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On the Homomorphisms between the Generalized Verma Modules Arising from Conformally Invariant Systems

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Abstract. It is shown by Barchini, Kable, and Zierau that conformally invariant systems of differential operators yield explicit homomorphisms between certain generalized Verma modules. In this paper we determine whether or not the homomorphisms arising from such systems of first and second order differential operators associated to maximal parabolic subalgebras of quasi-Heisenberg type are standard.
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1. Introduction

The main work of this paper concerns homomorphisms between the generalized Verma modules arising from conformally invariant systems of differential operators. As a conformally invariant system is a central object of this paper, we begin with introducing the definition of such systems of operators. Loosely speaking, a conformally invariant system is a system of differential operators that are equivariant under a Lie algebra action. To describe the equivariance condition precisely, let $\mathfrak{g}_0$ be a real Lie algebra. The definition of conformally invariant systems requires the notions of a $\mathfrak{g}_0$-manifold and $\mathfrak{g}_0$-bundle. First, a smooth manifold M is said to be a $\mathfrak{g}_0$*-manifold* if there exists a Lie algebra homomorphism $\pi_M : \mathfrak{g}_0 \to C^\infty(M) \oplus \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the space of smooth vector fields on M. Here, the Lie algebra structure of $C^\infty(M) \oplus \mathfrak{X}(M)$ is the standard one induced from the algebra structure of differential operators. Given $\mathfrak{g}_0$-manifold M, write $\pi_M(X) = \pi_0(X) + \pi_1(X)$ with $\pi_0(X) \in C^\infty(M)$ and $\pi_1(X) \in \mathfrak{X}(M)$. Next, let $\mathbb{D}(\mathcal{V})$ denote the space of differential operators on a vector bundle $\mathcal{V} \to M$. We regard any smooth functions f on M as elements in $\mathbb{D}(\mathcal{V})$ by identifying them with the multiplication operator that they induce. Then we say that a vector bundle $\mathcal{V} \to M$ is a $\mathfrak{g}_0$*-bundle* if there exists a Lie algebra homomorphism $\pi_{\mathcal{V}} : \mathfrak{g}_0 \to \mathbb{D}(\mathcal{V})$ so that in $\mathbb{D}(\mathcal{V})$ $[\pi_{\mathcal{V}}(X), f] = \pi_1(X)\bullet f$ for all $X \in \mathfrak{g}_0$ and

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all $f \in C^\infty(M)$, where the dot $\bullet$ denotes the action of the differential operator $\pi_1(X)$. Here, as for $C^\infty(M) \oplus \mathfrak{X}(M)$, the Lie algebra structure of $\mathbb{D}(\mathcal{V})$ is the standard one coming from its algebra structure of operators with composition. Now, given $\mathfrak{g}_0$-bundle $\mathcal{V} \to M$, a system of linearly independent differential operators $D_1, \ldots, D_m \in \mathbb{D}(\mathcal{V})$ is called a *conformally invariant system* on $\mathcal{V}$ with respect to $\pi_\mathcal{V}$ if, for all $X \in \mathfrak{g}_0$, it satisfies the bracket identity

$$[\pi_\mathcal{V}(X), D_j] = \sum_i^m C_{ij}^X D_i,$$

where C_{ij}^X are smooth functions on M. By extending the Lie algebra homomorphisms π_M and $\pi_\mathcal{V}$ $\mathbb{C}$-linearly, the definitions of a $\mathfrak{g}_0$-manifold, $\mathfrak{g}_0$-bundle, and conformally invariant system can be applied equally well to the complexified Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \otimes_\mathbb{R} \mathbb{C}$.

The Laplacian Δ on $\mathbb{R}^n$ and wave operator $\square$ on the Minkowski space $\mathbb{R}^{3,1}$ are two typical examples for conformally invariant systems consisting of one differential operator. The notion of conformally invariant systems generalizes that of Kostant's quasi-invariant differential operator ([16]). A systematic study of conformally invariant systems recently started with the work of Barchini-Kable-Zierau in [4] and [5], and the study of such systems of operators is continued in [11], [12], [13], [14], [15], [18], and [20].

While the works [4], [11]-[15], [18], and [20] mainly focus on the construction of conformally invariant systems or the solution spaces to such systems of operators, we in this paper study the homomorphisms between generalized Verma modules that arise from conformally invariant systems. Homomorphisms between generalized Verma modules (or equivalently intertwining differential operators between degenerate principal series representations) have received a lot of attentions from many points of views (see for example [6], [8], [10], [17], and [22]). It has been shown in [5] that a conformally invariant system yields a homomorphism between certain generalized Verma modules, one of which is non-scalar. In the present work we would like to understand the "standardness" of such homomorphisms. A homomorphism between generalized Verma modules is called *standard* if it is induced from a homomorphism between the corresponding (ordinary) Verma modules, and called *non-standard* otherwise. While standard homomorphisms are well-understood (see for example [6] and [21]), the classification of non-standard homomorphisms is still an open problem. See for instance [1], [2], and Section 11.5 of [3] for the classification of such maps for certain cases. We may want to note that much of the published work concerning non-standard homomorphisms is for the case that the nilpotent radical $\mathfrak{n}$ for parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is abelian.

In [18] we have built a number of conformally invariant systems of first and second order differential operators, that are associated to maximal parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ with nilpotent radical $\mathfrak{n}$ satisfying the conditions that $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$ and $\dim([\mathfrak{n}, \mathfrak{n}]) > 1$. We call such nilpotent algebra $\mathfrak{n}$ *quasi-Heisenberg* and such parabolic subalgebras $\mathfrak{q}$ *quasi-Heinseberg type*. Then, in this paper, we determine whether or not the homomorphisms between the generalized Verma modules arising from the systems of operators associated to maximal parabolic subalgebras $\mathfrak{q}$ of quasi-Heisenberg type are standard. As the nilpotent radical $\mathfrak{n}$ of $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is quasi-Heisenberg, this gives examples of non-standard maps beyond the scope of the case that $\mathfrak{n}$ is abelian.

To describe our work more precisely, we now briefly review the results of [18]. Let G be a complex, simple, connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$. Give a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ on $\mathfrak{g}$ so that $\mathfrak{q} = \mathfrak{g}(0) \oplus \bigoplus_{j>0} \mathfrak{g}(j) = \mathfrak{l} \oplus \mathfrak{n}$ is a parabolic subalgebra. Let $Q = N_G(\mathfrak{q}) = LN$. For a real form $\mathfrak{g}_0$ of $\mathfrak{g}$, define G_0 to be an analytic subgroup of G with Lie algebra $\mathfrak{g}_0$. Set $Q_0 = N_{G_0}(\mathfrak{q})$. Our manifold is $M = G_0/Q_0$ and we consider a line bundle $\mathcal{L}_s \to G_0/Q_0$ for each $s \in \mathbb{C}$. By the Bruhat theory, the homogeneous space G_0/Q_0 admits an open dense submanifold $\bar{N}_0 Q_0/Q_0$. We restrict our bundle to this submanifold. By slight abuse of notation we refer to the restricted bundle as $\mathcal{L}_s$. The systems that we construct act on smooth sections of the restricted bundle $\mathcal{L}_s$.

Our systems of operators are constructed from L-irreducible constituents W of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ for $1 \leq k \leq 2r$. We call the systems of operators Ω_k *systems*. (We shall describe the construction more precisely in Section 2.) It is not necessary that every L-irreducible constituent of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ contributes to the construction for Ω_k systems. Then we call irreducible constituents W *special* if they contribute to the systems of operators. Here, we should remark a certain discrepancy of the definition for special constituents between this paper and [18]. In [18], special constituents for Ω_2 systems are defined as irreducible constituents of $\mathfrak{g}(0) \otimes \mathfrak{g}(2)$ whose highest weights satisfy a certain technical condition. (See Definition 6.7 of [18].) In the paper we first observed that, if an irreducible constituent of $\mathfrak{g}(0) \otimes \mathfrak{g}(2)$ contributes to an Ω_2 system then its highest weight satisfies the technical condition. We then tried to show that the opposite direction also holds; namely, we tried to show that irreducible constituents with the highest weight condition contribute to Ω_2 systems. For all the cases but two, it is verified that such irreducible constituents do contribute to the construction. The difficulty for the two open cases is that there is a problem to apply to these cases the method that is used for any other cases. We do expect that also in the open cases the constituents with the highest weight condition contribute to the construction. Thus we redefined special constituents in the way introduced at the beginning of this paragraph, so that the definition works not only for Ω_2 systems but also for Ω_k systems for general k. We would like to verify the open cases elsewhere and so the two definitions for special constituents do agree.

There is no reason to expect that Ω_k systems are conformally invariant on $\mathcal{L}_s$ for arbitrary $s \in \mathbb{C}$; the conformal invariance of Ω_k systems depends on the complex parameter s for the line bundle $\mathcal{L}_s$. We then say that an Ω_k system has *special value* s_0 if the system is conformally invariant on the line bundle $\mathcal{L}_{s_0}$.

In [18], we found the special values of the Ω_1 system and certain Ω_2 systems associated to a maximal parabolic subalgebra $\mathfrak{q}$ of quasi-Heisenberg type. We may want to note that, to find the special values for Ω_2 systems, the technical condition on the highest weights for the special constituents plays a crucial role. (See Section 7 of [18].) See Theorem 5.1 and Table 4 for the special values of these systems. In Table 4, one notices that there are two missing cases, the cases with a question mark (?). These are the two open cases mentioned above. We would like to fill in the gaps in the future.

In this paper, with the special values determined in [18] in hand, for $k = 1, 2$, we classify the homomorphisms φ_{Ω_k} between the generalized Verma modules arising from the conformally invariant Ω_k systems as standard or non-standard. Our main tool is a well-known result due to Lepowsky (Theorem 4.3). It turns out that the map φ_{Ω_k} is non-standard if and only if the special value s_0 of an Ω_k system is a positive integer. See Theorem 5.3 for the result for the map φ_{Ω_1}. Table 5 summarizes the classification for φ_{Ω_2}.

Now we outline the rest of this paper. This paper consists of six sections with this introduction and one appendix. In Section 2 we recall from [18] the construction of the Ω_k systems. We also review maximal parabolic subalgebras $\mathfrak{q}$ of quasi-Heisenberg type in this section. Section 3 discusses the relationship between conformally invariant Ω_k systems and homomorphisms between generalized Verma modules. We start Section 4 with reviewing the general facts on the standard homomorphisms. We then specialize such facts to the situation that we concern.

In Sections 5 and 6, for $k = 1, 2$, we determine whether or not the homomorphisms φ_{Ω_k} arising from the Ω_k systems associated to the maximal parabolic subalgebra $\mathfrak{q}$ under consideration are standard. This is done in four theorems, namely, Theorem 5.3, Theorem 6.5, Theorem 6.6, and Theorem 6.38.

Finally, in Appendix A, we recall from [18] the miscellaneous useful data for the parabolic subalgebras under consideration. The data will be referred to in several proofs in this paper.

2. Preliminaries

The purpose of this section is to recall from [18] our construction of systems of differential operators. We also review the maximal parabolic subalgebras of quasi-Heisenberg type.

2.1. A specialization of the theory.

First we recall from Subsection 2.1 in [18] the $\mathfrak{g}$-manifold and $\mathfrak{g}$-bundle that we study in this paper. Let G be a complex, simple, connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$. Such G contains a maximal connected solvable subgroup B. Write $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ for its Lie algebra with $\mathfrak{h}$ the Cartan subalgebra and $\mathfrak{u}$ the nilpotent subalgebra. Let $\mathfrak{q} \supset \mathfrak{b}$ be a parabolic subalgebra of $\mathfrak{g}$. We define $Q = N_G(\mathfrak{q})$, a parabolic subgroup of G. Write $Q = LN$ for the Levi decomposition of Q.

Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$ in which the complex parabolic subalgebra $\mathfrak{q}$ has a real form $\mathfrak{q}_0$, and let G_0 be the analytic subgroup of G with Lie algebra $\mathfrak{g}_0$. Define $Q_0 = N_{G_0}(\mathfrak{q}) \subset Q$, and write $Q_0 = L_0N_0$. We will work with G_0/Q_0 for a class of maximal parabolic subgroup Q_0 whose Lie algebra $\mathfrak{q}_0$ is of two-step nilpotent type.

Next, let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let Δ^+ be the positive system attached to $\mathfrak{b}$ and denote by Π the set of simple roots. We write $\mathfrak{g}_\alpha$ for the root space for $\alpha \in \Delta$. For each subset $S \subset \Pi$, let $\mathfrak{q}_S$ be the corresponding standard parabolic subalgebra. Write $\mathfrak{q}_S = \mathfrak{l}_S \oplus \mathfrak{n}_S$ with Levi factor $\mathfrak{l}_S = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_S} \mathfrak{g}_\alpha$ and nilpotent radical $\mathfrak{n}_S = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_S} \mathfrak{g}_\alpha$, where $\Delta_S = \{\alpha \in \Delta \mid \alpha \in \mathrm{span}(\Pi \setminus S)\}$. If Q_0 is a maximal parabolic subgroup then there exists a unique simple root $\alpha_{\mathfrak{q}} \in \Pi$ so that $\mathfrak{q} = \mathfrak{q}_{\{\alpha_{\mathfrak{q}}\}}$. Let $\lambda_{\mathfrak{q}}$ be the fundamental weight of $\alpha_{\mathfrak{q}}$. The weight $\lambda_{\mathfrak{q}}$ is orthogonal to any roots α with $\mathfrak{g}_\alpha \subset [\mathfrak{l}, \mathfrak{l}]$. Hence it exponentiates to a character $\chi_{\mathfrak{q}}$ of L. As $\chi_{\mathfrak{q}}$ takes real values on L_0, for $s \in \mathbb{C}$, character $\chi^s = |\chi_{\mathfrak{q}}|^s$ is well-defined on L_0. Let $\mathbb{C}_{\chi^s}$ be the one-dimensional representation of L_0 with character χ^s. The representation χ^s is extended to a representation of Q_0 by making it trivial on N_0. It then deduces a line bundle $\mathcal{L}_s$ on G_0/Q_0 with fiber $\mathbb{C}_{\chi^s}$.

The group G_0 acts on the space

$$
\begin{aligned}
&C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^s})\\
&= \{F \in C^\infty(G_0, \mathbb{C}_{\chi^s}) \mid F(gq) = \chi^s(q^{-1})F(g) \text{ for all } q \in Q_0 \text{ and } g \in G_0\}
\end{aligned}
$$

by left translation. The action π_s of $\mathfrak{g}_0$ on $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^s})$ arising from this action is given by

$$
(\pi_s(Y)\bullet F)(g) = \frac{d}{dt}F(\exp(-tY)g)\big|_{t=0} \tag{2.1}
$$

for $Y \in \mathfrak{g}_0$. This action is extended $\mathbb{C}$-linearly to $\mathfrak{g}$ and then naturally to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. We use the same symbols for the extended actions.

Let $\bar{N}_0$ be the unipotent subgroup opposite to N_0. The natural infinitesimal action of $\mathfrak{g}$ on the image of the restriction map $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^s}) \to C^\infty(\bar{N}_0, \mathbb{C}_{\chi^s})$ induced by (2.1) gives an action of $\mathfrak{g}$ on the whole space $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^s})$. We also denote by π_s the induced action. Observe that we have the direct sum $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{q}$. If we write $Y = Y_{\bar{\mathfrak{n}}} + Y_{\mathfrak{q}}$ for the decomposition of $Y \in \mathfrak{g}$ in this direct sum then, for $Y \in \mathfrak{g}$ and $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^s})$, the derived action of $\mathfrak{g}$ on $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^s})$ is given by

$$
\big(\pi_s(Y)\bullet f\big)(\bar{n}) = s\lambda_{\mathfrak{q}}\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}\big)f(\bar{n}) - \big(R\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}}\big)\bullet f\big)(\bar{n}), \tag{2.2}
$$

where R is the infinitesimal right translation of $\mathfrak{g}$. The line bundle $\mathcal{L}_s \to G_0/Q_0$ restricted to $\bar{N}_0$ is the trivial bundle $\bar{N}_0 \times \mathbb{C}_{\chi^s} \to \bar{N}_0$. By slight abuse of notation, we refer to the trivial bundle over $\bar{N}_0$ as $\mathcal{L}_s$. It follows from the observation in Subsection 2.1 in [18] that $\bar{N}_0$ and $\mathcal{L}_s \to \bar{N}_0$ are a $\mathfrak{g}$-manifold and $\mathfrak{g}$-bundle, respectively.

2.2. The Ω_k systems.

In this subsection we briefly recall from Subsection 3.1 of [18] our construction of differential operators. For a subspace W of $\mathfrak{g}$, we write $\Delta(W) = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset W\}$ and $\Pi(W) = \Delta(W) \cap \Pi$. We keep the notation from the previous subsection, unless otherwise specified.

Let $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ be a $\mathbb{Z}$-grading on $\mathfrak{g}$ with $\mathfrak{g}(1) \neq 0$. For $1 \leq k \leq 2r$, we define a map $\tau_k : \mathfrak{g}(1) \to \mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ by $X \mapsto \frac{1}{k!}\big(\mathrm{ad}(X)^k \otimes \mathrm{Id}\big)\omega$ with $\omega = \sum_{\gamma_j \in \Delta(\mathfrak{g}(r))} X_{-\gamma_j} \otimes X_{\gamma_j}$, where X_{γ_j} are root vectors for γ_j so that $\{X_{\gamma_j}, X_{-\gamma_j}, [X_{\gamma_j}, X_{-\gamma_j}]\}$ is an $\mathfrak{sl}(2)$-triple. Take L to be the analytic subgroup of G with Lie algebra $\mathfrak{g}(0)$, and let W be an L-irreducible constituent of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$. Write $\mathcal{P}^k(\mathfrak{g}(1))$ for the space of polynomials on $\mathfrak{g}(1)$ of homogeneous degree k. If W^* is the dual space of W with respect to the Killing form κ then there exists an L-intertwining operator $\tilde{\tau}_k|_{W^*} \in \mathrm{Hom}_L(W^*, \mathcal{P}^k(\mathfrak{g}(1)))$ so that, for $Y^* \in W^*$, $\tilde{\tau}_k|_{W^*}(Y^*)(X) = Y^*(\tau_k(X))$. Here, we may want to note that $Y^*(\tau_k(X))$ is well-defined for $\tau_k(X) \notin W$. Indeed, observe that, as $\mathfrak{g}(-r+k)^* \otimes \mathfrak{g}(r)^* \cong \mathfrak{g}(r-k) \otimes \mathfrak{g}(-r)$ via the Killing form κ, the element $Y^* \in W^* \subset \mathfrak{g}(-r+k)^* \otimes \mathfrak{g}(r)^*$ is a linear combination of $\kappa(X_\alpha, \cdot) \otimes \kappa(X_\beta, \cdot)$ with constant coefficients, where X_α and X_β are root vectors for $\alpha \in \Delta(\mathfrak{g}(r-k))$ and $\beta \in \Delta(\mathfrak{g}(-r))$. If $Y^* = \sum_{\alpha,\beta} c_{\alpha,\beta}\, \kappa(X_\alpha, \cdot) \otimes \kappa(X_\beta, \cdot)$ with constants $c_{\alpha,\beta}$ then $Y^*(\tau_k(X))$ is given by $Y^*(\tau_k(X)) = (1/k!) \sum_{\gamma_j,\alpha,\beta} c_{\alpha,\beta}\, \kappa(X_\alpha, \mathrm{ad}(X)^k(X_{-\gamma_j}))\, \kappa(X_\beta, X_{\gamma_j})$.

If $\tilde{\tau}_k|_{W^*} \not\equiv 0$ then we call the irreducible constituent W **special** for τ_k. Given special constituent W for τ_k, we consider the following composition of linear maps:

$$W^* \stackrel{\tilde{\tau}_k|_{W^*}}{\to} \mathcal{P}^k(\mathfrak{g}(1)) \cong \mathrm{Sym}^k(\mathfrak{g}(-1)) \stackrel{\sigma}{\hookrightarrow} \mathcal{U}(\bar{\mathfrak{n}}) \stackrel{R}{\to} \mathbb{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}}. \tag{2.3}$$

Here, $\sigma : \mathrm{Sym}^k(\mathfrak{g}(-1)) \to \mathcal{U}(\bar{\mathfrak{n}})$ is the symmetrization operator and $\mathbb{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}}$ is the space of $\bar{\mathfrak{n}}$-invariant differential operators for $\mathcal{L}_s$. Let $\Omega_k|_{W^*} : W^* \to \mathbb{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}}$ be the composition of linear maps, namely, $\Omega_k|_{W^*} = R \circ \sigma \circ \tilde{\tau}_k|_{W^*}$. For simplicity we write $\Omega_k(Y^*) = \Omega_k|_{W^*}(Y^*)$ for the differential operator arising from $Y^* \in W^*$. Note that the linear operator $\Omega_k|_{W^*} : W^* \to \mathbb{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}}$ is an L_0-intertwining operator. (See the observation at the end of Section 3.1 of [18].)

Now, given basis $\{Y_1^*, \ldots, Y_m^*\}$ for W^*, we have a system of differential operators

$$\Omega_k(Y_1^*), \ldots, \Omega_k(Y_m^*). \tag{2.4}$$

We call such a system of operators the $\Omega_k|_{W^*}$ **system**. When the irreducible constituent W^* is not important, we simply refer to each $\Omega_k|_{W^*}$ system as an Ω_k system. We may want to note that Ω_k systems are independent of the choice for a basis for W^* up to some natural equivalence. (See Definition 3.5 of [18].)

It is important to notice that it is not necessary for Ω_k systems to be conformally invariant; their conformal invariance strongly depends on the complex parameter s for the line bundle $\mathcal{L}_s$. So we say that an Ω_k system has **special value** s_0 if the system is conformally invariant on the line bundle $\mathcal{L}_{s_0}$. In [18], we have found the special values for the Ω_1 system and certain Ω_2 systems associated to maximal parabolic subalgebras $\mathfrak{q}$ of quasi-Heinseberg type. We shall show the special values in Sections 5 and 6, respectively.

2.3. Maximal parabolic subalgebras of quasi-Heisenberg type.

In Sections 5 and 6, with the special values determined in [18] in hand, we shall determine whether or not the homomorphisms arising from the Ω_1 system and Ω_2 systems associated to maximal parabolic subalgebras $\mathfrak{q}$ of quasi-Heisenberg type are standard. Then, in this section, we recall from Section 4 of [18] such maximal parabolic subalgebras $\mathfrak{q}$.

First, we call a maximal parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ **quasi-Heisenberg type** if its nilradical $\mathfrak{n}$ satisfies the conditions that $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$ and $\dim([\mathfrak{n}, \mathfrak{n}]) > 1$. Let $\alpha_{\mathfrak{q}}$ be a simple root, so that the parabolic subalgebra $\mathfrak{q} = \mathfrak{q}_{\{\alpha_{\mathfrak{q}}\}} = \mathfrak{l} \oplus \mathfrak{n}$ determined by $\alpha_{\mathfrak{q}}$ is of quasi-Heisenberg type. Let $\langle \cdot, \cdot \rangle$ be the inner product induced on $\mathfrak{h}^*$ corresponding to the Killing form κ. Write $||\alpha||^2 = \langle \alpha, \alpha \rangle$ for $\alpha \in \Delta$. The coroot of α is $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$.

Recall from Subsection 2.1 that $\lambda_{\mathfrak{q}}$ denotes the fundamental weight for $\alpha_{\mathfrak{q}}$. If $H_{\lambda_{\mathfrak{q}}} \in \mathfrak{h}$ is defined by $\kappa(H, H_{\lambda_{\mathfrak{q}}}) = \lambda_{\mathfrak{q}}(H)$ for all $H \in \mathfrak{h}$ and if $H_{\mathfrak{q}} = (2/||\alpha_{\mathfrak{q}}||^2)H_{\lambda_{\mathfrak{q}}}$ then as $\mathfrak{q}$ has two-step nilpotent radical, for $\beta \in \Delta^+$, $\beta(H_{\mathfrak{q}})$ can only take the values of 0, 1, or 2. Therefore, if $\mathfrak{g}(j)$ denotes the j-eigenspace of $\mathrm{ad}(H_{\mathfrak{q}})$ then the action of $\mathrm{ad}(H_{\mathfrak{q}})$ on $\mathfrak{g}$ induces a 2-grading $\mathfrak{g} = \bigoplus_{j=-2}^{2} \mathfrak{g}(j)$ with parabolic subalgebra $\mathfrak{q} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$, where $\mathfrak{l} = \mathfrak{g}(0)$ and $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$. The subalgebra $\bar{\mathfrak{n}}$, the nilpotent radical opposite to $\mathfrak{n}$, is given by $\bar{\mathfrak{n}} = \mathfrak{g}(-1) \oplus \mathfrak{g}(-2)$. Here we have $\mathfrak{g}(0) = \mathfrak{l}$, $\mathfrak{g}(2) = \mathfrak{z}(\mathfrak{n})$ and $\mathfrak{g}(-2) = \mathfrak{z}(\bar{\mathfrak{n}})$, where $\mathfrak{z}(\mathfrak{n})$ (resp. $\mathfrak{z}(\bar{\mathfrak{n}})$) is the center of $\mathfrak{n}$ (resp. $\bar{\mathfrak{n}}$). Thus we denote the 2-grading on $\mathfrak{g}$ by

$$\mathfrak{g} = \mathfrak{z}(\bar{\mathfrak{n}}) \oplus \mathfrak{g}(-1) \oplus \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n}) \tag{2.5}$$

with parabolic subalgebra

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n}).$$

Therefore the maps τ_k associated to the grading (2.5) are given by

$$\tau_k : \mathfrak{g}(1) \to \mathfrak{g}(-2+k) \otimes \mathfrak{z}(\mathfrak{n}) \tag{2.6}$$

for $1 \leq k \leq 4$.

We next consider the structure of the Levi subalgebra $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus [\mathfrak{l}, \mathfrak{l}]$, where $\mathfrak{z}(\mathfrak{l})$ is the center of $\mathfrak{l}$. Observe that $\mathfrak{z}(\mathfrak{l})$ is one-dimensional. Indeed, we have $\mathfrak{z}(\mathfrak{l}) = \bigcap_{\alpha \in \Pi(\mathfrak{l})} \ker(\alpha)$ with $\Pi(\mathfrak{l}) = \Pi \backslash \{\alpha_{\mathfrak{q}}\}$. As $\mathfrak{l} = \mathfrak{g}(0)$, we have $\alpha(H_{\mathfrak{q}}) = 0$ for all $\alpha \in \Delta(\mathfrak{l})$. Thus, $H_{\mathfrak{q}}$ is an element of $\mathfrak{z}(\mathfrak{l})$, and so we have $\mathfrak{z}(\mathfrak{l}) = \mathbb{C}H_{\mathfrak{q}}$.

To observe the semisimple part $[\mathfrak{l}, \mathfrak{l}]$ of $\mathfrak{l}$, let γ be the highest root of $\mathfrak{g}$. If $\mathfrak{g}$ is not of type A_n then there is exactly one simple root that is not orthogonal to γ. Let α_γ be the unique simple root so that $\mathfrak{q}' = \mathfrak{q}_{\{\alpha_\gamma\}}$ is the parabolic subalgebra of Heisenberg type; that is, its nilradical $\mathfrak{n}'$ satisfies $\dim([\mathfrak{n}', \mathfrak{n}']) = 1$. Hence, if $\mathfrak{q} = \mathfrak{q}_{\{\alpha_{\mathfrak{q}}\}}$ is a parabolic subalgebra of quasi-Heisenberg type then α_γ is in $\Pi(\mathfrak{l}) = \Pi \backslash \{\alpha_{\mathfrak{q}}\}$. The semisimple part $[\mathfrak{l}, \mathfrak{l}]$ is either simple or the direct sum of two or three simple ideals with only one simple ideal containing the root space $\mathfrak{g}_{\alpha_\gamma}$ for α_γ. Given Dynkin type $\mathcal{T}$ of $\mathfrak{g}$, if we write $\mathcal{T}(i)$ for the Lie algebra together with the choice of maximal parabolic subalgebra $\mathfrak{q} = \mathfrak{q}_{\{\alpha_i\}}$ determined by α_i then the three simple factors occur only when $\mathfrak{q}$ is of type $D_n(n-2)$. So, if $\mathfrak{q}$ is not of type $D_n(n-2)$ then there are at most two simple factors. In this case we denote by $\mathfrak{l}_\gamma$ (resp. $\mathfrak{l}_{n\gamma}$) the simple ideal of $\mathfrak{l}$ that contains (resp. does not contain) $\mathfrak{g}_{\alpha_\gamma}$. Thus $\mathfrak{l}$ may decompose into

$$\mathfrak{l} = \mathbb{C}H_{\mathfrak{q}} \oplus \mathfrak{l}_\gamma \oplus \mathfrak{l}_{n\gamma}. \tag{2.7}$$

Note that when $[\mathfrak{l}, \mathfrak{l}]$ is a simple ideal, we have $\mathfrak{l}_{n\gamma} = \{0\}$. (See Appendix A.) The maximal parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ of quasi-Heisenberg type with the decomposition (2.7) are given as follows:

$$B_n(i)\ (3 \leq i \leq n), \quad C_n(i)\ (2 \leq i \leq n-1), \quad D_n(i)\ (3 \leq i \leq n-3), \tag{2.8}$$

and

$$E_6(3),\ E_6(5),\ E_7(2),\ E_7(6),\ E_8(1),\ F_4(4). \tag{2.9}$$

Here, the Bourbaki conventions [7] are used for the labels of the simple roots. Note that, in type A_n, any maximal parabolic subalgebra has abelian nilpotent radical, and also that, in type G_2, the two maximal parabolic subalgebras are of either 3-step nilpotent type or Heisenberg type.

3. The Ω_k systems and generalized Verma modules

The aim of this section is to show that conformally invariant Ω_k systems induce non-zero $\mathcal{U}(\mathfrak{g})$-homomorphisms between certain generalized Verma modules. The main idea is that conformally invariant Ω_k systems yield finite dimensional simple $\mathfrak{l}$-submodules of generalized Verma modules, on which $\mathfrak{n}$ acts trivially.

In general, to describe the relationship between conformally invariant systems and generalized Verma modules, we realize generalized Verma modules as the space of smooth distributions supported at the identity. However, in our setting that the $\mathfrak{g}$-bundle is a line bundle $\mathcal{L}_s$, it is not necessary to use such a realization. Thus, in this paper, we are going to describe the relationship without using the realization. For the general theory see Sections 3, 5, and 6 of [5].

A generalized Verma module $M_{\mathfrak{q}}[W] := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} W$ is a $\mathcal{U}(\mathfrak{g})$-module that is induced from a finite dimensional simple $\mathfrak{l}$-module W on which $\mathfrak{n}$ acts trivially. Observe that if $\mathbb{C}_{-s\lambda_{\mathfrak{q}}}$ is the $\mathfrak{q}$-module derived from the Q_0-representation $(\chi^{-s}, \mathbb{C})$ then the differential operators in $\mathbb{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}}$ can be described in terms of elements of $M_{\mathfrak{q}}[\mathbb{C}_{-s\lambda_{\mathfrak{q}}}]$. Indeed, by identifying $M_{\mathfrak{q}}[\mathbb{C}_{-s\lambda_{\mathfrak{q}}}]$ as $\mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{-s\lambda_{\mathfrak{q}}}$, the map $M_{\mathfrak{q}}[\mathbb{C}_{-s\lambda_{\mathfrak{q}}}] \to \mathcal{U}(\bar{\mathfrak{n}})$ given by $u \otimes 1 \mapsto u$ is an isomorphism of vector spaces. Then the composition

$$M_{\mathfrak{q}}[\mathbb{C}_{-s\lambda_{\mathfrak{q}}}] \to \mathcal{U}(\bar{\mathfrak{n}}) \xrightarrow{R} \mathbb{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}} \tag{3.1}$$

is a vector-space isomorphism.

Define

$$M_{\mathfrak{q}}[W]^{\mathfrak{n}} = \{v \in M_{\mathfrak{q}}[W] \mid X \cdot v = 0 \text{ for all } X \in \mathfrak{n}\}.$$

The following result is the specialization of Theorem 19 in [5] to the present situation. For the definitions for straight, homogeneous, L_0-stable conformally invariant systems, see p. 797, p. 804 and p. 806 of [5].

Theorem 3.2. *If $D = D_1, \ldots, D_m$ is a straight, homogeneous, L_0-stable conformally invariant system on the line bundle $\mathcal{L}_s$, and if ω_j denotes the element in $\mathcal{U}(\bar{\mathfrak{n}})$ that corresponds to D_j for $j = 1, \ldots, m$ via right differentiation R in (3.1) then the space*

$$F(D) = \mathit{span}_{\mathbb{C}}\{\omega_j \otimes 1 \mid j = 1, \ldots, m\}$$

is an L-submodule of $M_{\mathfrak{q}}[\mathbb{C}_{-s\lambda_{\mathfrak{q}}}]^{\mathfrak{n}}$.

Now, let W be a special constituent of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(k)$ for τ_k. Let $\omega_k|_{W^*} : W^* \to \mathcal{U}(\bar{\mathfrak{n}})$ be the linear operator so that $\omega_k|_{W^*}(Y^*)$ is the element in $\sigma(\mathrm{Sym}^k(\mathfrak{n})) \subset \mathcal{U}(\bar{\mathfrak{n}})$ that corresponds to the differential operator $\Omega_k(Y^*) = \Omega_k|_{W^*}(Y^*)$ in $\mathbb{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}}$, via right differentiation R in (2.3). As for $\Omega_k(Y^*)$, for simplicity, we write $\omega_k(Y^*) = \omega_k|_{W^*}(Y^*)$. Then, given basis $\{Y_1^*, \ldots, Y_m^*\}$ for

W^*, the space $F(\Omega_k|_{W^*})$ for the $\Omega_k|_{W^*}$ system $\Omega_k|_{W^*} = \Omega_k(Y_1^*), \ldots, \Omega_k(Y_m^*)$ is given by

$$F(\Omega_k|_{W^*}) = \operatorname{span}_{\mathbb{C}}\{\omega_k(Y_j^*) \otimes 1 \mid j = 1, \ldots, m\} \subset M_{\mathfrak{q}}[\mathbb{C}_{-s\lambda_{\mathfrak{q}}}]. \tag{3.3}$$

Proposition 3.4. *Suppose that special constituent W^* has highest weight ν.*

1. *The space $F(\Omega_k|_{W^*})$ is the simple L-submodule of $M_{\mathfrak{q}}[\mathbb{C}_{-s\lambda_{\mathfrak{q}}}]$ with highest weight $\nu - s\lambda_{\mathfrak{q}}$.*
2. *Moreover, if the $\Omega_k|_{W^*}$ system is conformally invariant on the line bundle $\mathcal{L}_{s_0}$ then $F(\Omega_k|_{W^*})$ is a simple L-submodule of $M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}]^{\mathfrak{n}}$ with highest weight $\nu - s_0\lambda_{\mathfrak{q}}$.*

Proof. First observe that, by the L_0-equivariance of the operator $\Omega_k|_{W^*} : W^* \to \mathbb{D}(\mathcal{L}_s)$, for $l \in L$ and $Y^* \in W^*$, we have

$$\omega_k(l \cdot Y^*) = \operatorname{Ad}(l)\omega_k(Y^*),$$

where the action $l \cdot Y^*$ is the standard action of L on W^*, which is induced from the adjoint action of L on W. This shows the L-invariance of $F(\Omega_k|_{W^*})$. To show the irreducibility observe that there exists a vector space isomorphism

$$F(\Omega_k|_{W^*}) \to W^* \otimes \mathbb{C}_{-s\lambda_{\mathfrak{q}}},$$

that is given by $\omega_k(Y_j^*)\otimes 1 \mapsto Y_j^* \otimes 1$. It is clear that this vector space isomorphism is L-equivariant with respect to the standard action of L on the tensor products $F(\Omega_k|_{W^*}) \subset \mathcal{U}(\bar{\mathfrak{n}})\otimes\mathbb{C}_{-s\lambda_{\mathfrak{q}}}$ and $W^*\otimes\mathbb{C}_{-s\lambda_{\mathfrak{q}}}$. In particular, if W^* has highest weight ν then $F(\Omega_k|_{W^*})$ is the simple L-module with highest weight $\nu - s\lambda_{\mathfrak{q}}$.

Note that, by Remark 3.8 in [18], if the $\Omega_k|_{W^*}$ system is conformally invariant then it is a straight, L_0-stable, and homogeneous system. Now the second assertion follows from the first and Theorem 3.2. ∎

Now, if the $\Omega_k|_{W^*}$ system is conformally invariant on $\mathcal{L}_{s_0}$ then, by Proposition 3.4, $F(\Omega_k|_{W^*})$ is a simple $\mathfrak{l}$-submodule of $M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}])$ on which $\mathfrak{n}$ acts trivially. Thus the inclusion map $\iota \in \operatorname{Hom}_L\big(F(\Omega_k|_{W^*}), M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}]\big)$ induces a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism

$$\varphi_{\Omega_k} \in \operatorname{Hom}_{\mathcal{U}(\mathfrak{g}),L}\big(M_{\mathfrak{q}}[F(\Omega_k|_{W^*})], M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}]\big)$$

between the generalized Verma modules, that is given by

$$\begin{aligned} M_{\mathfrak{q}}[F(\Omega_k|_{W^*})] &\overset{\varphi_{\Omega_k}}{\to} M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}] \\ u \otimes \big(\omega_k(Y) \otimes 1\big) &\mapsto u \cdot \iota\big(\omega_k(Y) \otimes 1\big). \end{aligned} \tag{3.5}$$

If $F(\Omega_k|_{W^*}) = \mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$ then the map in (3.5) is just the identity map. However, Proposition 3.6 below shows that it does not happen.

Proposition 3.6. *If the $\Omega_k|_{W^*}$ system is conformally invariant on the line bundle $\mathcal{L}_{s_0}$ then $F(\Omega_k|_{W^*}) \neq \mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$.*

Proof. Observe that if ν is the highest weight for W^* then $F(\Omega_k|_{W^*})$ has highest weight $\nu - s_0\lambda_{\mathfrak{q}}$. If $F(\Omega_k|_{W^*}) = \mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$ then $\nu = 0$, and so the irreducible constituent $W \subset \mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ would also have highest weight 0. As γ is the highest weight for $\mathfrak{g}(r)$, the highest weight of any irreducible constituent of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ is of the form $\gamma + \eta$ with η some weight for $\mathfrak{g}(-r+k)$. Thus, the highest weight 0 for W must be of the form $0 = \gamma + (-\gamma)$. However, since only $\mathfrak{g}(-r)$ has weight $-\gamma$, it cannot be a weight for $\mathfrak{g}(-r+k)$ unless $k = 0$. As $k = 1, \ldots, 2r$ (see Subsection 2.2), this shows that $F(\Omega_k|_{W^*}) \neq \mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$. ■

Corollary 3.7. *If the $\Omega_k|_{W^*}$ system is conformally invariant on the line bundle $\mathcal{L}_{s_0}$ then the generalized Verma module $M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}]$ is reducible.*

Proof. This immediately follows from (3.5) and Proposition 3.6. ■

The goal of this paper is to determine whether or not the maps φ_{Ω_k} are standard in the quasi-Heisenberg setting. To do so, it is convenient to parametrize generalized Verma modules by their infinitesimal characters. Therefore, for the rest of this paper, we write

$$M_{\mathfrak{q}}[F(\Omega_k|_{W^*})] = M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho) \tag{3.8}$$

and

$$M_{\mathfrak{q}}[\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}] = M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho), \tag{3.9}$$

where ρ is half the sum of the positive roots. Then (3.5) is expressed by

$$\begin{gathered} M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho) \overset{\varphi_{\Omega_k}}{\to} M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho) \\ u \otimes v \mapsto u \cdot \iota(v) \end{gathered} \tag{3.10}$$

with $v = \omega_k(Y^*) \otimes 1$.

4. Standard maps between generalized Verma modules

The aim of this sections is to discuss standard maps between generalized Verma modules and homomorphisms between (ordinary) Verma modules. In particular, we specialize a result of Lepowsky to the present situation.

We start with recalling the notion of standard maps. For $\eta \in \mathfrak{h}^*$, let $M(\eta)$ be the (ordinary) Verma module with highest weight $\eta - \rho$. Write

$$\mathbf{P}_{\mathfrak{l}}^{+} = \{\zeta \in \mathfrak{h}^* \mid \langle \zeta, \alpha^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Pi(\mathfrak{l})\}.$$

For $\eta, \zeta \in \mathbf{P}_{\mathfrak{l}}^{+}$, suppose that there exists a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism $\varphi : M(\eta) \to M(\zeta)$. If $K(\eta)$ is the kernel of the canonical projection map $\mathrm{pr}_\eta : M(\eta) \to M_{\mathfrak{q}}(\eta)$ then, by Proposition 3.1 in [21], we have $\varphi(K(\eta)) \subset K(\zeta)$. Thus the map φ induces a $\mathcal{U}(\mathfrak{g})$-homomorphism $\varphi_{std} : M_{\mathfrak{q}}(\eta) \to M_{\mathfrak{q}}(\zeta)$ so that the diagram

$$\begin{tikzcd} M(\eta) \arrow[r, "\varphi"] \arrow[d, "\mathrm{pr}_\eta"'] & M(\zeta) \arrow[d, "\mathrm{pr}_\zeta"] \\ M_{\mathfrak{q}}(\eta) \arrow[r, "\varphi_{std}"] & M_{\mathfrak{q}}(\zeta) \end{tikzcd}$$

commutes. The map φ_{std} is called the *standard map* from $M_{\mathfrak{q}}(\eta)$ to $M_{\mathfrak{q}}(\zeta)$. These maps were first studied by Lepowsky ([21]). As $\dim \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(M(\eta), M(\zeta)) \leq 1$, the standard maps φ_{std} are uniquely determined up to scalar multiples. Note that the standard maps φ_{std} could be zero and also that not every homomorphism between generalized Verma modules is standard. Any homomorphisms that are not standard are called non-standard maps.

If $\nu = -(1-s_0)\alpha_{\mathfrak{q}}$ in (3.10) with $1-s_0 \in 1+\mathbb{Z}_{\geq 0}$ then one can show that the standard map φ_{std} from $M_{\mathfrak{q}}(-(1-s_0)\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ is non-zero by computing $\varphi_{std}(1 \otimes v^+)$, where $1 \otimes v^+$ is a highest weight vector of $M_{\mathfrak{q}}(-(1-s_0)\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho)$ with weight $-(1-s_0)\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}}$. To prove it, we will use the following well-known result. (See for example [9, Proposition 1.4].)

Proposition 4.1. *Given $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Pi$, suppose that $n = \langle \lambda + \rho, \alpha^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0}$. If $1 \otimes v^+$ is a highest weight vector of weight λ in $M(\lambda + \rho)$ then $X^n_{-\alpha} \cdot (1 \otimes v^+)$ is a highest weight vector of weight $-n\alpha + \lambda$.*

Observe that, by (3.8) and (3.9), we have

$M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F(\Omega_k|_{W^*})$ and $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$.

Thus if v_h and $1_{-s_0\lambda_{\mathfrak{q}}}$ are highest weight vectors for $F(\Omega_k|_{W^*})$ and $\mathbb{C}_{-s_0\lambda_{\mathfrak{q}}}$, respectively, then $1 \otimes v_h$ and $1 \otimes 1_{-s_0\lambda_{\mathfrak{q}}}$ are highest weight vectors for $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ with highest weight $\nu - s_0\lambda_{\mathfrak{q}}$ and for $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ with highest weight $-s_0\lambda_{\mathfrak{q}}$, respectively.

Proposition 4.2. *If $1 - s_0 \in 1 + \mathbb{Z}_{\geq 0}$ then the standard map φ_{std} from $M_{\mathfrak{q}}(-(1-s_0)\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ maps*

$$1 \otimes v_h \mapsto cX^{1-s_0}_{-\alpha_{\mathfrak{q}}} \otimes 1_{-s_0\lambda_{\mathfrak{q}}} \neq 0$$

for some non-zero constant c. In particular, the standard map φ_{std} is non-zero.

Proof. Write $n = 1 - s_0$ and denote by $1 \otimes 1_{-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}}}$ a highest weight vector for $M(-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho)$ with highest weight $-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}}$. Observe that since $\langle \lambda_{\mathfrak{q}}, \alpha_0^\vee \rangle = \langle \rho, \alpha_0^\vee \rangle = 1$, we have $n = 1 - s_0 = \langle -s_0\lambda_{\mathfrak{q}} + \rho, \alpha_0^\vee \rangle$. Hence $-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho = s_{\alpha_{\mathfrak{q}}}(-s_0\lambda_{\mathfrak{q}} + \rho)$. By hypothesis, we have $n = 1 - s_0 \in 1 + \mathbb{Z}_{\geq 0}$. It then follows from Proposition 4.1 that the map $\varphi : M_{\mathfrak{q}}(-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho) \to M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ is given by

$$\varphi(1 \otimes 1_{-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}}}) = cX^n_{-\alpha_{\mathfrak{q}}} \otimes 1$$

with $c \neq 0$. As $\alpha_{\mathfrak{q}} \in \Pi \backslash \Pi(\mathfrak{l})$, if $\mathrm{pr}_{-s_0\lambda_{\mathfrak{q}}+\rho} : M(-s_0\lambda_{\mathfrak{q}} + \rho) \to M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ is the canonical projection map then $\mathrm{pr}_{-s_0\lambda_{\mathfrak{q}}+\rho}(X^n_{-\alpha_{\mathfrak{q}}} \otimes 1) \neq 0$. Then the universal property of $M_{\mathfrak{q}}(-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho)$ in the relative category $\mathcal{O}^{\mathfrak{q}}$ (see for example Section 9.4 in [9]) guarantees that $\mathrm{pr}_{-s_0\lambda_{\mathfrak{q}}+\rho} \circ \varphi$ factors through a non-zero map $\varphi_{std} : M_{\mathfrak{q}}(-n\alpha_{\mathfrak{q}} - s_0\lambda_{\mathfrak{q}} + \rho) \to M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$. ■

In order to determine if φ_{std} is non-zero in a more general setting, we will use the following theorem by Lepowsky. As usual, if there is a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism from $M(\eta)$ into $M(\zeta)$ then we write $M(\eta) \subset M(\zeta)$.

Theorem 4.3. *[21, Proposition 3.3] Let $\eta, \zeta \in \mathbf{P}_{\mathfrak{l}}^{+}$, and assume that $M(\eta) \subset M(\zeta)$. Then the standard map φ_{std} from $M_{\mathfrak{q}}(\eta)$ to $M_{\mathfrak{q}}(\zeta)$ is zero if and only if $M(\eta) \subset M(s_\alpha \zeta)$ for some $\alpha \in \Pi(\mathfrak{l})$.*

Theorem 4.3 reduces the existence problem of the non-zero standard map φ_{std} between generalized Verma modules to that of the non-zero map between appropriate Verma modules. It is well known when a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism between Verma modules exists. To describe the condition efficiently, we first introduce the definition of a *link* of two weights.

Definition 4.4. *(Bernstein-Gelfand-Gelfand) Let $\lambda, \delta \in \mathfrak{h}^*$ and $\beta_1, \ldots, \beta_t \in \Delta^+$. Set $\delta_0 = \delta$ and $\delta_i = s_{\beta_i} \cdots s_{\beta_1} \delta$ for $1 \leq i \leq t$. We say that the sequence $(\beta_1, \ldots, \beta_t)$* ***links*** *$\delta$ to λ if*

(1) $\delta_t = \lambda$ and

(2) $\langle \delta_{i-1}, \beta_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq t$.

Theorem 4.5. *(BGG-Verma) Let $\lambda, \delta \in \mathfrak{h}^*$. The following conditions are equivalent:*

(1) $M(\lambda) \subset M(\delta)$

(2) $L(\lambda)$ is a composition factor of $M(\delta)$

(3) There exists a sequence $(\beta_1, \ldots, \beta_t)$ with $\beta_i \in \Delta^+$ that links δ to λ,

where $L(\lambda)$ is the unique irreducible quotient of $M(\lambda)$.

Observe that if there is a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism (not necessarily standard) from $M_{\mathfrak{q}}(\eta)$ to $M_{\mathfrak{q}}(\zeta)$ then $M(\eta) \subset M(\zeta)$. By taking into account Theorem 4.5 and this observation, in our setting, Theorem 4.3 is equivalent to the following proposition.

Proposition 4.6. *Let $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ and $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ be the generalized Verma modules in (3.10). Then the standard map from $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ is zero if and only if there exists $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - s_0\lambda_{\mathfrak{q}} + \rho$ is linked to $\nu - s_0\lambda_{\mathfrak{q}} + \rho$.*

Proof. First observe that since there exists a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism φ_{Ω_k} from $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$, we have $M(\nu - s_0\lambda_{\mathfrak{q}} + \rho) \subset M(-s_0\lambda_{\mathfrak{q}} + \rho)$. Therefore, by Theorem 4.3 and Theorem 4.5, the standard map from $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ is zero if and only if there exists $\alpha \in \Pi(\mathfrak{l})$ so that $s_\alpha(-s_0\lambda_{\mathfrak{q}} + \rho)$ is linked to $\nu - s_0\lambda_{\mathfrak{q}} + \rho$. As $\langle \lambda_{\mathfrak{q}}, \alpha^\vee \rangle = 0$ and $\langle \rho, \alpha^\vee \rangle = 1$ for $\alpha \in \Pi(\mathfrak{l})$, we have $s_\alpha(-s_0\lambda_{\mathfrak{q}} + \rho) = -\alpha - s_0\lambda_{\mathfrak{q}} + \rho$. Now this proposition follows. ■

With Proposition 4.6 in hand, in the next two sections, we shall determine whether or not the homomorphisms φ_{Ω_k} that arise from the Ω_k system(s) for $k = 1, 2$ constructed in [18] are standard.

5. The homomorphism φ_{Ω_1} induced by the Ω_1 system

In this section we show that the homomorphism φ_{Ω_1} arising from the Ω_1 system associated to a maximal parabolic subalgebra $\mathfrak{q}$ of quasi-Heisenberg type is standard. For each $\alpha \in \Delta^+$, we define $\{X_\alpha, X_{-\alpha}, H_\alpha\}$ as an $\mathfrak{sl}(2)$-triple; in particular, we have $[X_\alpha, X_{-\alpha}] = H_\alpha$. For $\alpha, \beta \in \Delta$ with $\alpha+\beta \in \Delta$, we write a constant $N_{\alpha,\beta}$ for $[X_\alpha, X_\beta] = N_{\alpha,\beta}X_{\alpha+\beta}$. Recall from Subsection 2.2 that an irreducible constituent W of $\mathfrak{g}(-r+k)\otimes\mathfrak{g}(r)$ is called special for τ_k if $\tilde{\tau}_k|_{W^*} \not\equiv 0$.

It follows from (2.6) that the Ω_1 system is constructed from the map $\tau_1 : \mathfrak{g}(1) \to \mathfrak{g}(-1)\otimes\mathfrak{z}(\mathfrak{n})$ with $X \mapsto \big(\mathrm{ad}(X)\otimes \mathrm{Id}\big)\omega$, where $\omega = \sum_{\gamma_j \in \Delta(\mathfrak{z}(\mathfrak{n}))} X_{-\gamma_j}\otimes X_{\gamma_j}$. In Section 5 of [18], it is shown that irreducible constituent W of $\mathfrak{g}(-1)\otimes\mathfrak{z}(\mathfrak{n})$ is special if and only if $W \cong \mathfrak{g}(1)$ and also that there is only unique such a constituent. Via the composition of maps in (2.3), the Ω_1 system is given by $R(X_{-\alpha_1}), \ldots, R(X_{-\alpha_m})$ for $\Delta(\mathfrak{g}(1)) = \{\alpha_1, \ldots, \alpha_m\}$.

Theorem 5.1. *[18, Theorem 5.7] Let $\mathfrak{g}$ be a complex simple Lie algebra, and let $\mathfrak{q}$ be a maximal parabolic subalgebra of quasi-Heisenberg type. Then the Ω_1 system is conformally invariant on $\mathcal{L}_s$ if and only if $s = 0$.*

It follows from Proposition 3.4 and Theorem 5.1 that the Ω_1 system yields a finite dimensional simple $\mathfrak{l}$-submodule $F(\Omega_1)$ in $\big(\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{q})}\mathbb{C}_0\big)^{\mathfrak{n}} = M_\mathfrak{q}(\rho)^{\mathfrak{n}}$. If $\alpha_\mathfrak{q}$ is the simple root that determines the maximal parabolic subalgebra $\mathfrak{q}$ then, as it is the lowest weight for $\mathfrak{g}(1)$, $W^* \cong \mathfrak{g}(-1)$ has highest weight $-\alpha_\mathfrak{q}$. Thus, by Proposition 3.4, the simple $\mathfrak{l}$-module $F(\Omega_1)$ has highest weight $\nu - s_0\lambda_\mathfrak{q} = -\alpha_\mathfrak{q}$. Now, by (3.10), the inclusion map $F(\Omega_1) \hookrightarrow M_\mathfrak{q}(\rho)$ induces a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism

$$\varphi_{\Omega_1} : M_\mathfrak{q}(-\alpha_\mathfrak{q}+\rho) \to M_\mathfrak{q}(\rho).$$

Proposition 5.2. *If $\mathfrak{q}$ is a maximal parabolic subalgebra of quasi-Heisenberg type then the standard map $\varphi_{std} : M_\mathfrak{q}(-\alpha_\mathfrak{q}+\rho) \to M_\mathfrak{q}(\rho)$ is non-zero.*

Proof. This follows from Proposition 4.2 with $s_0 = 0$. ∎

Theorem 5.3. *If $\mathfrak{q}$ is a maximal parabolic subalgebra of quasi-Heisenberg type then the map φ_{Ω_1} is standard.*

Proof. Let v_h be a highest weight vector for $F(\Omega_1)$. Since $\varphi_{\Omega_1}(1\otimes v_h) = 1\cdot v_h = v_h$, to prove that φ_{Ω_1} is standard, by Propostion 4.2 and Proposition 5.2, it suffices to show that $v_h = cX_{-\alpha_\mathfrak{q}}\otimes 1_0$ with some non-zero constant c. To do so, as v_h is a highest weight vector for $F(\Omega_1)$, we show that $X_{-\alpha_\mathfrak{q}}\otimes 1_0$ is a highest weight vector for $F(\Omega_1)$. Since the Ω_1 system is $R(X_{-\alpha_1}), \ldots, R(X_{-\alpha_m})$ for $\Delta(\mathfrak{g}(1)) = \{\alpha_1, \ldots, \alpha_m\}$, it is clear that the elements $\omega_1(X_{-\alpha_j}) \in \sigma(\mathrm{Sym}^1(\bar{\mathfrak{n}})) = \bar{\mathfrak{n}}$ that correspond to $R(X_{-\alpha_j})$ under R are $\omega_1(X_{-\alpha_j}) = X_{-\alpha_j}$. Then it follows from (3.3) that

$$F(\Omega_1) = \mathrm{span}_\mathbb{C}\{X_{-\alpha}\otimes 1_0 \mid \alpha \in \Delta(\mathfrak{g}(1))\}.$$

Therefore $X_{-\alpha_\mathfrak{q}}\otimes 1_0$ is a highest weight vector for $F(\Omega_1)$. ∎

6. The homomorphisms φ_{Ω_2} induced by the Ω_2 systems

The aim of this section is to classify the homomorphisms φ_{Ω_2} that are induced by the Ω_2 systems associated to maximal parabolic subalgebras $\mathfrak{q}$ listed in (2.8) and (2.9) as standard or not.

We first recall from Section 6 of [18] some observation on special constituents. The Ω_2 systems are constructed from the map $\tau_2 : \mathfrak{g}(1) \to \mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ with $X \mapsto \frac{1}{2}\big(\mathrm{ad}(X)^2 \otimes \mathrm{Id}\big)\omega$. Observe that if $V(\nu)$ is a special constituent of $\mathfrak{g}(0) \otimes \mathfrak{z}(\mathfrak{n}) = \mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ with highest weight ν then, as $V(\nu)^*$ is embedded into $\mathcal{P}^2(\mathfrak{g}(1)) \cong \mathrm{Sym}^2(\mathfrak{g}(1))^* \subset \mathfrak{g}(1)^* \otimes \mathfrak{g}(1)^*$, we have $V(\nu) \hookrightarrow \mathfrak{g}(1) \otimes \mathfrak{g}(1)$. Thus the highest weight ν is of the form $\mu + \epsilon$, where μ is the highest weight for $\mathfrak{g}(1)$ and ϵ is some weight for $\mathfrak{g}(1)$.

Recall from (2.7) that we have $\mathfrak{l} = \mathbb{C}H_{\mathfrak{q}} \oplus \mathfrak{l}_\gamma \oplus \mathfrak{l}_{n\gamma}$. Thus the tensor product $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ may be written as $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n}) = \big(\mathbb{C}H_{\mathfrak{q}} \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})\big) \oplus \big(\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})\big)$. It is shown in Section 6 of [18] that, for $\mathfrak{q}$ under consideration in (2.8) and (2.9), there are exactly one or two special constituents of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$; one is an irreducible constituent of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ and the other is equal to $\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$. We denote by $V(\mu+\epsilon_\gamma)$ and $V(\mu+\epsilon_{n\gamma})$ the special constituents so that $V(\mu+\epsilon_\gamma) \subset \mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ and $V(\mu+\epsilon_{n\gamma}) = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$. We summarize the data on the special constituents in Table 1 and Table 2 below. We use the standard realizations for the roots for the classical algebras, while the Bourbaki conventions [7] are used for the exceptional algebras for the labels of the simple roots. A dash in the column for $V(\mu+\epsilon_{n\gamma})$ indicates that $\mathfrak{l}_{n\gamma} = \{0\}$ for the case. (So there is no special constituent $V(\mu+\epsilon_{n\gamma})$.)

Table 1: Highest Weights for Special Constituents (Classical Cases)

Type	$V(\mu+\epsilon_\gamma)$	$V(\mu+\epsilon_{n\gamma})$
$B_n(i),\ 3 \leq i \leq n-2$	$2\varepsilon_1$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_{i+1} + \varepsilon_{i+2}$
$B_n(n-1)$	$2\varepsilon_1$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_n$
$B_n(n)$	$2\varepsilon_1$	$-$
$C_n(i),\ 2 \leq i \leq n-1$	$\varepsilon_1 + \varepsilon_2$	$2\varepsilon_1 + 2\varepsilon_{i+1}$
$D_n(i),\ 3 \leq i \leq n-3$	$2\varepsilon_1$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_{i+1} + \varepsilon_{i+2}$

Definition 6.1. *[18, Definition 6.20] Let μ be the highest weight for $\mathfrak{g}(1)$, and let $\epsilon = \epsilon_\gamma$ or $\epsilon_{n\gamma}$. We say that a special constituent $V(\mu+\epsilon)$ is of*

*(1) **type 1a** if $\mu+\epsilon$ is not a root with $\epsilon \neq \mu$ and both μ and ϵ are long roots,*

*(2) **type 1b** if $\mu+\epsilon$ is not a root with $\epsilon \neq \mu$ and either μ or ϵ is a short root,*

*(3) **type 2** if $\mu+\epsilon = 2\mu$ is not a root, or*

*(4) **type 3** if $\mu+\epsilon$ is a root.*

Table 3 below shows the types of special constituents for each maximal parabolic subalgebra $\mathfrak{q}$. In [18] the special values for the type 1a and type 2 cases are determined.

For $\mu+\epsilon = \mu+\epsilon_\gamma$ or $\mu+\epsilon_{n\gamma}$, we write

$$\Delta_{\mu+\epsilon}(\mathfrak{g}(1)) = \{\alpha \in \Delta(\mathfrak{g}(1)) \mid \mu+\epsilon-\alpha \in \Delta(\mathfrak{g}(1))\}.$$

Table 2: Highest Weights for Special Constituents (Exceptional Cases)

Type	$V(\mu+\epsilon_\gamma)$
$E_6(3)$	$\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6$
$E_6(5)$	$2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+2\alpha_5+\alpha_6$
$E_7(2)$	$2\alpha_1+2\alpha_2+4\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7$
$E_7(6)$	$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+4\alpha_5+2\alpha_6+\alpha_7$
$E_8(1)$	$2\alpha_1+4\alpha_2+5\alpha_3+8\alpha_4+7\alpha_5+6\alpha_6+4\alpha_7+2\alpha_8$
$F_4(4)$	$2\alpha_1+4\alpha_2+6\alpha_3+2\alpha_4$
Type	$V(\mu+\epsilon_{n\gamma})$
$E_6(3)$	$2\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$
$E_6(5)$	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6$
$E_7(2)$	–
$E_7(6)$	$2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7$
$E_8(1)$	–
$F_4(4)$	–

Table 3: Types of Special Constituents

Type	$V(\mu+\epsilon_\gamma)$	$V(\mu+\epsilon_{n\gamma})$
$B_n(i),\ 3\le i\le n-2$	Type 1a	Type 1a
$B_n(n-1)$	Type 1a	Type 1b
$B_n(n)$	Type 2	–
$C_n(i),\ 2\le i\le n-1$	Type 3	Type 2
$D_n(i),\ 3\le i\le n-3$	Type 1a	Type 1a
$E_6(3)$	Type 1a	Type 1a
$E_6(5)$	Type 1a	Type 1a
$E_7(2)$	Type 1a	–
$E_7(6)$	Type 1a	Type 1a
$E_8(1)$	Type 1a	–
$F_4(4)$	Type 2	–

We denote by $|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|$ the number of elements in $\Delta_{\mu+\epsilon}(\mathfrak{g}(1))$.

Theorem 6.2. *[18, Theorem 7.16, Corollary 7.23] Suppose that $V(\mu+\epsilon)$ is a special constituent of type 1a or type 2.*

(1) If $V(\mu+\epsilon)$ is of type 1a then the $\Omega_2|_{V(\mu+\epsilon)^*}$ system is conformally invariant on $\mathcal{L}_s$ if and only if

$$s=\frac{|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|}{2}-1.$$

(2) If $V(\mu+\epsilon)$ is of type 2 then the $\Omega_2|_{V(\mu+\epsilon)^*}$ system is conformally invariant on $\mathcal{L}_s$ if and only if

$$s=-1.$$

Let λ_i be the fundamental weight for the simple root α_i that determines the maximal parabolic subalgebra $\mathfrak{q}$. Table 4 below summarizes the line bundles $\mathcal{L}_s=\mathcal{L}(s\lambda_i)$ on which the Ω_2 systems are conformally invariant. When $\mathfrak{q}$ is of

Table 4: Line bundles with special values

Parabolic $\mathfrak{q}$	$\Omega_2\|_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\|_{V(\mu+\epsilon_{n\gamma})^*}$
$B_n(i), 3 \leq i \leq n-2$	$\mathcal{L}\big((n-i-\frac{1}{2})\lambda_i\big)$	$\mathcal{L}(\lambda_i)$
$B_n(n-1)$	$\mathcal{L}\big(\frac{1}{2}\lambda_{n-1}\big)$	?
$B_n(n)$	$\mathcal{L}(-\lambda_n)$	–
$C_n(i), 2 \leq i \leq n-1$	?	$\mathcal{L}(-\lambda_i)$
$D_n(i), 3 \leq i \leq n-3$	$\mathcal{L}\big((n-i-1)\lambda_i\big)$	$\mathcal{L}(\lambda_i)$
$E_6(3)$	$\mathcal{L}(\lambda_3)$	$\mathcal{L}(2\lambda_3)$
$E_6(5)$	$\mathcal{L}(\lambda_5)$	$\mathcal{L}(2\lambda_5)$
$E_7(2)$	$\mathcal{L}(2\lambda_2)$	–
$E_7(6)$	$\mathcal{L}(\lambda_6)$	$\mathcal{L}(3\lambda_6)$
$E_8(1)$	$\mathcal{L}(3\lambda_1)$	–
$F_4(4)$	$\mathcal{L}(-\lambda_4)$	–

type $B_n(n-1)$, the constituent $V(\mu+\epsilon_{n\gamma})$ is of type 1b, and when $\mathfrak{q}$ is of type $C_n(i)$, the constituent $V(\mu+\epsilon_\gamma)$ is of type 3. Therefore, a question mark is put for these cases in the table.

Now, with the results in Table 4 in hand, we determine the standardness of φ_{Ω_2}. Observe from Table 3 and Table 4 that each $\Omega_2|_{V(\mu+\epsilon)^*}$ system satisfies exactly one of the following:

1. The special constituent $V(\mu+\epsilon)$ is of type 2.
2. The special value s_0 is a positive integer.
3. The parabolic subalgebra $\mathfrak{q}$ is of type $B_n(i)$ for $3 \leq i \leq n-1$ and $V(\mu+\epsilon) = V(\mu+\epsilon_\gamma)$.

We shall consider these three cases separately.

6.1. The type 2 case.

We first study the homomorphism attached to the special constituent $V(\mu+\epsilon)$ of type 2. By Table 3, we consider the following three cases:

$V(\mu+\epsilon_\gamma)$ for $B_n(n)$, $V(\mu+\epsilon_{n\gamma})$ for $C_n(i)(2 \leq i \leq n-1)$, and $V(\mu+\epsilon_\gamma)$ for $F_4(4)$.

If $V(\mu+\epsilon)$ is a type 2 special constituent then, by definition, $V(\mu+\epsilon) = V(2\mu)$. Thus, as μ and $\alpha_\mathfrak{q}$ are the highest and lowest weights for $\mathfrak{g}(1)$, respectively, we have $V(\mu+\epsilon)^* = V(2\mu)^* = V(-2\alpha_\mathfrak{q})$. Therefore ν in (3.10) is $\nu = -2\alpha_\mathfrak{q}$. Moreover, by Theorem 6.2, the $\Omega_2|_{V(2\mu)^*}$ system is conformally invariant on the line bundle $\mathcal{L}(-\lambda_\mathfrak{q})$. Thus $s_0 = -1$. Therefore it follows from (3.10) that we have

$$\varphi_{\Omega_2} : M_\mathfrak{q}(-2\alpha_\mathfrak{q} + \lambda_\mathfrak{q} + \rho) \to M_\mathfrak{q}(\lambda_\mathfrak{q} + \rho).$$

Proposition 6.3. *If $\mathfrak{q}$ is the maximal parabolic subalgebra of type $B_n(n)$, $C_n(i)$ for $2 \leq i \leq n-1$, or $F_4(4)$ then the standard map φ_{std} from $M_\mathfrak{q}(-2\alpha_\mathfrak{q} + \lambda_\mathfrak{q} + \rho)$ to $M_\mathfrak{q}(\lambda_\mathfrak{q} + \rho)$ is non-zero.*

Proof. This follows from Proposition 4.2 with $s_0 = -1$. ■

In Section 7.3 of [18], it is observed that if Y_l^* is a lowest weight vector for $V(2\mu)^*$ then the differential operator $\Omega_2(Y_l^*)$ is of the form

$$\Omega_2(Y_l^*) = aR(X_{-\mu})^2,$$

for some constant a. Therefore, the element $\omega_2(Y_l^*)$ in $\sigma(\mathrm{Sym}^2(\bar{\mathfrak{n}})) \subset \mathcal{U}(\bar{\mathfrak{n}})$ that corresponds to $\Omega_2(Y_l^*)$ under R in (2.3) is of the form

$$\omega_2(Y_l^*) = aX_{-\mu}^2. \tag{6.4}$$

Thus the simple $\mathfrak{l}$-submodule $F(\Omega_2|_{V(2\mu)^*})$ of $M_{\mathfrak{q}}(\lambda_{\mathfrak{q}}+\rho)^{\mathfrak{n}} = \big(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{\lambda_{\mathfrak{q}}}\big)^{\mathfrak{n}}$ has lowest weight $X_{-\mu}^2 \otimes 1_{\lambda_{\mathfrak{q}}}$.

Theorem 6.5. *Let $\mathfrak{q}$ be a maximal parabolic subalgebra of quasi-Heisenberg type, listed in (2.8) or (2.9). If the special constituent $V(\mu+\epsilon)$ is of type 2 then the map φ_{Ω_2} is standard.*

Proof. In order to prove that φ_{Ω_2} is standard, by Proposition 6.3, it suffices to show that $X_{-\alpha_{\mathfrak{q}}}^2 \otimes 1_{\lambda_{\mathfrak{q}}}$ is a highest weight vector for $F(\Omega_2|_{V(2\mu)^*})$. Since $F(\Omega_2|_{V(2\mu)^*})$ has highest weight $\nu - s_0\lambda_{\mathfrak{q}} = -2\alpha_{\mathfrak{q}} + \lambda_{\mathfrak{q}}$, it is enough to show that $X_{-\alpha_{\mathfrak{q}}}^2 \otimes 1_{\lambda_{\mathfrak{q}}}$ is in $F(\Omega_2|_{V(2\mu)^*})$. We know that a lowest weight vector for $F(\Omega_2|_{V(2\mu)^*})$ is $X_{-\mu}^2 \otimes 1_{\lambda_{\mathfrak{q}}}$. This will allow us to show that $X_{-\alpha_{\mathfrak{q}}}^2 \otimes 1_{\lambda_{\mathfrak{q}}}$ is in $F(\Omega_2|_{V(2\mu)^*})$. We do so in a case-by-case manner. Since the arguments are similar for each case, we show only the case $V(\mu+\epsilon_\gamma)$ for $B_n(n)$. (For the other cases see Section 8.3 in [19].) In the standard realization of the roots we have $\mu = \varepsilon_1$, $\alpha_{\mathfrak{q}} = \alpha_n = \varepsilon_n$, and

$$\Delta^+(\mathfrak{l}) = \{\varepsilon_j - \varepsilon_k \mid 1 \le j < k \le n\}$$

(see Appendix A). Thus,

$$X_{-\mu}^2 \otimes 1_{\lambda_{\mathfrak{q}}} = X_{-\varepsilon_1}^2 \otimes 1_{\lambda_n} \quad \text{and} \quad X_{-\alpha_{\mathfrak{q}}}^2 \otimes 1_{\lambda_{\mathfrak{q}}} = X_{-\varepsilon_n}^2 \otimes 1_{\lambda_n}.$$

A direct computation shows that

$$X_{\varepsilon_1-\varepsilon_n}^2 \cdot (X_{-\varepsilon_1}^2 \otimes 1_{\lambda_n}) = 2N_{\varepsilon_1-\varepsilon_n,-\varepsilon_1}^2 X_{-\varepsilon_n}^2 \otimes 1_{\lambda_n},$$

where $N_{\varepsilon_1-\varepsilon_n,-\varepsilon_1}$ is the constant so that $[X_{\varepsilon_1-\varepsilon_n}, X_{-\varepsilon_1}] = N_{\varepsilon_1-\varepsilon_n,-\varepsilon_1} X_{-\varepsilon_n}$. (See the beginning of Section 5.) Therefore, as $X_{\varepsilon_1-\varepsilon_n} \in \mathfrak{l}$, we have $X_{-\alpha_{\mathfrak{q}}}^2 \otimes 1_{\lambda_{\mathfrak{q}}} = X_{-\varepsilon_n}^2 \otimes 1_{\lambda_n} \in F(\Omega_2|_{V(2\mu)^*})$. ■

6.2. The positive integer special value case.

Next we handle the case that the special value s_0 is a positive integer.

Theorem 6.6. *Let $\mathfrak{q}$ be a maximal parabolic subalgebra of quasi-Heisenberg type, listed in (2.8) or (2.9). If the special value s_0 is a positive integer then the standard map from $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ is zero. Consequently, the map φ_{Ω_2} is non-standard.*

Proof. By Proposition 4.6, to show that the standard map is zero, it suffices to show that there exists $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - s_0\lambda_{\mathfrak{q}} + \rho$ is linked to $\nu - s_0\lambda_{\mathfrak{q}} + \rho$. We achieve it by a case-by-case observation. By Table 4, the following are the cases under consideration:

1. $V(\mu + \epsilon_{n\gamma})$ for $B_n(i)$ $(3 \leq i \leq n-2)$
2. $V(\mu + \epsilon_{\gamma})$ and $V(\mu + \epsilon_{n\gamma})$ for $D_n(i)$ $(3 \leq i \leq n-3)$
3. $V(\mu + \epsilon_{\gamma})$ and $V(\mu + \epsilon_{n\gamma})$ for $E_6(3)$
4. $V(\mu + \epsilon_{\gamma})$ and $V(\mu + \epsilon_{n\gamma})$ for $E_6(5)$
5. $V(\mu + \epsilon_{\gamma})$ for $E_7(2)$
6. $V(\mu + \epsilon_{\gamma})$ and $V(\mu + \epsilon_{n\gamma})$ for $E_7(6)$
7. $V(\mu + \epsilon_{\gamma})$ for $E_8(1)$

Our strategy is to first observe that the highest weight ν for $V(\mu+\epsilon)^*$ is of the form

$$\nu = -2\beta - \alpha' - \alpha''$$

for some $\beta \in \Delta(\mathfrak{g}(1))$ and $\alpha', \alpha'' \in \Pi(\mathfrak{l})$. We then show that the sequence (α', β) links $-\alpha'' - s_0\lambda_{\mathfrak{q}} + \rho$ to $(-2\beta - \alpha' - \alpha'') - s_0\lambda_{\mathfrak{q}} + \rho$. Here we only show three cases, namely, $V(\mu + \epsilon_{n\gamma})$ for $B_n(i)$, $V(\mu + \epsilon_{\gamma})$ for $D_n(i)$ $(3 \leq i \leq n-3)$, and $V(\mu + \epsilon_{\gamma})$ for $E_6(3)$. Other cases can be shown similarly. (For some details for the other cases see Section 8.3 in [19].)

<u>1. $V(\mu + \epsilon_{n\gamma})$ for $B_n(i)$ for $3 \leq i \leq n-2$</u>: Since, by Table 4, the special value s_0 is $s_0 = 1$, we wish to show that there is $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - \lambda_i + \rho$ is linked to $\nu - \lambda_i + \rho$. First we find the highest weight ν for $V(\mu + \epsilon_{n\gamma})^*$. Observe that we have $\Delta^+(\mathfrak{l}) = \Delta^+(\mathfrak{l}_\gamma) \cup \Delta^+(\mathfrak{l}_{n\gamma})$ with

$$\Delta^+(\mathfrak{l}_\gamma) = \{\varepsilon_j - \varepsilon_k \mid 1 \leq j < k \leq i\}$$

and

$$\Delta^+(\mathfrak{l}_{n\gamma}) = \{\varepsilon_j \pm \varepsilon_k \mid i+1 \leq j < k \leq n\} \cup \{\varepsilon_j \mid i+1 \leq j \leq n\}$$

in the standard realization of the roots (see Appendix A). Since

$$\Delta(\mathfrak{z}(\mathfrak{n})) = \{\varepsilon_j + \varepsilon_k \mid 1 \leq j < k \leq i\},$$

the simple $\mathfrak{l}$-module $\mathfrak{z}(\mathfrak{n})$ has lowest weight $\varepsilon_{i-1} + \varepsilon_i$. As $V(\mu + \epsilon_{n\gamma}) = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$, we have $V(\mu + \epsilon_{n\gamma})^* = \mathfrak{l}_{n\gamma}^* \otimes \mathfrak{z}(\mathfrak{n})^* = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})^*$. Since $\mathfrak{l}_{n\gamma}$ has highest weight $\varepsilon_{i+1} + \varepsilon_{i+2}$, this shows that the highest weight ν for $V(\mu + \epsilon_{n\gamma})^*$ is

$$\nu = (\varepsilon_{i+1} + \varepsilon_{i+2}) - (\varepsilon_{i-1} + \varepsilon_i) = -\varepsilon_{i-1} - \varepsilon_i + \varepsilon_{i+1} + \varepsilon_{i+2}.$$

We have

$$-\varepsilon_{i-1} - \varepsilon_i + \varepsilon_{i+1} + \varepsilon_{i+2} = -2(\varepsilon_i - \varepsilon_{i+1}) - (\varepsilon_{i-1} - \varepsilon_i) - (\varepsilon_{i+1} - \varepsilon_{i+2})$$

with $\varepsilon_i - \varepsilon_{i+1} \in \Delta(\mathfrak{g}(1))$ and $\varepsilon_{i-1} - \varepsilon_i$, $\varepsilon_{i+1} - \varepsilon_{i+2} \in \Pi(\mathfrak{l})$ (see Appendix A). Now we claim that $(\varepsilon_{i-1} - \varepsilon_i, \varepsilon_i - \varepsilon_{i+1})$ links $-(\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho$ to $-2(\varepsilon_i - \varepsilon_{i+1}) - (\varepsilon_{i-1} - \varepsilon_i) - (\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho$. This is to show that

$$s_{\varepsilon_i-\varepsilon_{i+1}} s_{\varepsilon_{i-1}-\varepsilon_i}(-(\varepsilon_{i+1}-\varepsilon_{i+2})-\lambda_i+\rho) = -2(\varepsilon_i-\varepsilon_{i+1})-(\varepsilon_{i-1}-\varepsilon_i)-(\varepsilon_{i+1}-\varepsilon_{i+2})-\lambda_i+\rho$$

with

$$\langle -(\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho, (\varepsilon_{i-1} - \varepsilon_i)^\vee \rangle \in \mathbb{Z}_{\geq 0}$$

and

$$\langle s_{\varepsilon_{i-1}-\varepsilon_i}(-(\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho), (\varepsilon_i - \varepsilon_{i+1})^\vee \rangle \in \mathbb{Z}_{\geq 0}.$$

(See Definition 4.4.) As $\varepsilon_{i-1} - \varepsilon_i \in \Pi(\mathfrak{l})$, we have $\langle \lambda_i, (\varepsilon_{i-1} - \varepsilon_i)^\vee \rangle = 0$. Since $\langle \rho, (\varepsilon_{i-1} - \varepsilon_i)^\vee \rangle = 1$, it follows that

$$\langle -(\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho, (\varepsilon_{i-1} - \varepsilon_i)^\vee \rangle = 1 \in \mathbb{Z}_{\geq 0}.$$

Thus,

$$s_{\varepsilon_{i-1}-\varepsilon_i}(-(\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho) = -(\varepsilon_{i-1} - \varepsilon_i) - (\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho.$$

Next, as $\varepsilon_i - \varepsilon_{i+1}$ is the simple root that determines the parabolic $\mathfrak{q}$, we have $\langle \lambda_i, (\varepsilon_i - \varepsilon_{i+1})^\vee \rangle = 1$. Since $\langle \rho, (\varepsilon_i - \varepsilon_{i+1})^\vee \rangle = 1$, it follows that

$$\begin{aligned}
&\langle s_{\varepsilon_{i-1}-\varepsilon_i}(-(\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho), (\varepsilon_i - \varepsilon_{i+1})^\vee \rangle \\
&= \langle -(\varepsilon_{i-1} - \varepsilon_i) - (\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho, (\varepsilon_i - \varepsilon_{i+1})^\vee \rangle \\
&= 2 \in \mathbb{Z}_{\geq 0}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&s_{\varepsilon_i-\varepsilon_{i+1}} s_{\varepsilon_{i-1}-\varepsilon_i}(-(\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho) \\
&= s_{\varepsilon_i-\varepsilon_{i+1}}(-(\varepsilon_{i-1} - \varepsilon_i) - (\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho) \\
&= -2(\varepsilon_i - \varepsilon_{i+1}) - (\varepsilon_{i-1} - \varepsilon_i) - (\varepsilon_{i+1} - \varepsilon_{i+2}) - \lambda_i + \rho.
\end{aligned}$$

2. $\underline{V(\mu + \epsilon_\gamma) \text{ for } D_n(i) \text{ for } 3 \leq i \leq n-3}$: Since, by Table 4, the special value s_0 is $s_0 = n - i - 1$, we want to show that there is $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - (n - i - 1)\lambda_i + \rho$ is linked to $\nu - (n - i - 1)\lambda_i + \rho$. By Table 1, we have $\mu + \epsilon_\gamma = 2\varepsilon_1$. Observe that if $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$ and $w_j = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_j}$ for $1 \leq j \leq i-1$ then the longest element w_0 of the Weyl group of type A_{i-1} may be expressed as $w_0 = w_{i-1} w_{i-2} \cdots w_1$. It is shown in Section 6 of [18] that $V(\mu + \epsilon_\gamma)$ is an $\mathfrak{l}_\gamma$-submodule of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$. Since $\mathfrak{l}_\gamma$ is of type A_{i-1} (see Appendix A), the highest weight ν for $V(\mu + \epsilon_\gamma)^*$ is then given by

$$\nu = -w_0(2\varepsilon_1) = -2\varepsilon_i.$$

We have

$$-2\varepsilon_i = -2(\varepsilon_i - \varepsilon_{n-1}) - (\varepsilon_{n-1} - \varepsilon_n) \ - (\varepsilon_{n-1} + \varepsilon_n)$$

with $\varepsilon_i - \varepsilon_{n-1} \in \Delta(\mathfrak{g}(1))$ and $\varepsilon_{n-1} - \varepsilon_n$, $\varepsilon_{n-1} + \varepsilon_n \in \Pi(\mathfrak{l})$. Then a direct computation shows that $(\varepsilon_{n-1} - \varepsilon_n, \varepsilon_i - \varepsilon_{n-1})$ links $-(\varepsilon_{n-1} + \varepsilon_n) - (n - i - 1)\lambda_i + \rho$ to $-2\varepsilon_i - (n - i - 1)\lambda_i + \rho$.

<u>3. $V(\mu+\epsilon_\gamma)$ for $E_6(3)$</u>: Since, by Table 4, the special value s_0 is $s_0 = 1$, we want to show that there is $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - \lambda_3 + \rho$ is linked to $\nu - \lambda_3 + \rho$. By Table 2, we have

$$\mu + \epsilon_\gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6.$$

As $V(\mu+\epsilon_\gamma)$ is a simple $\mathfrak{l}_\gamma$-submodule of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$, if w_0 is the longest element of the Weyl group of $\mathfrak{l}_\gamma$ then, by using LiE, the highest weight ν for $V(\mu+\epsilon_\gamma)^*$ is given by

$$\begin{aligned}\nu &= -w_0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6)\\ &= -2\alpha_3 - \alpha_1 - \alpha_4\end{aligned}$$

with $\alpha_3 \in \Delta(\mathfrak{g}(1))$ and $\alpha_1,\ \alpha_4 \in \Pi(\mathfrak{l})$. Now a direct computation shows that (α_1, α_3) links $-\alpha_4 - \lambda_3 + \rho$ to $(-2\alpha_3 - \alpha_1 - \alpha_4) - \lambda_3 + \rho$. ■

6.3. The $V(\mu+\epsilon_\gamma)$ case for $B_n(i)$ for $3 \le i \le n-1$.

Now we consider the case $V(\mu+\epsilon_\gamma)$ for $B_n(i)$ for $3 \le i \le n-1$. By Table 4, the special value s_0 is $s_0 = n - i - (1/2)$ for $1 \le i \le n-1$. (Note that when $i = n-1$, we have $s_0 = 1/2 = n-(n-1)-(1/2)$). By the same argument used for the case $V(\mu+\epsilon_\gamma)$ of $D_n(i)$ in the proof of Theorem 6.6, the highest weight ν for $V(\mu+\epsilon_\gamma)^*$ is $\nu = -2\varepsilon_i$. Therefore, we have

$$\varphi_{\Omega_2} : M_\mathfrak{q}(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho) \to M_\mathfrak{q}(-(n-i-(1/2))\lambda_i + \rho). \quad (6.7)$$

We first show that the standard map φ_{std} is non-zero. If $\beta = \sum_{\alpha\in\Pi} m_\alpha \alpha \in \sum_{\alpha\in\Pi} \mathbb{Z}\alpha$ then we say that $|m_\alpha|$ are the *multiplicities* of α in β.

Proposition 6.8. *If $\mathfrak{q}$ is the maximal parabolic subalgebra of type $B_n(i)$ with $3 \le i \le n-1$ then the standard map φ_{std} from $M_\mathfrak{q}(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho)$ to $M_\mathfrak{q}(-(n-i-(1/2))\lambda_i + \rho)$ is non-zero.*

Proof. First note that, as $s_0 = n-i-(1/2) \notin \mathbb{Z}$, Proposition 4.2 cannot be applied to this case. Then, to prove this proposition, we observe Proposition 4.6; we show that there is no $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha - (n-i-(1/2))\lambda_i + \rho$ is linked to $-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho$. For simplicity we write

$$\delta(i) = -(n-i-(1/2))\lambda_i + \rho.$$

Since $\varepsilon_i = \sum_{j=i}^n \alpha_j$ with α_j simple roots in the standard numbering, we want to show that there is no $\alpha \in \Pi(\mathfrak{l})$ so that $-\alpha + \delta(i)$ is linked to $-2\varepsilon_i + \delta(i) = -2\sum_{j=i}^n \alpha_j + \delta(i)$. Suppose that such $\alpha' \in \Pi(\mathfrak{l})$ exists. Let $(\beta_1, \ldots, \beta_m)$ be a link from $-\alpha' + \delta(i)$ to $-2\sum_{j=i}^n \alpha_j + \delta(i)$. Without loss of generality, we assume that for all $j = 1, \ldots, m$,

$$\langle s_{\beta_{j-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)), \beta_j^\vee \rangle \neq 0.$$

(If $j = 1$ then set $s_{\beta_0} = e$, the identity.) By the property (2) in Definition 4.4, this means that we assume that

$$\langle s_{\beta_{j-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)), \beta_j^\vee \rangle \in 1 + \mathbb{Z}_{\ge 0} \quad (6.9)$$

for all $j = 1, \ldots, m$. Observe that it follows from the property (2) in Definition 4.4 that any weight linked from $-\alpha' + \delta(i)$ is of the from

$$(-\sum_{\alpha \in \Pi} n_\alpha \alpha) - \alpha' + \delta(i) \text{ with } n_\alpha \in \mathbb{Z}_{\geq 0}. \tag{6.10}$$

We have $\Delta^+ = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{g}(1)) \cup \Delta(\mathfrak{z}(\mathfrak{n}))$, where $\Delta^+(\mathfrak{l})$, $\Delta(\mathfrak{g}(1))$, and $\Delta(\mathfrak{z}(\mathfrak{n}))$ are the sets of the positive roots in which α_i has multiplicity zero, one, and two, respectively. As $(\beta_1, \ldots, \beta_m)$ is a link from $-\alpha' + \delta(i)$ to $-2\sum_{j=i}^n \alpha_j + \delta(i)$, we have

$$s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i)) = -2\sum_{j=i}^n \alpha_j + \delta(i). \tag{6.11}$$

If $\beta_j \in \Delta^+(\mathfrak{l})$ for all j then we would have

$$-2\sum_{j=i}^n \alpha_j + \delta(i) = s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i)) = (-\sum_{\alpha \in \Pi(\mathfrak{l})} k_\alpha \alpha) - \alpha' + \delta(i)$$

for some $k_\alpha \in \mathbb{Z}_{\geq 0}$. This implies that

$$-2\alpha_i - 2\sum_{j=i+1}^n \alpha_j = (-\sum_{\alpha \in \Pi(\mathfrak{l})} k_\alpha \alpha) - \alpha'. \tag{6.12}$$

This is absurd, because, as $\Pi(\mathfrak{l}) = \Pi \backslash \{\alpha_i\}$ and $\alpha' \in \Pi(\mathfrak{l})$, the simple root α_i does not contribute to the right hand side of (6.12). Thus, there must exist at least one β_j in $(\beta_1, \ldots, \beta_m)$ with $\beta_j \in \Delta(\mathfrak{g}(1)) \cup \Delta(\mathfrak{z}(\mathfrak{n}))$.

Now we show that any β_j in $(\beta_1, \ldots, \beta_m)$ cannot belong to $\Delta(\mathfrak{g}(1)) \cup \Delta(\mathfrak{z}(\mathfrak{n}))$. First, suppose that there exists β_r in $(\beta_1, \ldots, \beta_m)$ with $\beta_r \in \Delta(\mathfrak{z}(\mathfrak{n}))$. Observe that $\Delta(\mathfrak{z}(\mathfrak{n}))$ consists of the positive roots $\varepsilon_j + \varepsilon_k$ for $1 \leq j < k \leq i$ (see Appendix A). So β_r is $\beta_r = \varepsilon_s + \varepsilon_t$ for some $1 \leq s < t \leq i$. Since each $\varepsilon_l = \sum_{j=l}^n \alpha_j$ with α_j simple roots, the positive root $\beta_r = \varepsilon_s + \varepsilon_t$ with $1 \leq s < t \leq i$ can be expressed as

$$\beta_r = \varepsilon_s + \varepsilon_t = \sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^n \alpha_j.$$

If $c = \langle s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)), \beta_r^\vee \rangle$ then

$$\begin{aligned} s_{\beta_r} \cdots s_{\beta_1}(-\alpha' + \delta(i)) &= s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)) - c\beta_r \\ &= s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)) - c\big(\sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^n \alpha_j\big). \end{aligned} \tag{6.13}$$

Observe that, by (6.10), $s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i))$ is of the form

$$s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)) = (-\sum_{\alpha \in \Pi} m_\alpha \alpha) - \alpha' + \delta(i) \tag{6.14}$$

for some $m_\alpha \in \mathbb{Z}_{\geq 0}$. Moreover, as $s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i))$ is a weight linked from $s_{\beta_r} \cdots s_{\beta_1}(-\alpha' + \delta(i))$, the weight $s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i))$ is of the form

$$s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i)) = (-\sum_{\alpha \in \Pi} m'_\alpha \alpha) + s_{\beta_r} \cdots s_{\beta_1}(-\alpha' + \delta(i)) \tag{6.15}$$

for some $m'_\alpha \in \mathbb{Z}_{\geq 0}$. By combining (6.13), (6.14), and (6.15), we have

$$\begin{aligned}
&s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i)) \\
&= (-\sum_{\alpha\in\Pi} m'_\alpha \alpha) + s_{\beta_r} \cdots s_{\beta_1}(-\alpha' + \delta(i)) \\
&= (-\sum_{\alpha\in\Pi} m'_\alpha \alpha) + s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)) - c\big(\sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^{n} \alpha_j\big) \\
&= (-\sum_{\alpha\in\Pi} m'_\alpha \alpha) + (-\sum_{\alpha\in\Pi} m_\alpha \alpha) - c\big(\sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^{n} \alpha_j\big) - \alpha' + \delta(i) \qquad (6.16)
\end{aligned}$$

with $m_\alpha, m'_\alpha \in \mathbb{Z}_{\geq 0}$. By (6.9), we have

$$c = \langle s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)), \beta_r^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0}.$$

Therefore, by (6.16), the weight $s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i))$ is of the form

$$s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i)) = -\sum_{\alpha\in\Pi} n_\alpha \alpha - \sum_{j=s}^{t-1} \alpha_j - 2\sum_{j=t}^{n} \alpha_j - \alpha' + \delta(i)$$

for some $n_\alpha \in \mathbb{Z}_{\geq 0}$. By (6.11), this implies that

$$2\sum_{j=i}^{n} \alpha_j = \sum_{\alpha\in\Pi} n_\alpha \alpha + \sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^{n} \alpha_j + \alpha'.$$

Since $s < t \leq i$, we then have

$$\begin{aligned}
0 &= \sum_{\alpha\in\Pi} n_\alpha \alpha + \sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^{n} \alpha_j + \alpha' - 2\sum_{j=i}^{n} \alpha_j \\
&= \begin{cases} \sum_{\alpha\in\Pi} n_\alpha \alpha + \sum_{j=s}^{t-1} \alpha_j + 2\sum_{j=t}^{i-1} \alpha_j + \alpha' & \text{if } t < i \\ \sum_{\alpha\in\Pi} n_\alpha \alpha + \sum_{j=s}^{t-1} \alpha_j + \alpha' & \text{if } t = i. \end{cases} \qquad (6.17)
\end{aligned}$$

This is a contradiction, because, as $n_\alpha \in \mathbb{Z}_{\geq 0}$, (6.17) cannot be zero. Therefore no β_j in $(\beta_1, \ldots, \beta_m)$ is a root in $\Delta(\mathfrak{z}(\mathfrak{n}))$.

Next we suppose that there exists β_r in $(\beta_1, \ldots, \beta_m)$ with $\beta_r \in \Delta(\mathfrak{g}(1))$. There are long roots and short roots in $\Delta(\mathfrak{g}(1))$. We handle these cases separately. We first suppose that β_r is a long root in $\Delta(\mathfrak{g}(1))$. The long roots in $\Delta(\mathfrak{g}(1))$ are $\varepsilon_j \pm \varepsilon_k$ for $1 \leq j \leq i$ and $i+1 \leq k \leq n$. (See Appendix A.) The roots $\varepsilon_j \pm \varepsilon_k$ may be expressed in terms of simple roots as

$$\varepsilon_j + \varepsilon_k = \sum_{l=j}^{n} \alpha_l + \sum_{l=k}^{n} \alpha_l = \sum_{l=j}^{i-1} \alpha_l + \alpha_i + \sum_{l=i+1}^{k-1} \alpha_l + 2\sum_{l=k}^{n-1} \alpha_l + 2\alpha_n$$

and

$$\varepsilon_j - \varepsilon_k = \sum_{l=j}^{n} \alpha_l - \sum_{l=k}^{n} \alpha_l = \sum_{l=j}^{i-1} \alpha_l + \alpha_i + \sum_{l=i+1}^{k-1} \alpha_l.$$

We show that if $\beta_r = \varepsilon_j \pm \varepsilon_k$ then $\langle s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)), \beta_r^\vee \rangle \notin \mathbb{Z}$. Observe that since α_n is the only short simple root, the coroot $(\varepsilon_j + \varepsilon_k)^\vee$ can be expressed as

$$\begin{aligned}
&(\varepsilon_j + \varepsilon_k)^\vee \\
&= \Big(\sum_{l=j}^{i-1} \alpha_l + \alpha_i + \sum_{l=i+1}^{k-1} \alpha_l + 2\sum_{l=k}^{n-1} \alpha_l + 2\alpha_n\Big)^\vee \\
&= \sum_{l=j}^{i-1} \frac{2\alpha_l}{||\varepsilon_j + \varepsilon_k||^2} + \frac{2\alpha_i}{||\varepsilon_j + \varepsilon_k||^2} + \sum_{l=i+1}^{k-1} \frac{2\alpha_l}{||\varepsilon_j + \varepsilon_k||^2} + 2\sum_{l=k}^{n-1} \frac{2\alpha_l}{||\varepsilon_j + \varepsilon_k||^2} + 2 \cdot \frac{2\alpha_n}{||\varepsilon_j + \varepsilon_k||^2} \\
&= \sum_{l=j}^{i-1} \alpha_l^\vee + \alpha_i^\vee + \sum_{l=i+1}^{k-1} \alpha_l^\vee + 2\sum_{l=k}^{n-1} \alpha_l^\vee + \alpha_n^\vee.
\end{aligned}$$

Similarly, we have

$$(\varepsilon_j - \varepsilon_k)^\vee = \sum_{l=j}^{i-1} \alpha_l^\vee + \alpha_i^\vee + \sum_{l=i+1}^{k-1} \alpha_l^\vee.$$

Now observe that, as λ_i is the fundamental weight for α_i, for $\alpha \in \Pi$, we have

$$\begin{aligned}
\langle \delta(i), \alpha^\vee \rangle &= \langle -(n - i - (1/2))\lambda_i + \rho, \alpha^\vee \rangle \\
&= \begin{cases} -n + i + (3/2) & \text{if } \alpha = \alpha_i \\ 1 & \text{otherwise.} \end{cases}
\end{aligned} \tag{6.18}$$

Thus,

$$\begin{aligned}
&\langle \delta(i), (\varepsilon_j + \varepsilon_k)^\vee \rangle \\
&= \langle \delta(i), \sum_{l=j}^{i-1} \alpha_l^\vee + \alpha_i^\vee + \sum_{l=i+1}^{k-1} \alpha_l^\vee + 2\sum_{l=k}^{n-1} \alpha_l^\vee + \alpha_n^\vee \rangle \\
&= \sum_{l=j}^{i-1} \langle \delta(i), \alpha_l^\vee \rangle + \langle \delta(i), \alpha_i^\vee \rangle + \sum_{l=i+1}^{k-1} \langle \delta(i), \alpha_l^\vee \rangle + 2\sum_{l=k}^{n-1} \langle \delta(i), \alpha_l^\vee \rangle + \langle \delta(i), \alpha_n^\vee \rangle \\
&= (i - 1 - (j - 1)) + (-n + i + (3/2)) + (k - 1 - i) + 2(n - 1 - (k - 1)) + 1 \\
&= n - k + i - j + (3/2).
\end{aligned}$$

Similarly,

$$\langle \delta(i), (\varepsilon_j - \varepsilon_k)^\vee \rangle = -n + k + i - j + (1/2).$$

Hence, for $\beta_r = \varepsilon_j \pm \varepsilon_k$, we have $\langle \delta(i), \beta_r^\vee \rangle \notin \mathbb{Z}$. Now, by (6.14), we have

$$\begin{aligned}
\langle s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)), \beta_r^\vee \rangle &= \langle (-\sum_{\alpha \in \Pi} m_\alpha \alpha) - \alpha' + \delta(i), \beta_r^\vee \rangle \\
&= -\sum_{\alpha \in \Pi} m_\alpha \langle \alpha, \beta_r^\vee \rangle - \langle \alpha', \beta_r^\vee \rangle + \langle \delta(i), \beta_r^\vee \rangle
\end{aligned}$$

with $m_\alpha \in \mathbb{Z}$. Since $m_\alpha, \langle \alpha, \beta_r^\vee \rangle, \langle \alpha', \beta_r^\vee \rangle \in \mathbb{Z}$ and $\langle \delta(i), \beta_r^\vee \rangle \notin \mathbb{Z}$, this shows that $\langle s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)), \beta_r^\vee \rangle \notin \mathbb{Z}$.

Finally, we suppose that β_r is a short root in $\Delta(\mathfrak{g}(1))$. The short roots in $\Delta(\mathfrak{g}(1))$ are ε_j for $1 \leq j \leq i$ (see Appendix A). Thus β_r is $\beta_r = \varepsilon_l$ for some $1 \leq l \leq i$. Since ε_l is of the form $\varepsilon_l = \sum_{j=l}^n \alpha_j$, (6.11) forces that $l = i$; otherwise, $s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i))$ would have a contribution from some $\alpha_j \in \Pi$ with $1 \leq j \leq i-1$. Thus $\beta_r = \varepsilon_i = \sum_{j=i}^n \alpha_j$. Since β_r is a short root, the coroot $\beta_r^\vee = (\sum_{j=i}^n \alpha_j)^\vee$ can be expressed as

$$\beta_r^\vee = \big(\sum_{j=i}^n \alpha_j\big)^\vee = \sum_{j=i}^n \frac{2\alpha_j}{||\beta_r||^2} = \frac{2\alpha_i}{||\beta_r||^2} + \sum_{j=i+1}^{n-1} \frac{2\alpha_j}{||\beta_r||^2} + \frac{2\alpha_n}{||\beta_r||^2} = 2\alpha_i^\vee + 2\sum_{j=i+1}^{n-1} \alpha_j^\vee + \alpha_n^\vee.$$

It then follows from (6.18) that

$$\begin{aligned}
\langle \delta(i), \beta_r^\vee \rangle &= \langle -(n-i-(1/2))\lambda_i + \rho, \big(\sum_{j=i}^n \alpha_j\big)^\vee \rangle \\
&= \langle -(n-i-(1/2))\lambda_i + \rho, 2\alpha_i^\vee + 2\sum_{j=i+1}^{n-1} \alpha_j^\vee + \alpha_n^\vee \rangle \\
&= 2\langle -(n-i-(1/2))\lambda_i + \rho, \alpha_i^\vee \rangle + 2\sum_{j=i+1}^{n-1} \langle -(n-i-(1/2))\lambda_i + \rho, \alpha_j^\vee \rangle \\
&\quad + \langle -(n-i-(1/2))\lambda_i + \rho, \alpha_n^\vee \rangle \\
&= 2(-n+i+(3/2)) + 2(n-1-i) + 1 \\
&= 2.
\end{aligned}$$

Thus, by (6.14), we have

$$\begin{aligned}
\langle s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)), \beta_r^\vee \rangle &= \langle (-\sum_{\alpha\in\Pi} m_\alpha \alpha) - \alpha' + \delta(i), \beta_r^\vee \rangle \\
&= \langle -\sum_{\alpha\in\Pi} m_\alpha \alpha - \alpha', \beta_r^\vee \rangle + 2 \qquad (6.19)
\end{aligned}$$

with $m_\alpha \in \mathbb{Z}_{\geq 0}$. Thus, as $\beta_r = \sum_{j=i}^n \alpha_j$, if $d = \langle -\sum_{\alpha\in\Pi} m_\alpha\alpha - \alpha', \beta_r^\vee \rangle + 2$ then $s_{\beta_r} \cdots s_{\beta_1}(-\alpha' + \delta(i))$ is of the form

$$s_{\beta_r} \cdots s_{\beta_1}(-\alpha' + \delta(i)) = s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)) - d\sum_{j=i}^n \alpha_j.$$

By (6.14) and (6.15), we have

$$\begin{aligned}
s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i)) &= (-\sum_{\alpha\in\Pi} m'_\alpha \alpha) + s_{\beta_r} \cdots s_{\beta_1}(-\alpha' + \delta(i)) \\
&= (-\sum_{\alpha\in\Pi} m'_\alpha \alpha) + s_{\beta_{r-1}} \cdots s_{\beta_1}(-\alpha' + \delta(i)) - d\sum_{j=i}^n \alpha_j \\
&= (-\sum_{\alpha\in\Pi} m'_\alpha \alpha) + (-\sum_{\alpha\in\Pi} m_\alpha \alpha) - d\sum_{j=i}^n \alpha_j - \alpha' + \delta(i)
\end{aligned}$$

with $m_\alpha, m'_\alpha \in \mathbb{Z}_{\geq}$. Therefore, $s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i))$ can be expressed as

$$s_{\beta_m} \cdots s_{\beta_1}(-\alpha' + \delta(i)) = -\sum_{\alpha\in\Pi} n_\alpha \alpha - d\sum_{j=i}^{n} \alpha_j - \alpha' + \delta(i)$$

for some $n_\alpha \in \mathbb{Z}_{\geq 0}$. By (6.11), this implies that

$$2\sum_{j=i}^{n} \alpha_j = \sum_{\alpha\in\Pi} n_\alpha \alpha + d\sum_{j=i}^{n} \alpha_j + \alpha'. \tag{6.20}$$

By comparing the coefficients of α_i in the both sides, we have

$$n_{\alpha_i} + d = 2. \tag{6.21}$$

By (6.9) and (6.19), we have $d = \langle -\sum_{\alpha\in\Pi} m_\alpha \alpha - \alpha', \beta_r^\vee\rangle + 2 \in 1 + \mathbb{Z}_{\geq 0}$. Since $n_{\alpha_i} \in \mathbb{Z}_{\geq 0}$, (6.21) forces that

$$d = 2 \text{ or } d = 1.$$

If $d = 2$ then (6.20) becomes

$$2\sum_{j=i}^{n} \alpha_j = \sum_{\alpha\in\Pi} n_\alpha \alpha + 2\sum_{j=i}^{n} \alpha_j + \alpha'.$$

Therefore,

$$\sum_{\alpha\in\Pi} n_\alpha \alpha + \alpha' = 0, \tag{6.22}$$

which is a contradiction, because as $\alpha' \in \Pi$ and $k'_\alpha \in \mathbb{Z}_{\geq 0}$, the left hand side of (6.22) cannot be zero. If $d = 1$ then, since $d = \langle -\sum_{\alpha\in\Pi} m_\alpha \alpha - \alpha', \beta_r^\vee\rangle + 2$, we have

$$\langle -\sum_{\alpha\in\Pi} m_\alpha \alpha - \alpha', \beta_r^\vee\rangle + 2 = 1.$$

Thus,

$$\langle \sum_{\alpha\in\Pi} m_\alpha \alpha + \alpha', \beta_r^\vee\rangle = 1. \tag{6.23}$$

Observe that, as $\beta_r = \varepsilon_i$ in the standard realization, if $\langle \alpha, \beta_r^\vee\rangle \neq 0$ for $\alpha \in \Pi$ then α must be $\alpha = \varepsilon_{i-1} - \varepsilon_i$ in $\Pi(\mathfrak{l})$ or $\alpha = \varepsilon_i - \varepsilon_{i+1}$ in $\Pi\backslash\Pi(\mathfrak{l})$. Since $\langle \varepsilon_{i-1} - \varepsilon_i, \varepsilon_i^\vee\rangle = -2$, $\langle \varepsilon_i - \varepsilon_{i+1}, \varepsilon_i^\vee\rangle = 2$, and $\alpha' \in \Pi(\mathfrak{l})$, the left hand side of (6.23) is

$$\begin{aligned}\langle \sum_{\alpha\in\Pi} m_\alpha \alpha + \alpha', \beta_r^\vee\rangle &= m_{\varepsilon_{i-1}-\varepsilon_i}\langle \varepsilon_{i-1} - \varepsilon_i, \varepsilon_i^\vee\rangle + m_{\varepsilon_i-\varepsilon_{i+1}}\langle \varepsilon_i - \varepsilon_{i+1}, \varepsilon_i^\vee\rangle + \langle \alpha', \varepsilon_i^\vee\rangle \\ &= -2m_{\varepsilon_{i-1}-\varepsilon_i} + 2m_{\varepsilon_i-\varepsilon_{i+1}} - 2\delta_{\alpha',\varepsilon_{i-1}-\varepsilon_i} \\ &= 2(m_{\varepsilon_i-\varepsilon_{i+1}} - m_{\varepsilon_{i-1}-\varepsilon_i} - \delta_{\alpha',\varepsilon_{i-1}-\varepsilon_i}),\end{aligned}$$

where $\delta_{\alpha',\varepsilon_{i-1}-\varepsilon_i}$ is the Kronecker delta. As $m_{\varepsilon_i-\varepsilon_{i+1}}$, $m_{\varepsilon_{i-1}-\varepsilon_i}$, and $\delta_{\alpha',\varepsilon_{i-1}-\varepsilon_i}$ are integers, this shows that $\langle \sum_{\alpha\in\Pi} m_\alpha \alpha + \alpha', \beta_r^\vee\rangle \neq 1$, which contradicts (6.23). Therefore, no β_r in $(\beta_1, \ldots, \beta_m)$ is a short root in $\Delta(\mathfrak{g}(1))$. Hence there is no link from $-\alpha' + \delta(i)$ to $-2\sum_{j=i}^{n} \alpha_j + \delta(i)$. ∎

Now we are going to show that the map

$$\varphi_{\Omega_2} : M_{\mathfrak{q}}(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho) \to M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i + \rho)$$

is standard. This is to show that, given highest weight vector v_h for $F(\Omega_2|_{V(\mu+\epsilon_\gamma)^*})$, the image $\varphi_{\Omega_2}(1 \otimes v_h)$ of $1 \otimes v_h$ is a non-zero scalar multiple of $\varphi_{std}(1 \otimes v_h)$, where $F(\Omega_2|_{V(\mu+\epsilon_\gamma)^*})$ is the finite dimensional simple $\mathfrak{l}$-submodule of

$$M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i + \rho)^{\mathfrak{n}}$$

induced by the $\Omega_2|_{V(\mu+\epsilon_\gamma)^*}$ system, so that

$$M_{\mathfrak{q}}(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F(\Omega_2|_{V(\mu+\epsilon_\gamma)^*}).$$

Observe that, by the definition of φ_{Ω_k}, we have $\varphi_{\Omega_2}(1 \otimes v_h) = 1 \cdot v_h = v_h$. On the other hand, if $1 \otimes v^+$ is a highest weight vector for $M(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho)$ with highest weight $-2\varepsilon_i - (n-i-(1/2))\lambda_i$ and if $\mathrm{pr} : M(-(n-i-(1/2))\lambda_i + \rho) \to M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i + \rho)$ is the canonical projection map then $\varphi_{std}(1 \otimes v_h) = (\mathrm{pr} \circ \varphi)(1 \otimes v^+)$, where φ is an embedding of $M(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho)$ into $M(-(n-i-(1/2))\lambda_i + \rho)$; in a diagram we have

$$\begin{tikzcd}
M(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho) \arrow[r, "\varphi"] \arrow[d, "\mathrm{pr}'"'] & M(-(n-i-(1/2))\lambda_i + \rho) \arrow[d, "\mathrm{pr}"] \\
M_{\mathfrak{q}}(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho) \arrow[r, "\varphi_{std}"] & M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i + \rho),
\end{tikzcd}$$

where $\mathrm{pr}' : M(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho) \to M_{\mathfrak{q}}(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho)$ is the canonical projection map. Note that, by Proposition 6.8, we have $(\mathrm{pr} \circ \varphi)(1 \otimes v^+) = \varphi_{std}(1 \otimes v_h) \neq 0$. Therefore, to show that φ_{Ω_2} is standard, we wish to show that $v_h = \varphi_{\Omega_2}(1 \otimes v_h)$ is a non-zero scalar multiple of $(\mathrm{pr} \circ \varphi)(1 \otimes v^+)$. Since $M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i + \rho) \cong \mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{-(n-i-(1/2))\lambda_i}$ as an $\mathfrak{l}$-module, we have

$$v_h = u_h \otimes 1_{-(n-i-(1/2))\lambda_i} \tag{6.24}$$

and

$$(\mathrm{pr} \circ \varphi)(1 \otimes v^+) = \tilde{u} \otimes 1_{-(n-i-(1/2))\lambda_i} \tag{6.25}$$

for some $u_h, \tilde{u} \in \mathcal{U}(\bar{\mathfrak{n}}) \backslash \{0\}$. Hence, to show that v_h is a non-zero scalar multiple of $(\mathrm{pr} \circ \varphi)(1 \otimes v^+)$, it suffices to show that u_h in (6.24) is a non-zero scalar multiple of $\tilde{u}$ in (6.25).

Observe that, as $v_h = u_h \otimes 1_{-(n-i-(1/2))\lambda_i}$ is a highest weight vector for the simple $\mathfrak{l}$-submodule $F(\Omega_2|_{V(\mu+\epsilon_\gamma)^*})$ of $\mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{-(n-i-(1/2))\lambda_i+\rho}$, for all $\alpha \in \Pi(\mathfrak{l})$, we have $X_\alpha \cdot (u_h \otimes 1_{-(n-i-(1/2))\lambda_i}) = 0$. Therefore $\mathrm{ad}(X_\alpha)(u_h) = 0$ for all $\alpha \in \Pi(\mathfrak{l})$. On the other hand, it follows from (3.3) that $F(\Omega_2|_{V(\mu+\epsilon_\gamma)^*})$ is spanned by the elements of the form $u \otimes 1_{-(n-i-(1/2))\lambda_i}$ with $u \in \sigma(\mathrm{Sym}^2(\bar{\mathfrak{n}}))$. Since $F(\Omega_2|_{V(\mu+\epsilon_\gamma)^*})$ has highest weight $-2\varepsilon_i - (n-i-(1/2))\lambda_i$, this shows that u_h is an element in $\sigma(\mathrm{Sym}^2(\bar{\mathfrak{n}}))$ with weight $-2\varepsilon_i$.

Definition 6.26. *For $u \in \mathcal{U}(\bar{\mathfrak{n}})$, we say that u satisfies* Condition (H) *if u satisfies following three conditions:*

(1) $u \in \sigma(Sym^2(\bar{\mathfrak{n}}))$,

(2) u has weight $-2\varepsilon_i$, and

(3) $ad(X_\alpha)(u) = 0$ for all $\alpha \in \Pi(\mathfrak{l})$.

It follows from the observation made before Definition 6.26 that $u_h \in \mathcal{U}(\bar{\mathfrak{n}})$ in (6.24) satisfies Condition (H). Our first goal is to show that any element in $\mathcal{U}(\bar{\mathfrak{n}})$ that satisfies Condition (H) is a scalar multiples of u_h.

Lemma 6.27. *For any $\beta \in \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{z}(\mathfrak{n}))$, we have $2\varepsilon_i - \beta \notin \Delta^+$.*

Proof. This lemma follows from a direct observation. (See Appendix A for $\Delta^+(\mathfrak{l}) = \Delta^+(\mathfrak{l}_\gamma) \cup \Delta^+(\mathfrak{l}_{n\gamma})$ and $\Delta(\mathfrak{z}(\mathfrak{n}))$.) ■

We write $\mathfrak{u} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ for the nilradical of $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ and we denote by $\bar{\mathfrak{u}}$ the opposite nilradical of $\mathfrak{u}$. Note that, as $\mathfrak{n}$ is the nilradical of the parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$, we have $\mathfrak{n} \subset \mathfrak{u}$.

Lemma 6.28. *If u is in $Sym^2(\bar{\mathfrak{u}})$ with weight $-2\varepsilon_i$ then u is of the form*

$$AX_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)}$$

for some constants A and B_k. In particular, we have $u \in Sym^2(\bar{\mathfrak{n}})$.

Proof. If $u \in \sigma(\mathrm{Sym}^2(\bar{\mathfrak{u}}))$ with weight $-2\varepsilon_i$ then u is of the from

$$u = \sum c_\beta X_{-\beta} X_{-2\varepsilon_i+\beta}$$

for some constants c_β, where the sum runs over the roots $\beta \in \Delta^+ = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{g}(1)) \cup \Delta(\mathfrak{z}(\mathfrak{n}))$ so that $2\varepsilon_i - \beta \in \Delta^+$. By Lemma 6.27, the roots β must be in $\Delta(\mathfrak{g}(1))$. Thus if $\Delta_{2\varepsilon_i}(\mathfrak{g}(1)) = \{\beta \in \Delta(\mathfrak{g}(1)) \mid 2\varepsilon_i - \beta \in \Delta\}$ then

$$u = \sum_{\beta \in \Delta_{2\varepsilon_i}(\mathfrak{g}(1))} c_\beta X_{-\beta} X_{-2\varepsilon_i+\beta}.$$

By Appendix A, we have

$$\Delta(\mathfrak{g}(1)) = \{\varepsilon_j \pm \varepsilon_k \mid 1 \leq j \leq i \text{ and } i+1 \leq k \leq n\} \cup \{\varepsilon_j \mid 1 \leq j \leq i\}.$$

Thus,

$$\Delta_{2\varepsilon_i}(\mathfrak{g}(1)) = \{\beta \in \Delta(\mathfrak{g}(1)) \mid 2\varepsilon_i - \beta \in \Delta\} = \{\varepsilon_i \pm \varepsilon_k \mid i+1 \leq k \leq n\} \cup \{\varepsilon_i\}.$$

Therefore u is of the form

$$\begin{aligned} u &= \sum_{\beta \in \Delta_{2\varepsilon_i}(\mathfrak{g}(1))} c_\beta X_{-\beta} X_{-2\varepsilon_i+\beta} \\ &= c_{\varepsilon_i} X_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} c_{\varepsilon_i+\varepsilon_k} X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)} + \sum_{k=i+1}^{n} c_{\varepsilon_i-\varepsilon_k} X_{-(\varepsilon_i-\varepsilon_k)} X_{-(\varepsilon_i+\varepsilon_k)} \\ &= c_{\varepsilon_i} X_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} (c_{\varepsilon_i+\varepsilon_k} + c_{\varepsilon_i-\varepsilon_k}) X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)}. \end{aligned}$$

If $A = c_{\varepsilon_i}$ and $B_k = c_{\varepsilon_i+\varepsilon_k} + c_{\varepsilon_i-\varepsilon_k}$ then u can be expressed as

$$u = AX_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)}.$$ ■

Proposition 6.29. *If $u \in \mathcal{U}(\bar{\mathfrak{n}})$ satisfies Condition (H) then u is a scalar multiple of u_h.*

Proof. As u_h satisfies Condition (H), to prove this proposition, it suffices to show that any element $u \in \mathcal{U}(\bar{\mathfrak{n}})$ that satisfies Condition (H) is a scalar multiple of

$$u_0 = X_{-\varepsilon_i}^2 + \sum_{j=i+1}^{n} b_j X_{-(\varepsilon_i+\varepsilon_j)} X_{-(\varepsilon_i-\varepsilon_j)}, \tag{6.30}$$

where

$$b_j = (-1)^{n-j} b_n \prod_{k=j}^{n-1} \frac{N_{\varepsilon_k-\varepsilon_{k+1},-(\varepsilon_i-\varepsilon_{k+1})}}{N_{\varepsilon_k-\varepsilon_{k+1},-(\varepsilon_i+\varepsilon_k)}} \tag{6.31}$$

for $j = i+1, \ldots, n-1$ and

$$b_n = -\frac{2N_{\varepsilon_n,-\varepsilon_i}}{N_{\varepsilon_n,-(\varepsilon_i+\varepsilon_n)}}. \tag{6.32}$$

Here, $N_{\alpha,\beta}$ are the constants so that $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$.

If $u \in \mathcal{U}(\bar{\mathfrak{n}})$ satisfies Condition (H) then $u \in \sigma(\mathrm{Sym}^2(\bar{\mathfrak{n}})) \subset \hat{\sigma}(\mathrm{Sym}^2(\bar{\mathfrak{u}}))$ and has weight $-2\varepsilon_i$, where $\hat{\sigma} : \mathrm{Sym}(\bar{\mathfrak{u}}) \to \mathcal{U}(\bar{\mathfrak{u}})$ is the symmetrization map for $\mathrm{Sym}(\bar{\mathfrak{u}})$. Thus it follows from Lemma 6.28 that u is of the from

$$u = AX_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)} \tag{6.33}$$

for some constants A and B_k. Now observe that, by the condition (3) in Definition 6.26, we have $\mathrm{ad}(X_\alpha)(u) = 0$ for all $\alpha \in \Pi(\mathfrak{l})$. Therefore, as $\varepsilon_j - \varepsilon_{j+1}$ and ε_n are in $\Pi(\mathfrak{l})$ for $j = i+1, \ldots, n-1$, we have

$$\mathrm{ad}(X_{\varepsilon_j-\varepsilon_{j+1}})(u) = 0 \quad \text{and} \quad \mathrm{ad}(X_{\varepsilon_n})(u) = 0$$

for $j = i+1, \ldots, n-1$. By (6.33), this means that for $j = i+1, \ldots, n-1$,

$$\mathrm{ad}(X_{\varepsilon_j-\varepsilon_{j+1}})\big(AX_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)}\big) = 0$$

and

$$\mathrm{ad}(X_{\varepsilon_n})\big(AX_{-\varepsilon_i}^2 + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)}\big) = 0,$$

which are

$$B_j\, \mathrm{ad}(X_{\varepsilon_j-\varepsilon_{j+1}})(X_{-(\varepsilon_i+\varepsilon_j)} X_{-(\varepsilon_i-\varepsilon_j)}) + B_{j+1}\, \mathrm{ad}(X_{\varepsilon_j-\varepsilon_{j+1}})(X_{-(\varepsilon_i+\varepsilon_{j+1})} X_{-(\varepsilon_i-\varepsilon_{j+1})}) = 0$$

and

$$A\, \mathrm{ad}(X_{\varepsilon_n})(X_{-\varepsilon_i}^2) + B_n\, \mathrm{ad}(X_{\varepsilon_n})(X_{-(\varepsilon_i+\varepsilon_n)} X_{-(\varepsilon_i-\varepsilon_n)}) = 0,$$

respectively. By solving the system of linear equations, we obtain $B_j = b_j A$ for $j = i+1, \ldots, n$ with b_j in (6.31) and (6.32). Therefore, by (6.30) and (6.33), we obtain $u = Au_0$. ∎

By Proposition 6.29, to prove that φ_{Ω_2} in (6.7) is standard, it suffices to show that $\tilde{u}$ in (6.25) satisfies Condition (H). As $(\mathrm{pr}\circ\varphi)(1\otimes v^+) = \tilde{u}\otimes 1_{-(n-i-(1/2))\lambda_i}$ is a highest weight vector with highest weight $-2\varepsilon_i - (n-i-(1/2))\lambda_i$, one can easily see that $\tilde{u}$ satisfies the conditions (2) and (3) in Definition 6.26. So we wish to show that $\tilde{u}$ is in $\sigma(\mathrm{Sym}^2(\bar{\mathfrak{n}}))$. To do so we need several technical lemmas.

Lemma 6.34. *No polynomial in Sym*$^r(\bar{\mathfrak{n}})$ *for* $r \geq 3$ *has weight* $-2\varepsilon_i$.

Proof. Observe that the simple root $\alpha_{\mathfrak{q}} = \alpha_i$ has multiplicity ≥ 1 in any roots $\beta \in \Delta(\mathfrak{n})$. Therefore, in the weights for any polynomials in $\mathrm{Sym}^r(\bar{\mathfrak{n}})$, the simple root α_i has multiplicity greater than or equal to r. Since α_i has multiplicity 2 in $-2\varepsilon_i = -2\sum_{j=i}\alpha_j$, no polynomial in $\mathrm{Sym}^r(\bar{\mathfrak{n}})$ for $r \geq 3$ has weight $-2\varepsilon_i$. ∎

Corollary 6.35. *Any non-zero polynomials in Sym*$^r(\bar{\mathfrak{u}})$ *with weight* $-2\varepsilon_i$ *for* $r \geq 3$ *have contributions from root vectors* $X_{-\alpha}$ *for* $\alpha \in \Delta^+(\mathfrak{l})$.

Proof. Since $\Delta(\mathfrak{u}) = \Delta^+(\mathfrak{l})\cup\Delta(\mathfrak{n})$, this is an immediate consequence of Lemma 6.34. ∎

Lemma 6.36. *If* $u \in \mathcal{U}(\bar{\mathfrak{u}})$ *has weight* $-2\varepsilon_i$ *then*

$$u = AX^2_{-\varepsilon_i} + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)}X_{-(\varepsilon_i-\varepsilon_k)} + \sum_{\alpha\in\Delta^+(\mathfrak{l})} u^\alpha X_{-\alpha} \tag{6.37}$$

for some constants A *and* B_k, *and some elements* $u^\alpha \in \mathcal{U}(\bar{\mathfrak{u}})$.

Proof. If

$$\mathcal{U}_r(\bar{\mathfrak{u}}) = \{u \in \mathcal{U}(\bar{\mathfrak{u}}) \mid u \text{ has degree at most } r\}$$

then $\mathcal{U}(\bar{\mathfrak{u}}) = \bigcup_{r=1}^\infty \mathcal{U}_r(\bar{\mathfrak{u}})$ and $\mathcal{U}_{r+1}(\bar{\mathfrak{u}})/\mathcal{U}_r(\bar{\mathfrak{u}}) \cong \mathrm{Sym}^{r+1}(\bar{\mathfrak{u}})$. We show this lemma by induction on the degree r for $\mathcal{U}_r(\bar{\mathfrak{u}})$. First observe that since $-2\varepsilon_i \notin \Delta$, the element u cannot be in $\mathcal{U}_1(\bar{\mathfrak{u}}) = \mathbb{C}\oplus\bar{\mathfrak{u}}$. Thus if $u \in \mathcal{U}_2(\bar{\mathfrak{u}})$ then $u \in \mathrm{Sym}^2(\bar{\mathfrak{u}}) \cong \mathcal{U}_2(\bar{\mathfrak{u}})/\mathcal{U}_1(\bar{\mathfrak{u}})$. Thus, by Lemma 6.28, if $u \in \mathcal{U}_2(\bar{\mathfrak{u}})$ then $u = AX^2_{-\varepsilon_i} + \sum_{k=i+1}^n B_k X_{-(\varepsilon_i+\varepsilon_k)}X_{-(\varepsilon_i-\varepsilon_k)}$ for some constants A and B_k. Now assume that this lemma holds for $u \in \mathcal{U}_r(\bar{\mathfrak{u}})$ for $3 \leq r \leq t$, and suppose that $u \in \mathcal{U}_{t+1}(\bar{\mathfrak{u}})$. By Corollary 6.35, any polynomials in $\mathcal{U}_{t+1}(\bar{\mathfrak{u}})/\mathcal{U}_t(\bar{\mathfrak{u}}) \cong \mathrm{Sym}^{t+1}(\bar{\mathfrak{u}})$ with weight $-2\varepsilon_i$ have contributions from root vectors in $\mathfrak{l}$. By permuting the root vectors, in $\mathcal{U}_{t+1}(\bar{\mathfrak{u}})$, those polynomials can be expressed as

$$(\text{some polynomial in } \mathcal{U}_t(\bar{\mathfrak{u}})) + \sum_{\alpha\in\Delta^+(\mathfrak{l})} v^\alpha X_{-\alpha}$$

with some $v^\alpha \in \mathcal{U}_t(\bar{\mathfrak{u}})$. Therefore the element $u \in \mathcal{U}_{t+1}(\bar{\mathfrak{u}})$ is of the form

$$u = p + \sum_{\alpha\in\Delta^+(\mathfrak{l})} v^\alpha X_{-\alpha}$$

for some $p, v^\alpha \in \mathcal{U}_t(\bar{\mathfrak{u}})$. By the induction hypothesis, the polynomial $p \in \mathcal{U}_t(\bar{\mathfrak{u}})$ can be then expressed as

$$p = AX^2_{-\varepsilon_i} + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)}X_{-(\varepsilon_i-\varepsilon_k)} + \sum_{\alpha\in\Delta^+(\mathfrak{l})} \check{u}^\alpha X_{-\alpha}$$

for some constants A and B_k, and some elements $\breve{u}^\alpha \in \mathcal{U}_{t-1}(\bar{\mathfrak{u}})$. If $u^\alpha = \breve{u}^\alpha + v^\alpha$ then u is of the form in (6.37). By induction, this lemma follows. ■

Now we are ready to show that the map φ_{Ω_2} in (6.7) is standard. Recall that if $1 \otimes v^+$ is a highest weight vector for $M(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho)$ with highest weight $-2\varepsilon_i - (n-i-(1/2))\lambda_i$ and if $\mathrm{pr} : M(-(n-i-(1/2))\lambda_i + \rho) \to M_{\mathfrak{q}}(-(n-i-(1/2))\lambda_i + \rho)$ is the canonical projection map then $\varphi_{std}(1 \otimes v_h) = (\mathrm{pr} \circ \varphi)(1 \otimes v^+)$, where φ is an embedding of $M(-2\varepsilon_i - (n-i-(1/2))\lambda_i + \rho)$ into $M(-(n-i-(1/2))\lambda_i + \rho)$. By Proposition 6.8, we have $(\mathrm{pr} \circ \varphi)(1 \otimes v^+) = \varphi_{std}(1 \otimes v_h) \neq 0$.

Theorem 6.38. *If $\mathfrak{q}$ is the maximal parabolic subalgebra of type $B_n(i)$ for $3 \leq i \leq n-1$ then the map φ_{Ω_2} induced by the $\Omega_2|_{V(\mu+\epsilon_\gamma)^*}$ system is standard.*

Proof. Observe that, as $M(-(n-i-(1/2))\lambda_i + \rho) \cong \mathcal{U}(\bar{\mathfrak{u}}) \otimes \mathbb{C}_{-(n-i-(1/2))\lambda_i}$, the vector $\varphi(1 \otimes v^+)$ is of the form $\varphi(1 \otimes v^+) = u' \otimes 1_{-(n-i-(1/2))\lambda_i}$ for some $u' \in \mathcal{U}(\bar{\mathfrak{u}})$. Since $\varphi(1 \otimes v^+)$ has weight $-2\varepsilon_i - (n-i-(1/2))\lambda_i$, the element u' has weight $-2\varepsilon_i$. Thus, by Lemma 6.36, we have

$$u' = AX^2_{-\varepsilon_i} + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)} + \sum_{\alpha \in \Delta^+(\mathfrak{l})} u^\alpha X_{-\alpha}$$

for some constants A and B_k, and some elements $u^\alpha \in \mathcal{U}(\bar{\mathfrak{u}})$. As $X_{-\varepsilon_i}$, $X_{-(\varepsilon_i+\varepsilon_k)}$, and $X_{-(\varepsilon_i-\varepsilon_k)}$ are not in $\mathfrak{l}$, $\varphi_{std}(1 \otimes v_h)$ is given by

$$\begin{aligned}
&\varphi_{std}(1 \otimes v_h) = (\mathrm{pr} \circ \varphi)(1 \otimes v^+) \\
&= \mathrm{pr}\Big(\big(AX^2_{-\varepsilon_i} + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)} + \sum_{\alpha \in \Delta^+(\mathfrak{l})} u^\alpha X_{-\alpha} \big) \otimes 1_{-(n-i-(1/2))\lambda_i} \Big) \\
&= \big(AX^2_{-\varepsilon_i} + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)} \big) \otimes 1_{-(n-i-(1/2))\lambda_i}.
\end{aligned}$$

Write $\tilde{u} = AX^2_{-\varepsilon_i} + \sum_{k=i+1}^{n} B_k X_{-(\varepsilon_i+\varepsilon_k)} X_{-(\varepsilon_i-\varepsilon_k)}$. Clearly $\tilde{u}$ satisfies Condition (H). Thus, by Proposition 6.29, there exists a constant c so that $\tilde{u} = cu_h$ with u_h in (6.24). By Proposition 6.8, we have $\tilde{u} \neq 0$; thus $c \neq 0$. Since $\varphi_{\Omega_2}(1 \otimes v_h) = v_h = u_h \otimes 1_{-(n-i-(1/2))\lambda_i}$, we obtain

$$\varphi_{\Omega_2}(1 \otimes v_h) = u_h \otimes 1_{-(n-i-(1/2))\lambda_i} = (1/c)\varphi_{std}(1 \otimes v_h). \qquad ■$$

In Table 5 below we summarize the classification of the maps φ_{Ω_2}.

A. Miscellenious Data

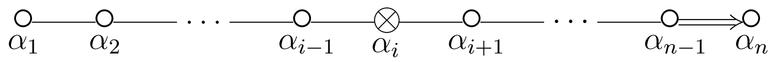
In this appendix we recall from [18] the miscellenious data for the maximal parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{z}(\mathfrak{n})$ of quasi-Heisenberg type shown in (2.8) and (2.9) in Section 2. For the definition of the deleted Dynkin diagram see Subsection 4.1 of [18].

Table 5: The Homomorphism φ_{Ω_2} for the Non-Heisenberg Case

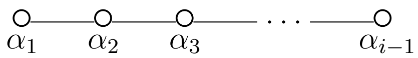
Parabolic subalgebra $\mathfrak{q}$	$\Omega_2\vert_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\vert_{V(\mu+\epsilon_{n\gamma})^*}$
$B_n(i), 3 \leq i \leq n-2$	standard	non-standard
$B_n(n-1)$	standard	?
$B_n(n)$	standard	–
$C_n(i), 2 \leq i \leq n-1$	?	standard
$D_n(i), 3 \leq i \leq n-3$	non-standard	non-standard
$E_6(3)$	non-standard	non-standard
$E_6(5)$	non-standard	non-standard
$E_7(2)$	non-standard	–
$E_7(6)$	non-standard	non-standard
$E_8(1)$	non-standard	–
$F_4(4)$	standard	–

§B$_n(i)$, $3 \leq i \leq n-2$

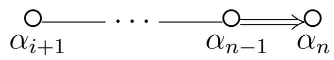
1. The deleted Dynkin diagram:



2. The subgraph for $\mathfrak{l}_\gamma$:



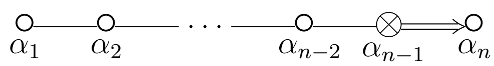
3. The subgraph for $\mathfrak{l}_{n\gamma}$:



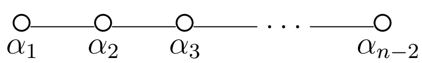
We have $\alpha_\gamma = \alpha_2$. The highest weight μ and the set of roots $\Delta(\mathfrak{g}(1))$ for $\mathfrak{g}(1)$ are $\mu = \varepsilon_1 + \varepsilon_{i+1}$ and $\Delta(\mathfrak{g}(1)) = \{\varepsilon_j \pm \varepsilon_k \mid 1 \leq j \leq i \text{ and } i+1 \leq k \leq n\} \cup \{\varepsilon_j \mid 1 \leq j \leq i\}$. The highest weight γ and the set of roots $\Delta(\mathfrak{z}(\mathfrak{n}))$ for $\mathfrak{z}(\mathfrak{n})$ are $\gamma = \varepsilon_1 + \varepsilon_2$ and $\Delta(\mathfrak{z}(\mathfrak{n})) = \{\varepsilon_j + \varepsilon_k \mid 1 \leq j < k \leq i\}$. The highest root ξ_γ and the set of positive roots $\Delta^+(\mathfrak{l}_\gamma)$ for $\mathfrak{l}_\gamma$ are $\xi_\gamma = \varepsilon_1 - \varepsilon_i$ and $\Delta^+(\mathfrak{l}_\gamma) = \{\varepsilon_j - \varepsilon_k \mid 1 \leq j < k \leq i\}$. The highest root $\xi_{n\gamma}$ and the set of positive roots $\Delta^+(\mathfrak{l}_{n\gamma})$ for $\mathfrak{l}_{n\gamma}$ are $\xi_{n\gamma} = \varepsilon_{i+1} + \varepsilon_{i+2}$ and $\Delta^+(\mathfrak{l}_{n\gamma}) = \{\varepsilon_j \pm \varepsilon_k \mid i+1 \leq j < k \leq n\} \cup \{\varepsilon_j \mid i+1 \leq j \leq n\}$.

§B$_n(n-1)$

1. The deleted Dynkin diagram:



2. The subgraph for $\mathfrak{l}_\gamma$:

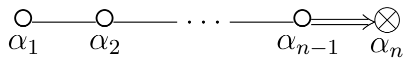


3. The subgraph for $\mathfrak{l}_{n\gamma}$:

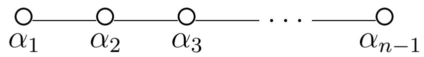
We have $\alpha_\gamma = \alpha_2$. The highest weight μ and the set of weights $\Delta(\mathfrak{g}(1))$ for $\mathfrak{g}(1)$ are $\mu = \varepsilon_1 + \varepsilon_n$ and $\Delta(\mathfrak{g}(1)) = \{\varepsilon_j \pm \varepsilon_n \mid 1 \le j \le n-1\} \cup \{\varepsilon_j \mid 1 \le j \le n-1\}$. The highest weight γ and the set of weights $\mathfrak{g}(\mathfrak{z}(\mathfrak{n}))$ for $\mathfrak{z}(\mathfrak{n}))$ are $\gamma = \varepsilon_1 + \varepsilon_2$ and $\Delta(\mathfrak{z}(\mathfrak{n})) = \{\varepsilon_j + \varepsilon_k \mid 1 \le j < k \le n-1\}$. The highest root ξ_γ and the set of positive roots $\Delta^+(\mathfrak{l}_\gamma)$ for $\mathfrak{l}_\gamma$ are $\xi_\gamma = \varepsilon_1 - \varepsilon_{n-1}$ and $\Delta^+(\mathfrak{l}_\gamma) = \{\varepsilon_j - \varepsilon_k \mid 1 \le j < k \le n-1\}$. The highest root $\xi_{n\gamma}$ and the set of positive roots $\Delta^+(\mathfrak{l}_{n\gamma})$ for $\mathfrak{l}_{n\gamma}$ are $\xi_{n\gamma} = \varepsilon_n$ and $\Delta^+(\mathfrak{l}_{n\gamma}) = \{\varepsilon_n\}$.

§$\mathrm{B}_n(n)$

1. The deleted Dynkin diagram:



2. The subgraph for $\mathfrak{l}_\gamma$:

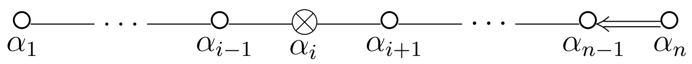


3. No subgraph for $\mathfrak{l}_{n\gamma}$ ($\mathfrak{l}_{n\gamma} = \{0\}$)

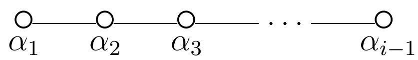
We have $\alpha_\gamma = \alpha_2$. The highest weight μ and the set of weights $\Delta(\mathfrak{g}(1))$ are $\mu = \varepsilon_1$ and $\Delta(\mathfrak{g}(1)) = \{\varepsilon_j \mid 1 \le j \le n\}$. The highest weight γ and the set of weights $\Delta(\mathfrak{z}(\mathfrak{n}))$ for $\mathfrak{z}(\mathfrak{n})$ are $\gamma = \varepsilon_1 + \varepsilon_2$ and $\Delta(\mathfrak{z}(\mathfrak{n})) = \{\varepsilon_j + \varepsilon_k \mid 1 \le j < k \le n\}$. The highest root ξ_γ and the set of positive roots for $\mathfrak{l}_\gamma$ are $\xi_\gamma = \varepsilon_1 - \varepsilon_n$ and $\Delta^+(\mathfrak{l}_\gamma) = \{\varepsilon_j - \varepsilon_k \mid 1 \le j < k \le n\}$.

§$\mathrm{C}_n(i)$, $2 \le i \le n-1$

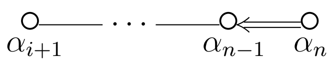
1. The deleted Dynkin diagram:



2. The subgraph for $\mathfrak{l}_\gamma$:



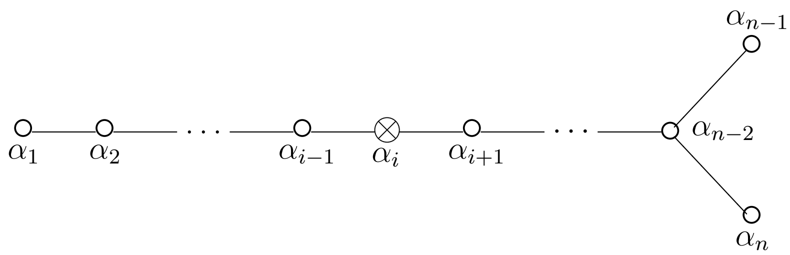
3. The subgraph for $\mathfrak{l}_{n\gamma}$:



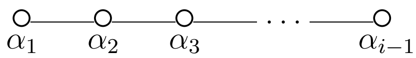
We have $\alpha_\gamma = \alpha_1$. The highest weight μ and the set of weights $\Delta(\mathfrak{g}(1))$ for $\mathfrak{g}(1)$ are $\mu = \varepsilon_1 + \varepsilon_{i+1}$ and $\Delta(\mathfrak{g}(1)) = \{\varepsilon_j \pm \varepsilon_k \mid 1 \le j \le i \text{ and } i+1 \le k \le n\}$. The highest weight γ and the set of weights $\Delta(\mathfrak{z}(\mathfrak{n}))$ for $\mathfrak{z}(\mathfrak{n})$ are $\gamma = 2\varepsilon_1$ $\Delta(\mathfrak{z}(\mathfrak{n})) = \{\varepsilon_j + \varepsilon_k \mid 1 \le j < k \le i\} \cup \{2\varepsilon_j \mid 1 \le j \le i\}$. The highest root ξ_γ and the set of positive roots $\Delta^+(\mathfrak{l}_\gamma)$ for $\mathfrak{l}_\gamma$ are $\xi_\gamma = \varepsilon_1 - \varepsilon_i$ and $\Delta^+(\mathfrak{l}_\gamma) = \{\varepsilon_j - \varepsilon_k \mid 1 \le j < k \le i\}$ The highest root $\xi_{n\gamma}$ and the set of positive roots $\Delta(\mathfrak{l}_{n\gamma})$ for $\mathfrak{l}_{n\gamma}$ are $\xi_{n\gamma} = 2\varepsilon_{i+1}$ and $\Delta^+(\mathfrak{l}_{n\gamma}) = \{\varepsilon_j \pm \varepsilon_k \mid i+1 \le j < k \le n\} \cup \{2\varepsilon_j \mid i+1 \le j \le n\}$.

§$\mathrm{D}_n(i)$, $3 \le i \le n-3$

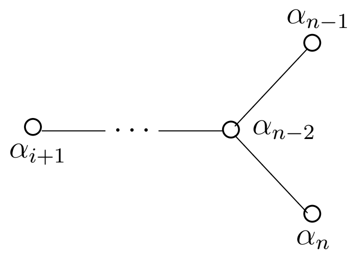
1. The deleted Dynkin diagram:



2. The subgraph for $\mathfrak{l}_\gamma$:



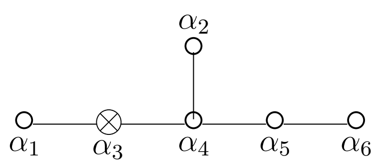
3. The subgraph for $\mathfrak{l}_{n\gamma}$:



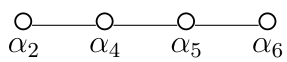
We have $\alpha_\gamma = \alpha_2$. The highest weight μ and the set of weights $\Delta(\mathfrak{g}(1))$ for $\mathfrak{g}(1)$ are $\mu = \varepsilon_1 + \varepsilon_{i+1}$ and $\Delta(\mathfrak{g}(1)) = \{\varepsilon_j \pm \varepsilon_k \mid 1 \le j \le i \text{ and } i+1 \le k \le n\}$. The highest weight γ and the set of weights $\Delta(\mathfrak{z}(\mathfrak{n}))$ for $\mathfrak{z}(\mathfrak{n}))$ are $\gamma = \varepsilon_1 + \varepsilon_2$ and $\Delta(\mathfrak{z}(\mathfrak{n})) = \{\varepsilon_j + \varepsilon_k \mid 1 \le j < k \le i\}$. The highest root ξ_γ and the set of positive roots $\Delta^+(\mathfrak{l}_\gamma)$ for $\mathfrak{l}_\gamma$ are $\xi_\gamma = \varepsilon_1 - \varepsilon_i$ and $\Delta^+(\mathfrak{l}_\gamma) = \{\varepsilon_j - \varepsilon_k \mid 1 \le j < k \le i\}$. The highest root $\xi_{n\gamma}$ and the set of positive roots $\Delta^+(\mathfrak{l}_{n\gamma})$ for $\mathfrak{l}_{n\gamma}$ are $\xi_{n\gamma} = \varepsilon_{i+1} + \varepsilon_{i+2}$ $\Delta^+(\mathfrak{l}_{n\gamma}) = \{\varepsilon_j \pm \varepsilon_k \mid i+1 \le j < k \le n\}$.

§$\mathrm{E}_6(3)$

1. The deleted Dynkin diagram:



2. The subgraph for $\mathfrak{l}_\gamma$:

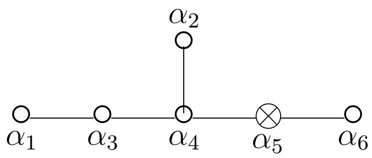


3. The subgraph for $\mathfrak{l}_{n\gamma}$:

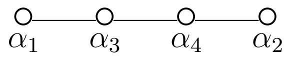
We have $\alpha_\gamma = \alpha_2$. The highest weight μ for $\mathfrak{g}(1)$ is $\mu = \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$. The highest weight γ for $\mathfrak{z}(\mathfrak{n})$ is $\gamma = \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$. The highest root ξ_γ for $\mathfrak{l}_\gamma$ is $\xi_\gamma = \alpha_2+\alpha_4+\alpha_5+\alpha_6$. The highest root $\xi_{n\gamma}$ for $\mathfrak{l}_{n\gamma}$ is $\xi_{n\gamma} = \alpha_1$.

§E$_6$(5)

1. The deleted Dynkin diagram:



2. The subgraph for $\mathfrak{l}_\gamma$:

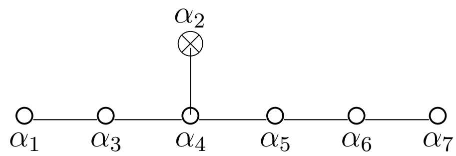


3. The subgraph for $\mathfrak{l}_{n\gamma}$:

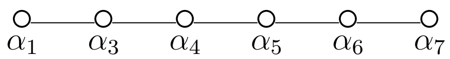
We have $\alpha_\gamma = \alpha_2$. The highest weight μ for $\mathfrak{g}(1)$ is $\mu = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6$. The highest weight γ for $\mathfrak{z}(\mathfrak{n})$ is $\gamma = \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$. The highest weight ξ_γ for $\mathfrak{l}_\gamma$ is $\xi_\gamma = \alpha_1+\alpha_2+\alpha_3+\alpha_4$. The highest weight $\xi_{n\gamma}$ for $\mathfrak{l}_{n\gamma}$ is $\xi_{n\gamma} = \alpha_6$.

§E$_7$(2)

1. The deleted Dynkin diagram:



2. The subgraph for $\mathfrak{l}_\gamma$:

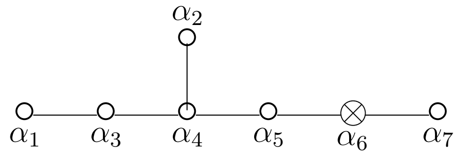


3. No subgraph for $\mathfrak{l}_{n\gamma}$ ($\mathfrak{l}_{n\gamma} = \{0\}$)

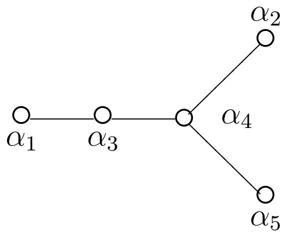
We have $\alpha_\gamma = \alpha_1$. The highest weight μ for $\mathfrak{g}(1)$ is $\mu = \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$. The highest weight γ for $\mathfrak{z}(\mathfrak{n})$ is $\gamma = 2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$. The highest root ξ_γ for $\mathfrak{l}_\gamma$ is $\xi_\gamma = \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7$.

§E$_7$(6)

1. The deleted Dynkin diagram:



2. The subgraph for $\mathfrak{l}_\gamma$:

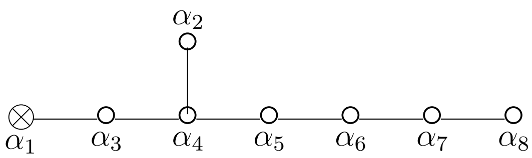


3. The subgraph for $\mathfrak{l}_{n\gamma}$:

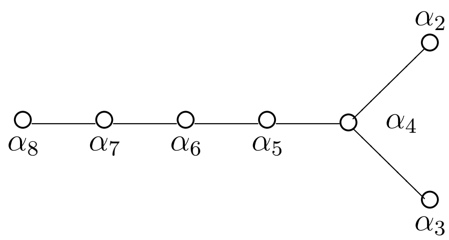
We have $\alpha_\gamma = \alpha_1$. The highest weight μ for $\mathfrak{g}(1)$ is $\mu = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$. The highest weight γ for $\mathfrak{z}(\mathfrak{n})$ is $\gamma = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. The highest root ξ_γ for $\mathfrak{l}_\gamma$ is $\xi_\gamma = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$. The highest root $\xi_{n\gamma}$ for $\mathfrak{l}_{n\gamma}$ is $\xi_{n\gamma} = \alpha_7$.

§E$_8$(1)

1. The deleted Dynkin diagram:



2. The subgraph for $\mathfrak{l}_\gamma$:

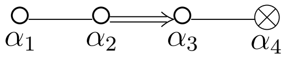


3. No subgraph for $\mathfrak{l}_{n\gamma}$ ($\mathfrak{l}_{n\gamma} = \{0\}$)

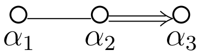
We have $\alpha_\gamma = \alpha_8$. The highest weight μ for $\mathfrak{g}(1)$ is $\mu = \alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$. The highest weight γ for $\mathfrak{z}(\mathfrak{n})$ is $\gamma = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$. The highest root ξ_γ for $\mathfrak{l}_\gamma$ is $\xi_\gamma = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$.

§F$_4$(4)

1. The deleted Dynkin diagram:



2. The subgraph for $\mathfrak{l}_\gamma$:



3. No subgraph for $\mathfrak{l}_{n\gamma}$ ($\mathfrak{l}_{n\gamma} = \{0\}$)

We have $\alpha_\gamma = \alpha_1$. The highest weight μ for $\mathfrak{g}(1)$ is $\mu = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$. The highest weight γ for $\mathfrak{z}(\mathfrak{n})$ is $\gamma = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. The highest root for ξ_γ for $\mathfrak{l}_\gamma$ is $\xi_\gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3$.

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