# ℤ-Graded Oscillator Generalizations of the Classical Theorem on Harmonic Polynomials

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**Abstract.** Classical harmonic analysis says that the spaces of homogeneous harmonic polynomials (solutions of Laplace equation) are irreducible modules of the corresponding orthogonal Lie group (algebra) and the whole polynomial algebra is a free module over the invariant polynomials generated by harmonic polynomials. Algebraically, this gives an $(sl(2,\mathbb{R}), o(n,\mathbb{R}))$ Howe duality. In this paper, we study two-parameter oscillator variations of the above theorem associated with noncanonical oscillator representations of $o(n,\mathbb{C})$. We find the condition when the homogeneous solution spaces of the variated Laplace equation are irreducible modules of $o(n,\mathbb{C})$ and the homogeneous subspaces are direct sums of the images of these solution subspaces under the powers of the dual differential operator. This establishes an $(sl(2,\mathbb{C}), o(n,\mathbb{C}))$ Howe duality on some homogeneous subspaces. In generic case, the obtained irreducible $o(n,\mathbb{C})$-modules are infinite-dimensional non-unitary modules without highest-weight vectors. When both parameters are equal to the maximal allowed value, we obtain explicit irreducible $(\mathcal{G}, \mathcal{K})$-modules for $o(n,\mathbb{C})$.
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## 1. Introduction

Harmonic polynomials are important objects in analysis, differential geometry and physics. A fundamental theorem in classical harmonic analysis says that the spaces of homogeneous harmonic polynomials (solutions of Laplace equation) are irreducible modules of the corresponding orthogonal Lie group (algebra) and the whole polynomial algebra is a free module over the invariant polynomials generated by harmonic polynomials. Bases of these irreducible modules can be obtained easily (e.g., cf. [16]). The algebraic beauty of the above theorem is that Laplace equation characterizes the irreducible submodules of the polynomial algebra and the corresponding quadratic invariant gives a decomposition of the polynomial algebra into a direct sum of irreducible submodules. This actually forms an $(sl(2,\mathbb{R}), o(n,\mathbb{R}))$ Howe duality (cf. [6–9]).

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Cao [1] proved that the subspaces of homogeneous polynomial vector solutions of the $n$-dimensional Navier equations in elasticity are exactly direct sums of three explicitly given irreducible submodules when $n \neq 4$ and direct sums of four explicitly given irreducible submodules if $n = 4$ of the corresponding orthogonal Lie group (algebra), and the whole polynomial vector space is also a free module over the invariant polynomials generated these solutions. Moreover, he solved the initial value problem for the Navier equations. In particular, Cao's work can be viewed as a supplement to Olver's well known work [15] on algebraic study of linear elasticity. It is a quadratic vector generalization of the classical theorem on harmonic polynomials.

In [17], the second author proved that the space of homogeneous polynomial solutions with degree $m$ for the dual cubic Dickson invariant differential operator is exactly a direct sum of $[\![m/2]\!]+1$ explicitly determined irreducible $E_6$-submodules and the whole polynomial algebra is a free module over the polynomial algebra in the Dickson invariant generated by these solutions. This gave a cubic $E_6$-generalization of the above classical theorem on harmonic polynomials.

Lie algebras (Lie groups) serve as the symmetries in quantum physics (e.g., cf. [3, 5, 11, 12, 14]). Their various representations provide distinct concrete physical models. Many important physical phenomena have been interpreted as the consequences of symmetry breaking (e.g., cf. [14]). Harmonic oscillators are basic objects in quantum mechanics (e.g., cf. [3, 5]). Oscillator representations of finite-dimensional simple Lie algebras are the most fundamental ones in quantum physics (e.g., cf. [2, 4]). Howe [9] determined the multiplicity-free such representations. In [16], the second author found the methods of solving flag partial differential equations for polynomial solutions. Moreover, we [13] used a result in [16] to prove certain $\mathbb{Z}^2$-graded oscillator generalizations of the classical theorem on harmonic polynomials for $sl(n, \mathbb{C})$.

The aim of this work is to establish certain two-parameter oscillator variations of the classical theorem on harmonic polynomials, associated with non-canonical oscillator representations of orthogonal Lie algebras, which are obtained by swapping differential operators and multiplication operators in the canonical oscillator representations induced from the natural representations. The Howe duality does not hold on the whole polynomial algebras. But we find the condition when the homogeneous solution spaces of the variated Laplace equation are irreducible modules of $o(n, \mathbb{C})$ and the homogeneous subspaces are direct sums of the images of these solution subspaces under the powers of the dual differential operator. This establishes an $(sl(2, \mathbb{C}), o(n, \mathbb{C}))$ Howe duality on some homogeneous subspaces. In particular, we obtain explicit infinite-dimensional non-unitary modules of orthogonal Lie algebras that are not of highest-weight type. Explicit bases of all the above irreducible modules in generic case are obtained.

Let $\mathcal{G}$ be a semisimple Lie algebra and let $\mathcal{K}$ be a maximal proper reductive Lie subalgebra of $\mathcal{G}$. An infinite-dimensional irreducible $\mathcal{G}$-module is said of $(\mathcal{G}, \mathcal{K})$-type if it is a direct sum of finite-dimensional irreducible $\mathcal{K}$-submodules. When both parameters are equal to the maximal allowed value, we obtain an infinite family of explicit irreducible $(\mathcal{G}, \mathcal{K})$-modules for orthogonal Lie algebras. Below we give a technical introduction.

For convenience, we will use the notion $\overline{i, i+j} = \{i, i+1, i+2, ..., i+j\}$ for integers $i$ and $j$ with $i \leq j$. Denote by $\mathbb{N}$ the additive semigroup of nonnegative integers. Let $E_{r,s}$ be the square matrix with 1 as its $(r, s)$-entry and 0 as the others.

The compact orthogonal Lie algebra $o(n, \mathbb{R}) = \sum_{1 \leq r < s \leq n} \mathbb{R}(E_{r,s} - E_{s,r})$, whose representation on the polynomial algebra $\mathcal{A} = \mathbb{R}[x_1, ..., x_n]$ is given by $(E_{r,s} - E_{s,r})|_{\mathcal{A}} = x_r \partial_{x_s} - x_s \partial_{x_r}$. Denote by $\mathcal{A}_k$ the subspace of homogeneous polynomials in $\mathcal{A}$ with degree $k$. Recall that the Laplace operator $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ and its corresponding invariant $\eta = x_1^2 + x_2^2 + \cdots + x_n^2$. When $n \geq 3$, it is well known that the subspaces of harmonic polynomials

$$\mathcal{H}_k = \{f \in \mathcal{A}_k \mid \Delta(f) = 0\} \tag{1.1}$$

form irreducible $o(n, \mathbb{R})$-submodules and

$$\mathcal{A} = \bigoplus_{i,k=0}^{\infty} \eta^i \mathcal{H}_k \tag{1.2}$$

is a direct sum of irreducible submodules. In other words, the irreducible submodules are characterized by the Laplace operator $\Delta$ and its dual invariant $\eta$ give the complete reducibility of the polynomial algebra $\mathcal{A}$. Since the space $\mathbb{R}\Delta + \mathbb{R}[\Delta, \eta] + \mathbb{R}\eta$ forms an operator Lie algebra isomorphic to $sl(2, \mathbb{R})$ (e.g., cf. [10]), the above conclusion gives an $(sl(2, \mathbb{R}), o(n, \mathbb{R}))$ Howe duality (cf. [6–9]).

Note that the split

$$o(2n, \mathbb{C}) = \sum_{i,j=1}^{n} \mathbb{C}(E_{i,j} - E_{n+j,n+i}) + \sum_{1 \leq i < j \leq n} [\mathbb{C}(E_{i,n+j} - E_{j,n+i}) + \mathbb{C}(E_{n+j,i} - E_{n+i,j})]. \tag{1.3}$$

Denote $\mathcal{B} = \mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]$. The canonical oscillator representation of $o(2n, \mathbb{C})$ on $\mathcal{B}$ is given by

$$(E_{i,j} - E_{n+j,n+i})|_{\mathcal{B}} = x_i \partial_{x_j} - y_j \partial_{y_i}, \quad (E_{i,n+j} - E_{j,n+i})|_{\mathcal{B}} = x_i \partial_{y_j} - x_j \partial_{y_i}, \tag{1.4}$$

$$(E_{n+i,j} - E_{n+j,i})|_{\mathcal{B}} = y_i \partial_{x_j} - y_j \partial_{x_i} \tag{1.5}$$

for $i, j \in \overline{1, n}$. The Laplace operator becomes $\Delta = \sum_{i=1}^{n} \partial_{x_i} \partial_{y_i}$ and its dual invariant is $\eta = \sum_{i=1}^{n} x_i y_i$.

Fix $n_1, n_2 \in \overline{1, n}$ with $n_1 \leq n_2$. Changing operators $\partial_{x_r} \mapsto -x_r$, $x_r \mapsto \partial_{x_r}$ for $r \in \overline{1, n_1}$ and $\partial_{y_s} \mapsto -y_s$, $y_s \mapsto \partial_{y_s}$ for $s \in \overline{n_2 + 1, n}$ in the above canonical oscillator representation, we get another noncanonical oscillator representation of $o(2n, \mathbb{C})$ on $\mathcal{B}$ determined by

$$(E_{i,j} - E_{n+j,n+i})|_{\mathcal{B}} = E_{i,j}^x - E_{j,i}^y \tag{1.6}$$

with

$$E_{i,j}^x|_{\mathcal{B}} = \begin{cases} -x_j \partial_{x_i} - \delta_{i,j} & \text{if } i, j \in \overline{1, n_1}; \\ \partial_{x_i} \partial_{x_j} & \text{if } i \in \overline{1, n_1}, \ j \in \overline{n_1 + 1, n}; \\ -x_i x_j & \text{if } i \in \overline{n_1 + 1, n}, \ j \in \overline{1, n_1}; \\ x_i \partial_{x_j} & \text{if } i, j \in \overline{n_1 + 1, n} \end{cases} \tag{1.7}$$

and

$$E_{i,j}^{y}|_{\mathcal{B}} = \begin{cases} y_i\partial_{y_j} & \text{if } i, j \in \overline{1, n_2}; \\ -y_i y_j & \text{if } i \in \overline{1, n_2}, \ j \in \overline{n_2 + 1, n}; \\ \partial_{y_i}\partial_{y_j} & \text{if } i \in \overline{n_2 + 1, n}, \ j \in \overline{1, n_2}; \\ -y_j\partial_{y_i} - \delta_{i,j} & \text{if } i, j \in \overline{n_2 + 1, n}, \end{cases} \quad (1.8)$$

and

$$E_{i,n+j}|_{\mathcal{B}} = \begin{cases} \partial_{x_i}\partial_{y_j} & \text{if } i \in \overline{1, n_1}, \ j \in \overline{1, n_2}, \\ -y_j\partial_{x_i} & \text{if } i \in \overline{1, n_1}, \ j \in \overline{n_2 + 1, n}, \\ x_i\partial_{y_j} & \text{if } i \in \overline{n_1 + 1, n}, \ j \in \overline{1, n_2}, \\ -x_i y_j & \text{if } i \in \overline{n_1 + 1, n}, \ j \in \overline{n_2 + 1, n} \end{cases} \quad (1.9)$$

and

$$E_{n+i,j}|_{\mathcal{B}} = \begin{cases} -x_j y_i & \text{if } j \in \overline{1, n_1}, \ i \in \overline{1, n_2}, \\ -x_j\partial_{y_i} & \text{if } j \in \overline{1, n_1}, \ i \in \overline{n_2 + 1, n}, \\ y_i\partial_{x_j} & \text{if } j \in \overline{n_1 + 1, n}, \ i \in \overline{1, n_2}, \\ \partial_{x_j}\partial_{y_i} & \text{if } j \in \overline{n_1 + 1, n}, \ i \in \overline{n_2 + 1, n}. \end{cases} \quad (1.10)$$

Correspondingly we obtain a variation of the Laplace operator:

$$\mathcal{D} = -\sum_{i=1}^{n_1} x_i\partial_{y_i} + \sum_{r=n_1+1}^{n_2} \partial_{x_r}\partial_{y_r} - \sum_{s=n_2+1}^{n} y_s\partial_{x_s} \quad (1.11)$$

and its dual

$$\eta = \sum_{i=1}^{n_1} y_i\partial_{x_i} + \sum_{r=n_1+1}^{n_2} x_r y_r + \sum_{s=n_2+1}^{n} x_s\partial_{y_s}. \quad (1.12)$$

Set

$$\mathcal{B}_{\langle k \rangle} = \text{Span}\{x^\alpha y^\beta \mid \alpha, \beta \in \mathbb{N}^n; \sum_{r=n_1+1}^{n} \alpha_r - \sum_{i=1}^{n_1} \alpha_i + \sum_{i=1}^{n_2} \beta_i - \sum_{r=n_2+1}^{n} \beta_r = k\} \quad (1.13)$$

for $k \in \mathbb{Z}$. Define

$$\mathcal{H}_{\langle k \rangle} = \{f \in \mathcal{B}_{\langle k \rangle} \mid \mathcal{D}(f) = 0\}. \quad (1.14)$$

Below we always take $\mathcal{K} = \sum_{i,j=1}^{n} \mathbb{C}(E_{i,j} - E_{n+j,n+i})$. Our first result is:

**Theorem 1**. *For any $n_1 - n_2 + 1 - \delta_{n_1,n_2} \geq k \in \mathbb{Z}$, $\mathcal{H}_{\langle k \rangle}$ is an irreducible $o(2n, \mathbb{F})$-submodule and $\mathcal{B}_{\langle k \rangle} = \bigoplus_{i=0}^{\infty} \eta^i(\mathcal{H}_{\langle k-2i \rangle})$ is a decomposition of irreducible submodules. The module $\mathcal{H}_{\langle k \rangle}$ under the assumption is of highest-weight type only if $n_2 = n$. When $n_1 = n_2 = n$, all the irreducible modules $\mathcal{H}_{\langle k \rangle}$ with $0 \geq k \in \mathbb{Z}$ are of $(\mathcal{G}, \mathcal{K})$-type.*

It can be verified that the space $\mathbb{C}\mathcal{D} + \mathbb{C}[\mathcal{D}, \eta] + \mathbb{C}\eta$ forms an operator Lie algebra isomorphic to $sl(2, \mathbb{C})$. The above theorem establishes an $(sl(2, \mathbb{C}), o(2n, \mathbb{C}))$ Howe duality on the homogeneous subspaces $\mathcal{B}_{\langle k \rangle}$ with $n_1 - n_2 + 1 - \delta_{n_1,n_2} \geq k \in \mathbb{Z}$. When $n_1 - n_2 + 1 - \delta_{n_1,n_2} < k \in \mathbb{Z}$, $\mathcal{H}_{\langle k \rangle}$ is an indecomposable module and contains the proper nonzero submodule $\mathcal{H}_{\langle k \rangle} \bigcap \eta(\mathcal{B})$ by [15]. This shows that our representation is not unitary. Due to the mixture of differential operators with multiplications operators, the modules $\mathcal{H}_{\langle k \rangle}$ are not of highest-weight type when $n_2 < n$.

Observe that the split

$$o(2n+1, \mathbb{C}) = o(2n, \mathbb{C}) \oplus \bigoplus_{i=1}^{n}[\mathbb{C}(E_{0,i} - E_{n+i,0}) + \mathbb{C}(E_{0,n+i} - E_{i,0})]. \qquad (1.15)$$

Let $\mathcal{B}' = \mathbb{C}[x_0, x_1, ..., x_n, y_1, ..., y_n]$. We define a noncanonical oscillator representation of $o(2n+1, \mathbb{C})$ on $\mathcal{B}'$ by the differential operators in (1.6)-(1.10) with $|_{\mathcal{B}}$ replaced by $|_{\mathcal{B}'}$ and

$$E_{0,i}|_{\mathcal{B}'} = \begin{cases} -x_0 x_i & \text{if } i \in \overline{1, n_1}, \\ x_0 \partial_{x_i} & \text{if } i \in \overline{n_1 + 1, n}, \\ x_0 \partial_{y_{i-n}} & \text{if } i \in \overline{n + 1, n + n_2}, \\ -x_0 y_{i-n} & \text{if } i \in \overline{n + n_2 + 1, 2n} \end{cases} \qquad (1.16)$$

and

$$E_{i,0}|_{\mathcal{B}'} = \begin{cases} \partial_{x_0} \partial_{x_i} & \text{if } i \in \overline{1, n_1}, \\ x_i \partial_{x_0} & \text{if } i \in \overline{n_1 + 1, n}, \\ y_{i-n} \partial_{x_0} & \text{if } i \in \overline{n + 1, n + n_2}, \\ \partial_{x_0} \partial_{y_{i-n}} & \text{if } i \in \overline{n + n_2 + 1, 2n}. \end{cases} \qquad (1.17)$$

Correspondingly we obtain a variation of the Laplace operator:

$$\mathcal{D}' = \partial_{x_0}^2 - 2\sum_{i=1}^{n_1} x_i \partial_{y_i} + 2\sum_{r=n_1+1}^{n_2} \partial_{x_r} \partial_{y_r} - 2\sum_{s=n_2+1}^{n} y_s \partial_{x_s} \qquad (1.18)$$

and its dual operator

$$\eta' = x_0^2 + 2\sum_{i=1}^{n_1} y_i \partial_{x_i} + 2\sum_{r=n_1+1}^{n_2} x_r y_r + 2\sum_{s=n_2+1}^{n} x_s \partial_{y_s}. \qquad (1.19)$$

Set

$$\mathcal{B}'_{\langle k \rangle} = \sum_{i=0}^{\infty} \mathcal{B}_{\langle k-i \rangle} x_0^i, \qquad \mathcal{H}'_{\langle k \rangle} = \{f \in \mathcal{B}'_{\langle k \rangle} \mid \mathcal{D}'(f) = 0\}. \qquad (1.20)$$

The following is our second result:

**Theorem 2**. *For any $k \in \mathbb{Z}$, $\mathcal{H}'_{\langle k \rangle}$ is an irreducible $o(2n+1, \mathbb{C})$-submodule. Moreover, $\mathcal{B}' = \bigoplus_{k \in \mathbb{Z}} \bigoplus_{i=0}^{\infty} (\eta')^i (\mathcal{H}'_{\langle k \rangle})$ is a decomposition of irreducible submodules. The module $\mathcal{H}'_{\langle k \rangle}$ is of highest-weight type only if $n_2 = n$. When $n_1 = n_2 = n$, all the irreducible modules $\mathcal{H}'_{\langle k \rangle}$ are of $(\mathcal{G}, \mathcal{K})$-type.*

It can be verified that the space $\mathbb{C}\mathcal{D}' + \mathbb{C}[\mathcal{D}', \eta'] + \mathbb{C}\eta'$ forms an operator Lie algebra isomorphic to $sl(2, \mathbb{C})$. The above theorem establishes an $(sl(2, \mathbb{C}), o(2n, \mathbb{C}))$ Howe duality on the whole polynomial algebra $\mathcal{B}'$.

The explicit expressions for all the above irreducible modules Theorems 1 and 2 are given. In the case of highest-weight type, the highest-weight vector and its weight of the corresponding irreducible modules are also presented. Since the representations with parameters $(n_1, n_2)$ are contragredient to those with parameters $(n - n_2, n - n_1)$, the case $n_2 < n_1$ has virtually been handled.

In Section 2, we present some preparatory works. In Section 3, we prove Theorem 1. Section 4 is devoted to the proof of Theorem 2.

## 2. Preparation

In our proofs of Theorem 1 and Theorem 2, the following result from [16] play a key role.

**Lemma 2.1.** *Let $\mathcal{B}$ be a commutative associative algebra and let $\mathcal{A}$ be a free $\mathcal{B}$-module generated by a filtrated subspace $V = \bigcup_{r=0}^{\infty} V_r$ (i.e., $V_r \subset V_{r+1}$). Let $T_1$ be a linear operator on $\mathcal{B} \oplus \mathcal{A}$ with a right inverse $T_1^-$ such that*

$$T_1(\mathcal{B}, \mathcal{A}),\ T_1^-(\mathcal{B}, \mathcal{A}) \subset (\mathcal{B}, \mathcal{A}),\quad T_1(\eta_1\eta_2) = T_1(\eta_1)\eta_2,\quad T_1^-(\eta_1\eta_2) = T_1^-(\eta_1)\eta_2 \quad (2.1)$$

*for $\eta_1 \in \mathcal{B}$, $\eta_2 \in V$, and let $T_2$ be a linear operator on $\mathcal{A}$ such that*

$$T_2(V_0) = \{0\},\quad T_2(V_{r+1}) \subset \mathcal{B}V_r,\quad T_2(f\zeta) = fT_2(\zeta) \quad (2.2)$$

*for $r \in \mathbb{N}$, $f \in \mathcal{B}$, $\zeta \in \mathcal{A}$. Then we have*

$$\{g \in \mathcal{A} \mid (T_1 + T_2)(g) = 0\}$$
$$= Span\{\sum_{i=0}^{\infty}(-T_1^-T_2)^i(hg) \mid g \in V,\ h \in \mathcal{B};\ T_1(h) = 0\}. \quad (2.3)$$

Set

$$\epsilon_i = (0, ..., 0, \overset{i}{1}, 0, ..., 0) \in \mathbb{N}^n. \quad (2.4)$$

For each $i \in \overline{1, n}$, we define the linear operator $\int_{(x_i)}$ on $\mathcal{A}$ by:

$$\int_{(x_i)}(x^\alpha) = \frac{x^{\alpha+\epsilon_i}}{\alpha_i + 1} \quad \text{for } \alpha \in \mathbb{N}^n. \quad (2.5)$$

Furthermore, we let

$$\int_{(x_i)}^{(0)} = 1, \qquad \int_{(x_i)}^{(m)} = \overbrace{\int_{(x_i)} \cdots \int_{(x_i)}}^{m} \qquad \text{for } 0 < m \in \mathbb{Z} \quad (2.6)$$

and denote

$$\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n},\quad \int^{(\alpha)} = \int_{(x_1)}^{(\alpha_1)}\int_{(x_2)}^{(\alpha_2)}\cdots\int_{(x_n)}^{(\alpha_n)} \qquad \text{for } \alpha \in \mathbb{N}^n. \quad (2.7)$$

Obviously, $\int^{(\alpha)}$ is a right inverse of $\partial^\alpha$ for $\alpha \in \mathbb{N}^n$. We remark that $\int^{(\alpha)}\partial^\alpha \neq 1$ if $\alpha \neq 0$ due to $\partial^\alpha(1) = 0$. In this paper, our $T_1$'s are of the type $\partial^\alpha$ and the right inverse $T_1^- = \int^{(\alpha)}$.

Note that

$$\mathcal{K} = \sum_{i,j=1}^{n} \mathbb{C}(E_{i,j} - E_{n+j,n+i}) \tag{2.8}$$

forms a Lie subalgebra of $o(2n, \mathbb{C})$, which is isomorphic to $gl(n, \mathbb{C})$. Take the Cartan subalgebra $H = \sum_{i=1}^{n} \mathbb{C}(E_{i,i} - E_{n+i,n+i})$ and the subalgebra spanned by positive root vectors

$$\mathcal{K}_+ = \sum_{1 \le i < j \le n} \mathbb{C}(E_{i,j} - E_{n+j,n+i}). \tag{2.9}$$

Recall $\mathcal{B} = \mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]$ and the action of $o(2n, \mathbb{C})$ on $\mathcal{B}$ given in (1.6)-(1.10). A $\mathcal{K}$-*singular vector* $v$ in $\mathcal{B}$ is a weight vector such that $\mathcal{K}_+(v) = \{0\}$. According to [13], we have:

**Lemma 2.2.** *Suppose $n_1 + 1 < n_2$. If $n_2 < n$, the set of all homogeneous $\mathcal{K}$-singular vectors are*

$$\{\eta^m(x_i^{m_1} y_j^{m_2}) \mid m, m_1, m_2 \in \mathbb{N}; i = n_1, n_1 + 1; j = n_2, n_2 + 1\}. \tag{2.10}$$

*When $n_2 = n$, the set of all homogeneous $\mathcal{K}$-singular vectors are*

$$\{\eta^m(x_i^{m_1} y_n^{m_2}) \mid m, m_1, m_2 \in \mathbb{N}; i = n_1, n_1 + 1\}. \tag{2.11}$$

**Lemma 2.3.** *Suppose $n_1 + 1 = n_2$. If $n_2 < n$, the set of all homogeneous $\mathcal{K}$-singular vectors are*

$$\{\eta^{m_2}(x_i^{m_1} y_j^{m_3}), x_{n_1+1}^{m_1} y_{n_1+1}^{m_2}, \eta^{m_1+m_2}(x_{n_1}^{m_2} y_{n_1+1}^{m_3-m_1}), \eta^{m_1+m_2}(y_{n_1+2}^{m_2} x_{n_1+1}^{m_3-m_1})$$
$$\mid m_r \in \mathbb{N}; (i,j) = (n_1, n_1 + 1), (n_1, n_1 + 2), (n_1 + 1, n_1 + 2)\}. \tag{2.12}$$

*When $n_2 = n$, the set of all homogeneous $\mathcal{K}$-singular vectors are*

$$\{\eta^{m_2}(x_{n-1}^{m_1} y_n^{m_3}), x_n^{m_1} y_n^{m_2}, \eta^{m_1+m_2}(x_{n-1}^{m_2} y_n^{m_3-m_1}) \mid m_i \in \mathbb{N}\}. \tag{2.13}$$

Denote

$$\zeta_1 = x_{n_1-1} y_{n_1} - x_{n_1} y_{n_1-1}, \quad \zeta_2 = x_{n_2+1} y_{n_2+2} - x_{n_2+2} y_{n_2+1}. \tag{2.14}$$

**Lemma 2.4.** *Suppose $n_1 = n_2$. If $1 < n_1 < n - 1$, the set of all homogeneous $\mathcal{K}$-singular vectors are*

$$\{x_{n_1}^{m_1} y_{n_1}^{m_2} \zeta_1^{m_3+1}, x_{n_1+1}^{m_1} y_{n_1+1}^{m_2} \zeta_2^{m_3+1}, \eta^{m_3}(x_{n_1}^{m_1} y_{n_1+1}^{m_2}) \mid m_i \in \mathbb{N}\}. \tag{2.15}$$

*When $n_1 = n_2 = 1$ and $n \ge 3$, all homogeneous $\mathcal{K}$-singular vectors are*

$$\{x_2^{m_1} y_2^{m_2} \zeta_2^{m_3+1}, \eta^{m_3}(x_1^{m_1} y_2^{m_2}) \mid m_i \in \mathbb{N}\}. \tag{2.16}$$

*Assuming $n_1 = n_2 = n - 1$ and $n \ge 3$, we have the following set of all homogeneous $\mathcal{K}$-singular vectors:*

$$\{x_{n-1}^{m_1} y_{n-1}^{m_2} \zeta_1^{m_3+1}, \eta^{m_3}(x_{n-1}^{m_1} y_n^{m_2}) \mid m_i \in \mathbb{N}\}. \tag{2.17}$$

*In the case $n_1 = n_2 = 1$ and $n = 2$, all homogeneous $\mathcal{K}$-singular vectors are*

$$\{\eta^{m_3}(x_1^{m_1} y_2^{m_2}) \mid m_i \in \mathbb{N}\}. \tag{2.18}$$

*Assume $n_1 = n_2 = n$. All homogeneous $\mathcal{K}$-singular vectors are*

$$\{x_n^{m_1} y_n^{m_2} \zeta_1^{m_3} \mid m_i \in \mathbb{N}\}. \tag{2.19}$$

An element $g \in \mathcal{B}$ is called *nilpotent with respect to $\mathcal{K}_+$* if there exist a positive integer $m$ such that

$$\xi_1 \cdots \xi_m(g) = 0 \qquad \text{for any } \xi_1, ..., \xi_m \in \mathcal{K}_+. \tag{2.20}$$

**Lemma 2.5.**    *Any element in $\mathcal{B}$ is nilpotent with respect to $\mathcal{K}_+$. In particular, any nonzero $\mathcal{K}$-submodule of $\mathcal{B}$ must contain a $\mathcal{K}$-singular vector.*

For $\ell_1, \ell_2 \in \mathbb{Z}$, We write

$$\mathcal{B}_{\langle \ell_1, \ell_2 \rangle} = \operatorname{Span}\{x^\alpha y^\beta \mid \alpha, \beta \in \mathbb{N}^n; \sum_{r=n_1+1}^{n} \alpha_r - \sum_{i=1}^{n_1} \alpha_i = \ell_1; \sum_{i=1}^{n_2} \beta_i - \sum_{r=n_2+1}^{n} \beta_r = \ell_2\} \tag{2.21}$$

and

$$\mathcal{H}_{\langle \ell_1, \ell_2 \rangle} = \{f \in \mathcal{B}_{\langle \ell_1, \ell_2 \rangle} \mid \mathcal{D}(f) = 0\}. \tag{2.22}$$

Then

$$\mathcal{B}_{\langle k \rangle} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{B}_{\langle \ell, k-\ell \rangle}, \quad \mathcal{H}_{\langle k \rangle} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{H}_{\langle \ell, k-\ell \rangle} \tag{2.23}$$

for $k \in \mathbb{Z}$. The following is the main result in [13].

**Lemma 2.6.**    *Let $\ell_1, \ell_2 \in \mathbb{Z}$ such that $\ell_1 + \ell_2 \leq n_1 - n_2 + 1 - \delta_{n_1,n_2}$. For $m \in \mathbb{N}$, all nonzero subspaces $\eta^m(\mathcal{H}_{\langle \ell_1-m, \ell_2-m \rangle})$ are irreducible highest-weight $sl(n, \mathbb{C})$-submodules. Moreover, $\mathcal{B}_{\langle \ell_1, \ell_2 \rangle} = \bigoplus_{m=0}^{\infty} \eta^m(\mathcal{H}_{\langle \ell_1-m, \ell_2-m \rangle})$.*

Note

$$[\mathcal{D}, \eta] = n_2 - n_1 - \sum_{i=1}^{n_1} x_i \partial_{x_i} + \sum_{r=n_1+1}^{n} x_r \partial_{x_r} + \sum_{j=1}^{n_2} y_j \partial_{y_j} - \sum_{s=n_2+1}^{n} y_s \partial_{y_s} \tag{2.24}$$

by (1.11) and (1.12). This shows that the space $\mathbb{C}\mathcal{D} + \mathbb{C}[\mathcal{D}, \eta] + \mathbb{C}\eta$ forms an operator Lie algebra isomorphic to $sl(2, \mathbb{C})$. Furthermore, for any $g \in \mathcal{H}_{\langle \ell_1, \ell_2 \rangle}$ and $m \in \mathbb{N}$, we have $\eta^m(g) \in \mathcal{B}_{\ell_1+m, \ell_2+m}$ and

$$\mathcal{D}(\eta^m(g)) = m(n_2 - n_1 + \ell_1 + \ell_2 + m - 1)\eta^{m-1}(g). \tag{2.25}$$

## 3.    Proof of Theorem 1

In this section, we will prove it case by case.

*Case 1.* $n_1 + 1 < n_2$ *and* $n_1 - n_2 + 1 \geq k \in \mathbb{Z}$.

According to Lemma 2.2 and (2.24), the $\mathcal{K}$-singular vectors in $\mathcal{H}_{\langle k \rangle}$ are: for $m_1, m_2 \in \mathbb{N}$,

$$x_{n_1}^{m_1} y_{n_2+1}^{m_2} \qquad \text{with} \ -(m_1 + m_2) = k, \tag{3.1}$$

$$x_{n_1+1}^{m_1} y_{n_2+1}^{m_2} \qquad \text{with} \ m_1 - m_2 = k, \tag{3.2}$$

$$x_{n_1}^{m_1} y_{n_2}^{m_2} \qquad \text{with} \ -m_1 + m_2 = k. \tag{3.3}$$

Note

$$(E_{n+n_2+1,n_1} - E_{n+n_1,n_2+1})|_{\mathcal{B}} = -x_{n_1} \partial_{y_{n_2+1}} - y_{n_1} \partial_{x_{n_2+1}} \tag{3.4}$$

by (1.10). So

$$(E_{n+n_2+1,n_1} - E_{n+n_1,n_2+1})^{m_2}(x_{n_1}^{m_1} y_{n_2+1}^{m_2}) = (-1)^{m_2} m_2! x_{n_1}^{-k} \tag{3.5}$$

for the vectors in (3.1). Moreover,

$$(E_{n+n_2+1,n_1+1} - E_{n+n_1+1,n_2+1})|_{\mathcal{B}} = \partial_{x_{n_1+1}} \partial_{y_{n_2+1}} - y_{n_1+1} \partial_{x_{n_2+1}} \tag{3.6}$$

again by (1.10), which implies

$$(E_{n+n_2+1,n_1+1} - E_{n+n_1+1,n_2+1})^{m_2}(x_{n_1+1}^{m_1} y_{n_2+1}^{m_2}) = m_1! [\prod_{r=0}^{m_1-1}(m_2 - r)] y_{n_2+1}^{-k} \tag{3.7}$$

for the vectors in (3.2). Furthermore,

$$(E_{n_1,n+n_2} - E_{n_2,n+n_1})|_{\mathcal{B}} = \partial_{x_{n_1}} \partial_{y_{n_2}} - x_{n_2} \partial_{y_{n_1}} \tag{3.8}$$

by (1.9), which implies

$$(E_{n_1,n+n_2} - E_{n_2,n+n_1})^{m_2}(x_{n_1}^{m_1} y_{n_2}^{m_2}) = m_2! [\prod_{r=0}^{m_2-1}(m_1 - r)] x_{n_1}^{-k} \tag{3.9}$$

for the vectors in (3.3).

On the other hand,

$$(E_{n_1,n+n_2+1} - E_{n_2+1,n+n_1})|_{\mathcal{B}} = -y_{n_2+1} \partial_{x_{n_1}} - x_{n_2+1} \partial_{y_{n_1}} \tag{3.10}$$

by (1.9), which implies

$$(E_{n_1,n+n_2+1} - E_{n_2+1,n+n_1})^{m_2}(x_{n_1}^{-k}) = (-1)^{m_2} [\prod_{r=0}^{m_2-1}(-k - r)] x_{n_1}^{m_1} y_{n_2+1}^{m_2} \tag{3.11}$$

for the vectors in (3.1). Moreover,

$$(E_{n_1+1,n+n_2+1} - E_{n_2+1,n+n_1+1})|_{\mathcal{B}} = -x_{n_1+1} y_{n_2+1} - x_{n_2+1} \partial_{y_{n_1+1}} \tag{3.12}$$

by (1.9), which implies

$$(E_{n_1+1,n+n_2+1} - E_{n_2+1,n+n_1+1})^{m_2}(y_{n_2+1}^{-k}) = (-1)^{m_2} x_{n_1}^{m_1} y_{n_2+1}^{m_2} \tag{3.13}$$

for the vectors in (3.2). Furthermore,

$$(E_{n+n_2,n_1} - E_{n+n_1,n_2})|_{\mathcal{B}} = -x_{n_1} y_{n_2} - y_{n_1} \partial_{x_{n_2}} \tag{3.14}$$

by (1.10), which implies

$$(E_{n+n_2,n_1} - E_{n+n_1,n_2})^{m_2}(x_{n_1}^{-k}) = (-1)^{m_2} x_{n_1}^{m_1} y_{n_2}^{m_2} \tag{3.15}$$

for the vectors in (3.3). Thus for any two vectors in (3.1)-(3.3), there exists an element in the universal enveloping algebra $U(o(2n,\mathbb{C}))$ which carries one to another. On the other hand, the vectors in (3.1)-(3.3) have distinct weights. Thus any nonzero $o(2n,\mathbb{C})$-submodule of $\mathcal{H}_{\langle k \rangle}$ must contain one of the vectors in (3.1)-(3.3) by Lemmas 2.2 and 2.5. Hence all the vectors in (3.1)-(3.3) are in the submodule by (3.4)-(3.15). Therefore, the submodule must contains all $\mathcal{H}_{\langle \ell, k-\ell \rangle}$ for $\ell \in \mathbb{Z}$ by Lemma 2.6, equivalently, it is equal to $\mathcal{H}_{\langle k \rangle}$ due to (2.23). So $\mathcal{H}_{\langle k \rangle}$ is irreducible. By (3.12) and (3.14), $\mathcal{H}_{\langle k \rangle}$ is not of highest-weight type. For any $m \in \mathbb{N}$, $n_1 - n_2 + 1 \geq k - m$ and so $\mathcal{H}_{\langle k-m \rangle}$ is an irreducible $o(2n,\mathbb{C})$-submodule.

Since $\mathcal{D}$ is locally nilpotent, for any $0 \neq u \in \mathcal{B}_{\langle k \rangle}$, there exists an element $\kappa(u) \in \mathbb{N}$ such that

$$\mathcal{D}^{\kappa(u)}(u) \neq 0 \text{ and } \mathcal{D}^{\kappa(u)+1}(u) = 0. \tag{3.16}$$

Set

$$\Psi = \sum_{i=0}^{\infty} \eta^i(\mathcal{H}_{\langle k-2i \rangle}). \tag{3.17}$$

Given $0 \neq u \in \mathcal{B}_{\langle k \rangle}$, $\kappa(u) = 1$ implies $u \in \mathcal{H}_{\langle k \rangle} \subset \Psi$. Suppose that $u \in \Psi$ whenever $\kappa(u) < r$ for some positive integer $r$. Assume $\kappa(u) = r$. First

$$v = \mathcal{D}^r(u) \in \mathcal{H}_{\langle k-2r \rangle} \subset \Psi. \tag{3.18}$$

Note

$$\mathcal{D}^r[\eta^r(v)] = r![\prod_{i=1}^{r}(n_2 - n_1 + k - r - i)]v \tag{3.19}$$

by (2.25). Thus we have either

$$u = \frac{1}{r![\prod_{i=1}^{r}(n_2 - n_1 + k - r - i)]} \eta^r(v) \in \Psi \tag{3.20}$$

or

$$\kappa\left(u - \frac{1}{r![\prod_{i=1}^{r}(n_2 - n_1 + k - r - i)]} \eta^r(v)\right) < r. \tag{3.21}$$

By induction,

$$u - \frac{1}{r![\prod_{i=1}^{r}(n_2 - n_1 + k - r - i)]} \eta^r(v) \in \Psi, \tag{3.22}$$

which implies $u \in \Psi$. Therefore, we have $\Psi = \mathcal{B}_{\langle k \rangle}$. Since the weight of any $\mathcal{K}$-singular vector in $\eta^i(\mathcal{H}_{\langle k-2i \rangle})$ is different from the weight of any $\mathcal{K}$-singular vector in $\eta^j(\mathcal{H}_{\langle k-2j \rangle})$ when $i \neq j$, the sums in Theorem 1 are direct sums.

*Case 2. $n_1 + 1 = n_2$ and $0 \geq k \in \mathbb{Z}$.*

Assume $n_1 + 1 = n_2 < n$. By (2.12) and (2.24), the $\mathcal{K}$-singular vectors in $\mathcal{H}_{\langle k \rangle}$ are those in (3.1)-(3.3). So Theorem 1 holds by the arguments in Case 1. Suppose $n_1 < n_2 = n$. According to (2.13) and (2.24), the $\mathcal{K}$-singular vectors in $\mathcal{H}_{\langle k \rangle}$ are those in (3.3). Expressions (3.9) and (3.15)-(3.22) imply the conclusions in the Theorem 1.

*Case 3. $n_1 = n_2$ and $0 \geq k \in \mathbb{Z}$.*

Recall

$$\zeta_1 = x_{n_1-1}y_{n_1} - x_{n_1}y_{n_1-1}, \quad \zeta_2 = x_{n_1+1}y_{n_1+2} - x_{n_1+2}y_{n_1+1}. \tag{3.23}$$

Suppose $n_1 = n_2 < n - 1$. Lemma 2.4 and (2.24) tell us that the $\mathcal{K}$-singular vectors in $\mathcal{H}_{\langle k \rangle}$ are those in (3.1) and

$$x_{n_1}^{-k}\zeta_1^{m+1} \qquad \text{for } m \in \mathbb{N}, \tag{3.24}$$

$$y_{n_1+1}^{-k}\zeta_2^{m+1} \qquad \text{for } m \in \mathbb{N}. \tag{3.25}$$

Again all the singular vectors have distinct weights. If $\mathcal{N}$ is a nonzero submodule of $\mathcal{H}_{\langle k \rangle}$, then $\mathcal{N}$ must contain one of the above $\mathcal{K}$-singular vectors by Lemma 2.5. If $\mathcal{N}$ contains a singular vector in (3.1), then $x_{n_1}^{-k} \in N$ by (3.5). Suppose $x_{n_1}^{-k}\zeta_1^{m+1} \in N$ for some $m \in \mathbb{N}$. Note

$$(E_{n_1-1,n+n_1} - E_{n_1,n+n_1-1})|_{\mathcal{B}} = \partial_{x_{n_1-1}}\partial_{y_{n_1}} - \partial_{x_{n_1}}\partial_{y_{n_1-1}} \tag{3.26}$$

by (1.9). Thus

$$(E_{n_1-1,n+n_1} - E_{n_1,n+n_1-1})^{m+1}(x_{n_1}^{-k}\zeta_1^{m+1})$$
$$= \left[\sum_{r=0}^{m+1}(-1)^r\binom{m+1}{r}(\partial_{x_{n_1-1}}\partial_{y_{n_1}})^{m+1-r}(\partial_{x_{n_1}}\partial_{y_{n_1-1}})^r\right]$$
$$\times \left[\sum_{s=0}^{m+1}(-1)^s\binom{m+1}{s}(x_{n_1-1}y_{n_1})^{m+1-s}x_{n_1}^{-k+s}y_{n_1-1}^s\right]$$
$$= \left(\sum_{r=0}^{m+1}\binom{m+1}{r}^2[(m+1-r!)]^2r![\prod_{i=1}^{r}(-k+i)]\right)x_{n_1}^{-k}$$
$$= [(m+1)!]^2\left(\sum_{r=0}^{m+1}\binom{-k+r}{r}\right)x_{n_1}^{-k} \in N. \tag{3.27}$$

So we have $x_{n_1}^{-k} \in \mathcal{N}$ again. Symmetrically, $y_{n_1+1}^{-k} \in \mathcal{N}$ if $y_{n_1+1}^{-k}\zeta_2^{m+1} \in \mathcal{N}$ for some $m \in \mathbb{N}$. Observe

$$(E_{n+n_1,n_1+1} - E_{n+n_1+1,n_1})|_{\mathcal{B}} = y_{n_1}\partial_{x_{n_1+1}} + x_{n_1}\partial_{y_{n_1+1}} \tag{3.28}$$

by (1.10). Thus

$$(E_{n+n_1,n_1+1} - E_{n+n_1+1,n_1})^{-k}(y_{n_1+1}^{-k}) = (-k)!x_{n_1}^{-k} \in \mathcal{N}. \qquad (3.29)$$

Thus we always have $x_{n_1}^{-k} \in \mathcal{N}$.

According to (3.11), $\mathcal{N}$ contains all the singular vectors in (3.1). Observe

$$(E_{n+n_1-1,n_1} - E_{n+n_1,n_1-1})|_{\mathcal{B}} = \zeta_1, \quad (E_{n_1+2,n+n_1+1} - E_{n_1+1,n+n_1+2})|_{\mathcal{B}} = \zeta_2 \quad (3.30)$$

as multiplication operators on $\mathcal{B}$ by (1.9) and (1.10). Thus

$$(E_{n+n_1-1,n_1} - E_{n+n_1,n_1-1})^{m+1}(x_{n_1}^{-k}) = x_{n_1}^{-k}\zeta_1^{m+1}, \qquad (3.31)$$

$$(E_{n_1+2,n+n_1+1} - E_{n_1+1,n+n_1+2})^{m+1}(x_{n_1}^{-k}) = x_{n_1}^{-k}\zeta_2^{m+1} \in \mathcal{N}. \qquad (3.32)$$

Note

$$(E_{n_1+1,n+n_1} - E_{n_1,n+n_1+1})|_{\mathcal{B}} = x_{n_1+1}\partial_{y_{n_1}} + y_{n_1+1}\partial_{x_{n_1}} \qquad (3.33)$$

by (1.9). So

$$\frac{1}{(-k)!}(E_{n_1+1,n+n_1} - E_{n_1,n+n_1+1})^{-k}(x_{n_1}^{-k}\zeta_2^{m+1}) = y_{n_1+1}^{-k}\zeta_2^{m+1} \in \mathcal{N}. \qquad (3.34)$$

Thus $\mathcal{N}$ contains all the $\mathcal{K}$-singular vectors in $\mathcal{H}_{\langle k \rangle}$, which implies that it contains all $\mathcal{H}_{\langle \ell, k-\ell \rangle} \subset \mathcal{H}_{\langle k \rangle}$ by Lemma 2.6. So $\mathcal{N} = \mathcal{H}_{\langle k \rangle}$ by (2.23), that is, $\mathcal{H}_{\langle k \rangle}$ is an irreducible $o(2n, \mathbb{C})$-module. By (3.16)-(3.22), the direct sums in Theorem 1 holds. Observe that Theorem 1 under the subcase $n_1 = n_2 = n-1, n$ is implied by (2.17)-(2.19) and the related partial arguments in the above. This completes the proof of Theorem 1. ∎

Suppose $n_1 < n_2$. Taking $T_1 = \partial_{x_{n_1+1}}\partial_{y_{n_1+1}}$, $T_1^- = \int_{(x_{n_1+1})}\int_{(y_{n_1+1})}$ and $T_2 = \mathcal{D} - \partial_{x_{n_1+1}}\partial_{y_{n_1+1}}$ in Lemma 2.1, $\mathcal{H}_{\langle k \rangle}$ has a basis

$$\Big\{ \sum_{i=0}^{\infty} \frac{(x_{n_1+1}y_{n_1+1})^i(\mathcal{D} - \partial_{x_{n_1+1}}\partial_{y_{n_1+1}})^i(x^\alpha y^\beta)}{\prod_{r=1}^{i}(\alpha_{n_1+1} + r)(\beta_{n_1+1} + r)} \mid \alpha, \beta \in \mathbb{N}^n;$$

$$\alpha_{n_1+1}\beta_{n_1+1} = 0; -\sum_{i=1}^{n_1}\alpha_i + \sum_{r=n_1+1}^{n}\alpha_r + \sum_{i=1}^{n_2}\beta_i - \sum_{r=n_2+1}^{n}\beta_r = k \Big\}. \quad (3.35)$$

Finally, we want to find an expression for $\mathcal{H}_{\langle k \rangle}$ for $0 \geq k \in \mathbb{Z}$ when $n_1 = n_2$. First $n_1 = n_2 = 1$ and $n = 2$. According to (4.30) and (4.38) in [13],

$$\mathcal{H}_{\langle -k \rangle} = \text{Span}\{[x_1^r y_2^s(x_1 x_2 - y_1 y_2)^l \mid r, s, l \in \mathbb{N}; r + s = k\}. \qquad (3.36)$$

Next we consider the subcase $n_1 = n_2 = 1$ and $n \geq 3$. According to (4.26), (4.38) and (4.41) in [13]

$$\mathcal{H}_{\langle -k \rangle}$$
$$= \text{Span}\{[\prod_{r=2}^{n} y_r^{\widehat{l_r}}][\prod_{2 \leq p < q \leq n}(x_p y_q - x_q y_p)^{\widehat{k_{p,q}}}][\prod_{s=2}^{n}(x_1 x_s - y_1 y_s)^{\widehat{l_s}}], x_1^l[\prod_{s=2}^{n} y_s^{k_s}]$$
$$\times[\prod_{s=2}^{n}(x_1 x_s - y_1 y_s)^{l_s}] \mid l, k_s, l_s, \widehat{l}, \widehat{k_{p,q}}, \widehat{l_s} \in \mathbb{N}; l + \sum_{s=2}^{n} k_s = \sum_{r=2}^{n} \widehat{l_r} = k\}. \quad (3.37)$$

Assume $1 < n_1 = n_2 < n-1$. By (4.23), (4.38), (4.39) (note $\mathcal{H}'_{\langle -m_1-m_2,m_2\rangle} = \mathcal{H}_{\langle -m_1-m_2,m_2\rangle}$) and (4.41) in [13], we have

$$\mathcal{H}_{\langle -k\rangle}$$
$$= \mathrm{Span}\{[\prod_{r=1}^{n_1} x_r^{l'_r}][\prod_{1\le p<q\le n_1}(x_py_q - x_qy_p)^{k'_{p,q}}][\prod_{r=1}^{n_1}\prod_{s=n_1+1}^{n}(x_rx_s - y_ry_s)^{l'_{r,s}}],$$

$$[\prod_{r=1}^{n-n_1} y_{n_1+r}^{\widehat{l_r}}][\prod_{n_1+1\le p<q\le n}(x_py_q - x_qy_p)^{\widehat{k}_{p,q}}][\prod_{r=1}^{n_1}\prod_{s=n_1+1}^{n}(x_rx_s - y_ry_s)^{\widehat{l}_{r,s}}],$$

$$[\prod_{r=1}^{n_1} x_r^{l_r}][\prod_{s=1}^{n-n_1} y_{n_1+s}^{k_s}][\prod_{r=1}^{n_1}\prod_{s=1}^{n-n_1}(x_rx_{n_1+s} - y_ry_{n_1+s})^{l_{r,s}}] \mid l_r, k_s, l_{r,s}, l'_r, k'_{p,q},$$

$$l'_{r,s}, \widehat{l}_r, \widehat{k}_{p,q}, \widehat{l}_{r,s} \in \mathbb{N}; \sum_{r=1}^{n_1} l_r + \sum_{s=1}^{n-n_1} k_s = \sum_{r=1}^{n_1} l'_r = \sum_{r=1}^{n-n_1} \widehat{l}_r = k\}. \tag{3.38}$$

Consider the subcase $n_1 = n_2 = n - 1$ and $n \ge 3$. By (4.28), (4.38) and (4.39) (note $\mathcal{H}'_{\langle -m_1-m_2,m_2\rangle} = \mathcal{H}_{\langle -m_1-m_2,m_2\rangle}$) in [13], we obtain

$$\mathcal{H}_{\langle -k\rangle}$$
$$= \mathrm{Span}\{[\prod_{r=1}^{n-1} x_r^{l'_r}][\prod_{1\le p<q\le n-1}(x_py_q - x_qy_p)^{k'_{p,q}}][\prod_{r=1}^{n-1}(x_rx_n - y_ry_n)^{\bar{l}'_r}], [\prod_{r=1}^{n-1} x_r^{l_r}]y_n^{\widehat{k}}$$

$$\times [\prod_{r=1}^{n-1}(x_rx_n - y_ry_n)^{\bar{l}_r}] \mid l_r, \widehat{k}, \bar{l}_r, l'_r, k'_{p,q}, \bar{l}'_r \in \mathbb{N}; \sum_{r=1}^{n-1} l_r + \widehat{k} = \sum_{r=1}^{n-1} l'_r = k\}. \tag{3.39}$$

At last, we assume $n_1 = n_2 = n$. By (4.32) and (4.39) (note $\mathcal{H}'_{\langle -m_1-m_2,m_2\rangle} = \mathcal{H}_{\langle -m_1-m_2,m_2\rangle}$) in [13],

$$\mathcal{H}_{\langle -k\rangle} = \mathrm{Span}\{\prod_{r=1}^{n} x_r^{l_r}][\prod_{1\le p<q\le n}(x_py_q - x_qy_p)^{k_{p,q}}] \mid l_r, k_{p,q} \in \mathbb{N}; \sum_{r=1}^{n} l_r = k\}. \tag{3.40}$$

## 4. Proof of Theorem 2

In this section, we prove Theorem 2.

Recall

$$o(2n+1,\mathbb{C}) = o(2n,\mathbb{C}) \oplus \bigoplus_{i=1}^{n}[\mathbb{C}(E_{0,i} - E_{n+i,0}) + \mathbb{C}(E_{0,n+i} - E_{i,0})] \tag{4.1}$$

and $\mathcal{B}' = \mathbb{C}[x_0, x_1, ..., x_n, y_1, ..., y_n]$. Fix $n_1, n_2 \in \overline{1,n}$ such that $n_1 \le n_2$. The representation of $o(2n+1,\mathbb{C})$ on $\mathcal{B}'$ by the differential operators in (1.6)-(1.10) with $|_\mathcal{B}$ replaced by $|_{\mathcal{B}'}$, (1.16) and (1.17). Recall $\mathcal{B}'_{\langle k\rangle} = \sum_{i=0}^{\infty} \mathcal{B}_{\langle k-i\rangle}x_0^i$. Then all $\mathcal{B}'_{\langle k\rangle}$ with $k \in \mathbb{Z}$ are $o(2n+1,\mathbb{C})$-submodules and $\mathcal{B}' = \bigoplus_{k\in\mathbb{Z}} \mathcal{B}'_{\langle k\rangle}$ forms a $\mathbb{Z}$-graded algebra. Moreover, the related Laplace operator $\mathcal{D}' = \partial_{x_0}^2 + 2\mathcal{D}$ by (1.18) and its dual $\eta' = x_0^2 + 2\eta$ by (1.19). A straightforward verification shows

$$\mathcal{D}'\xi = \xi\mathcal{D}', \ \xi\eta' = \eta'\xi \text{ on } \mathcal{B}' \qquad \text{for } \xi \in o(2n+1,\mathbb{C}). \tag{4.2}$$

As in the introduction, $\mathcal{H}'_{\langle k \rangle} = \{f \in \mathcal{B}'\langle k \rangle \mid \mathcal{D}'(f) = 0\}$. According to (4.2), $\mathcal{H}'_{\langle k \rangle}$ is an $o(2n+1, \mathbb{C})$-submodule.

For $\iota = 0, 1$, we define

$$T_\iota = \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i+\iota}}{(2i+\iota)!} \mathcal{D}^i. \tag{4.3}$$

By Lemma 2.1 with $T_1 = \partial_{x_0}^2$, $T_1^- = \int_{(x_0)}^{(2)}$ and $T_2 = 2\mathcal{D}$, we obtain

$$\mathcal{H}'_{\langle k \rangle} = T_0(\mathcal{B}_{\langle k \rangle}) \oplus T_1(\mathcal{B}_{\langle k-1 \rangle}). \tag{4.4}$$

**Lemma 4.1.** *Suppose $n_1 < n_2$. For $k \in \mathbb{N}$, $\mathcal{H}'_{\langle k \rangle}$ is an $o(2n+1, \mathbb{C})$-submodule generated by $x_{n_1+1}^k$ and $\mathcal{H}'_{\langle -k \rangle}$ is an $o(2n+1, \mathbb{C})$-submodule generated by $x_{n_1}^k$.*

**Proof.** First we assume $n_1 < n$. Let $V$ be an $o(2n+1, \mathbb{C})$-submodule generated by $x_{n_1+1}^k$. Since $x_{n_1+1}^k \in \mathcal{H}'_{\langle k \rangle}$ that is an $o(2n+1, \mathbb{C})$-submodule, we have $V \subset \mathcal{H}'_{\langle k \rangle}$.

Observe

$$(E_{n_1+1,n+j} - E_{j,n+n_1+1})|_{\mathcal{B}'} = -x_{n_1+1}y_j - x_j\partial_{y_{n_1+1}} \qquad \text{for } j \in \overline{n_2+1, n} \tag{4.5}$$

by (1.9) with $|_{\mathcal{B}}$ replaced by $|_{\mathcal{B}'}$. Repeatedly applying (4.5) to $x_{n_1+1}^k$ with various $j \in \overline{n_2+1, n}$, we obtain

$$x_{n_1+1}^{k+\ell} \prod_{s=n_2+1}^{n} y_s^{\beta_s} \in V \qquad \text{for } \ell, \beta_{n_2+1}, ..., \beta_n \in \mathbb{N}, \ \sum_{s=n_2+1}^{n} \beta_s = \ell. \tag{4.6}$$

Note

$$(E_{n+i,n+n_1+1} - E_{n_1+1,i})|_{\mathcal{B}'} = y_i\partial_{y_{n_1+1}} + x_ix_{n_1+1} \qquad \text{for } i \in \overline{1, n_1} \tag{4.7}$$

by (1.6)-(1.8) with $|_{\mathcal{B}}$ replaced by $|_{\mathcal{B}'}$. Repeatedly applying (4.7) to (4.6) with various $i \in \overline{1, n_1}$, we have

$$[\prod_{i=1}^{n_1+1} x_i^{\alpha_i}][\prod_{s=n_2+1}^{n} y_s^{\beta_s}] \in V \text{ for } \alpha_1, ..., \alpha_{n_1+1}, \beta_{n_2+1}, ..., \beta_n \in \mathbb{N} \tag{4.8}$$

such that $k + \sum_{r=1}^{n_1} \alpha_r + \sum_{s=n_2+1}^{n} \beta_s = \alpha_{n_1+1}$.

Denote

$$I = \{0, \overline{n_1+1, n_2}, \overline{n+n_1+1, n+n_2}\}. \tag{4.9}$$

Then the Lie subalgebra

$$\mathcal{G} = o(2n+1, \mathbb{C}) \bigcap (\sum_{i,j \in I} \mathbb{C}E_{i,j}) \cong o(2(n_2 - n_1) + 1, \mathbb{C}). \tag{4.10}$$

Set

$$\bar{\mathcal{H}}_{\langle \ell \rangle} = \mathcal{H}'_{\langle \ell \rangle} \bigcap \mathbb{C}[x_0, x_{n_1+1}, ..., x_{n_2}, y_{n_1+1}, ..., y_{n_2}] \qquad \text{for } \ell \in \mathbb{N}. \tag{4.11}$$

Then $\bar{\mathcal{H}}_{\langle\ell\rangle}$ forms an irreducible $\mathcal{G}$-module by an equivalent form of the classical theorem on harmonic polynomials. Repeatedly applying $\mathcal{G}|_{\mathcal{B}'}$ to (4.8), we get

$$T_\iota(x^\alpha y^\beta) \in V \tag{4.12}$$

for $\alpha, \beta \in \mathbb{N}^n$ such that $\alpha_i = 0$ if $i > n_2$, $\beta_j = 0$ if $j \leq n_1$, and

$$\sum_{r=n_1+1}^{n_2} \alpha_r - \sum_{i=1}^{n_1} \alpha_i + \sum_{s=n_1+1}^{n_2} \beta_s - \sum_{j=n_2+1}^{n} \beta_j = k. \tag{4.13}$$

Repeatedly applying (4.7) to (4.13) satisfying the above conditions, we get that (4.13) holds for $\alpha, \beta \in \mathbb{N}^n$ such that $\alpha_i = 0$ if $i > n_2$, and

$$\sum_{r=n_1+1}^{n_2} \alpha_r - \sum_{i=1}^{n_1} \alpha_i + \sum_{s=1}^{n_2} \beta_s - \sum_{j=n_2+1}^{n} \beta_j = k. \tag{4.14}$$

According (1.6)-(1.8) with $|_\mathcal{B}$ replaced by $|_{\mathcal{B}'}$,

$$(E_{i,n_1+1} - E_{n+n_1+1,n+i})|_{\mathcal{B}'} = x_i \partial_{x_{n_1+1}} + y_{n_1+1} y_i \qquad \text{for } i \in \overline{n_2 + n}. \tag{4.15}$$

Repeatedly applying (4.15) to (4.13) satisfying (4.14), we get (4.13) for any $\alpha, \beta \in \mathbb{N}^n$ such that

$$\sum_{r=n_1+1}^{n} \alpha_r - \sum_{i=1}^{n_1} \alpha_i + \sum_{s=1}^{n_2} \beta_s - \sum_{j=n_2+1}^{n} \beta_j = k. \tag{4.16}$$

Thus $V = \mathcal{H}'_{\langle k\rangle}$.

Let $U$ be an $o(2n+1, \mathbb{C})$-submodule generated by $x_{n_1}^k$. Since $x_{n_1}^k \in \mathcal{H}'_{\langle -k\rangle}$ that is an $o(2n+1, \mathbb{C})$-submodule, we have $U \subset \mathcal{H}'_{\langle -k\rangle}$. Repeatedly applying (4.5) and (4.7) to $x_{n_1}^k$ with various $i \in \overline{1, n_1}$ and $j \in \overline{n_2 + 1, n}$, we have

$$x_{n_1}^k [\prod_{i=1}^{n_1+1} x_i^{\alpha_i}][\prod_{s=n_2+1}^{n} y_s^{\beta_s}] \in U \text{ for } \alpha_1, ..., \alpha_{n_1+1}, \beta_{n_2+1}, ..., \beta_n \in \mathbb{N} \tag{4.17}$$

such that $\sum_{r=1}^{n_1} \alpha_r + \sum_{s=n_2+1}^{n} \beta_s = \alpha_{n_1+1}$. Note

$$(E_{n_1,i} - E_{n+i,n+n_1})|_{\mathcal{B}'} = -x_i \partial_{x_{n_1}} - y_i \partial_{y_{n_1}} \qquad \text{for } i \in \overline{n_1 - 1} \tag{4.18}$$

by (1.6)-(1.8) with $|_\mathcal{B}$ replaced by $|_{\mathcal{B}'}$. Repeatedly applying (4.18) to (4.17), we have

$$[\prod_{i=1}^{n_1+1} x_i^{\alpha_i}][\prod_{s=n_2+1}^{n} y_s^{\beta_s}] \in U \text{ for } \alpha_1, ..., \alpha_{n_1+1}, \beta_{n_2+1}, ..., \beta_n \in \mathbb{N} \tag{4.19}$$

such that $\sum_{r=1}^{n_1} \alpha_r + \sum_{s=n_2+1}^{n} \beta_s = \alpha_{n_1+1} + k$.

Repeatedly applying $\mathcal{G}|_{\mathcal{B}'}$ to (4.19), we get

$$T_\iota(x^\alpha y^\beta) \in U \tag{4.20}$$

for $\alpha, \beta \in \mathbb{N}^n$ such that $\alpha_i = 0$ if $i > n_2$, $\beta_j = 0$ if $j \leq n_1$, and

$$\sum_{r=n_1+1}^{n_2} \alpha_r - \sum_{i=1}^{n_1} \alpha_i + \sum_{s=n_1+1}^{n_2} \beta_s - \sum_{j=n_2+1}^{n} \beta_j = -k. \tag{4.21}$$

Repeatedly applying (4.7) to (4.20) satisfying the above conditions, we get that (4.20) holds for $\alpha, \beta \in \mathbb{N}^n$ such that $\alpha_i = 0$ if $i > n_2$, and

$$\sum_{r=n_1+1}^{n_2} \alpha_r - \sum_{i=1}^{n_1} \alpha_i + \sum_{s=1}^{n_2} \beta_s - \sum_{j=n_2+1}^{n} \beta_j = -k. \tag{4.22}$$

Repeatedly applying (4.15) to (4.20) satisfying (4.22), we get (4.20) for any $\alpha, \beta \in \mathbb{N}^n$ such that

$$\sum_{r=n_1+1}^{n} \alpha_r - \sum_{i=1}^{n_1} \alpha_i + \sum_{s=1}^{n_2} \beta_s - \sum_{j=n_2+1}^{n} \beta_j = -k. \tag{4.23}$$

Thus $U = \mathcal{H}'_{\langle -k \rangle}$.                                                                 ∎

**Lemma 4.2.** *Suppose $n_1 = n_2$. For $k \in \mathbb{N}$, $\mathcal{H}'_{\langle -k \rangle}$ is an $o(2n+1, \mathbb{C})$-submodule generated by $x_{n_1}^k$ and $\mathcal{H}'_{\langle k \rangle}$ is an $o(2n+1, \mathbb{C})$-submodule generated by $T_1(y_{n_1}^{k-1})$ when $k > 0$.*

**Proof.**     In this case,

$$\mathcal{D} = -\sum_{i=1}^{n_1} x_i \partial_{y_i} - \sum_{s=n_1+1}^{n} y_s \partial_{x_s} \tag{4.24}$$

Note

$$(E_{n+n_1,0} - E_{0,n_1})|_{\mathcal{B}'} = y_{n_1} \partial_{x_0} + x_0 x_{n_1} \tag{4.25}$$

by (1.16) and (1.17) . Thus for $u \in \mathcal{B}$,

$$(E_{n+n_1,0} - E_{0,n_1})[T_0(u)]$$

$$= (y_{n_1} \partial_{x_0} + x_0 x_{n_1})(\sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i}}{(2i)!} \mathcal{D}^i(u))$$

$$= \sum_{i=1}^{\infty} \frac{(-2)^i x_0^{2i-1}}{(2i-1)!} y_{n_1} \mathcal{D}^i(u) + \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i+1}}{(2i)!} \mathcal{D}^i(x_{n_1} u)$$

$$= \sum_{i=1}^{\infty} \frac{(-2)^i i x_0^{2i-1}}{(2i-1)!} \mathcal{D}^{i-1}(x_{n_1} u) + \sum_{i=1}^{\infty} \frac{(-2)^i x_0^{2i-1}}{(2i-1)!} \mathcal{D}^{i-1}[\mathcal{D}(y_{n_1} u)]$$

$$+ \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i+1}}{(2i)!} \mathcal{D}^i(x_{n_1} u)$$

$$= -\sum_{i=0}^{\infty} \frac{(-2)^i 2(i+1) x_0^{2i+1}}{(2i+1)!} \mathcal{D}^i(x_{n_1} u) - 2\sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i+1}}{(2i+1)!} \mathcal{D}^i[\mathcal{D}(y_{n_1} u)]$$

$$+ \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i+1}}{(2i)!} \mathcal{D}^i(x_{n_1} u) = -T_1(x_{n_1} u) - 2T_1[\mathcal{D}(y_{n_1} u)] \tag{4.26}$$

and

$$(E_{n+n_1,0} - E_{0,n_1})[T_1(u)]$$
$$= (y_{n_1}\partial_{x_0} + x_0 x_{n_1})(\sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i+1}}{(2i+1)!} \mathcal{D}^i(u))$$
$$= \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i}}{(2i)!} y_{n_1} \mathcal{D}^i(u) + \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i+2}}{(2i+1)!} \mathcal{D}^i(x_{n_1} u)$$
$$= -\sum_{i=1}^{\infty} \frac{(-2)^{i-1} x_0^{2i}}{(2i-1)!} \mathcal{D}^{i-1}(x_{n_1} u) + \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i}}{(2i)!} \mathcal{D}^i(y_{n_1} u)$$
$$+ \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i+2}}{(2i+1)!} \mathcal{D}^i(x_{n_1} u)$$
$$= \sum_{i=0}^{\infty} \frac{(-2)^i x_0^{2i}}{(2i)!} \mathcal{D}^i(y_{n_1} u) = T_0(y_{n_1} u). \tag{4.27}$$

Symmetrically, we have,

$$(E_{n_1+1,0} - E_{0,n+n_1+1})|_{\mathcal{B}'} = x_{n_1+1}\partial_{x_0} + x_0 y_{n_1+1} \tag{4.28}$$

by (1.16) and (1.17). So

$$(E_{n_1+1,0} - E_{0,n+n_1+1})[T_0(u)] = -T_1(y_{n_1+1}u) - 2T_1[\mathcal{D}(x_{n_1+1}u)] \tag{4.29}$$

and

$$(E_{n_1+1,0} - E_{0,n+n_1+1})[T_1(u)] = T_0(x_{n_1+1}u). \tag{4.30}$$

Let $V$ be an $o(2n+1, \mathbb{C})$-submodule generated by $x_{n_1}^k$. Since $x_{n_1}^k \in \mathcal{H}'_{\langle -k \rangle}$ that is an $o(2n+1, \mathbb{C})$-submodule, we have $V \subset \mathcal{H}'_{\langle -k \rangle}$. Note

$$(E_{n_1+1,n+n_1} - E_{n_1,n+n_1+1})|_{\mathcal{B}'} = x_{n_1+1}\partial_{y_{n_1}} + y_{n_1+1}\partial_{x_{n_1}} \tag{4.31}$$

by (1.9) with $|_{\mathcal{B}}$ replaced by $|_{\mathcal{B}'}$. Repeatedly applying (4.31) to $x_{n_1}^k$, we have

$$x_{n_1}^{k-r} y_{n_1+1}^r = T_0(x_{n_1}^{k-r} y_{n_1+1}^r) \in V \qquad \text{for } r \in \overline{0, k}. \tag{4.32}$$

According to (4.26),

$$(E_{n+n_1,0} - E_{0,n_1})[T_0(x_{n_1}^{k-r} y_{n_1+1}^r)]$$
$$= -T_1(x_{n_1}^{k-r+1} y_{n_1+1}^r)) - 2T_1[\mathcal{D}(x_{n_1}^{k-r} y_{n_1} y_{n_1+1}^r)] = T_1(x_{n_1}^{k-r+1} y_{n_1+1}^r) \in V. \tag{4.33}$$

Similarly, (4.29) shows

$$(E_{n_1+1,0} - E_{0,n+n_1+1})[T_0(x_{n_1}^{k-r} y_{n_1+1}^r)] = T_1(x_{n_1}^{k-r} y_{n_1+1}^{r+1}) \in V. \tag{4.34}$$

Suppose that

$$T_1(x_{n_1}^{m_1} y_{n_1}^{m_2} x_{n_1+1}^{m_3} y_{n_1+1}^{m_4}) \in V \qquad \text{for } m_i \in \mathbb{N}, \ m_1 + m_4 - m_2 - m_3 = k+1 \tag{4.35}$$

and $\sum_{i=1}^{4} m_i = \ell$. Then (4.27) implies

$$(E_{n+n_1,0} - E_{0,n_1})[T_1(x_{n_1}^{m_1} y_{n_1}^{m_2} x_{n_1+1}^{m_3} y_{n_1+1}^{m_4})] = T_0(x_{n_1}^{m_1} y_{n_1}^{m_2+1} x_{n_1+1}^{m_3} y_{n_1+1}^{m_4}) \in V \quad (4.36)$$

and (4.30) yields

$$(E_{n_1+1,0} - E_{0,n+n_1+1})[T_1(x_{n_1}^{m_1} y_{n_1}^{m_2} x_{n_1+1}^{m_3} y_{n_1+1}^{m_4})] = T_0(x_{n_1}^{m_1} y_{n_1}^{m_2} x_{n_1+1}^{m_3+1} y_{n_1+1}^{m_4}) \in V. \quad (4.37)$$

For $i_1, i_2 \in \mathbb{N}$ such that $i_1 + i_2 = 1$, (4.24), (4.26), (4.36) and (4.37) give

$$\begin{aligned}
&(E_{n+n_1,0} - E_{0,n_1})[T_0(x_{n_1}^{m_1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_2} y_{n_1+1}^{m_4})] \\
&= -T_1(x_{n_1}^{m_1+1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_2} y_{n_1+1}^{m_4}) - 2T_1[\mathcal{D}(x_{n_1}^{m_1} y_{n_1}^{m_2+i_1+1} x_{n_1+1}^{m_3+i_2} y_{n_1+1}^{m_4})] \\
&= (2(m_2 + i_1) + 1)T_1(x_{n_1}^{m_1+1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_2} y_{n_1+1}^{m_4}) \\
&\quad + 2(m_3 + i_2)T_1(x_{n_1}^{m_1} y_{n_1}^{m_2+i_1+1} x_{n_1+1}^{m_3+i_2-1} y_{n_1+1}^{m_4+1}) \in V.
\end{aligned} \quad (4.38)$$

On the other hand,

$$(E_{n_1+1,n_1} - E_{n+n_1,n+n_1+1})|_{\mathcal{B}'} = -x_{n_1} x_{n_1+1} + y_{n_1} y_{n_1+1} \quad (4.39)$$

by (1.6)-(1.8) with $|_{\mathcal{B}}$ replaced by $|_{\mathcal{B}'}$. Thus

$$\begin{aligned}
&(E_{n_1+1,n_1} - E_{n+n_1,n+n_1+1})[T_1(x_{n_1}^{m_1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_2-1} y_{n_1+1}^{m_4})] \\
&= -T_1(x_{n_1}^{m_1+1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_2} y_{n_1+1}^{m_4}) + T_1(x_{n_1}^{m_1} y_{n_1}^{m_2+i_1+1} x_{n_1+1}^{m_3+i_1-1} y_{n_1+1}^{m_4+1}) \in V. \quad (4.40)
\end{aligned}$$

Solving the system (4.38) and (4.40) for $T_1(x_{n_1}^{m_1+1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_2} y_{n_1+1}^{m_4})$, we get

$$T_1(x_{n_1}^{m_1+1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_2} y_{n_1+1}^{m_4}) \in V. \quad (4.41)$$

Symmetrically, (4.24), (4.29), (4.36) and (4.37) give

$$\begin{aligned}
&(E_{n_1+1,0} - E_{0,n+n_1+1})[T_0(x_{n_1}^{m_1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_2} y_{n_1+1}^{m_4})] \\
&= -T_1(x_{n_1}^{m_1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_2} y_{n_1+1}^{m_4+1}) - 2T_1[\mathcal{D}(x_{n_1}^{m_1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_2+1} y_{n_1+1}^{m_4})] \\
&= (2(m_3 + i_2) + 1)T_1(x_{n_1}^{m_1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_2} y_{n_1+1}^{m_4+1}) \\
&\quad + 2(m_2 + i_1)T_1(x_{n_1}^{m_1+1} y_{n_1}^{m_2+i_1-1} x_{n_1+1}^{m_3+i_2+1} y_{n_1+1}^{m_4}) \in V
\end{aligned} \quad (4.42)$$

and (4.39) yields

$$\begin{aligned}
&(E_{n_1+1,n_1} - E_{n+n_1,n+n_1+1})[T_1(x_{n_1}^{m_1} y_{n_1}^{m_2+i_1-1} x_{n_1+1}^{m_3+i_2} y_{n_1+1}^{m_4})] \\
&= -T_1(x_{n_1}^{m_1+1} y_{n_1}^{m_2+i_1-1} x_{n_1+1}^{m_3+i_2+1} y_{n_1+1}^{m_4}) + T_1(x_{n_1}^{m_1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_1} y_{n_1+1}^{m_4+1}) \in V. \quad (4.43)
\end{aligned}$$

Solving the system (4.42) and (4.43) for $T_1(x_{n_1}^{m_1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_2} y_{n_1+1}^{m_4+1})$, we find

$$T_1(x_{n_1}^{m_1} y_{n_1}^{m_2+i_1} x_{n_1+1}^{m_3+i_2} y_{n_1+1}^{m_4+1}) \in V. \quad (4.44)$$

By induction on $\sum_{i=1}^{4} m_i$, we conclude

$$T_\iota(x_{n_1}^{m_1} y_{n_1}^{m_2} x_{n_1+1}^{m_3} y_{n_1+1}^{m_4}) \in V \qquad \text{for} \ \ m_i \in \mathbb{N}, \ m_1 + m_4 - m_2 - m_3 = k + \delta_{\iota,1}. \quad (4.45)$$

Set

$$\mathcal{A}' = \mathbb{C}[x_1, ..., x_{n_1}, y_1, ..., y_{n_1}], \quad \mathcal{A}^* = \mathbb{C}[x_{n_1+1}, ..., x_n, y_{n_1+1}, ..., y_n], \qquad (4.46)$$

$$\mathcal{K}_1 = \sum_{i,j=1}^{n_1} [\mathbb{C}(E_{i,j} - E_{n+j,n+i}) + \mathbb{C}(E_{i,n+j} - E_{j,n+i}) + \mathbb{C}(E_{n+i,j} - E_{n+j,i})] \quad (4.47)$$

and

$$\mathcal{K}_2 = \sum_{i,j=n_1+1}^{n} [\mathbb{C}(E_{i,j} - E_{n+j,n+i}) + \mathbb{C}(E_{i,n+j} - E_{j,n+i}) + \mathbb{C}(E_{n+i,j} - E_{n+j,i})]. \quad (4.48)$$

Then $\mathcal{K}_1 \cong gl(n_1, \mathbb{C})$ and $\mathcal{K}_2 \cong gl(n - n_1, \mathbb{C})$ are Lie subalgebras of $\mathcal{K}$ and $\mathcal{B} = \mathcal{A}'\mathcal{A}^*$. Since

$$\mathcal{A}'_{\langle \ell_1, \ell_2 \rangle} = \mathcal{A}' \bigcap \mathcal{B}_{\langle \ell_1, \ell_2 \rangle}, \quad \mathcal{A}^*_{\langle \ell_1, \ell_2 \rangle} = \mathcal{A}^* \bigcap \mathcal{B}_{\langle \ell_1, \ell_2 \rangle} \qquad (4.49)$$

are of finite-dimensional for any $\ell_1, \ell_2 \in \mathbb{Z}$ (cf. (2.21)) and

$$\mathcal{A}' = \bigoplus_{\ell_1, \ell_2 \in \mathbb{Z}} \mathcal{A}'_{\langle \ell_1, \ell_2 \rangle}, \quad \mathcal{A}^* = \bigoplus_{\ell_1, \ell_2 \in \mathbb{Z}} \mathcal{A}^*_{\langle \ell_1, \ell_2 \rangle}, \qquad (4.50)$$

$\mathcal{A}'$ is a $\mathcal{K}_1$-module generated by its $\mathcal{K}_1$-singular vectors and $\mathcal{A}^*$ is a $\mathcal{K}_2$-module generated by its $\mathcal{K}_2$-singular vectors. According to Lemma 2.4,

$$\{x_{n_1}^{m_1} y_{n_1}^{m_2} \zeta_1^{\delta_{1,n_1} m_3} \mid m_i \in \mathbb{N}\} \qquad (4.51)$$

is the set of homogeneous $\mathcal{K}_1$-singular vectors in $\mathcal{A}'$ and

$$\{x_{n_1+1}^{m_1} y_{n_1+1}^{m_2} \zeta_2^{\delta_{1,n-n_1} m_3} \mid m_i \in \mathbb{N}\} \qquad (4.52)$$

is the set of homogeneous $\mathcal{K}_2$ singular vectors in $\mathcal{A}^*$. Therefore, $\mathcal{A}$ is a $(\mathcal{K}_1 + \mathcal{K}_2)$-module generated by

$$\{x_{n_1}^{m_1} y_{n_1}^{m_2} \zeta_1^{\delta_{1,n_1} m_3} x_{n_1+1}^{m_4} y_{n_1+1}^{m_5} \zeta_2^{\delta_{1,n-n_1} m_6} \mid m_i \in \mathbb{N}\}. \qquad (4.53)$$

Observe

$$(E_{n+n_1-1,n_1} - E_{n+n_1,n_1-1})|_{\mathcal{B}'} = \zeta_1, \quad (E_{n_1+2,n+n_1+1} - E_{n_1+1,n+n_1+2})|_{\mathcal{B}'} = \zeta_2 \quad (4.54)$$

as multiplication operators on $\mathcal{B}'$ by (1.9) and (1.10) with $|_{\mathcal{B}}$ replaced by $|_{\mathcal{B}'}$. Repeatedly applying (4.54) to (4.45), we obtain

$$T_\iota(x_{n_1}^{m_1} y_{n_1}^{m_2} \zeta_1^{\delta_{1,n_1} m_5} x_{n_1+1}^{m_3} y_{n_1+1}^{m_4} \zeta_2^{\delta_{1,n-n_1} m_6}) \in V \qquad (4.55)$$

for $m_i \in \mathbb{N}$ such that $m_1 + m_4 - m_2 - m_3 = k + \delta_{\iota,1}$. Applying $U(\mathcal{K}_1 + \mathcal{K}_2)$ to (4.55), we have

$$T_0(\mathcal{A}_{\langle -k \rangle}), T_1(\mathcal{A}_{\langle -k-1 \rangle}) \subset V. \qquad (4.56)$$

According to (4.4), $\mathcal{H}'_{\langle -k \rangle} \subset V$. Thus $\mathcal{H}'_{\langle -k \rangle} = V$.

Let $U$ be an $o(2n + 1, \mathbb{C})$-submodule generated by $T_0(y_{n_1}^k)$ with $k > 0$. Since $T_1(y_{n_1}^{k-1}) \in \mathcal{H}'_{\langle k \rangle}$ that is an $o(2n + 1, \mathbb{C})$-submodule, we have $U \subset \mathcal{H}'_{\langle k \rangle}$. Repeatedly applying (4.31) to $T_1(y_{n_1}^{k-1})$, we have

$$T_1(x_{n_1+1}^r y_{n_1}^{k-r-1}) \in U \qquad \text{for } r \in \overline{0, k-1}. \tag{4.57}$$

By the same arguments as (4.35)-(4.56) with $V$ replaced by $U$ and $k$ replaced by $-k$, we prove $\mathcal{H}'_{\langle k \rangle} = U$. ∎

**Proof of Theorem 2**.

First we prove that $\mathcal{H}'_{\langle m \rangle}$ is an irreducible $o(2n + 1, \mathbb{C})$-submodule for any $m \in \mathbb{Z}$. We divide it into two cases.

*Case 1.* $n_1 + 1 \leq n_2$.

First we consider $\mathcal{H}'_{\langle k \rangle}$ with $k \in \mathbb{N}$. Let $U$ be any nonzero $o(2n + 1, \mathbb{C})$-submodule of $\mathcal{H}'_{\langle k \rangle}$. Note

$$(E_{n+1,n+n_2} - E_{n_2,1})|_{\mathcal{B}'} = y_1 \partial_{y_{n_2}} + x_1 x_{n_2} \tag{4.58}$$

by (1.6)-(1.8) with $|_{\mathcal{B}}$ replaced by $|_{\mathcal{B}'}$. Moreover,

$$(E_{n_1+1,n+1} - E_{1,n+n_1+1})|_{\mathcal{B}'} = x_{n_1+1} \partial_{y_1} - x_1 \partial_{y_{n_1+1}} \tag{4.59}$$

by (1.9) with $|_{\mathcal{B}}$ replaced by $|_{\mathcal{B}'}$. By Lemma 2.5, $U$ must contain a $\mathcal{K}$-singular vector. If $T_\iota(\eta^\ell(x_j^{k_1} y_{n_2}^{k_2})) \in U$ with $j = n_1, n_1 + 1$, we have

$$(k_2!)^{-2}(E_{n_1+1,n+1} - E_{1,n+n_1+1})^{k_2}(E_{n+1,n+n_2} - E_{n_2,1})^{k_2}[T_\iota(\eta^\ell(x_j^{k_1} y_{n_2}^{k_2}))]$$
$$= T_\iota(\eta^\ell(x_j^{n_1} x_{n_1+1}^{k_2})) \in U. \tag{4.60}$$

We use (4.59) because $n_2$ may be equal to $n_1 + 1$. Next we consider the case $T_\iota(\eta^\ell(x_j^{k_1} y_{n_2+1}^{k_2})) \in U$ with $j = n_1, n_1 + 1$. Observe

$$(E_{n+n_1,n_2+1} - E_{n+n_2+1,n_1})_{\mathcal{B}'} = y_{n_1} \partial_{x_{n_2+1}} + x_{n_1} \partial_{y_{n_2+1}} \tag{4.61}$$

by (1.10) with $|_{\mathcal{B}}$ replaced by $|_{\mathcal{B}'}$. Thus

$$\frac{1}{k_2!}(E_{n+n_1,n_2+1} - E_{n+n_2+1,n_1})^{k_2}[T_\iota(\eta^\ell(x_j^{k_1} y_{n_2+1}^{k_2}))] = T_\iota(\eta^\ell(x_j^{k_1} x_{n_1}^{k_2})) \in U. \tag{4.62}$$

Thus Lemma 2.2 and Lemma 2.3 show that some

$$T_\iota(\eta^\ell(x_{n_1}^{k_1} x_{n_1+1}^{k_2})) \in U. \tag{4.63}$$

Note

$$(E_{n_1+1,0} - E_{0,n+n_1+1})|_{\mathcal{B}} = x_{n_1+1} \partial_{x_0} - x_0 \partial_{y_{n_1+1}} \tag{4.64}$$

by (1.16) and (1.17). Thus (1.12) and (2.24) give

$$(E_{n_1+1,0} - E_{0,n+n_1+1})[T_0(\eta^\ell(x_{n_1}^{k_1}x_{n_1+1}^{k_2}))]$$

$$= (x_{n_1+1}\partial_{x_0} - x_0\partial_{y_{n_1+1}})(\sum_{i=0}^\infty \frac{(-2)^i x_0^{2i}}{(2i)!}\mathcal{D}^i(\eta^\ell(x_{n_1}^{k_1}x_{n_1+1}^{k_2})))$$

$$= \sum_{i=1}^\infty \frac{(-2)^i x_0^{2i-1}}{(2i-1)!}x_{n_1+1}\mathcal{D}^i(\eta^\ell(x_{n_1}^{k_1}x_{n_1+1}^{k_2}))$$

$$- \sum_{i=0}^\infty \frac{(-2)^i x_0^{2i+1}}{(2i)!}\mathcal{D}^i(\partial_{y_{n_1+1}}(\eta^\ell(x_{n_1}^{k_1}x_{n_1+1}^{k_2})))$$

$$= \sum_{i=1}^\infty \frac{(-2)^i x_0^{2i-1}}{(2i-1)!}\mathcal{D}^i(\eta^\ell(x_{n_1}^{k_1}x_{n_1+1}^{k_2+1})) - \sum_{i=1}^\infty \frac{(-2)^i i x_0^{2i-1}}{(2i-1)!}\mathcal{D}^{i-1}(\partial_{y_{n_1+1}}\eta^\ell(x_{n_1}^{k_1}x_{n_1+1}^{k_2}))$$

$$-\ell\sum_{i=0}^\infty \frac{(-2)^i x_0^{2i+1}}{(2i)!}\mathcal{D}^i(\eta^{\ell-1}(x_{n_1}^{k_1}x_{n_1+1}^{k_2+1})))$$

$$= \ell(n_2 - n_1 + \ell + k_2 - k_1)\sum_{i=1}^\infty \frac{(-2)^i x_0^{2i-1}}{(2i-1)!}\mathcal{D}^{i-1}(\eta^{\ell-1}(x_{n_1}^{k_1}x_{n_1+1}^{k_2+1}))$$

$$+\ell T_1((\eta^{\ell-1}(x_{n_1}^{k_1}x_{n_1+1}^{k_2+1})))$$

$$= \ell[1 - 2(n_2 - n_1 + \ell + k_2 - k_1)]T_1((\eta^{\ell-1}(x_{n_1}^{k_1}x_{n_1+1}^{k_2+1}))) \qquad (4.65)$$

and

$$(E_{n_1+1,0} - E_{0,n+n_1+1})[T_1(\eta^\ell(x_{n_1}^{k_1}x_{n_1+1}^{k_2}))]$$

$$= (x_{n_1+1}\partial_{x_0} - x_0\partial_{y_{n_1+1}})(\sum_{i=0}^\infty \frac{(-2)^i x_0^{2i+1}}{(2i+1)!}\mathcal{D}^i(\eta^\ell(x_{n_1}^{k_1}x_{n_1+1}^{k_2})))$$

$$= \sum_{i=0}^\infty \frac{(-2)^i x_0^{2i}}{(2i)!}x_{n_1+1}\mathcal{D}^i(\eta^\ell(x_{n_1}^{k_1}x_{n_1+1}^{k_2+1})) - \sum_{i=0}^\infty \frac{(-2)^i x_0^{2i+2}}{(2i+1)!}\mathcal{D}^i(\partial_{y_{n_1+1}}(\eta^\ell(x_{n_1}^{k_1}x_{n_1+1}^{k_2})))$$

$$= \sum_{i=0}^\infty \frac{(-2)^i x_0^{2i}}{(2i)!}\mathcal{D}^i(\eta^\ell(x_{n_1}^{k_1}x_{n_1+1}^{k_2+1})) + \sum_{i=1}^\infty \frac{(-2)^{i-1} x_0^{2i}}{(2i-1)!}\mathcal{D}^{i-1}(\partial_{y_{n_1+1}}(\eta^\ell(x_{n_1}^{k_1}x_{n_1+1}^{k_2})))$$

$$-\ell\sum_{i=0}^\infty \frac{(-2)^i x_0^{2i+2}}{(2i+1)!}\mathcal{D}^i(\eta^{\ell-1}(x_{n_1}^{k_1}x_{n_1+1}^{k_2+1}))) = T_0(\eta^\ell(x_{n_1}^{k_1}x_{n_1+1}^{k_2+1})). \qquad (4.66)$$

By (4.63), (4.65) and (4.66), we get

$$T_0(x_{n_1}^{k_1}x_{n_1+1}^{k_2}) \in U \qquad \text{for some } k_1, k_2 \in \mathbb{N} \text{ such that } k_2 - k_1 = k. \qquad (4.67)$$

Observe

$$(E_{n_1,n+1} - E_{n+n_1+1,n+n_1})_{\mathcal{B}'} = \partial_{x_{n_1}}\partial_{x_{n_1+1}} - y_{n_1+1}\partial_{y_{n_1}} \qquad (4.68)$$

by (1.6)-(1.8) with $|_\mathcal{B}$ replaced by $|_{\mathcal{B}'}$. Repeatedly applying (4.68) to (4.67), we obtain $x_{n_1+1}^k = T_0(x_{n_1+1}^k) \in U$. Thanks to Lemma 4.1, $U = \mathcal{H}'_{\langle k \rangle}$. So $\mathcal{H}'_{\langle k \rangle}$ is an irreducible $o(2n+1, \mathbb{C})$-submodule.

Let $V$ be any nonzero $o(2n+1,\mathbb{C})$-submodule of $\mathcal{H}'_{\langle-k\rangle}$ with $k \in \mathbb{N}+1$. By the above arguments, $x_{n_1}^k \in V$. Thanks to Lemma 4.1, $V = \mathcal{H}'_{\langle-k\rangle}$. So $\mathcal{H}'_{\langle-k\rangle}$ is an irreducible $o(2n+1,\mathbb{C})$-submodule.

*Case 2.* $n_1 = n_2$.

In this case

$$\eta = \sum_{i=1}^{n_1} y_i \partial_{x_i} + \sum_{s=n_1+1}^{n} x_s \partial_{y_s}. \tag{4.69}$$

Thus

$$x_{n_1}^{m_1} y_{n_1}^{m_2} \zeta_1^{m_3+1} = \frac{1}{\prod_{i=1}^{m_2}(m_1+i)} \eta^{m_2}(x_{n_1}^{m_1+m_2}\zeta_1^{m_3+1}) \tag{4.70}$$

and

$$x_{n_1+1}^{m_1} y_{n_1+1}^{m_2} \zeta_2^{m_3+1} = \frac{1}{\prod_{i=1}^{m_1}(m_2+i)} \eta^{m_1}(y_{n_1+1}^{m_1+m_2}\zeta_2^{m_3+1}). \tag{4.71}$$

First we consider $\mathcal{H}'_{\langle k\rangle}$ with $k \in \mathbb{N}+1$. Let $U$ be any nonzero $o(2n+1,\mathbb{C})$-submodule of $\mathcal{H}'_{\langle k\rangle}$. Then $U$ must contain a $\mathcal{K}$-singular vector by Lemma 2.5. Moreover, (4.61) becomes

$$(E_{n+n_1,n_1+1} - E_{n+n_1+1,n_1})_{\mathcal{B}'} = y_{n_1}\partial_{x_{n_1+1}} + x_{n_1}\partial_{y_{n_1+1}}. \tag{4.72}$$

If $T_\iota(\eta^\ell(x_{n_1}^{k_1}y_{n_1+1}^{k_2})) \in U$, then

$$\frac{1}{k_2!}(E_{n+n_1,n_1+1} - E_{n+n_1+1,n_1})^{k_2}[T_\iota(\eta^\ell(x_{n_1}^{k_1}y_{n_1+1}^{k_2}))] = T_\iota(\eta^\ell(x_{n_1}^{k_1+k_2})) \in U. \tag{4.73}$$

When $T_\iota(\eta^\ell(x_{n_1}^{k_1}\zeta_1^{k_2})) \in U$, we have $T_\iota(\eta^\ell(x_{n_1}^{k_1})) \in U$ by (3.27). Symmetrically, we have $T_\iota(\eta^\ell(y_{n_1+1}^{k_1})) \in U$ if $T_\iota(\eta^\ell(y_{n_1+1}^{k_1}\zeta_2^{k_2})) \in U$, and (4.73) implies $0 \neq T_\iota(\eta^\ell(x_{n_1}^{k_1})) \in U$. By Lemma 2.4, we always have some $T_\iota(\eta^\ell(x_{n_1}^{k_1})) \in U$. According to (4.69),

$$\eta^m(x_{n_1}^{m_1}) = 0 \qquad \text{for } m, m_1 \in \mathbb{N} \text{ such that } m > m_1. \tag{4.74}$$

Thus $\ell \leq k_1$.

Note that

$$(E_{n_1,0} - E_{0,n+n_1})|_{\mathcal{B}'} = \partial_{x_0}\partial_{x_{n_1}} - x_0\partial_{y_{n_1}} \tag{4.75}$$

by (1.16) and (1.17). Moreover, (4.24), (4.69) and (4.75) imply that

$$
\begin{aligned}
&(E_{n_1,0} - E_{0,n+n_1})[T_0(\eta^\ell(x_{n_1}^{k_1}))] \\
=\ & (\partial_{x_0}\partial_{x_{n_1}} - x_0\partial_{y_{n_1}})(\sum_{i=0}^\infty \frac{(-2)^i x_0^{2i}}{(2i)!} \mathcal{D}^i(\eta^\ell(x_{n_1}^{k_1}))) \\
=\ & -\sum_{i=1}^\infty \frac{(-2)^i i x_0^{2i-1}}{(2i-1)!} \mathcal{D}^{i-1}(\partial_{y_{n_1}}(\eta^\ell(x_{n_1}^{k_1}))) + k_1\sum_{i=1}^\infty \frac{(-2)^i x_0^{2i-1}}{(2i-1)!} \mathcal{D}^i(\eta^\ell(x_{n_1}^{k_1-1}))) \\
& -\sum_{i=0}^\infty \frac{(-2)^i x_0^{2i+1}}{(2i)!} \mathcal{D}^i(\partial_{y_{n_1}}(\eta^\ell(x_{n_1}^{k_1}))) \\
=\ & \ell k_1 \sum_{i=0}^\infty \frac{(-2)^i x_0^{2i+1}}{(2i+1)!} \mathcal{D}^i(\eta^{\ell-1}(x_{n_1}^{k_1-1})) \\
& -2\ell k_1(\ell - k_1)\sum_{i=0}^\infty \frac{(-2)^i x_0^{2i+1}}{(2i+1)!} \mathcal{D}^i(\eta^{\ell-1}(x_{n_1}^{k_1-1}))) \\
=\ & \ell k_1[1 - 2(\ell - k_1)]T_1(\eta^{\ell-1}(x_{n_1}^{k_1-1})), \qquad (4.76)
\end{aligned}
$$

$$
\begin{aligned}
&(E_{n_1,0} - E_{0,n+n_1})[T_1(\eta^\ell(x_{n_1}^{k_1}))] \\
=\ & (\partial_{x_0}\partial_{x_{n_1}} - x_0\partial_{y_{n_1}})(\sum_{i=0}^\infty \frac{(-2)^i x_0^{2i+1}}{(2i+1)!} \mathcal{D}^i(\eta^\ell(x_{n_1}^{k_1}))) \\
=\ & \sum_{i=1}^\infty \frac{(-2)^{i-1} x_0^{2i}}{(2i-1)!} \mathcal{D}^{i-1}(\partial_{y_{n_1}}(\eta^\ell(x_{n_1}^{k_1}))) + \sum_{i=0}^\infty \frac{(-2)^i x_0^{2i}}{(2i)!} \mathcal{D}^i(\partial_{x_{n_1}}(\eta^\ell(x_{n_1}^{k_1}))) \\
& -\sum_{i=0}^\infty \frac{(-2)^i x_0^{2i+2}}{(2i+1)!} \mathcal{D}^i(\partial_{y_{n_1}}(\eta^\ell(x_{n_1}^{k_1}))) = k_1 T_0(\eta^\ell(x_{n_1}^{k_1-1})). \qquad (4.77)
\end{aligned}
$$

Therefore, we have

$$
\frac{1}{(k-1)!}T_1(\eta^{k-1}(x_{n_1}^{k-1})) = T_1(y_{n_1}^{k-1}) \in U. \qquad (4.78)
$$

By Lemma 4.2, $U = \mathcal{H}'_{\langle k \rangle}$. Thus $\mathcal{H}'_{\langle k \rangle}$ is an irreducible $o(2n+1, \mathbb{C})$-submodule.

Let $V$ be any nonzero $o(2n+1, \mathbb{C})$-submodule of $\mathcal{H}'_{\langle -k \rangle}$ with $k \in \mathbb{N}$. By the above arguments (4.69)-(4.77), $x_{n_1}^k \in V$. Thanks to Lemma 4.2, $V = \mathcal{H}'_{\langle -k \rangle}$. So $\mathcal{H}'_{\langle -k \rangle}$ is an irreducible $o(2n+1, \mathbb{C})$-submodule.

Fix $k \in \mathbb{Z}$. Since $\mathcal{D}'$ in (1.18) is locally nilpotent, for any $0 \neq u \in \mathcal{B}'_{\langle k \rangle}$, there exists an element $\kappa(u) \in \mathbb{N}$ such that

$$
(\mathcal{D}')^{\kappa(u)}(u) \neq 0 \text{ and } (\mathcal{D}')^{\kappa(u)+1}(u) = 0. \qquad (4.79)
$$

Set

$$
\Psi' = \sum_{i=0}^\infty (\eta')^i(\mathcal{H}'_{\langle k-2i \rangle}). \qquad (4.80)
$$

Given $0 \neq u \in \mathcal{B}'_{\langle k \rangle}$, $\kappa(u) = 1$ implies $u \in \mathcal{H}'_{\langle k \rangle} \subset \Psi'$. Suppose that $u \in \Psi'$ whenever $\kappa(u) < r$ for some positive integer $r$. Assume $\kappa(u) = r$. First

$$v = (\mathcal{D}')^r(u) \in \mathcal{H}'_{\langle k-2r \rangle} \subset \Psi'. \tag{4.81}$$

Note that $[\partial_{x_0}^2, x_0^2] = 2 + 4x_0 \partial_{x_0}$ and so

$$(\mathcal{D}')^r[(\eta')^r(v)] = 2^r r! [\prod_{i=1}^{r} [1 + 2(n_2 - n_1 + k - r - i)]]v \tag{4.82}$$

by (1.18), (1.19) and (2.24). Thus we have either

$$u = \frac{1}{2^r r! \prod_{i=1}^{r} [1 + 2(n_2 - n_1 + k - r - i)]} (\eta')^r(v) \in \Psi' \tag{4.83}$$

or

$$\kappa \left( u - \frac{1}{2^r r! \prod_{i=1}^{r} [1 + 2(n_2 - n_1 + k - r - i)]} \eta^r(v) \right) < r. \tag{4.84}$$

By induction,

$$u - \frac{1}{2^r r! \prod_{i=1}^{r} [1 + 2(n_2 - n_1 + k - r - i)]} \eta^r(v) \in \Psi', \tag{4.85}$$

which implies $u \in \Psi'$. Therefore, we have $\Psi' = \mathcal{B}'_{\langle k \rangle}$. Since the weight of any $\mathcal{K}$-singular vector in $(\eta')^i(\mathcal{H}'_{\langle k-2i \rangle})$ is different from the weight of any $\mathcal{K}$-singular vector in $(\eta')^j(\mathcal{H}'_{\langle k-2j \rangle})$ when $i \neq j$, the sums in Theorem 2 are direct sums. This completes the proof of Theorem 2. ∎

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