Minimal Faithful Representation of the Heisenberg Lie Algebra with Abelian Factor

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Abstract. For a finite dimensional Lie algebra $\mathfrak{g}$ over a field k of characteristic zero, the μ-function (respectively μ_{nil}-function) is defined to be the minimal dimension of V such that $\mathfrak{g}$ admits a faithful representation (respectively a faithful nilrepresentation) on V. Let $\mathfrak{h}_m$ be the Heisenberg Lie algebra of dimension $2m+1$ and let $\mathfrak{a}_n$ be the abelian Lie algebra of dimension n. The aim of this paper is to compute $\mu(\mathfrak{h}_m \oplus \mathfrak{a}_n)$ and $\mu_{\mathrm{nil}}(\mathfrak{h}_m \oplus \mathfrak{a}_n)$ for all $m, n \in \mathbb{N}$. We also give a faithful representation and faithful nilrepresentation of $\mathfrak{h}_m \oplus \mathfrak{a}_n$ of minimal dimension for all $m, n \in \mathbb{N}$.
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1. Introduction

In this paper, all Lie algebras and representations are finite dimensional over a field k of characteristic zero. Given a Lie algebra $\mathfrak{g}$, we denote by $\mu(\mathfrak{g})$ the minimal dimension of a faithful representation and by $\mu_{\mathrm{nil}}(\mathfrak{g})$ the minimal dimension of a faithful nilrepresentation. By Ado's Theorem (or its proof) these numbers are well defined and they are integers invariants of $\mathfrak{g}$ (see for instance [12, page 202]). Clearly, $\mu(\mathfrak{g}) \leq \mu_{\mathrm{nil}}(\mathfrak{g})$, when $\mathfrak{g}$ is nilpotent.

In the theory of the affine crystallographic groups and finitely-generated torsion-free nilpotent groups, the invariant μ plays an important role (for details and reference see [4]). The following results are well-known:

Proposition. *[4, Propositions 2.31 and 3.8] Let G be an Lie group of dimension n with $\mathfrak{g} = Lie(G)$ the Lie algebra of G. If G admits a left-invariant affine structure then $\mu(\mathfrak{g}) \leq n+1$.*

The study of left-invariant affine structures over a Lie group G is linked to the existence of left symmetric structures on the corresponding Lie algebra $\mathfrak{g}$. Milnor stated that all real solvable Lie algebra would admit a left symmetric

structure. There are many articles with positive results in this direction (see for instance the original paper of Milnor [15] and [1, 9, 11]). However, Y. Benoist [2] and Burde and Grunewald [6] gave the first examples of nilpotent Lie groups not admitting any left-invariant affine structure, they constructed nilpotent Lie algebras $\mathfrak{g}$ such that $\mu(\mathfrak{g}) > \dim \mathfrak{g} + 1$.

On the other hand, this problem is also related to the representation theory of finitely generated nilpotent groups, since it is important to have methods for finding integer matrix representations of small dimension for this class of groups (see [10, 17]).

Computing μ and μ_{nil} (or finding bounds for them) for a given Lie algebra is acknowledged to be a very difficult task and the goal has been achieved for very few families (see for instance [20, 7, 8, 14]).

Let $\mathfrak{g}$ be a Lie algebra. Assume that the center $\mathfrak{z}(\mathfrak{g})$ is trivial then the adjoint representation is faithful, it follows that $\mu(\mathfrak{g}) \leq \dim \mathfrak{g}$. If $\mathfrak{g}$ is k-step nilpotent whith $k = 2$ or 3 then

$$\mu(\mathfrak{g}) \leq \dim \mathfrak{g} + 1, \tag{1}$$

(see [19]). In general, if $\mathfrak{g}$ is nilpotent this result is not even true. In this direction, it is known that

$$\mu(\mathfrak{g}) < \frac{3}{\sqrt{\dim \mathfrak{g}}} 2^{\dim \mathfrak{g}}, \tag{2}$$

(see [3]). Since the classification of representations of nilpotent Lie algebra is a *wild* problem, it reasonable to expect difficulties in obtaining $\mu(\mathfrak{g})$.

Let $\mathfrak{a}_n$ be the abelian Lie algebra of dimension n and let $\mathfrak{h}_m$ be the Heisenberg Lie algebra of dimension $2m + 1$ with a basis $\{X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z\}$ such that the only non-zero brackets are

$$[X_i, Y_i] = Z.$$

It is clear that the center of $\mathfrak{h}_m$ is $\mathfrak{z}(\mathfrak{h}_m) = \mathrm{k}\{Z\}$. In the rest of the paper, we denote by $\lceil x \rceil$ the ceiling of x. The following results are know for $\mathfrak{h}_m$ and $\mathfrak{a}_n$.

Theorem 1.1.   *Let* $m, n \in \mathbb{N}$

(1) $\mu_{\mathrm{nil}}(\mathfrak{h}_m) = \mu(\mathfrak{h}_m) = m + 2$ *(see [3]);*

(2) $\mu(\mathfrak{a}_n) = \lceil 2\sqrt{n-1} \rceil$ *and* $\mu_{\mathrm{nil}}(\mathfrak{a}_n) = \lceil 2\sqrt{n} \rceil$ *(see [20], [13], [16]).*

Let $\mathfrak{h}_m \oplus \mathfrak{a}_n$ be the direct sum of the Heisenberg Lie algebra and $\mathfrak{a}_n$. The upper bound for $\mu(\mathfrak{h}_m \oplus \mathfrak{a}_n)$ given by equation (1) is

$$\mu(\mathfrak{h}_m \oplus \mathfrak{a}_n) < 2m + n + 2.$$

Our main result in this paper are the value of μ and μ_{nil} for the Lie algebra $\mathfrak{h}_m \oplus \mathfrak{a}_n$ for all $m, n \in \mathbb{N}$. In this direction, we prove the following theorem.

Theorem.   *Let* $m \in \mathbb{N}_0$ *and* $n \in \mathbb{N}$. *Then*

$$\mu_{\mathrm{nil}}(\mathfrak{h}_m \oplus \mathfrak{a}_n) = m + \lceil 2\sqrt{n+1} \rceil \quad \text{and} \quad \mu(\mathfrak{h}_m \oplus \mathfrak{a}_n) = m + \lceil 2\sqrt{n} \rceil.$$

The paper is organized as follows. In §2, the upper bounds for $\mu(\mathfrak{h}_m \oplus \mathfrak{a}_n)$ and $\mu_{\mathrm{nil}}(\mathfrak{h}_m \oplus \mathfrak{a}_n)$ are obtained by explicitly constructing faithful representations and faithful nilrepresentations of $\mathfrak{h}_m \oplus \mathfrak{a}_n$ (which are shown to be of minimal dimension, as we will see later). In §3 provides a detailed exposition of the proof of the lower bound of $\mu(\mathfrak{h}_m \oplus \mathfrak{a}_n)$ and $\mu_{\mathrm{nil}}(\mathfrak{h}_m \oplus \mathfrak{a}_n)$, here we use a theorem of Zassenhaus and some results obtained in [8].

Our result for $m = 0$ coincides with the statement of Theorem 1.1.(2) and it is straightforward to see that our value of μ_{nil} coincides with the statement of Theorem 1.1.(1) if $n = 0$.

In general

$$\mu(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \le \mu(\mathfrak{g}_1) + \mu(\mathfrak{g}_2) \tag{3}$$

for arbitrary Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ is a semisimple Lie algebra over $\mathbb{C}$, where $\mathfrak{g}_j$ is a simple ideal of $\mathfrak{g}$, then $\mu(\mathfrak{g}) = \mu(\mathfrak{g}_1) + \cdots + \mu(\mathfrak{g}_r)$ (see [7]). But the equality in (3) seldom occurs for nilpotent Lie algebras, for instance $\mu(\mathfrak{a}_n) = \lceil 2\sqrt{n-1} \rceil < \sum_{i=1}^{n} \mu(\mathfrak{a}_1) = n$. As far as we know, there is no example of two nilpotent Lie algebra $\mathfrak{g}_i$, $\dim \mathfrak{g}_i \ge 2$, such that $\mu(\mathfrak{g}_1 \oplus \mathfrak{g}_2) = \mu(\mathfrak{g}_1) + \mu(\mathfrak{g}_2)$.

In this context, the Lie algebra $\mathfrak{h}_m \oplus \mathfrak{a}_n$ can be viewed as an extreme case in this sense, since

$$\mu_{\mathrm{nil}}(\mathfrak{h}_m \oplus \mathfrak{a}_n) = \mu_{\mathrm{nil}}(\mathfrak{h}_m) + \mu_{\mathrm{nil}}(\mathfrak{a}_n) - \epsilon$$
$$\mu(\mathfrak{h}_m \oplus \mathfrak{a}_n) = \mu(\mathfrak{h}_m) + \mu(\mathfrak{a}_n) - \epsilon.$$

with $\epsilon = 1$ or 2, depending on the value of n.

This paper is used in [18] to calculate the μ-function and μ_{nil}-function for all 6-dimensional nilpotent Lie algebra over k.

2. A family of representations of $\mathfrak{h}_m \oplus \mathfrak{a}_n$

We start with a simple proposition.

Proposition 2.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and let (π, V) be a representation of $\mathfrak{g}$. Then (π, V) is faithful if and only if $(\pi\,|_{\mathfrak{z}(\mathfrak{g})}, V)$ is faithful on $\mathfrak{z}(\mathfrak{g})$.*

Proof. Assume that $\mathfrak{g}$ is k-step nilpotent and $\ker \pi$ is non trivial and let $X_0 \in \ker \pi, X_0 \neq 0$. Since $(\pi\,|_{\mathfrak{z}(\mathfrak{g})}, V)$ is faithful on $\mathfrak{z}(\mathfrak{g})$, we know that $X_0 \notin \mathfrak{z}(\mathfrak{g})$ and thus there exists $X_1 \in \mathfrak{g}$ such that $[X_0, X_1] \neq 0$. Since $\ker \pi$ is an ideal of $\mathfrak{g}$ and $(\pi\,|_{\mathfrak{z}(\mathfrak{g})}, V)$ is faithful, $[X_0, X_1] \in \ker \pi, [X_0, X_1] \notin \mathfrak{z}(\mathfrak{g})$. Hence, there exits $X_2 \in \mathfrak{g}$ such that $[[X_0, X_1], X_2] \neq 0$ and hence, as before $[[X_0, X_1], X_2] \in \ker \pi, [[X_0, X_1], X_2] \notin \mathfrak{z}(\mathfrak{g})$. We now apply this argument again k-times, to obtain $X_{k-1} \in \mathfrak{g}$ such that $[[\ldots, [[X_0, X_1], X_2], \ldots], X_k] \neq 0$ but this contradicts the fact that $\mathfrak{g}$ is k-step nilpotent.

The converse is clear.   ■

The Heisenberg Lie algebra $\mathfrak{h}_m$ has a canonical faithful representation

$(\pi_0, \mathrm{k}^{m+2})$ which in terms of the canonical basis of k^{m+2} is given by

$$\pi_0 \left(\sum_{i=1}^{m} x_i X_i + \sum_{i=1}^{m} y_i Y_i + zZ \right) = \begin{bmatrix} 0 & x_1 & \dots & x_m & z \\ & & & & y_1 \\ & & 0 & & \vdots \\ & & & & y_m \\ & & & & 0 \end{bmatrix} \qquad x_i, y_i, z \in \mathrm{k}. \qquad (4)$$

Let $\{A_1, \dots, A_n\}$ be a basis of $\mathfrak{a}_n$, and let $\{Z, A_1, \dots, A_n\}$ be a basis of the center $\mathfrak{z}(\mathfrak{h}_m \oplus \mathfrak{a}_n)$.

We shall now construct a family of nilrepresentations of $\mathfrak{h}_m \oplus \mathfrak{a}_n$ that contains a faithful nilrepresentation of minimal dimension.

For $a, b \in \mathbb{N}$, define the linear maps

- $\tau_a : \mathrm{k}\{X_1, \dots, X_m\} \to M_{a,m}(\mathrm{k}), \quad \tau_a \left(\sum_{i=1}^{m} x_i X_i \right) = \begin{bmatrix} x_1 & \dots & x_m \\ & 0 & \end{bmatrix}.$

- $\tau_b : \mathrm{k}\{Y_1, \dots, Y_m\} \to M_{m,b}(\mathrm{k}), \quad \tau_b \left(\sum_{i=1}^{m} y_i Y_i \right) = \begin{bmatrix} y_1 & & \\ \vdots & 0 & \\ y_m & & \end{bmatrix}.$

- $\tau_{a,b} : \mathrm{k}\{Z, A_1, \dots, A_n\} \to M_{a,b}(\mathrm{k})$

$$\tau_{a,b} \left(zZ + \sum_{r=1}^{n} a_r A_r \right)_{ij} = \begin{cases} z, & \text{if } i = j = 1; \\ a_{j-1}, & \text{if } i = 1, j \geq 2; \\ a_{j-1+(i-1)b}, & \text{if } 2 \leq i \leq \min\left\{a, \left\lceil \frac{n-b+1}{b} \right\rceil + 1\right\}; \\ 0, & \text{otherwise}. \end{cases}$$

For instance, if $n = 10, a = 5, b = 3$ we have

$$\tau_{4,3} \left(zZ + \sum_{r=1}^{10} a_i A_i \right) = \begin{bmatrix} z & a_1 & a_2 \\ a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 \\ a_9 & a_{10} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, we define the representation $(\pi_{a,b}, \mathrm{k}^{m+a+b})$ of $\mathfrak{h}_m \oplus \mathfrak{a}_n$, in terms of the canonical basis of k^{m+a+b}. Let $X = \sum_{i=1}^{m} x_i X_i, Y = \sum_{i=1}^{m} y_i Y_i$ and $A = \sum_{i=1}^{n} a_i A_i$. Then $\pi_{a,b}$ is given by the following block matrix

$$\pi_{a,b} \left(X + Y + zZ + A \right) = \begin{bmatrix} \overbrace{}^{a} & \overbrace{\tau_a(X)}^{m} & \overbrace{\tau_{a,b}(zZ + A)}^{b} \\ 0 & \tau_a(X) & \tau_{a,b}(zZ + A) \\ 0 & 0 & \tau_b(Y) \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \}a \\ \}m \\ \}b \end{matrix}$$

Example 2.2. Let $m = 2, n = 4$ and $X = \sum_{i=1}^{2} x_i X_i + \sum_{i=1}^{2} y_i Y_i + zZ + \sum_{i=1}^{4} a_i A_i$;

a) $a = 2, b = 3$ then $\pi_{2,3}(X) = \begin{bmatrix} 0 & 0 & x_1 & x_2 & z & a_1 & a_2 \\ 0 & 0 & 0 & 0 & a_3 & a_4 & 0 \\ 0 & 0 & 0 & 0 & y_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$

b) $a = 1, b = 3$ then $\pi_{1,3}(X) = \begin{bmatrix} 0 & x_1 & x_2 & z & a_1 & a_2 \\ 0 & 0 & 0 & y_1 & 0 & 0 \\ 0 & 0 & 0 & y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

By straightforward calculation we have

- $\left(\pi_{a,b}, \mathrm{k}^{m+a+b}\right)$ is a nilrepresentation of $\mathfrak{h}_m \oplus \mathfrak{a}_n$.

- If $a = b = 1$ then $\pi_{1,1}|_{\mathfrak{h}_m} = \pi_0$.

- By Proposition 2.1, $\left(\pi_{a,b}, \mathrm{k}^{m+a+b}\right)$ is faithful if and only if $\tau_{a,b}$ is injective. Then $\left(\pi_{a,b}, \mathrm{k}^{m+a+b}\right)$ is faithful if and only if $ab \geq n + 1$.

Since
$$\min\{a + b : ab \geq n + 1\} = \left\lceil 2\sqrt{n+1} \right\rceil$$

for every $n \in \mathbb{N}$, we obtain the following result.

Proposition 2.3. *Let $m, n \in \mathbb{N}$ then*

$$\mu_{\mathrm{nil}}(\mathfrak{h}_m \oplus \mathfrak{a}_n) \leq m + \left\lceil 2\sqrt{n+1} \right\rceil.$$

Analogously, we introduce a family of representation of $\mathfrak{h}_m \oplus \mathfrak{a}_n$ that contains a faithful representation of minimal dimension.

Let $X = \sum_{i=1}^{m} x_i X_i + \sum_{i=1}^{m} y_i Y_i + zZ + \sum_{i=1}^{n} a_i A_i \in \mathfrak{h}_m \oplus \mathfrak{a}_n$ and let $\left(\pi_{a,b}, \mathrm{k}^{m+a+b}\right)$ be the representation of $\mathfrak{h}_m \oplus \mathfrak{a}_n$ that is given by

$$\widetilde{\pi}_{a,b}(X) = \pi_{a,b}\left(\sum_{i=1}^{m} x_i X_i + \sum_{i=1}^{m} y_i Y_i + zZ + \sum_{i=1}^{n-1} a_i A_i \right) + a_n I$$

where I is an identity matrix of size $m + a + b$.

It is easy to see that $\left(\widetilde{\pi}_{a,b}, \mathrm{k}^{m+a+b}\right)$ is a representation of $\mathfrak{h}_m \oplus \mathfrak{a}_n$ and $\widetilde{\pi}$ is faithful if $ab \geq n$. Then we obtain the following proposition.

Proposition 2.4. *Let $m, n \in \mathbb{N}$ then*

$$\mu(\mathfrak{h}_m \oplus \mathfrak{a}_n) \leq m + \left\lceil 2\sqrt{n} \right\rceil.$$

3. The lower bound

In this section we prove the lower bound of $\mu(\mathfrak{h}_m \oplus \mathfrak{a}_n)$ and $\mu_{\mathrm{nil}}(\mathfrak{h}_m \oplus \mathfrak{a}_n)$.

We will need the following facts:

Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{h}_m \oplus \mathfrak{a}_n$ such that $Z \notin \mathfrak{g}$, from an argument similar to [8, Lemma 3] it follows that

$$\dim \mathfrak{g} \leq m + \dim \mathfrak{g} \cap \mathfrak{z}(\mathfrak{h}_m \oplus \mathfrak{a}_n). \tag{5}$$

We will also need the following theorem.

Theorem 3.1.   *[8, Theorem 4.2] Let V be a finite-dimensional vector space and let $\mathcal{N}$ be a non-zero abelian subspace of $\mathrm{End}(V)$ consisting of nilpotent operators. Then there exists a linearly independent set $B = \{v_1, \ldots, v_s\} \subset V$ and a decomposition $\mathcal{N} = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_s$, with $\mathcal{N}_i \neq 0$ for all i, such that the maps $F_i : \mathcal{N} \to V$ defined by $F_i(N) = N(v_i)$ satisfy*

(1) *$F_i \mid_{\mathcal{N}_i}$ is injective for all $i = 1, \ldots, s$.*

(2) *$\mathcal{N}_j \subset \ker F_i$ for all $1 \leq i < j \leq s$;*

(3) *$\mathcal{N}_j V \subset \mathrm{Im}\, F_i \mid_{\mathcal{N}_i}$ for all $1 \leq i < j \leq s$.*

Furthermore, given a finite subset $\{N_1, \ldots, N_q\}$ of non-zero operators in $\mathcal{N}$, the vector v_1 can be chosen so that $N_k(v_1) \neq 0$ for all $k = 1, \ldots, k$.

We are now ready to prove the first main result of this section.

Theorem 3.2.   Let $m, n \in \mathbb{N}$ then

$$\mu_{\mathrm{nil}}(\mathfrak{h}_m \oplus \mathfrak{a}_n) \geq m + \left\lceil 2\sqrt{n+1} \right\rceil.$$

Proof.   Let (π, V) be a faithful nilrepresentation of $\mathfrak{h}_m \oplus \mathfrak{a}_n$ and let $\mathfrak{z} = \mathrm{k}\{Z\} \oplus \mathfrak{a}_n$. We apply Theorem 3.1 to the subspace $\mathcal{N} = \pi(\mathfrak{z})$. We obtain a linearly independent set $B = \{v_1, \ldots, v_s\} \subset V$ and a decomposition $\mathcal{N} = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_s$, with $\mathcal{N}_i \neq 0$ for all i, such that the maps $F_i : \mathcal{N} \to V$ defined by $F_i(N) = N(v_i)$ satisfy $(1), (2)$ and (3) of Theorem 3.1. We additionally require that $\pi(Z)(v_1) \neq 0$.
    Let ϕ be a linear map such that

$$\phi : \mathfrak{h}_m \oplus \mathfrak{a}_n \to V, \quad \phi(X) = \pi(X)v_1.$$

Note that $\phi \mid_{\mathfrak{z}} = F_1 \circ \pi$. We claim that

(i) $\dim \mathrm{Im}\, \phi + \dim \ker F_1 \geq m + n + 1$.

(ii) $\mathrm{Im}\, \phi \cap \mathrm{k}B = 0$, and thus $\dim V \geq s + \dim \mathrm{Im}\, \phi$.

(iii) $n + 1 \leq s \dim \mathrm{Im}\, F_1$, and thus $\left\lceil 2\sqrt{n+1} \right\rceil \leq s + \dim \mathrm{Im}\, F_1$.

Proof of (i). It is clear that $\ker \phi$ is a subalgebra of $\mathfrak{h}_m \oplus \mathfrak{a}_n$ such that $Z \notin \ker \phi$. Since $\pi(\ker \phi \cap \mathfrak{z}) = \ker F_1$, it follows from (5) that

$$\dim \ker \phi \leq m + \dim \ker F_1.$$

Since $\dim \ker \phi + \dim \mathrm{Im}\, \phi = 2m + 1 + n$, we obtain (i).

Proof of (ii). Let $v \in \mathrm{Im}\, \phi \cap \mathrm{k}B$ then $v = \sum_{i=1}^{s} a_i v_i$ and since $v \in \mathrm{Im}\, \phi$ there is $X \in \mathfrak{h}_m \oplus \mathfrak{a}_n$ such that $\pi(X)(v_1) = v$. Hence

$$\pi(X)(v_1) = \sum_{i=1}^{s} a_i v_i. \tag{6}$$

We must prove that $a_i = 0$ for all i. Assume that $a_i \neq 0$ for some i and let $i_0 = \max\{i : a_i \neq 0\}$. Since $\pi(X)$ is nilpotent endomorphism on V, its only eigenvalue is zero and thus $i_0 > 1$. Let $N \in \mathcal{N}_0$, $N \neq 0$ and let us apply N to both sides of equation (6). We obtain zero on the left hand side, since $N \in \pi(\mathfrak{z})$, $i_0 > 1$. But from (1) and (2) of the Theorem 3.1, we obtain on the right hand side $a_{i_0} N(v_{i_0}) \neq 0$, which is a contradiction.

Proof of (iii). Part (1) and (3) combined imply that $\dim \mathcal{N}_x \geq \dim \mathcal{N}_y$ if $x < y$. In particular $\dim \mathcal{N}_1 \geq \mathcal{N}_j$ for all $j = 1, \ldots, s$ and thus

$$n + 1 = \dim \mathcal{N} = \sum_{j=1}^{s} \dim \mathcal{N}_j \leq s \dim \mathcal{N}_1 = s \dim \operatorname{Im} F_1.$$

Since $\min\{a + b : a, b \in \mathbb{N} \text{ and } ab \geq n + 1\} = \left\lceil 2\sqrt{n+1} \right\rceil$ for all $n \in \mathbb{N}$, we obtain (iii).

From (i) and (ii) it follows that

$$\dim V + \dim \ker F_1 \geq m + n + 1 + s,$$

and combining it with (iii) we obtain

$$\dim V + \dim \ker F_1 + \dim \operatorname{Im} F_1 \geq m + n + 1 + \left\lceil 2\sqrt{n+1} \right\rceil.$$

Finally, since $\dim \ker F_1 + \dim \operatorname{Im} F_1 = n + 1$ we obtain

$$\dim V \geq m + \left\lceil 2\sqrt{n+1} \right\rceil$$

and this completes the proof. ∎

First we recall a theorem due to Zassenhaus (see [12, page 41]) to prove the lower bound for $\mu(\mathfrak{h}_m \oplus \mathfrak{a}_n)$.

Theorem 3.3. *[8, Theorem 2.1] Let $\mathfrak{g}$ be finite dimensional nilpotent Lie algebra and let (π, V) be a finite-dimensional representation of $\mathfrak{g}$. If $\Bbbk$ is algebraically closed then*

$$V = V_1 \oplus \cdots \oplus V_s,$$

such that $\pi(X) \mid_{V_i}$ is a scalar λ_i plus a nilpotent operator $N_i(X)$ on V_i for all $X \in \mathfrak{g}$ and $i = 1, \ldots, s$. Moreover, (N_i, V_i) is a nilrepresentation of $\mathfrak{g}$ for all $i = 1, \ldots, s$.

We end the paper with the second main result of this section.

Theorem 3.4. *Let $m, n \in \mathbb{N}$ then*

$$\mu(\mathfrak{h}_m \oplus \mathfrak{a}_n) \geq m + \left\lceil 2\sqrt{n} \right\rceil.$$

Proof.   Let (π, V) a faithful representation of $\mathfrak{h}_m \oplus \mathfrak{a}_n$, by Zassenhaus theorem we have

$$V = V_1 \oplus \cdots \oplus V_s,$$

such that $\pi(X)\mid_{V_i} = \lambda_i(X)I + N_i(X)$ for all $X \in \mathfrak{g}$.

Since (π, V) is a faithful representation there is $v \in V$ such that $\pi(Z)(v) \neq 0$. It follows that there exist $i_0 = 1, \ldots, s$, such that $\pi(Z)(v_{i_0}) \neq 0$.

Since $Z \in \mathfrak{z}(\mathfrak{h}_m \oplus \mathfrak{a}_n) \cap [\mathfrak{h}_m \oplus \mathfrak{a}_n, \mathfrak{h}_m \oplus \mathfrak{a}_n]$, we have $\lambda_{i_0}(Z) = 0$ and therefore $N_{i_0}(Z)(v_{i_0}) \neq 0$. By Proposition 2.1 we have $(N_{i_0}\mid_{\mathfrak{h}_m}, V_{i_0})$ is a faithful nilrepresentation of $\mathfrak{h}_m$. Therefore, if $\mathfrak{c} = \ker N_{i_0}$, it follows that $\mathfrak{c} \cap \mathfrak{h}_m = 0$ and thus $\mathfrak{c}$ is an abelian Lie subalgebra of $\mathfrak{h}_m \oplus \mathfrak{a}_n$. Let $\mathfrak{b}$ be a complementary subspace of $\mathfrak{h}_m \oplus \mathfrak{c}$ in $\mathfrak{h}_m \oplus \mathfrak{a}_n$, thus $\mathfrak{h}_m \oplus \mathfrak{a}_n = \mathfrak{h}_m \oplus \mathfrak{b} \oplus \mathfrak{c}$. It is clear that $\mathfrak{h}_m \oplus \mathfrak{b}$ is a Lie subalgebra of $\mathfrak{h}_m \oplus \mathfrak{a}_n$. By construction of $\mathfrak{b}$, we have $(N_{i_0}\mid_{\mathfrak{h}_m \oplus \mathfrak{b}}, V_{i_0})$ is a faithful nilrepresentation of $\mathfrak{h}_m \oplus \mathfrak{b}$. We conclude, by Theorem 3.2, that

$$\dim V_{i_0} \geq m + \left\lceil 2\sqrt{\dim \mathfrak{b} + 1} \right\rceil.$$

Let $\mathfrak{c}_0 = \ker \lambda_{i_0} \cap \mathfrak{c}$ then $\dim \mathfrak{c}_0 = \dim \mathfrak{c} - 1$ and $(\pi\mid_{\mathfrak{c}_0}, \oplus_{i \neq i_0} V_i)$ is a faithful representation of $\mathfrak{c}_0$. By Theorem 1.1(2), we have

$$\dim \oplus_{i \neq i_0} V_i \geq \left\lceil 2\sqrt{\dim \mathfrak{c}_0 - 1} \right\rceil$$

Since $\dim V = \dim V_{i_0} + \dim \oplus_{i \neq i_0} V_i$, it follows that

$$\begin{aligned}
\dim V &\geq m + \left\lceil 2\sqrt{\dim \mathfrak{b} + 1} \right\rceil + \left\lceil 2\sqrt{\dim \mathfrak{c}_0 - 1} \right\rceil \\
&\geq m + \left\lceil 2\sqrt{\dim \mathfrak{b} + \dim \mathfrak{c}_0 + 1} \right\rceil \\
&\geq m + \left\lceil 2\sqrt{n} \right\rceil
\end{aligned}$$

and this completes the proof. ∎

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