Journal of Lie Theory
Volume **23** (2013) 1101–1103

A Note on Orthogonal Lie Algebras in Dimension 4 Viewed as Current Lie Algebras

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Communicated by E. B. Vinberg

Abstract. Orthogonal Lie algebras in dimension 4 are identified as current Lie algebras, thus producing a natural decomposition for them over any field.
Mathematics Subject Classification 2010: 17B05.
Key Words and Phrases: Classical Lie algebras, current Lie algebras.

Let $f : V \times V \to F$ be a non-degenerate symmetric bilinear form defined on a vector space V of dimension 4 over a field F. If $\mathrm{char}(F) = 2$ we further assume that f is not alternating. Let D be the discriminant of f relative to a basis of V. Let $L(f)$ be the subalgebra of $\mathfrak{gl}(V)$ associated to f, formed by all $x \in \mathfrak{gl}(V)$ that are skew-adjoint relative to f. Let $M = [L(f), L(f)]$, which coincides with $L(f)$ if and only if $\mathrm{char}(F) \neq 2$. In any case, M is a 6-dimensional orthogonal Lie algebra.

According to [B], Chapter 1, §6, Exercise 26, if $\mathrm{char}(F) \neq 2$ then M is either the direct sum of two 3-dimensional simple ideals or simple, depending on whether D is a square in F or not. On the other hand, if $\mathrm{char}(F) = 2$ then either $M = N \ltimes R$ or M is simple, depending, again, on whether D is a square in F or not; here N is a simple 3-dimensional simple subalgebra and R is the solvable radical of M.

We may view these orthogonal Lie algebras as current Lie algebras, and in this way explain all cases described above in a uniform manner.

Theorem. We have $M \cong [L(f|_W), L(f|_W)] \otimes F[X]/(X^2 - D)$, where W is an arbitrary 3-dimensional subspace of V such that $f|_W$ is non-degenerate.

Proof. It follows from [K], Theorems 4 and 20, that V admits an orthogonal basis $B = \{v_1, v_2, v_3, v_4\}$. Thus, the Gram matrix of f relative to B is diagonal with non-zero entries a, b, c, d and $D = abcd$. Let

$$f_1 = be_{12} - ae_{21},\ f_2 = ce_{23} - be_{32},\ f_3 = ce_{13} - ae_{31},$$

$$h_1 = ab(de_{34} - ce_{43}),\ h_2 = bc(de_{14} - ae_{41}),\ h_3 = ac(be_{42} - de_{24}).$$

ISSN 0949–5932 / \$2.50

Then $f_1, f_2, f_3, h_1, h_2, h_3$ is a basis of M, with multiplication table

$$[f_1, f_2] = bf_3,\ [f_2, f_3] = cf_1,\ [f_3, f_1] = af_2,$$

$$[f_1, h_2] = bh_3,\ [f_2, h_3] = ch_1,\ [f_3, h_1] = ah_2,$$
$$[f_2, h_1] = -bh_3,\ [f_3, h_2] = -ch_1,\ [f_1, h_3] = -ah_2,$$

and

$$[h_1, h_2] = Dbf_3,\ [h_2, h_3] = Dcf_1,\ [h_3, h_1] = Daf_2.$$

Thus M has the same multiplication table as $[L(f|_W), L(f|_W)] \otimes F[X]/(X^2 - D)$, where W is the span by v_1, v_2, v_3. By [K], Theorems 4 and 20, W can be replaced by any 3-dimensional subspace of V where the restriction of f is non-degenerate. ■

It is obvious from the Theorem that M decomposes exactly as prescribed above if D is a square. Suppose D is not a square. Then $K = F[X]/(X^2 - D)$ is a quadratic field extension of F. Since $[L(f|_W), L(f|_W)]$ is a perfect 3-dimensional Lie algebra over F, it follows that $[L(f|_W), L(f|_W)] \otimes K$ is a simple Lie algebra over K, and hence over F, as seen below. Thus, by the Theorem, M is a simple Lie algebra over F.

Lemma. (cf. [LP], Lemma 2.7) Let L be a simple Lie algebra over a field K. Suppose that F is a subfield of K. Then L is simple over F.

Proof. Let I be a non-zero ideal of L over F. The K-span of I is a non-zero ideal J of L over K, so $J = L = [L, L] = [L, J] = [L, I] \subseteq I$. ■

A simple Lie algebra need not remain simple, or even semisimple, upon field extension. For instance, let L be an absolutely simple Lie algebra over an imperfect field F of characteristic p, and let $s \in F \setminus F^p$ and $K = F[X]/(X^p - s)$, which has an element t satisfying $t^p = s$. Then $P = L \otimes K$ is simple over K and hence over F, but $P \otimes K \cong L \otimes K[X]/(X^p - s) \cong L \otimes K[X]/(X - t)^p$ is not semisimple over K.

Acknowledgment. We are grateful to A. Pianzola for pointing out reference [LP], which replaced a more convoluted argument.

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Received March 6, 2013
and in final form March 18, 2018