Journal of Lie Theory
Volume **23** (2013) 885–897

The Group Structure for Jet Bundles over Lie Groups*

Cornelia Vizman

Communicated by K.-H. Neeb

Abstract. The jet bundle J^kG of k-jets of curves in a Lie group G has a natural Lie group structure. We present an explicit formula for the group multiplication in the right trivialization and for the group 2-cocycle describing the abelian Lie group extension $\mathfrak{g} \to J^kG \to J^{k-1}G$.
Mathematics Subject Classification 2010: 58A20, 20K35, 05A18.
Key Words and Phrases: Jet bundle, group cocycle, ordered partition, Leibniz algebra, near-ring.

1. Introduction

The jet bundles $J^kM \to M$ and higher order tangent bundles $T^kM \to M$ over a smooth manifold M are examples of natural operations in the sense of [4]. When applied to a Lie group G, these constructions provide new Lie groups J^kG and T^kG. In this article we provide a formula for the group structure on J^kG in terms of the Lie bracket and we compare it with a similar formula for the group structure on T^kG [1]. The tangent functor possesses a natural section for $T^kG \to T^{k-1}G$, so T^kG is a semidirect product of $T^{k-1}G$, while the jet functor provides an abelian extension $J^kG \to J^{k-1}G$.

The manifold of k-jets of smooth curves in a Lie group G is a fiber bundle $J^kG \to G$, called the k-th order jet bundle [4]. Denoting by j^kc the k-jet at 0 of the curve c, the multiplication $(j^kc)(j^kb) := j^k(cb)$ in J^kG doesn't depend on the representing curves and defines a Lie group structure on J^kG. Its Lie algebra $J^k\mathfrak{g}$ is isomorphic to $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{R}[X]/(X^{k+1})$, the scalar extension of the Lie algebra $\mathfrak{g}$ by the truncated polynomial ring $\mathbb{R}[X]/(X^{k+1})$. As a vector space it is isomorphic to $(k+1)$ copies of $\mathfrak{g}$.

The main result of the paper is Theorem 2.5 which gives the expression of the group multiplication on the jet bundle J^kG in the right trivialization [3]

$$j^kc \in J^kG \mapsto \big(c(0), (\delta^r c)(0), (\delta^r c)'(0), \dots, (\delta^r c)^{(k-1)}(0)\big) \in G \times \mathfrak{g}^k,$$

* This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS UEFISCDI, project number PN-II-ID-PCE-2011-3-0921.

ISSN 0949–5932 / \$2.50

where c is a smooth curve in G with right logarithmic derivative $\delta^r c = c'c^{-1}$. One gets the multiplication law

$$(g,(x_n)_{1\le n\le k})\,(h,(y_n)_{1\le n\le k}) = \left(gh, \left(x_n + \sum_{\lambda\in\mathcal{P}_n} \mathrm{ad}_{x_{i_{\ell-1}}}\dots\mathrm{ad}_{x_{i_1}}\mathrm{Ad}_g\, y_{i_\ell}\right)_{1\le n\le k}\right)$$

and the inverse

$$(g,(x_n)_{1\le n\le k})^{-1} = \left(g^{-1}, \left(\sum_{\lambda\in\mathcal{P}_n}(-1)^\ell\,\mathrm{Ad}_{g^{-1}}\mathrm{ad}_{x_{i_1}}\dots\mathrm{ad}_{x_{i_{\ell-1}}} x_{i_\ell}\right)_{1\le n\le k}\right),$$

both involving sums over all anti-lexicographically ordered partitions $\lambda_1\cup\dots\cup\lambda_\ell = \{1,\dots,n\}$ (disjoint union) with i_r the cardinality of the subset λ_r for $r=1,\dots,\ell$. The number of anti-lexicographically ordered partitions with fixed cardinalities $(i_1,\dots,i_\ell)$ is

$$N_{(i_1,\dots,i_\ell)} = \binom{i_1+\dots+i_\ell-1}{i_\ell-1}\binom{i_1+\dots i_{\ell-1}-1}{i_{\ell-1}-1}\dots\binom{i_1+i_2-1}{i_2-1}, \tag{1}$$

hence the n-th component in the above formulas can be written respectively as

$$z_n = x_n + \sum_{i_1+\dots+i_\ell=n} N_{(i_1,\dots,i_\ell)}\,\mathrm{ad}_{x_{i_{\ell-1}}}\dots\mathrm{ad}_{x_{i_1}}\mathrm{Ad}_g\, y_{i_\ell}$$
$$w_n = \sum_{i_1+\dots+i_\ell=n}(-1)^\ell N_{(i_1,\dots,i_\ell)}\,\mathrm{Ad}_{g^{-1}}\mathrm{ad}_{x_{i_1}}\dots\mathrm{ad}_{x_{i_{\ell-1}}} x_{i_\ell}.$$

The formulas involving partitions remind of the expressions for the multiplication and the inverse for the k-th order tangent group T^kG in the right trivialization [1]. The reason is that the k-th order jet bundle J^kG can be identified with the subgroup $(T^kG)^{S_k}$ of T^kG, the fixed point set under the natural action of the permutation group S_k (see Theorem 3.3).

All these formulas have no denominators, so one can work with geometric objects over general fields [1]. The identity fibers $J_k(\mathfrak{g})$ and $G_k(\mathfrak{g})$ of the fiber bundles J^kG and T^kG are polynomial groups depending only on the Lie bracket on $\mathfrak{g}$. The dimension of $J_k(\mathfrak{g})$ is $k\dim\mathfrak{g}$, while the dimension of $G_k(\mathfrak{g})$ is $(2^k-1)\dim\mathfrak{g}$. Both assignments give functorial maps from Lie algebras to polynomial groups, even in the more general case when $\mathfrak{g}$ is a Leibniz algebra [2]. Moreover, the group multiplication is in both cases affine in the second argument, so we actually get affine near-ring structures, slight generalizations of near-ring structures (where the group multiplication is linear in one of the arguments) [6].

In the right trivialization, the first order jet bundle J^1G coincides with the tangent bundle TG, so it is the semidirect product $G\ltimes\mathfrak{g}$ for the adjoint action. The second order jet bundle J^2G is an abelian extension of $G\ltimes\mathfrak{g}$ by $\mathfrak{g}$ with characteristic group cocycle

$$c((g,x),(h,y)) = \mathrm{ad}_x\mathrm{Ad}_g\, y,$$

and non-trivial Lie algebra cocycle

$$\sigma((\xi,x),(\eta,y)) = 2[x,y].$$

This is a special case of Proposition 4.1, where the characteristic cocycles for abelian extensions $\mathfrak{g} \to J^kG \to J^{k-1}G$ associated to higher order jet bundles are computed.

2. Group multiplication in jet bundles

This section contains the main theorem of this paper, namely the expression of the multiplication on the k-th order jet bundle J^kG in right trivialization.

Theorem 2.1. [3] *The right trivialization of the jet bundle $J^kG \to G$:*

$$j^kc \in J^kG \mapsto \big(c(0), (\delta^r c)(0), (\delta^r c)'(0), \dots, (\delta^r c)^{(k-1)}(0)\big) \in G \times \mathfrak{g}^k, \tag{2}$$

where c is a smooth curve in G and j^kc its k-jet at 0, is an isomorphism of bundles, whose inverse assigns to $(g, x_1, x_2, \dots, x_k) \in G \times \mathfrak{g}^k$ the k-jet of the curve c in G, uniquely defined by $c(0) = g$ and $\delta^r c(t) = x'(t)$ for the Lie algebra curve

$$x(t) = tx_1 + \frac{t^2}{2!}x_2 + \dots + \frac{t^k}{k!}x_k.$$

Remark 2.2. When $k > 2$, the k-jet j^kc with right trivialization $(g, x_1, \dots, x_k)$ doesn't coincide in general with the k-jet j^kb of the curve $b(t) = (\exp x(t))g$, where x is the Lie algebra curve defined above. Indeed, the right logarithmic derivative of b is

$$\delta^r b(t) = \delta^r \exp x(t) = \sum_{j=0}^{\infty} \frac{1}{(j+1)!} \operatorname{ad}_{x(t)}^j x'(t),$$

so $\delta^r b(0) = x_1$, $(\delta^r b)'(0) = x_2$, but the higher order derivatives get additional terms $(\delta^r b)''(0) = x_3 + \frac{1}{2}\operatorname{ad}_{x_1} x_2$, $(\delta^r b)'''(0) = x_4 + \operatorname{ad}_{x_1} x_3 + \frac{1}{2}\operatorname{ad}_{x_1}^2 x_2$.

The multiplication $(j^kc)(j^kb) := j^k(cb)$ doesn't depend on the representing curves and defines a Lie group structure on the k-th order jet bundle $J^kG \to G$. Denoting the identity fiber by $J_k(\mathfrak{g})$, the Lie group J^kG is a semidirect product of G and $J_k(\mathfrak{g})$.

Let us introduce some notation needed for expressing the group multiplication on J^kG in the right trivialization.

Let $\mathcal{P}_n$ denote the set of all partitions $\lambda = \lambda_1|\dots|\lambda_\ell$ of $\{1,\dots,n\}$, which means that $\{1,2,\dots,n\} = \lambda_1 \cup \dots \cup \lambda_\ell$ disjoint union of sets. We call $\ell(\lambda) = \ell$ the length of the partition. We order each partition anti-lexicographically: the ordering is done from right to left, always choosing the subset that contains the highest available number. For $n = 3$ there are 5 such anti-lexicographically ordered partitions $\mathcal{P}_3 = \{1|2|3, 12|3, 2|13, 1|23, 123\}$.

The cardinality of λ_r is denoted by $i_r = |\lambda_r|$. Of course $n = i_1 + \dots + i_\ell$, so each element $\lambda \in \mathcal{P}_n$ determines an ordered decomposition of the number n.

There are elements in $\mathcal{P}_n$ that determine the same ordered decomposition of n, e.g. $2|13$ and $1|23$ both determine the decomposition $3 = 1 + 2$.

Lemma 2.3. *The number of anti-lexicographically ordered partitions $\lambda = \lambda_1 | \dots | \lambda_\ell$ of $\{1, \dots, n\}$ with fixed cardinalities $i_1, \dots, i_\ell$ is $N_{(i_1,\dots,i_\ell)}$ from* (1).

Proof. Because of the anti-lexicographic ordering, the subset λ_ℓ must contain the element n, while the other $i_\ell - 1$ elements can be chosen in $\binom{n-1}{i_\ell - 1}$ ways. The subset $\lambda_{\ell-1}$ must contain the biggest of the remaining elements, so we have $\binom{n-i_\ell-1}{i_{\ell-1}-1}$ choices. Similarly, we get $\binom{n-i_\ell-\dots-i_{r+1}-1}{i_r-1}$ choices for the elements of λ_r, $r = 1, \dots, \ell$. Their product gives the expression of $N_{(i_1,\dots,i_\ell)}$ from (1) because of the identity $i_1 + \dots + i_\ell = n$. ∎

Remark 2.4. ¿From each anti-lexicographically ordered partition $\lambda = \lambda_1 | \dots | \lambda_\ell$ in $\mathcal{P}_n$ one derives new anti-lexicographically ordered partitions

$$\lambda^{[m]} \in \mathcal{P}_{n+1}, \quad m = 0, \dots, \ell$$

by the following procedure: first we increase by 1 each element of λ, thus getting a partition of the set $\{2, \dots, n+1\}$, which we denote λ^+, then we adjoin the element 1 to the partition λ^+. There are $m + 1$ different ways to do this, namely by placing the element 1 in the m-th subset of λ^+ one obtains the partition $\lambda^{[m]}$ of length ℓ, while by placing the element 1 in a separate subset (in front), one obtains the partition $\lambda^{[0]}$ of length $\ell + 1$. In all cases, the new partition is again anti-lexicographically ordered. For instance, the three derived partitions of $\lambda = 2|13$ are $\lambda^{[0]} = 1|3|24$, $\lambda^{[1]} = 3|124$, and $\lambda^{[2]} = 13|24$.

Each anti-lexicographically ordered partition in $\mathcal{P}_{n+1}$ is a derived partition for a unique anti-lexicographically ordered partition in $\mathcal{P}_n$, namely the one obtained by removing the element 1 and lowering each remaining element by 1, e.g. $2|13 = \lambda^{[2]} \in \mathcal{P}_3$ for $\lambda = 1|2 \in \mathcal{P}_2$.

Theorem 2.5. *The multiplication law in J^kG, identified with $G \times \mathfrak{g}^k$ in the right trivialization* (2), *is $(g, x_1, \dots, x_k)(h, y_1, \dots, y_k) = (gh, z_1, \dots, z_k)$, where each $z_n \in \mathfrak{g}$ is given by*

$$\begin{aligned} z_n &= x_n + \sum_{\lambda \in \mathcal{P}_n} \operatorname{ad}_{x_{i_{\ell-1}}} \dots \operatorname{ad}_{x_{i_1}} \operatorname{Ad}_g y_{i_\ell} \\ &= x_n + \sum_{\ell=1}^{n} \sum_{i_1+\dots+i_\ell=n} N_{(i_1,\dots,i_\ell)} \operatorname{ad}_{x_{i_{\ell-1}}} \dots \operatorname{ad}_{x_{i_1}} \operatorname{Ad}_g y_{i_\ell}, \end{aligned} \tag{3}$$

and the inverse in J^kG is $(g, x_1, \dots, x_k)^{-1} = (g^{-1}, w_1, \dots, w_k)$, where each $w_n \in \mathfrak{g}$ is given by

$$\begin{aligned} w_n &= \sum_{\lambda \in \mathcal{P}_n} (-1)^\ell \operatorname{Ad}_{g^{-1}} \operatorname{ad}_{x_{i_1}} \dots \operatorname{ad}_{x_{i_{\ell-1}}} x_{i_\ell} \\ &= \sum_{\ell=1}^{n} \sum_{i_1+\dots+i_\ell=n} (-1)^\ell N_{(i_1,\dots,i_\ell)} \operatorname{Ad}_{g^{-1}} \operatorname{ad}_{x_{i_1}} \dots \operatorname{ad}_{x_{i_{\ell-1}}} x_{i_\ell}. \end{aligned} \tag{4}$$

$\mathcal{P}_n$ denotes the set of anti-lexicographically ordered partitions $\lambda = \lambda_1|\dots|\lambda_\ell$ of $\{1,\dots,n\}$, i_r denotes the cardinality of λ_r, and $N_{(i_1,\dots,i_\ell)}$ is given by (1).

Proof. In order to prove the multiplication formula (3) we consider the curves x and y in the Lie algebra $\mathfrak{g}$,

$$x(t) = tx_1 + \frac{t^2}{2!}x_2 + \dots + \frac{t^k}{k!}x_k, \text{ resp. } y(t) = ty_1 + \frac{t^2}{2!}y_2 + \dots + \frac{t^k}{k!}y_k,$$

and the curves c and b in the Lie group G, uniquely defined by $\delta^r c = x'$ and $c(0) = g$, resp. $\delta^r b = y'$ and $b(0) = h$. In the right trivialization (2) we identify $j^k c = (g, x'(0), \dots, x^{(k)}(0)) = (g, x_1, \dots, x_k)$ resp. $j^k b = (h, y_1, \dots, y_k)$, so the multiplication is

$$(g, x_1, \dots, x_k)(h, y_1, \dots, y_k) = j^k(cb) = (gh, z'(0), z''(0), \dots, z^{(k)}(0))$$

for $z' = \delta^r(cb) = \delta^r c + \mathrm{Ad}_c \delta^r b = x' + \mathrm{Ad}_c y'$.

It remains to prove that $z_n = z^{(n)}(0)$. To each anti-lexicographically ordered partition $\lambda \in \mathcal{P}_n$ with $|\lambda_r| = i_r$ we assign a curve F_λ in $\mathfrak{g}$ defined by

$$F_\lambda(t) := \mathrm{ad}_{x^{(i_{\ell-1})}(t)} \dots \mathrm{ad}_{x^{(i_1)}(t)} \mathrm{Ad}_{c(t)} y^{(i_\ell)}(t).$$

Its derivative can be written as a sum of terms of the same type:

$$\begin{aligned} F'_\lambda &= \left(\mathrm{ad}_{x^{(i_{\ell-1})}} \dots \mathrm{ad}_{x^{(i_1)}} \mathrm{Ad}_c y^{(i_\ell)}\right)' = \mathrm{ad}_{x^{(i_{\ell-1})}} \dots \mathrm{ad}_{x^{(i_1)}} \mathrm{ad}_{x'} \mathrm{Ad}_c y^{(i_\ell)} \\ &+ \sum_{m=1}^{\ell-1} \mathrm{ad}_{x^{(i_{\ell-1})}} \dots \mathrm{ad}_{x^{(i_m+1)}} \dots \mathrm{ad}_{x^{(i_1)}} \mathrm{Ad}_c y^{(i_\ell)} + \mathrm{ad}_{x^{(i_{\ell-1})}} \dots \mathrm{ad}_{x^{(i_1)}} \mathrm{Ad}_c y^{(i_\ell+1)}, \end{aligned} \tag{5}$$

using at step two the fact that

$$(\mathrm{Ad}_c y^{(i)})' = \mathrm{ad}_{\delta^r c} \mathrm{Ad}_c y^{(i)} + \mathrm{Ad}_c y^{(i+1)} = \mathrm{ad}_{x'} \mathrm{Ad}_c y^{(i)} + \mathrm{Ad}_c y^{(i+1)}.$$

Since the ordered decomposition of the number n induced by the anti-lexicographic ordered partition $\lambda \in \mathcal{P}_n$ is $n = i_1 + \dots + i_\ell$, the ordered decompositions of $n+1$ induced by the derived partitions $\lambda^{[0]}, \dots, \lambda^{[m]}, \dots, \lambda^{[\ell]} \in \mathcal{P}_{n+1}$ are $n+1 = 1 + i_1 + \dots + i_\ell$, $\dots, n+1 = i_1 + \dots + (i_m + 1) + \dots + i_\ell$, $\dots$, $n+1 = i_1 + \dots + (i_\ell + 1)$. This ensures that each term in the expression (5) of F'_λ corresponds to one of the derived partitions of λ, so

$$F'_\lambda(t) = \sum_{m=0}^{\ell} F_{\lambda^{[m]}}(t).$$

We are now ready to compute the higher order derivatives of $z' = x' + \mathrm{Ad}_c y'$. We will prove by induction that:

$$z^{(n)}(t) = x^{(n)}(t) + \sum_{\lambda \in \mathcal{P}_n} F_\lambda(t). \tag{6}$$

For $n=2$ the identity holds because $z''=(x'+\mathrm{Ad}_c y')'=x''+\mathrm{ad}_{x'}\mathrm{Ad}_c y'+\mathrm{Ad}_c y''$ and $\mathcal{P}_2$ consists of only two partitions $1|2$ and 12. Assuming that (6) holds for $n-1$, we compute $z^{(n)}$:

$$z^{(n)}=x^{(n)}+\sum_{\lambda\in\mathcal{P}_{n-1}}F'_\lambda=x^{(n)}+\sum_{\lambda\in\mathcal{P}_{n-1}}\sum_{m=0}^{|\lambda|}F_{\lambda[m]}=x^{(n)}+\sum_{\rho\in\mathcal{P}_n}F_\rho.$$

In the last step we use Remark 2.4: each element $\rho\in\mathcal{P}_n$ is obtained from a unique element $\lambda\in\mathcal{P}_{n-1}$ through derivation. This ends the proof of (6) by induction, and the first part of (3) follows by evaluation at 0. For the second part of (3) we apply Lemma 2.3.

To show the inversion formula (4), we consider again a Lie algebra curve $x(t)=tx_1+\frac{t^2}{2!}x_2+\cdots+\frac{t^k}{k!}x_k$, and the Lie group curve c uniquely defined by $\delta^r c=x'$ and $c(0)=g$. In the right trivialization (2) we have $j^kc=(g,x_1,\dots,x_k)$, so

$$(g,x_1,\dots,x_k)^{-1}=j^k(c^{-1})=(g^{-1},w'(0),w''(0),\dots,w^{(k)}(0))$$

for $w'=\delta^r(c^{-1})=-\mathrm{Ad}_{c^{-1}}\delta^r c=-\mathrm{Ad}_{c^{-1}}x'$. We show by induction that

$$w^{(n)}=\sum_{\lambda\in\mathcal{P}_n}(-1)^\ell\,\mathrm{Ad}_{c^{-1}}\mathrm{ad}_{x^{(i_1)}}\dots\mathrm{ad}_{x^{(i_{\ell-1})}}x^{(i_\ell)}. \tag{7}$$

We denote by E_λ the Lie algebra curve $(-1)^\ell\,\mathrm{Ad}_{c^{-1}}\mathrm{ad}_{x^{(i_1)}}\dots\mathrm{ad}_{x^{(i_{\ell-1})}}x^{(i_\ell)}$. For $n=2$ we compute $w''=-(\mathrm{Ad}_{c^{-1}}x')'=-\mathrm{Ad}_{c^{-1}}x''+\mathrm{Ad}_{c^{-1}}\mathrm{ad}_{x'}x'=E_{12}+E_{1|2}$. Assuming that (7) holds for $n-1$, we compute:

$$w^{(n)}=\sum_{\lambda\in\mathcal{P}_{n-1}}E'_\lambda=\sum_{\lambda\in\mathcal{P}_{n-1}}\sum_{m=0}^{|\lambda|}E_{\lambda[m]}=\sum_{\rho\in\mathcal{P}_n}E_\rho.$$

where we use at step two the identity

$$\begin{aligned}(E_\lambda)'&=\big((-1)^\ell\,\mathrm{Ad}_{c^{-1}}\mathrm{ad}_{x^{(i_1)}}\dots\mathrm{ad}_{x^{(i_{\ell-1})}}x^{(i_\ell)}\big)'\\&=(-1)^{\ell+1}\mathrm{Ad}_{c^{-1}}\mathrm{ad}_{x'}\mathrm{ad}_{x^{(i_1)}}\dots\mathrm{ad}_{x^{(i_{\ell-1})}}x^{(i_\ell)}\\&+\sum_{m=1}^{\ell}(-1)^\ell\,\mathrm{Ad}_{c^{-1}}\mathrm{ad}_{x^{(i_1)}}\dots\mathrm{ad}_{x^{(i_m+1)}}\dots\mathrm{ad}_{x^{(i_{\ell-1})}}x^{(i_\ell)}=\sum_{m=0}^{\ell}E_{\lambda[m]}.\end{aligned}$$

This ends the proof of (7) by induction.

Evaluation at 0 gives us the first part of (4), while the second part of (4) follows immediately with Lemma 2.3. ∎

Example 2.6. For convenience of the reader we expand the multiplication in the identity fiber $J_4(\mathfrak{g})$ of J^4G:

$$\begin{aligned}(x_1,x_2,x_3,x_4)(y_1,y_2,y_3,y_4)=(&x_1+y_1,x_2+y_2+\mathrm{ad}_{x_1}y_1,\\&x_3+y_3+2\,\mathrm{ad}_{x_1}y_2+\mathrm{ad}_{x_2}y_1+\mathrm{ad}_{x_1}^2y_1,\\&x_4+y_4+3\,\mathrm{ad}_{x_1}y_3+3\,\mathrm{ad}_{x_2}y_2+3\,\mathrm{ad}_{x_1}^2y_2\\&+\mathrm{ad}_{x_3}y_1+2\,\mathrm{ad}_{x_2}\mathrm{ad}_{x_1}y_1+\mathrm{ad}_{x_1}\mathrm{ad}_{x_2}y_1+\mathrm{ad}_{x_1}^3y_1).\end{aligned}$$

Remark 2.7. The counterpart of (3) and (4) in the left trivialization of J^kG (the bijection (2) with the right logarithmic derivative $\delta^r c$ replaced by the left logarithmic derivative $\delta^l c = c^{-1}c'$) are

$$z_n = y_n + \sum_{i_1+\dots+i_\ell=n} (-1)^{\ell-1} N_{(i_1,\dots,i_\ell)} \operatorname{ad}_{y_{i_{\ell-1}}} \dots \operatorname{ad}_{y_{i_1}} \operatorname{Ad}_{h^{-1}} x_{i_\ell},$$

and

$$w_n = -\sum_{i_1+\dots+i_\ell=n} N_{(i_1,\dots,i_\ell)} \operatorname{Ad}_g \operatorname{ad}_{x_{i_1}} \dots \operatorname{ad}_{x_{i_{\ell-1}}} x_{i_\ell}.$$

A *pure element* of J^kG is a k-jet whose expression (g, x_i) in the right trivialization contains a single non-zero Lie algebra element. The inverse of the pure element (e, x_i) is $(e, -x_i)$. The following identity concerning multiplication of pure elements was already obtained in [1]

$$(e, x_1)(e, y_1) = (e, x_1 + y_1, \operatorname{ad}_{x_1} y_1, \operatorname{ad}_{x_1}^2 y_1, \dots, \operatorname{ad}_{x_1}^{k-1} y_1).$$

It is not hard to see that the multiplication with a pure element (e, y_i), where $i > 1$, gives

$$(e, x_1)(e, y_i) = \left(e, x_1, y_i, \binom{i}{1} \operatorname{ad}_{x_1} y_i, \binom{i+1}{2} \operatorname{ad}_{x_1}^2 y_i, \dots, \binom{k-1}{k-i} \operatorname{ad}_{x_1}^{k-i} y_i\right).$$

The multiplication of two arbitrary pure elements is the content of the following corollary.

Corollary 2.8. *In the right trivialized jet bundle J^kG, the product of two pure elements (e, x_i) and (e, y_j) with $1 \le i < j \le k$ is*

$$(e, x_i)(e, y_j) = (e, x_i, y_j, \frac{(i+j-1)!}{i!(j-1)!} \operatorname{ad}_{x_i} y_j, \dots, \frac{(ni+j-1)!}{n!(i!)^n(j-1)!} \operatorname{ad}_{x_i}^n y_j),$$

where n is the biggest natural number that satisfies $ni + j \le k$. The components known to be zero were not written. Similar formulas hold for $i = j$ and $i > j$.

Proof. It is enough to compute

$$N_{(i,\dots,i,j)} = \binom{ni+j-1}{j-1}\binom{ni-1}{i-1}\dots\binom{2i-1}{i-1}$$
$$= \frac{(i+1)\dots(2i-1)(2i+1)\dots(ni-1)(ni+1)\dots(ni+j-1)}{((i-1)!)^{n-1}(j-1)!} = \frac{(ni+j-1)!}{n!(i!)^n(j-1)!}$$

and to apply Theorem 2.5. ∎

3. Group multiplication in T^kG

In this section we present the multiplication on the trivialized k-th order tangent group T^kG [1] and we show that the k-th order jet group J^kG is isomorphic to the subgroup of fixed points of T^kG under the obvious action of the permutation group S_k.

The structure of higher order tangent groups was investigated in [1] Section 24. In the right trivialization we have a tower of semidirect products:

$$T^{k-1}\mathfrak{g} \to T^kG = T(T^{k-1}G) \to T^{k-1}G,$$

where $T^{k-1}\mathfrak{g}$ denotes the Lie algebra of $T^{k-1}G$. Using infinitesimal units $\varepsilon_1, \dots, \varepsilon_k$ to keep track of each extension in the tower, i.e. $T^kG = T^{k-1}G \times \varepsilon_k T^{k-1}\mathfrak{g}$, we get the right trivialization of the k-th order tangent bundle T^kG

$$\begin{aligned}
TG &= G \times \varepsilon_1\mathfrak{g}\\
T^2G &= TG \times \varepsilon_2 T\mathfrak{g} = G \times \varepsilon_1\mathfrak{g} \times \varepsilon_2\mathfrak{g} \times \varepsilon_2\varepsilon_1\mathfrak{g}\\
&\dots\\
T^kG &= G \times \oplus_{\alpha\in I_k^*}\varepsilon^\alpha\mathfrak{g}
\end{aligned}$$

where I_k denotes the power set of $\{1, \dots, k\}$ and $I_k^* = I_k - \{\emptyset\}$. For any multi-index $\alpha = \{\alpha_1, \dots, \alpha_n\} \in I_k^*$ we denote $\varepsilon^\alpha := \varepsilon_{\alpha_n} \dots \varepsilon_{\alpha_1}$. In particular $T^k\mathfrak{g} = \oplus_{\alpha\in I_k}\varepsilon^\alpha\mathfrak{g}$ as a vector space.

An alternative notation is used in [1]: the multi-index $\alpha \in I_k$ is identified with its characteristic function in $\{0,1\}^k$, written as a string of 0's and 1's, for instance the string corresponding to the multi-index $\{2,3\} \in I_3$ is (110). We state below the Theorem 24.7 from [1] in our notation.

Theorem 3.1. [1] *The group multiplication for the k-th order tangent group T^kG in the right trivialization is*

$$(g, (\varepsilon^\alpha x_\alpha)_{\alpha\in I_k^*})(h, (\varepsilon^\alpha y_\alpha)_{\alpha\in I_k^*}) = (gh, (\varepsilon^\alpha z_\alpha)_{\alpha\in I_k^*})$$

with

$$z_\alpha = x_\alpha + \sum_{\lambda\in\mathcal{P}(\alpha)} \mathrm{ad}_{x_{\lambda_{\ell-1}}} \dots \mathrm{ad}_{x_{\lambda_1}} \mathrm{Ad}_g\, y_{\lambda_\ell} \tag{8}$$

where $\mathcal{P}(\alpha)$ denotes the set of all anti-lexicographically ordered partitions of the subset $\alpha \subset \{1, \dots, k\}$ and ℓ is the length of the partition λ. The inverse is given by the formula

$$(g, (\varepsilon^\alpha x_\alpha)_{\alpha\in I_k^*})^{-1} = (g^{-1}, (\varepsilon^\alpha w_\alpha)_{\alpha\in I_k^*})$$

with

$$w_\alpha = \sum_{\lambda\in\mathcal{P}(\alpha)} (-1)^\ell\, \mathrm{Ad}_{g^{-1}}\, \mathrm{ad}_{x_{\lambda_{\ell-1}}} \dots \mathrm{ad}_{x_{\lambda_1}} x_{\lambda_\ell}. \tag{9}$$

Remark 3.2. The analogues of (8) and (9) in the left trivialization of T^kG are

$$z_\alpha = y_\alpha + \sum_{\lambda\in\mathcal{P}(\alpha)} (-1)^{\ell-1}\, \mathrm{ad}_{y_{\lambda_{\ell-1}}} \dots \mathrm{ad}_{y_{\lambda_1}} \mathrm{Ad}_{h^{-1}}\, x_{\lambda_\ell} \tag{10}$$

and

$$w_\alpha = -\sum_{\lambda\in\mathcal{P}(\alpha)} \mathrm{Ad}_g\, \mathrm{ad}_{x_{\lambda_1}} \dots \mathrm{ad}_{x_{\lambda_{\ell-1}}} x_{\lambda_\ell}, \tag{11}$$

We consider the obvious left action of the symmetric group S_k on T^kG by permuting the infinitesimal units $\varepsilon_1, \dots, \varepsilon_k$ [2]:

$$\sigma \cdot (g, (\varepsilon^\alpha x_\alpha)_{\alpha \in I_k^*}) = (g, (\varepsilon^{\sigma(\alpha)} x_\alpha)_{\alpha \in I_k^*}), \quad \sigma \in S_k. \tag{12}$$

In contrast to the action (24.4) from [1], this action is not compatible with the group multiplication (8) (a permutation $\sigma \in S_k$ might perturb the anti-lexicographic ordering of $\lambda \in \mathcal{P}_n$), but we still have the following result:

Proposition 3.3. *With respect to the action of S_k given by* (12)*, the fixed point set $(T^kG)^{S_k}$ is a subgroup of the tangent group T^kG, isomorphic to the jet group J^kG.*

Proof. Elements $(g, (\varepsilon^\alpha x_\alpha)_{\alpha \in I_k^*})$ of $(T^kG)^{S_k}$ are characterized by $x_\alpha = x_\beta$ for all $\alpha, \beta \in I_k^*$ with the same cardinality, hence there exist $x_1, \dots, x_n \in \mathfrak{g}$ such that $x_\alpha = x_n$ for $|\alpha| = n$. Let $(g, (\varepsilon^\alpha x_\alpha)_{\alpha \in I_k^*}), (h, (\varepsilon^\alpha y_\alpha)_{\alpha \in I_k^*}) \in (T^kG)^{S_k}$ with $x_\alpha = x_{|\alpha|}$ and $y_\alpha = y_{|\alpha|}$. ¿From the group multiplication (8) on T^kG we get

$$z_\alpha = x_{|\alpha|} + \sum_{\lambda \in \mathcal{P}(\alpha)} \operatorname{ad}_{x_{|\lambda_{\ell-1}|}} \dots \operatorname{ad}_{x_{|\lambda_1|}} \operatorname{Ad}_g y_{|\lambda_\ell|}.$$

For each $\alpha \in I_k^*$ with $|\alpha| = n$, there exists a unique strictly increasing bijection $\varphi : \{1, \dots, n\} \to \alpha = \{\alpha_1, \dots, \alpha_n\}$. It induces canonically a 1-1 correspondence between the sets $\mathcal{P}_n$ and $\mathcal{P}(\alpha)$ of anti-lexicographically ordered partitions, bijection that preserves the length of the partition as well as the cardinality of each subset. We get that

$$z_\alpha = x_n + \sum_{\lambda \in \mathcal{P}_n} \operatorname{ad}_{x_{|\lambda_{\ell-1}|}} \dots \operatorname{ad}_{x_{|\lambda_1|}} \operatorname{Ad}_g y_{|\lambda_\ell|}, \tag{13}$$

hence each z_α depends only on $|\alpha| = n$, thus showing that

$$(g, (\varepsilon^\alpha x_\alpha)_{\alpha \in I_k^*})(h, (\varepsilon^\alpha y_\alpha)_{\alpha \in I_k^*}) = (gh, (\varepsilon^\alpha z_\alpha)_{\alpha \in I_k^*}) \in (T^kG)^{S_k}.$$

In a similar manner, using (9), one sees that the inverse $(g, (\varepsilon^\alpha x_\alpha)_{\alpha \in I_k^*})^{-1} \in (T^kG)^{S_k}$, so $(T^kG)^{S_k}$ is a subgroup of T^kG.

The identification of an element $(g, (\varepsilon^\alpha x_\alpha)_{\alpha \in I_k^*})$ in $(T^kG)^{S_k}$ with the corresponding element $(g, (x_n)_{1 \le n \le k})$ in J^kG, where $x_n = x_\alpha$ for any $\alpha \in I_k^*$ with $|\alpha| = n$, provides a bijection between these groups. ¿From the two expressions (3) and (13) of the group multiplication in the right trivialization, it is easy to see that this bijection is a group isomorphism. ∎

Example 3.4. The multiplication on T^3G is

$$\begin{gathered}
(e, (\varepsilon^\alpha x_\alpha)_{\alpha \in I_3^*})(e, (\varepsilon^\alpha y_\alpha)_{\alpha \in I_3^*}) \\
= (e, \varepsilon^1(x_1 + y_1), \varepsilon^2(x_2 + y_2), \varepsilon^{12}(x_{12} + y_{12} + \operatorname{ad}_{x_1} y_2), \\
\varepsilon^3(x_3 + y_3), \varepsilon^{13}(x_{13} + y_{13} + \operatorname{ad}_{x_1} y_3), \varepsilon^{23}(x_{23} + y_{23} + \operatorname{ad}_{x_2} y_3), \\
\varepsilon^{123}(x_{123} + y_{123} + \operatorname{ad}_{x_1} y_{23} + \operatorname{ad}_{x_2} y_{13} + \operatorname{ad}_{x_{12}} y_3 + \operatorname{ad}_{x_2} \operatorname{ad}_{x_1} y_3))
\end{gathered}$$

while the multiplication on J^3G can be written as

$$(e,x_1,x_2,x_3)(e,y_1,y_2,y_3)$$
$$= (e, x_1+y_1, x_2+y_2+\mathrm{ad}_{x_1} y_1, x_3+y_3+2\,\mathrm{ad}_{x_1} y_2+\mathrm{ad}_{x_2} y_1+\mathrm{ad}_{x_1}^2 y_1).$$

We observe that all elements of T^3G can be decomposed in products of pure elements

$$(e,(\varepsilon^\alpha x_\alpha)_{\alpha\in I_3^*}) = (e,\varepsilon^{123}x_{123})(e,\varepsilon^{23}x_{23})(e,\varepsilon^{13}x_{13})(e,\varepsilon^{3}x_{3})(e,\varepsilon^{12}x_{12})(e,\varepsilon^{2}x_{2})(e,\varepsilon^{1}x_{1}),$$

a property that holds for all T^kG [1]. There is no analogous decomposition for J^3G:

$$(e,x_3)(e,x_2)(e,x_1) = (e,x_1,x_2,x_3+\mathrm{ad}_{x_2} x_1]).$$

Jet functor. The identity fiber of $T^kG\to G$ is a subgroup of T^kG, denoted by $G_k(\mathfrak{g})$ and identified with $\oplus_{\alpha\in I_k^*}\varepsilon^\alpha\mathfrak{g}$, so its dimension is $(2^k-1)\dim\mathfrak{g}$. The identity fiber $J_k(\mathfrak{g})$ of $J^kG\to G$ is a subgroup of J^kG, identified with the direct sum of k copies of $\mathfrak{g}$, so its dimension is $k\dim\mathfrak{g}$.

As before we consider the natural action of the symmetric group S_k on $G_k(\mathfrak{g})$ by permuting the infinitesimal units $\varepsilon_1,\dots,\varepsilon_k$:

$$\sigma\cdot(\varepsilon^\alpha x_\alpha)_{\alpha\in I_k^*} = (\varepsilon^{\sigma(\alpha)}x_\alpha)_{\alpha\in I_k^*},\quad \sigma\in S_k.$$

The fixed point set $G_k(\mathfrak{g})^{S_k}$, identified with $J_k(\mathfrak{g})$, is a subgroup of $G_k(\mathfrak{g})$, Elements $(\varepsilon^\alpha x_\alpha)_{\alpha\in I_k^*}$ of the fixed point set are characterized by $x_\alpha = x_\beta$ for all $\alpha,\beta\in I_k^*$ with the same cardinality.

Theorem 3.1 resp. Theorem 2.5 provide the expressions for the group multiplication and the inverse of an element on $G_k(\mathfrak{g})$:

$$(\varepsilon^\alpha x_\alpha)_{\alpha\in I_k^*}(\varepsilon^\alpha y_\alpha)_{\alpha\in I_k^*} = \left(\varepsilon^\alpha\left(x_\alpha+\sum_{\lambda\in\mathcal{P}(\alpha)}\mathrm{ad}_{x_{\lambda_{\ell-1}}}\dots\mathrm{ad}_{x_{\lambda_1}} y_{\lambda_\ell}\right)\right)_{\alpha\in I_k^*} \tag{14}$$

$$(\varepsilon^\alpha x_\alpha)^{-1}_{\alpha\in I_k^*} = \left(\varepsilon^\alpha\left(\sum_{\lambda\in\mathcal{P}(\alpha)}(-1)^\ell\,\mathrm{ad}_{x_{\lambda_{\ell-1}}}\dots\mathrm{ad}_{x_{\lambda_1}} x_{\lambda_\ell}\right)\right)_{\alpha\in I_k^*},$$

resp. on $J_k(\mathfrak{g})$: $(x_n)_{1\le n\le k}(y_n)_{1\le n\le k} = (z_n)_{1\le n\le k}$ and $(x_n)^{-1}_{1\le n\le k} = (w_n)_{1\le n\le k}$,

$$z_n = x_n+\sum_{\lambda\in\mathcal{P}_n}\mathrm{ad}_{x_{i_{\ell-1}}}\dots\mathrm{ad}_{x_{i_1}} y_{i_\ell} = x_n+\sum_{i_1+\dots+i_\ell=n}N_{(i_1,\dots,i_\ell)}\,\mathrm{ad}_{x_{i_{\ell-1}}}\dots\mathrm{ad}_{x_{i_1}} y_{i_\ell} \tag{15}$$

$$w_n = \sum_{\lambda\in\mathcal{P}_n}(-1)^\ell\,\mathrm{ad}_{x_{i_1}}\dots\mathrm{ad}_{x_{i_{\ell-1}}} x_{i_\ell} = \sum_{i_1+\dots+i_\ell=n}(-1)^\ell N_{(i_1,\dots,i_\ell)}\,\mathrm{ad}_{x_{i_1}}\dots\mathrm{ad}_{x_{i_{\ell-1}}} x_{i_\ell},$$

where λ is an anti-lexicographically ordered partition $\lambda_1|\dots|\lambda_\ell$ of $\alpha\in I_k^*$, resp. of $\{1,\dots,n\}$, of length ℓ and $i_r = |\lambda_r|$ for $r=1,\dots,\ell$.

The maps sending the Lie algebra $\mathfrak{g}$ to the groups $G_k(\mathfrak{g})$ and $J_k(\mathfrak{g})$ are functorial. Both are polynomial groups [2]. Moreover, the group multiplication is affine in the second argument, so we actually get affine near-ring structures, slight generalizations of near-ring structures [6].

Leibniz algebras. A left Leibniz algebra [5] is a vector space $\mathfrak{g}$ endowed with a bilinear map $[\ ,\ ]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that the left Leibniz identity holds

$$[x,[y,z]]=[[x,y],z]+[y,[x,z]],\quad x,y,z\in\mathfrak{g}.$$

This condition is equivalent to the fact that the left ad-actions are derivations of the bracket. A Lie algebra is the same as a Leibniz algebra with antisymmetric bracket.

Theorem 3.5. [2] *For any left Leibniz algebra $\mathfrak{g}$, the multiplication* (14) *endows $G_k(\mathfrak{g})$ with a group structure. Its fixed point set $J_k(\mathfrak{g})$ under the permutation action is a subgroup.*

The multiplication formula (15) on $J_k(\mathfrak{g})$ holds also for a Leibniz algebra $\mathfrak{g}$.

4. Abelian extension

This section is concerned with the group extension $\mathfrak{g}\to J^kG\to J^{k-1}G$ assigned to the right trivialization. The following result is a direct consequence of Theorem 2.5.

Proposition 4.1. *The right trivialized k-th order jet bundle $J^kG=G\times\mathfrak{g}^k$ is an abelian Lie group extension of the right trivialized $(k-1)$-th order jet bundle $J^{k-1}G=G\times\mathfrak{g}^{k-1}$ by $\mathfrak{g}$, characterized by the group cocycle*

$$\begin{aligned}c_k\left((g,x_1,\dots,x_{k-1}),(h,y_1,\dots,y_{k-1})\right)&=\sum_{\lambda\in\mathcal{P}_k-\{12\dots k\}}\mathrm{ad}_{x_{i_{\ell-1}}}\dots\mathrm{ad}_{x_{i_1}}\mathrm{Ad}_g\,y_{i_\ell}\\&=\sum_{\ell=2}^{k}\sum_{i_1+\dots i_\ell=k}N_{(i_1,\dots,i_\ell)}\,\mathrm{ad}_{x_{i_{\ell-1}}}\dots\mathrm{ad}_{x_{i_1}}\mathrm{Ad}_g\,y_{i_\ell}\end{aligned}$$

with $\lambda=\lambda_1|\dots|\lambda_\ell$ anti-lexicographically ordered and $i_r=|\lambda_r|$.

Its Lie algebra $J^k\mathfrak{g}$ is an abelian extension of $J^{k-1}\mathfrak{g}$ by $\mathfrak{g}$ with Lie algebra cocycle

$$\sigma_k\left((\xi,x_1,\dots,x_{k-1}),(\eta,y_1,\dots,y_{k-1})\right)=\sum_{i=1}^{k-1}\binom{k}{i}[x_i,y_{k-i}].$$

Proof. The expression of the group cocycle c_k follows immediately from the multiplication rule (3) rewritten as

$$z_n=x_n+\mathrm{Ad}_g\,y_n+\sum_{\lambda\in\mathcal{P}_k-\{12\dots k\}}\mathrm{ad}_{x_{i_{\ell-1}}}\dots\mathrm{ad}_{x_{i_1}}\mathrm{Ad}_g\,y_{i_\ell}.$$

The Lie algebra cocycle σ_k can be obtained from the group cocycle c_k by derivation:

$$\sigma_k=\partial_1\partial_2c_k(e,e)-\partial_2\partial_1c_k(e,e).\tag{16}$$

We will see that only the terms of the Lie group cocycle that correspond to partitions $\lambda\in\mathcal{P}_k$ of length two provide non-zero terms for the Lie algebra cocycle.

There are $\binom{k-1}{i}$ anti-lexicographically ordered partitions $\lambda \in \mathcal{P}_k$ of length 2 such that $|\lambda_1| = i$ and $|\lambda_2| = k-i$, because the only requirement is that $k \in \lambda_2$. Thus, splitting $\mathcal{P}_k - \{12\dots k\}$ into ordered partitions of length 2 and ordered partitions of length bigger than 2, the group cocycle can be written as

$$\begin{aligned} c_k\left((g,x_1,\dots,x_{k-1}),(h,y_1,\dots,y_{k-1})\right) & \\ &= \sum_{i=1}^{k-1}\binom{k-1}{i}\operatorname{ad}_{x_i}\operatorname{Ad}_g y_{k-i} \\ &+ \sum_{p+q<k}\operatorname{ad}_{x_p}\operatorname{ad}_{x_q} R^k_{pq}, \end{aligned}$$

where R^k_{pq} is of the form $\operatorname{ad}_{x_{j_m}}\dots\operatorname{ad}_{x_{j_1}}\operatorname{Ad}_g y_{j_0}$ with $m \ge 0$. Then

$$\partial_1\partial_2 c_k(e,e)\left((\xi,x_1,\dots,x_{k-1}),(\eta,y_1,\dots,y_{k-1})\right) = \sum_{i=1}^{k-1}\binom{k-1}{i}\operatorname{ad}_{x_i} y_{k-i}$$

and by (16) the Lie algebra cocycle is

$$\begin{aligned} &\sigma_k\left((\xi,x_1,\dots,x_{k-1}),(\eta,y_1,\dots,y_{k-1})\right) \\ &= \sum_{i=1}^{k-1}\binom{k-1}{i}\operatorname{ad}_{x_i} y_{k-i} - \sum_{i=1}^{k-1}\binom{k-1}{i}\operatorname{ad}_{y_i} x_{k-i} \\ &= \sum_{i=1}^{k-1}\left(\binom{k-1}{i}+\binom{k-1}{k-i}\right)\operatorname{ad}_{x_i} y_{k-i} = \sum_{i=1}^{k-1}\binom{k}{i}\left[x_i, y_{k-i}\right], \end{aligned}$$

as requested. ∎

Remark 4.2. Given two $\mathfrak{g}$-modules V and W, each $\mathfrak{g}$-invariant element γ in $\Lambda^2 V^* \otimes W$ determines canonically a Lie algebra 2-cocycle on the semidirect product $\mathfrak{g} \ltimes V$

$$\sigma_\gamma((\xi,u),(\eta,v)) = \gamma(u,v) \in W.$$

Its cohomology class in $H^2(\mathfrak{g}\ltimes V, W)$ is non-zero. A special case is the $\mathfrak{g}$-invariant element $[\ ,\ ]$ in $\Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}$, which determines the Lie algebra 2-cocycle σ_2 that characterizes $J^2\mathfrak{g}$.

In the right trivialization, the $\varepsilon^{12\dots k}\mathfrak{g}$ component of T^kG with addition is a normal abelian subgroup of T^kG, hence we get an abelian Lie group extension

$$\mathfrak{g} \to T^kG \to T^kG/\mathfrak{g}.$$

The jet group $J^{k-1}G = (T^{k-1}G)^{S_{k-1}}$ is isomorphic to the subgroup $(T^kG/\mathfrak{g})^{S_k}$ of fixed points, since the only multi-index with cardinality k, namely $12\dots k$, was divided out. The pull-back of the abelian extension above by the inclusion $J^{k-1}G \subset T^kG/\mathfrak{g}$ is nothing else but the abelian extension

$$\mathfrak{g} \to J^kG \to J^{k-1}G.$$

Acknowledgment. The author is grateful to Wolfgang Bertram and to the referee for extremely helpful suggestions.

References

[1] Bertram, W., "Differential geometry, Lie groups, and symmetric spaces over general base fields and rings," Memoirs of the Amer. Math. Soc. **192**, no. 900, 2008.

[2] Didry, M., *Construction of groups associated to Lie- and to Leibniz-algebras*, J. Lie Theory **17** (2007), 399–426.

[3] Gay-Balmaz, F., D. Holm, D. Meier, T. Ratiu, and F. Vialard, *Invariant higher-order variational problems*, Comm. Math. Phys. **309** (2012), 413–458.

[4] Kolár, I., P. W. Michor, and J. Slovák, "Natural operations in differential geometry," Springer-Verlag, Berlin, Heidelberg, New York, 1993.

[5] Loday, J.-L., *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Enseign. Math. **39** (1993), 269–293.

[6] Pilz, G., "Near-rings, the theory and its applications," Mathematics Studies **23**, North-Holland, Amsterdam, 1977.

Cornelia Vizman
West University of Timisoara
Department of Mathematics
Bd. V. Pârvan 4
300223-Timisoara, Romania
vizman@math.uvt.ro

Received October 26, 2012
and in final form January 29, 2013