof. Let $f(t)$ be the distance from x_0 to $\varphi_{W;t}\varphi_{X;t_0}(x_0)$ with respect to h. Obviously $f(t)$ is continuous and non-negative for all $t \geq 0$ and $f(0) < t_0$ because the geodesic $\varphi_{X;t}(x_0)$ from $t=0$ to $t=t_0$ is not minimizing. There exists a $t' \in [0,t_0)$ such that $f(t_0-t') = t'$. The completeness of M implies there is a minimizing geodesic from x_0 to $\varphi_{W;t_0-t'}\varphi_{X;t_0}(x_0)$. Our earlier arguments indicate that this minimizing geodesic is the flow curve of a Killing vector field X' of constant length 1 with respect to h with $[X',W]=0$, i.e.

$$\varphi_{X;t_0}(x_0) = \varphi_{W;t_0-t'}\varphi_{X';t'}(x_0). \tag{6}$$

This completes the proof of the lemma as well as of the theorem. ■

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