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Projections of Orbital Measures, Gelfand–Tsetlin Polytopes, and Splines

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Abstract. The unitary group $U(N)$ acts by conjugations on the space $\mathscr{H}(N)$ of $N\times N$ Hermitian matrices, and every orbit of this action carries a unique invariant probability measure called an orbital measure. Consider the projection of the space $\mathscr{H}(N)$ onto the real line assigning to an Hermitian matrix its $(1,1)$-entry. Under this projection, the density of the pushforward of a generic orbital measure is a spline function with N knots. This fact was pointed out by Andrei Okounkov in 1996, and the goal of the paper is to propose a multidimensional generalization. Namely, it turns out that if instead of the $(1,1)$-entry we cut out the upper left matrix corner of arbitrary size $K\times K$, where $K=2,\dots,N-1$, then the pushforward of a generic orbital measure is still computable: its density is given by a $K\times K$ determinant composed from one-dimensional splines. The result can also be reformulated in terms of projections of the Gelfand–Tsetlin polytopes.
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1. Introduction

Orbital measures. Let $\mathscr{H}(N)$ be the space of $N\times N$ Hermitian matrices. For $K=1,\dots,N-1$, we denote by $p^N_K:\mathscr{H}(N)\to\mathscr{H}(K)$ the linear projection consisting in deleting from the matrix $H\in\mathscr{H}(N)$ its last $N-K$ rows and columns. We call $p^N_K(H)$, the image of H under this projection, the $K\times K$ *corner* of H.

The unitary group $U(N)$ acts on $\mathscr{H}(N)$ by conjugations, and because $U(N)$ is compact, each orbit of this action carries a unique invariant probability measure, which we call the *orbital measure*. Given an orbital measure μ on $\mathscr{H}(N)$, denote by $p^N_K(\mu)$ its pushforward under projection p^N_K. Our goal is to describe $p^N_K(\mu)$.

The orbits in $\mathscr{H}(N)$ (and hence the orbital measures) can be indexed by N-tuples of weakly increasing real numbers $X=(x_1\le\dots\le x_N)$, the matrix

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eigenvalues. Let $\mathscr{X}(N) \subset \mathbb{R}^N$ denote the set of all such X's. Given $X \in \mathscr{X}(N)$, we write O_X and μ_X for the corresponding orbit and orbital measure, respectively.

Since $p^N_K(\mu_X)$ is a $U(K)$-invariant probability measure on $\mathscr{H}(K)$, it can be uniquely decomposed into a continual convex combination of orbital measures, governed by a probability measure $\nu_{X,K}$ on the parameter space $\mathscr{X}(K)$. That is, $\nu_{X,K}$ is characterized by the property that, for an arbitrary Borel subset $S \subseteq \mathscr{X}(N)$,

$$(p^N_K(\mu_X))(S) = \int_{Y \in \mathscr{X}(K)} \mu_Y(S)\nu_{X,K}(dY).$$

The measure $\nu_{X,K}$ can be called the *radial part* of measure $p^N_K(\mu_X)$.

Main result. Denote by $\mathscr{X}^0(N)$ the interior of $\mathscr{X}(N)$; that is, $\mathscr{X}^0(N)$ consists of N-tuples of *strictly* increasing real numbers. If $X \in \mathscr{X}^0(N)$, then $\nu_{X,K}$ is absolutely continuous with respect to Lebesgue measure on $\mathscr{X}(K) \subset \mathbb{R}^K$, and the main result, Theorem 3.3, gives an explicit formula for the density of $\nu_{X,K}$.

In the case $K = 1$ the target space of the projection is the real line, and the density in question coincides with a *B-spline*, a certain piecewise polynomial function on $\mathbb{R}$ (this fact was observed by Andrei Okounkov). In the general case, it turns out that the density of $\nu_{X,K}$ is expressed through a $K \times K$ determinant composed from some B-splines.

As the reader will see, the proof of Theorem 3.3 is straightforward and elementary. The main reason why I believe the result may be of interest is the very appearance of splines, which are objects of classical and numerical analysis, in a problem of representation-theoretic origin.

Gelfand–Tsetlin polytopes. Before explaining a connection with representation theory I want to give a different interpretation of the measure $\nu_{X,K}$.

For $X \in \mathscr{X}(N)$ and $Y \in \mathscr{X}(N-1)$, write $Y \prec X$ or $X \succ Y$ if the coordinates of X and Y *interlace*, that is

$$x_1 \le y_1 \le x_2 \le \cdots \le x_{N-1} \le y_{N-1} \le x_N.$$

Given $X \in \mathscr{X}(N)$, the corresponding *Gelfand–Tsetlin polytope* P_X is the compact convex subset in the vector space

$$\mathbb{R}^{N-1} \times \mathbb{R}^{N-2} \times \cdots \times \mathbb{R} = \mathbb{R}^{N(N-1)/2},$$

formed by triangular arrays subject to the interlacement constraints:

$$P_X := \{(Y^{(N-1)}, \dots, Y^{(1)}) \in R^{N(N-1)/2} : X \succ Y^{(N-1)} \succ \cdots \succ Y^{(1)}\}.$$

Consider the map assigning to a matrix $H \in O_X$ the array formed by the collections of eigenvalues of its corners $p^N_{N-1}(H), p^N_{N-2}(H), \dots, p^N_1(H)$. It is well known (see Corollary 3.2 below) that this map projects the orbit O_X onto the polytope P_X and takes μ_X to the uniform measure on P_X (that is, the normalized Lebesgue measure). Next, given $K = 1, \dots, N-1$, consider the natural projection $P_X \to \mathscr{X}(K)$ extracting from the array $(Y^{(N-1)}, \dots, Y^{(1)})$ its Kth component $Y^{(K)}$. The measure $\nu_{X,K}$ is nothing else than the pushforward of the uniform measure under the latter projection.

Discrete version of the problem: relative dimension in Gelfand–Tsetlin graph. Let $\mathbb{GT}_N := \mathscr{X}(N) \cap \mathbb{Z}^N$ be the set of weakly increasing N-tuples of integers. The elements of $\mathbb{GT}_N$ are in bijection with the irreducible representations of the group $U(N)$: with $X = (x_1, \dots, x_N) \in \mathbb{GT}_N$ we associate the irreducible representation T_X with signature (=highest weight) $\widehat{X} := (x_N, \dots, x_1)$. Here we pass from X to $\widetilde{X}$, because the coordinates of signatures are usually written in the descending order, see Weyl [13].

Let $X \in \mathbb{GT}_N$ and consider the finite set $P_X^{\mathbb{Z}} := P_X \cap Z^{N(N-1)/2}$ consisting of integral points in the polytope P_X. Let us replace the uniform measure on P_X by the uniform measure on $P_X^{\mathbb{Z}}$ (that is, the normalized counting measure). Next, given $K = 1, \dots, N-1$, we consider again the same projection $P_X \to \mathscr{X}(K)$ as before and denote by $\nu_{X,K}^{\mathbb{Z}}$ the pushforward of the uniform measure on $P_X^{\mathbb{Z}}$. Evidently, $\nu_{X,K}^{\mathbb{Z}}$ is a probability measure with finite support.

Elements of $P_X^{\mathbb{Z}}$ are the *Gelfand–Tsetlin schemes* (also called Gelfand–Tsetlin patterns) with top row X; they parameterize the elements of Gelfand–Tsetlin basis in T_X. By the very definition of $\nu_{X,K}^{\mathbb{Z}}$, for $Y \in \mathbb{GT}_K$, the quantity $\nu_{X,K}^{\mathbb{Z}}(Y)$ (the mass assigning by $\nu_{X,K}^{\mathbb{Z}}$ to Y) equals the fraction of the schemes with the Kth row equal to Y. This quantity is the same as the relative dimension of the isotypic component of T_Y in the restriction of T_X to the subgroup $U(K) \subset U(N)$.

The *Gelfand–Tsetlin graph* has the vertex set $\mathbb{GT}_1 \sqcup \mathbb{GT}_2 \sqcup \dots$ and the edges formed by couples $Y \prec X$. In the terminology of Borodin–Olshanski [3], $\nu_{X,K}^{\mathbb{Z}}(Y)$ is the *relative dimension* of the vertex $Y \in \mathbb{GT}_K$ with respect to the vertex $X \in \mathbb{GT}_N$. In [3], we derived a determinantal formula for the relative dimension (see also Petrov [11] for a different proof). That formula can be viewed as a discrete version of the formula of Theorem 3.3.

I first guessed the formula of Theorem 3.3 by degenerating the "discrete" formula of [3]. However, this is not an optimal way of derivation, because the discrete case is much more difficult than the continuous one. I am grateful to Alexei Borodin for the suggestion to study the degeneration of the "discrete" formula. Note that from the comparison of the measures $\nu_{N,K}$ and $\nu_{X,K}^{\mathbb{Z}}$ it is seen that the former should be related to the latter by a scaling limit transition.

2. Preliminaries

The *fundamental spline* with $n \geq 2$ knots $y_1 < \dots < y_n$ can be characterized as the only function $a \mapsto M(a; y_1 \dots, y_n)$ on $\mathbb{R}$ of class C^{n-3}, vanishing outside the interval (y_1, y_n), equal to a polynomial of degree $\leq n-2$ on each interval (y_i, y_{i+1}), and normalized by the condition

$$\int_{-\infty}^{+\infty} M(a; y_1, \dots, y_n) da = 1.$$

Here is an explicit expression:

$$M(a; y_1, \dots, y_n) := (n-1) \sum_{i:\, y_i > a} \frac{(y_i - a)^{n-2}}{\prod\limits_{r:\, r \neq i} (y_i - y_r)}. \tag{1}$$

In particular, for $n = 2$

$$M(a; y_1, y_2) = \frac{\mathbf{1}_{y_1 \le a \le y_2}}{y_2 - y_1}.$$

Remark 2.1. The above definition is taken from Curry–Schoenberg [4]. In the subsequent publications, Schoenberg changed the term to *B-spline*. The latter term became commonly used. However, in the modern literature, it more often refers to the function

$$B(a; y_1, \dots, y_n) := (y_n - y_1) \sum_{i:\, y_i > a} \frac{(y_i - a)^{n-2}}{\prod_{r:\, r \ne i} (y_i - y_r)}, \tag{2}$$

which differs from $M(a; y_1, \dots, y_n)$ by the numerical factor $(y_n - y_1)/(n-1)$; see, e.g., de Boor [2] or Phillips [12]. The normalization in (2) has its own advantages, but we will not use it. Note also that $M(a; y_1, \dots, y_n)$ is a special case of *Peano kernel*, see Davis [5], Faraut [7].

We need two well-known formulas relating $M(a; y_1, \dots, y_n)$ to divided differences (see, e.g., [4], [7]).

Recall that the *divided difference* of a function $f(x)$ on points $y_1, \dots, y_n$ is defined recursively by

$$f[y_1, y_2] = \frac{f(y_2) - f(y_1)}{y_2 - y_1}, \quad f[y_1, y_2, y_3] = \frac{f[y_2, y_3] - f[y_1, y_2]}{y_3 - y_1},$$

and so on; the final step is

$$f[y_1, \dots, y_n] = \frac{f[y_2, \dots, y_n] - f[y_1, \dots, y_{n-1}]}{y_n - y_1}. \tag{3}$$

Next, set

$$x_+^s = \begin{cases} x^s, & x > 0 \\ 0, & x \le 0. \end{cases}$$

In this notation, the first formula in question is

$$M(a; y_1, \dots, y_n) = (n-1) f[y_1, \dots, y_n], \qquad \text{where } f(x) := (x - a)_+^{n-2}, \tag{4}$$

and the second formula is

$$f[y_1, \dots, y_n] = \frac{1}{(n-1)!} \int M(a; y_1, \dots, y_n) f^{(n-1)}(a) da. \tag{5}$$

In (5), f is assumed being a function on $\mathbb{R}$ with piecewise continuous derivative of order $n-1$. In particular, (5) is applicable to $f(x) = (x - t)_+^{n-1}$, which is used in the lemma below.

To shorten the notation, let us abbreviate $Y := (y_1 < \dots < y_n)$.

Lemma 2.2. *Fix* $n = 2, 3, \dots$ *and an* n*-tuple* $Y = (y_1 < \cdots < y_n) \in \mathscr{X}^0(N)$. *For an arbitrary* $b \in \mathbb{R}$ *set*

$$f_b(x) := (x-b)_+^{n-1}, \qquad x \in \mathbb{R}.$$

One has

$$\int_{-\infty}^{c} M(a;Y)da = 1 - f_c[Y], \quad c \in \mathbb{R}, \tag{6}$$

$$\int_{b}^{c} M(a;Y)da = f_b[Y] - f_c[Y], \quad b < c, \tag{7}$$

$$\int_{b}^{+\infty} M(a;Y)da = f_b[Y], \quad b \in \mathbb{R}. \tag{8}$$

Proof. To check (7), we apply (5) to $f(x) = f_b(x) - f_c(x)$, which is justified, see the comment just after (5). Then in the left-hand side of (5) we get $f_b[Y] - f_c[Y]$. Next, observe that the $(n-1)$th derivative of $f_t(x)$ equals $(n-1)!\mathbf{1}_{x \ge b}$, so that

$$f^{(n-1)}(a) = (n-1)!(\mathbf{1}_{a \ge b} - \mathbf{1}_{a \ge c}) = (n-1)!\mathbf{1}_{b \le a < c}.$$

Therefore, in the right-hand side we get $\int_b^c M(a;Y)da$, which proves (7).

Now (8) follows from (7) by setting $c = +\infty$, and (6) follows from (8), because the total integral of the $M(a;Y)$ equals 1. ∎

3. Projections of orbital measures

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We keep to the notation of Sections 1 and 2

Given $X \in \mathscr{X}(N)$, the pushforward of the orbital measure μ_X under the map

$$O_X \ni H \mapsto \text{the spectrum of } p^N_{N-1}(H)$$

can be viewed as a probability measure on $\mathscr{X}(N-1)$ depending on X as a parameter; let us denote it by $\Lambda^N_{N-1}(X, \cdot\,)$ or $\Lambda^N_{N-1}(X, dY)$. We regard $\Lambda(X, dY)$ as a Markov kernel.

By classical Rayleigh's theorem, the eigenvalues of a matrix $H \in \mathscr{X}(N)$ and its corner $p^N_{N-1}(H)$ interlace. Therefore, the measure $\Lambda^N_{N-1}(X, \cdot\,)$ is concentrated on the subset

$$\{Y \in \mathscr{X}(N-1) : Y \prec X\} \subset R^{N-1}. \tag{9}$$

Proposition 3.1. *Assume* $X = (x_1, \dots, x_N) \in \mathscr{X}^0(N)$. *Then the measure* $\Lambda^N_{N-1}(X, \cdot\,)$ *is absolutely continuous with respect to Lebesgue measure on the set* (9), *and the density of* $\Lambda^N_{N-1}(X, \cdot\,)$, *denoted by* $\Lambda^N_{N-1}(X, Y)$, *is given by*

$$\Lambda^N_{N-1}(X, Y) = (N-1)! \frac{V(Y)}{V(X)} \mathbf{1}_{Y \prec X}, \tag{10}$$

where we use the notation

$$V(X) = \prod_{j>i} (x_j - x_i)$$

and the symbol $\mathbf{1}_{Y \prec X}$ *equals* 1 *or* 0 *depending on whether* $Y \prec X$ *or not.*

Proof. To the best of my knowledge, a published proof first appeared in Baryshnikov [1, Proposition 4.2]. However, the argument given in [1] was known earlier: it is hidden in the first computation of the spherical functions of the groups $SL(N, \mathbb{C})$ due to Gelfand and Naimark, see [8, §9]. Note also that a more general result can be found in Neretin [9].

Here is a different proof. Consider the Laplace transform of the orbital measure μ_X:

$$\widehat{\mu}_X(Z) := \int e^{\mathrm{Tr}(ZH)} \mu_X(dH), \tag{11}$$

where Z is a complex $N \times N$ matrix. The integral in the right-hand side is often called the *Harish-Chandra–Itzykson–Zuber integral*. Its value is given by a well-known formula (see, e.g., Olshanski–Vershik [10, Corollary 5.2]):

$$\widehat{\mu}_X(Z) = c_N \frac{\det[e^{z_i x_j}]_{i,j=1}^N}{\prod\limits_{j>i} (z_j - z_i)(x_j - x_i)}, \tag{12}$$

where $z_1, \dots, z_N$ are the eigenvalues of Z and

$$c_N = (N-1)!(N-2)! \dots 0!$$

(note that the right-hand side of (12) does not depend on the enumeration of the eigenvalues of Z).

The claim of the proposition is equivalent to the following equality: Assume that the entries in the last row and column of Z equal 0, so that Z has the form

$$Z = \begin{bmatrix} \widetilde{Z} & 0 \\ 0 & 0 \end{bmatrix}, \tag{13}$$

where $\widetilde{Z}$ is a complex matrix of size $(N-1) \times (N-1)$; then

$$\widehat{\mu}_X(Z) = \frac{(N-1)!}{V(X)} \int_{Y \prec X} V(Y) \widehat{\mu}_Y(\widetilde{Z}) dY. \tag{14}$$

To prove (14), consider the matrix $T := [e^{z_i x_j}]$ in the right-hand side of (12). Since Z has the form (13), at least one of the eigenvalues $z_1, \dots, z_N$ equals 0. It is convenient to slightly change the enumeration and denote the eigenvalues as $z_0 = 0, z_1, \dots, z_{N-1}$. In accordance to this we will assume that the row number i of T ranges over $\{0, \dots, N-1\}$ while the column index j ranges over $\{1, \dots, N\}$. Since $z_0 = 0$, the 0th row of T is $(1, \dots, 1)$. Let us subtract the $(N-1)$th column from the Nth one, then subtract the $(N-2)$th column from the $(N-1)$th one, etc. This gives $\det T = \det \widetilde{T}$, where $\widetilde{T}$ stands for the matrix of order $N-1$ with the entries

$$\widetilde{T}_{i,j} = e^{z_i x_{j+1}} - e^{z_i x_j} = z_i \int_{x_i}^{x_j} e^{z_i y_j} dy_j, \qquad i, j = 1, \dots, N-1.$$

It follows

$$\det \widetilde{T} = z_1 \dots z_N \int_{Y \prec X} dY \det[e^{z_i y_j}]_{i,j=1}^{N-1},$$

so that

$$\widehat{\mu}_X(Z) = c_N \frac{z_1 \dots z_N \int_{Y \prec X} dY \det[e^{z_i y_j}]_{i,j=1}^{N-1}}{\prod\limits_{N-1 \ge j > i \ge 0} (z_j - z_i) \cdot V(X)}.$$

Next, because $z_0 = 0$, the product over $j > i$ in the denominator equals

$$z_1 \dots z_N \prod_{N-1 \ge j > i \ge 1} (z_j - z_i),$$

so that the product $z_1 \dots z_N$ is cancelled out. Taking into account the fact that $\widehat{\mu}_Y(\widetilde{Z})$ is given by the determinantal formula similar to (12) and using the obvious relation $c_N = (N-1)! c_{N-1}$ we finally get the desired equality (14). ■

From Proposition 3.1 it is easy to deduce the following corollary (see also [1, Proposition 4.7] and Defosseux [6]).

Corollary 3.2. *Fix $X \in \mathscr{X}^0(N)$ and let H range over O_X. The map assigning to H the collection of the eigenvalues of the corners $p_K^N(H)$, where $K = N-1, N-2, \dots, 1$, projects O_X onto the Gelfand–Tsetlin polytope P_X and takes the measure μ_X to the Lebesgue measure multiplied by the constant*

$$\frac{(N-1)!(N-2)! \dots 0!}{V(X)}.$$

In particular, the volume of P_X in the natural coordinates is equal to the inverse of the above quantity.

Recall that $\nu_{X,K}$ stands for the radial part of the $K \times K$ corner of the random matrix $H \in O_X$, driven by the orbital measure μ_X (see Section 1), and $M(a; y_1, \dots, y_n)$ denotes the fundamental spline with n knots $y_1, \dots, y_n$ (see (1) and (4)).

Theorem 3.3. *Fix $X = (x_1, \dots, x_N) \in \mathscr{X}^0(N)$. For any $K = 1, \dots, N-1$, the measure $\nu_{X,K}$ on $\mathscr{X}(K)$ is absolutely continuous with respect to Lebesgue measure and has the density*

$$M(a_1, \dots, a_K; x_1, \dots, x_N) := c_{N,K} \frac{V(A) \det\left[M(a_j; x_i, \dots, x_{N-K+i})\right]_{i,j=1}^K}{\prod\limits_{(j,i):\, j-i \ge N-K+1} (x_j - x_i)}, \quad (15)$$

where

$$c_{N,K} = \prod_{i=1}^{K-1} \binom{N-K+i}{i}.$$

Note that for $K = 1$ the right-hand side reduces to the fundamental spline with knots $x_1, \dots, x_N$. Thus, in the case $K = 1$ the theorem says that the density of the measure $\nu_{N,1}$ on $\mathbb{R}$ coincides with the spline $M(a; x_1, \dots, x_N)$. This simple but important claim is due to Andrei Okounkov, see [10, Proposition 8.2].

Proof. We argue by induction on K, starting with $K = N - 1$ and ending at $K = 1$.

Step 1. Examine the case $K = N - 1$, which is the base of induction. We have $\nu_{X,N-1}(dA) = \Lambda^N_{N-1}(X, dA)$. By proposition 3.1, the measure $\Lambda^N_{N-1}(X, \cdot)$ on $\mathscr{X}(N-1)$ is absolutely continuous with respect to Lebesgue measure and has density $\Lambda^N_{N-1}(X, A)$ given by (10). Thus, we have to check that $\Lambda^N_{N-1}(X, A)$ coincides with the quantity $M(a_1, \dots, a_{N-1}; x_1, \dots, x_N)$ given by the right-hand side of (15), where we have to take $K = N - 1$. That is, the desired equality has the form

$$(N-1)! \frac{V(A)}{V(X)} \mathbf{1}_{A \prec X} = c_{N,N-1} \frac{V(A) \det [M(a_j; x_i, x_{i+1})]_{i,j=1}^{N-1}}{\prod\limits_{(j,i):\, j-i \geq 2} (x_j - x_i)}.$$

Since $c_{N,N-1} = (N-1)!$, the desired equality reduces to

$$\det [M(a_j; x_i, x_{i+1})]_{i,j=1}^{N-1} = \frac{\mathbf{1}_{A \prec X}}{(x_2 - x_1)(x_3 - x_2) \dots (x_N - x_{N-1})}.$$

Observe that the (i, j)-entry in the determinant is the quantity

$$M(a_j; x_i, x_{i+1}) = \frac{\mathbf{1}_{x_i \leq a_j \leq x_{i+1}}}{x_{i+1} - x_i},$$

which vanishes unless $a_j \in [x_i, x_{i+1}]$. Since $a_1 \leq \dots \leq a_{N-1}$, the determinant vanishes unless $A \prec X$. Furthermore, if $A \prec X$, then the matrix under the sign of determinant is diagonal, so the determinant equals the product of the diagonal entries, which equals

$$\frac{1}{(x_2 - x_1)(x_3 - x_2) \dots (x_N - x_{N-1})},$$

as required.

Step 2. Given $K = 1, \dots, N - 1$, we consider the superposition of Markov kernels

$$\Lambda^N_K := \Lambda^N_{N-1} \Lambda^{N-1}_{N-2} \dots \Lambda^{K+1}_K.$$

In more detail, the result is a Markov kernel on $\mathscr{X}(N) \times \mathscr{X}(K)$ given by

$$\Lambda^N_K(X, dA) = \int \Lambda^N_{N-1}(X, dY^{(N-1)}) \Lambda^{N-1}_{N-2}(Y^{(N-1)}, dY^{(N-2)}) \dots \Lambda^{K+1}_K(Y^{(K+1)}, dA),$$

where the integral is taken over variables $Y^{(N-1)}, \dots, Y^{(K+1)}$. Obviously, $\Lambda^N_K(X, dA) = \nu_{X,K}(dA)$, which entails the recurrence relation

$$\nu_{X,K-1} = \nu_{X,K} \Lambda^K_{K-1}, \qquad K = N-1, N-2, \dots, 2, \tag{16}$$

where, by definition, $\nu_{X,K}\Lambda^K_{K-1}$ is the measure on $\mathscr{X}(K-1)$ given by

$$(\nu_{X,K}\Lambda^K_{K-1})(dB) = \int_{A\in\mathscr{X}(K)} \nu_{X,K}(dA)\Lambda^K_{K-1}(A,dB). \tag{17}$$

Step 3. Assume now that the claim of the theorem holds for some $K \geq 2$ and deduce from this that it also holds for $K-1$. To do this we employ (16) and (17).

First of all, (16) and (17) imply that $\nu_{X,K-1}$ is absolutely continuous with respect to Lebesgue measure on $\mathscr{X}(K-1)$ and has the density

$$(\nu_{X,K}\Lambda^K_{K-1})(B) = \int_{A\in\mathscr{X}(K)} \nu_{X,K}(dA)\Lambda^K_{K-1}(A,B), \qquad B\in\mathscr{X}(K-1). \tag{18}$$

Let us compute the integral explicitly. By the induction assumption, $\nu_{X,K}$ is absolutely continuous and has density (15). Therefore, integral (18) can be written in the form

$$\int_{A\in\mathscr{X}^0(K)} M(a_1,\dots,a_K;x_1,\dots,x_N)\Lambda^K_{K-1}(A,B)da_1\dots da_K.$$

Write $B=(b_1,\dots,b_K)$. Substituting the explicit expression for $\Lambda^K_{K-1}(A,B)$ given by Proposition 3.1 we rewrite this as

$$(K-1)!V(B)\int_A \frac{M(a_1,\dots,a_K;x_1,\dots,x_N)}{V(A)}da_1\dots da_K, \tag{19}$$

where the integration domain is

$$-\infty < a_1 \leq b_1, \quad \dots, \quad b_i \leq a_{i+1} \leq b_{i+1}, \quad \dots, \quad b_{k-1} \leq a_K < +\infty. \tag{20}$$

Next, plug in into (19) the explicit expression for $M(a_1,\dots,a_K;x_1,\dots,x_N)$ given by (15). Then the factor $V(A)$ is cancelled out and we get

$$\frac{c_{N,K}(K-1)!V(B)}{\prod\limits_{(j,i):\, j-i\geq N-K+1}(x_j-x_i)}\int_A \det\left[M(a_j;x_i,\dots,x_{N-K+i})\right]_{i,j=1}^K da_1\dots da_K \tag{21}$$

with the same integration domain (20).

Put aside the pre-integral factor in (21) and examine the integral itself. It can be written as a $K\times K$ determinant,

$$\det[F(i,j)]_{i,j=1}^K,$$

where

$$F(i,j) := \int_{b_{j-1}}^{b_j} M(a;Y_i)da$$

and

$$Y_i := (x_i,\dots,x_{N-K+i})$$

with the understanding that $b_0=-\infty$ and $b_K=+\infty$.

We are going to prove that

$$\det[F(i,j)]_{i,j=1}^{K} = (N-K+1)^{K-1}\prod_{i=1}^{K-1}(x_{N-K+i+1}-x_i) \times \det[M(b_j;x_i,\dots,x_{N-K+i+1})]_{i,j=1}^{K-1}. \quad (22)$$

This will justify the induction step, because

$$c_{N,K} = c_{N,K-1}\cdot\frac{(N-K+1)^{K-1}}{(K-1)!}$$

and

$$\frac{\prod_{i=1}^{K-1}(x_{N-K+i+1}-x_i)}{\prod_{(j,i):\,j-i\ge N-K+1}(x_j-x_i)} = \frac{1}{\prod_{(j,i):\,j-i\ge N-K+2}(x_j-x_i)}.$$

Step 4. It remains to prove (22). We evaluate the quantities $F(i,j)$ using Lemma 2.2, where we substitute $n=N-K+1$ and $Y=Y_i$. Then we get that the matrix entries $F(i,j)$ are given by the following formulas:

- The entries of the first column have the form
$$F(i,1) = 1 - f_{b_1}[Y_i] \qquad \text{by (6)}.$$
- The entries of the jth column, $2\le j\le K-1$, have the form
$$F(i,j) = f_{b_{j-1}}[Y_i] - f_{b_j}[Y_i] \qquad \text{by (7)}.$$
- The entries of the last column have the form
$$F(i,K) = f_{b_K}[Y_i] \qquad \text{by (8)}.$$

We have $\det F = \det G$, where the $K\times K$ matrix G is defined by

$$G(i,j) := F(i,j) + \cdots + F(i,K).$$

The entries of the matrix G are

$$G(i,1) = 1, \qquad G(i,j) = f_{b_{j-1}}[Y_i], \quad 2\le j\le K.$$

Next, we get $\det G = \det H$ with the $(K-1)\times(K-1)$ matrix H defined by

$$H(i,j) := F(i+1,j+1) - F(i,j), \qquad 1\le i,j\le K-1.$$

Observe now that

$$H(i,j) = f_{b_j}[Y_{i+1}] - f_{b_j}[Y_i],$$

which can be rewritten as

$$H(i,j) = (x_{N-K+i+1}-x_i)\frac{f_{b_j}[x_{i+1},\dots x_{N-K+i+1}] - f_{b_j}[x_i,\dots,x_{N-K+i}]}{x_{N-K+i+1}-x_i} \quad (23)$$

$$= (x_{N-K+i+1}-x_i) f_{b_j}[x_i,\dots,x_{N-K+i+1}] \quad \text{by (3)} \quad (24)$$

$$= \frac{1}{N-K+1}(x_{N-K+i+1}-x_i) M(b_j;x_i,\dots,x_{N-K+i+1}) \quad \text{by (4)}. \quad (25)$$

This shows that the determinant $\det H = \det[H(i,j)]_{i,j=1}^{K-1}$ equals the right-hand side of (22). Since $\det H = \det F$, this completes the proof. ∎

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References

[1] Baryshnikov, Yu, *GUEs and queues*, Prob. Theory Rel. Fields **119** (2001), 256–274.

[2] Boor, C. de, " "A practical guide to splines," Springer, 1978.

[3] Borodin, A., and G. Olshanski, *The boundary of the Gelfand–Tsetlin graph: A new approach*, Adv. Math. **230** (2012), 1738–1779.

[4] Curry, H. B., and I. J. Schoenberg, *On Polya frequency functions IV: The fundamental spline functions and their limits*, J. Analyse Math. **17** (1966), 71–107.

[5] Davis, P. J., "Interpolation and approximation," Dover, 1975.

[6] Defosseux, M., *Orbit measures, random matrix theory and interlaced determinantal processes*, Ann. Inst. Henri Poincaré (B) Prob. Stat. **46** (2010), 209–249.

[7] Faraut, J., *Noyau de Peano et intégrales orbitales*, Glob. J. Pure Appl. Math. **1** (2005), 306–320.

[8] Gelfand, I. M., and M. A. Naimark, *Unitary representations of classical groups*, Proc. Steklov Math. Institute **36** (1950) (Russian); German translation: I. M. Gelfand, M. A. Neumark, „Unitäre Darstellungen der klassischen Gruppen“, Mathematische Lehrbücher und Monographien. II. Abt. Band **6**. Berlin: Akademie–Verlag (1957).

[9] Neretin, Yu. A., *Rayleigh triangles and nonmatrix interpolation of matrix beta integrals*, Sbornik: Mathematics **194** (2003), 515–540.

[10] Olshanski, G., and A. Vershik, *Ergodic unitarily invariant measures on the space of infinite Hermitian matrices*, in: "Contemporary Mathematical Physics. F. A. Berezin's memorial volume", American Mathematical Society Translations, Series 2, Vol. **175** (Advances in the Mathematical Sciences—31), R. L. Dobrushin, R. A. Minlos, M. A. Shubin, A. M. Vershik, eds., Amer. Math. Soc., Providence, RI, 1996, 137–175.

[11] Petrov, L., *The boundary of the Gelfand-Tsetlin graph: New proof of Borodin-Olshanski's formula, and its q-analogue*, Moscow Math. J., to appear. arXiv:1208.3443.

[12] Phillips, G. M., "Approximation and interpolation by polynomials," CMS Books in Math. bf14, Springer, 2003.

[13] Weyl, H., "The classical groups. Their invariants and representations," Princeton Univ. Press, 1939; 1997 (fifth edition).

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