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Irreducible Representations of a Product of Real Reductive Groups

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Abstract. Let G_1, G_2 be real reductive groups and (π, V) be a smooth admissible representation of $G_1 \times G_2$. We prove that (π, V) is irreducible if and only if it is the completed tensor product of (π_i, V_i), $i = 1, 2$, where (π_i, V_i) is a smooth, irreducible, admissible representation of moderate growth of G_i, $i = 1, 2$. We deduce this from the analogous theorem for Harish-Chandra modules, for which one direction was proven in A. Aizenbud and D. Gourevitch, Multiplicity one theorem for $(GL_{n+1}(\mathbb{R}), GL_n(\mathbb{R}))$, Selecta Mathematica N. S. **15** (2009), 271–294, and the other direction we prove here. As a corollary, we deduce that strong Gelfand property for a pair $H \subset G$ of real reductive groups is equivalent to the usual Gelfand property of the pair $\Delta H \subset G \times H$.
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1. Introduction

Let G_1, G_2 be reductive Lie groups , $\mathfrak{g}_i$ be the Lie algebra of G_i. Fix K_i - a maximal compact subgroup of G_i $(i = 1, 2)$. Let $\mathcal{M}(\mathfrak{g}_i, K_i)$ be the category of Harish-Chandra $(\mathfrak{g}_i, K_i)$-modules and $\mathcal{M}(G_i)$ be the category of smooth admissible Fréchet representations of moderate growth (see [4, 10]). We also denote by $Irr(G_i)$ and $Irr(\mathfrak{g}_i, K_i)$ the isomorphism classes or irreducible objects in the above categories.
In this note we prove

Theorem 1.1. *Let $M \in Irr(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$. Then there exist $M_i \in Irr(\mathfrak{g}_i, K_i)$ such that $M = M_1 \otimes M_2$.*

The converse statement, saying that for irreducible $M_i \in \mathcal{M}(\mathfrak{g}_i, K_i)$, $M_1 \otimes M_2$ is irreducible is [1, Proposition A.0.6]. By the Casselman-Wallach equivalence of categories $\mathcal{M}(\mathfrak{g}, K) \simeq \mathcal{M}(G)$, these two statements imply

Theorem 1.2. *A representation $(\pi, V) \in \mathcal{M}(G_1 \times G_2)$ is irreducible if and only if there exist irreducible $(\pi_i, V_i) \in \mathcal{M}(G_i)$ such that $(\pi, V) \simeq (\pi_1, V_1)\hat{\otimes}(\pi_2, V_2)$.*

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Finally, we deduce a consequence of this theorem concerning Gelfand pairs. A pair (G,H) of reductive groups is called a *Gelfand pair* if $H \subset G$ is a closed subgroup and the space $(\pi^*)^H$ of H-invariant continuous functionals on any $\pi \in Irr(G)$ has dimension zero or one. It is called a *strong Gelfand pair* or a *multiplicity-free pair* if $\dim \mathrm{Hom}_H(\pi|_H, \tau) \leq 1$ for any $\pi \in Irr(G)$, $\tau \in Irr(H)$.

Corollary 1.3. *Let $H \subset G$ be reductive groups and let $\Delta H \subset G \times H$ denote the diagonal. Then (G,H) is a multiplicity-free pair if and only if $(G \times H, \Delta H)$ is a Gelfand pair.*

An analog of Corollary 1.3 was proven in [7] for generalized Gelfand property of arbitrary Lie groups, with smooth representations replaced by smooth vectors in unitary representations.

An analog of Theorem 1.2 for p-adic groups was proven in [3, §§2.16] and in [5]. For a more detailed exposition see [6, §§10.5].

2. Preliminaries

2.1. Harish-Chandra modules and smooth representations.

In this subsection we fix a real reductive group G and a maximal compact subgroup $K \subset G$. Let $\mathfrak{g}, \mathfrak{k}$ denote the complexified Lie algebras of G, K.

Definition 2.1. *A $(\mathfrak{g}, K)$-module is a $\mathfrak{g}$-module π with a locally finite action of K such the two induced actions of $\mathfrak{k}$ coincide and $\pi(ad(k)(X)) = \pi(k)\pi(X)\pi(k^{-1})$ for any $k \in K$ and $X \in \mathfrak{g}$.*

A finitely-generated $(\mathfrak{g}, K)$-module is called admissible if any representation of K appears in it with finite (or zero) multiplicity. In this case we also call it a Harish-Chandra module.

Lemma 2.2 ([9], §§4.2)**.** *Any Harish-Chandra module π has finite length.*

Theorem 2.3 (Casselman-Wallach, see [10], §§§11.6.8)**.** *The functor of taking K-finite vectors $HC : \mathcal{M}(G) \to \mathcal{M}(\mathfrak{g}, K)$ is an equivalence of categories.*

In fact, Casselman and Wallach construct an inverse functor $\Gamma : \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(G)$, that is called Casselman-Wallach globalization functor (see [10, Chapter 11] or [4] or, for a different approach, [2]).

Corollary 2.4.

(i) The category $\mathcal{M}(G)$ is abelian.

(ii) Any morphism in $\mathcal{M}(G)$ has closed image.

Proof. (i) $\mathcal{M}(\mathfrak{g}, K)$ is clearly abelian and by the theorem is equivalent to $\mathcal{M}(G)$.

(ii) Let $\phi : \pi \to \tau$ be a morphism in $\mathcal{M}(G)$. Let $\tau' = \overline{Im\phi}$, $\pi' = \pi / \ker \phi$ and $\phi' : \pi' \to \tau'$ be the natural morphism. Clearly ϕ' is monomorphic and epimorphic in the category $\mathcal{M}(G)$. Thus by (i) it is an isomorphism. On the other hand, $Im\phi' = Im\phi \subset \overline{Im\phi} = \tau'$. Thus $Im\phi = \overline{Im\phi}$. ■

We will also use the embedding theorem of Casselman.

Theorem 2.5. *Any irreducible $(\mathfrak{g}, K)$-module can be imbedded into a $(\mathfrak{g}, K)$-module of principal series.*

Lemma 2.2, Theorems 2.3 and 2.5 and Corollary 2.4 have the following corollary.

Corollary 2.6. *The underlying topological vector space of any admissible smooth Fréchet representation of moderate growth is a nuclear Fréchet space.*

Definition 2.7. *Let G_1 and G_2 be real reductive groups. Let $(\pi_i, V_i) \in \mathcal{M}(G_i)$ be admissible smooth Fréchet representations of moderate growth of G_i. We define $\pi_1 \otimes \pi_2$ to be the natural representation of $G_1 \times G_2$ on the space $V_1 \widehat{\otimes} V_2$.*

Proposition 2.8 ([1], Proposition A.0.6)**.** *Let G_1 and G_2 be real reductive groups. Let $\pi_i \in Irr(\mathfrak{g}_i, K_i)$ be irreducible Harish-Chandra modules of G_i. Then $\pi_1 \otimes \pi_2 \in Irr(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$.*

We will use the classical statement on irreducible representations of compact groups.

Lemma 2.9. *Let K_1, K_2 be compact groups. A representation τ of $K_1 \times K_2$ is irreducible if and only if there exist irreducible representations τ_i of K_i such that $\tau \simeq \tau_1 \otimes \tau_2$. Note that τ_i are finite-dimensional, and $\otimes$ is the usual tensor product.*

Corollary 2.10. *Let G_1 and G_2 be real reductive groups and $(\pi_i, V_i) \in \mathcal{M}(G_i)$. Then we have a natural isomorphism $(\pi_1 \otimes \pi_2)^{HC} \simeq \pi_1^{HC} \otimes \pi_2^{HC}$.*

3. Proof of Theorem 1.1

Throughout the section ρ_i always denote irreducible representations of K_1, σ_j always denote irreducible representations of K_2. For a representation V of K_1 (or of K_2) we will denote by V^ρ (resp. by V^σ) the corresponding isotypic component. Let $K := K_1 \times K_2$ and $\mathfrak{g} := \mathfrak{g}_1 \times \mathfrak{g}_2$.

Let (π, V) be an irreducible admissible $(\mathfrak{g}, K)$ - module. We show that there exist non-zero irreducible and admissible $(\mathfrak{g}_1, K_1)$-module V_1 and $(\mathfrak{g}_2, K_2)$-module V_2 and a non-zero morphism $V_1 \bigotimes V_2 \to V$. From the irreducibility of V and $V_1 \bigotimes V_2$, we obtain that $V \simeq V_1 \bigotimes V_2$.

Let's first find the module V_1. Choose $\tau \in Irr(K)$ such that the isotypic component V^τ is non-zero. By Lemma 2.9 $\tau \simeq \rho \otimes \sigma$ for some $\rho \in Irr(K_1)$, $\sigma \in$

$Irr(K_2)$. Let W be the $(\mathfrak{g}_1, K_1)$-module generated by V^τ. Note that since the actions of $(\mathfrak{g}_1, K_1)$ and $(\mathfrak{g}_2, K_2)$ commute, W is also a K_2-module and $W = W^\sigma$. We claim that W is an admissible $(\mathfrak{g}_1, K_1)$-module. Indeed, let ρ_1 be an irreducible representation of K_1. Then $W^{\rho_1} \subseteq V^{\rho_1 \otimes \sigma}$ and as a corollary

$$\dim(W^{\rho_1}) \leq \dim\left(V^{\rho_1 \otimes \sigma}\right) < \infty,$$

since V is an admissible $(\mathfrak{g}, K)$-module.

Now by Lemma 2.2 W has finite length and thus there is an irreducible admissible $(\mathfrak{g}_1, K_1)$-submodule $V_1 \subseteq W$. Thus, we finished the first stage of the proof.
Let

$$W_2' := \mathrm{Hom}_{(\mathfrak{g}_1, K_1)}(V_1, V).$$

Clearly, $W_2' \neq 0$. Since actions of $(\mathfrak{g}_1, K_1)$ and $(\mathfrak{g}_2, K_2)$ on V commute, W_2' has a natural structure of $(\mathfrak{g}_2, K_2)$-module. Take any non-zero morphism $L \in W_2'$ and let $W_2 \subset W_2'$ be the $(\mathfrak{g}_2, K_2)$-module generated by L.

Let us show that W_2 is admissible. Choose $\sigma_2 \in Irr(K_2)$. Let $\rho_2 \in Irr(K_1)$ such that $V_1^{\rho_2} \neq 0$. Then $V_1^{\rho_2}$ generates V_1 and thus for any $L', L'' \in W_2^{\sigma_2}$ if L' agrees with L'' on $V_1^{\rho_2}$ then $L' = L''$. This gives a linear embedding from $W_2^{\sigma_2}$ into the finite-dimensional space $\mathrm{Hom}_{\mathbb{C}}(V_1^{\rho_2}, V^{\rho_2 \otimes \sigma_2})$. Thus W_2 is an admissible $(\mathfrak{g}_2, K_2)$-module.

Thus W_2 has finite length and therefore there is an irreducible admissible submodule $V_2 \subseteq W_2$. Define a linear map $\phi : V_1 \bigotimes V_2 \to V$ by the formula

$$\phi(v \otimes l) := l(v)$$

on the pure tensors. Clearly, this is a non-zero $(\mathfrak{g}, K)$-map.

The result $V_1 \bigotimes V_2 \simeq V$ follows now from the irreducibility of V and of $V_1 \bigotimes V_2$ (Proposition 2.8).

Remark 3.1. An alternative way to prove this theorem is to remark that the category $\mathcal{M}(\mathfrak{g}, K)$ is equivalent to the category of admissible modules over the idempotented algebra $\mathcal{H}(\mathfrak{g}, K)$ of K-finite distributions on G supported in K (see [5]), then show that this algebra is the tensor product of $\mathcal{H}(\mathfrak{g}_i, K_i)$ and thus the proofs from [3, 5] extend to this case. We estimate that such proof would be of similar length, but slightly less elementary.

4. Proof of Theorem 1.2 and Corollary 1.3

Proof. [Proof of Theorem 1.2] First take $\pi_i \in Irr(G_i)$, for $i = 1, 2$. Then $\pi_i^{HC} \in Irr(\mathfrak{g}_i, K_i)$ and by Proposition 2.8 $\pi_1^{HC} \otimes \pi_2^{HC} \in Irr(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$. By Corollary 2.10 $(\pi_1 \otimes \pi_2)^{HC} \simeq \pi_1^{HC} \otimes \pi_2^{HC} \in Irr(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$. This implies $\pi_1 \otimes \pi_2 \in Irr(G_1 \times G_2)$.

Now take $\pi \in Irr(G_1 \times G_2)$. Then $\pi^{HC} \in Irr(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$ and by Theorem 1.1 there exist $(M_i) \in Irr(\mathfrak{g}_i, K_i)$ such that $\pi^{HC} \simeq M_1 \otimes M_2$. By Theorem 2.3 there exist $\pi_i \in Irr(G_i)$ such that $\pi_i^{HC} \simeq M_i$. Then $\pi^{HC} \simeq \pi_1^{HC} \otimes \pi_2^{HC} \simeq (\pi_1 \otimes \pi_2)^{HC}$ and by Theorem 2.3 this implies $\pi \simeq \pi_1 \otimes \pi_2$. ∎

Corollary 1.3 follows from Theorem 1.2 and the following lemma.

Lemma 4.1. *Let $H \subset G$ be real reductive groups. Let (π, E) and (τ, W) be admissible smooth Fréchet representations of moderate growth of G and H respectively. Then $Hom_H(\pi, \tau)$ is canonically isomorphic to $Hom_{\Delta H}(\pi \otimes \widetilde{\tau}, \mathbb{C})$, where $\widetilde{\tau}$ denotes the contragredient representation.*

Proof. For a nuclear Fréchet space V we denote by V' its dual space equipped with the strong topology. Let $\widetilde{W} \subset W'$ denote the underlying space of $\widetilde{\tau}$. By the theory of nuclear Fréchet spaces ([8, Chapter 50], we know $Hom_{\mathbb{C}}(E, W) \cong E' \widehat{\otimes} W$ and $Hom_{\mathbb{C}}(E \widehat{\otimes} \widetilde{W}, \mathbb{C}) \cong E' \widehat{\otimes} \widetilde{W}'$. Thus we have canonical embeddings

$$Hom_H(\pi, \tau) \hookrightarrow Hom_{\Delta H}(\pi \otimes \widetilde{\tau}, \mathbb{C}) \hookrightarrow Hom_H(\pi, \widetilde{\tau}')$$

Since the image of any H-equivariant map from π to $\widetilde{\tau}'$ lies in the space of smooth vectors $\widetilde{\widetilde{\tau}}$, which is canonically isomorphic to τ, the lemma follows. ■

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