Journal of Lie Theory
Volume **23** (2013) 1115–1128

Hom-Lie Superalgebra Structures on Finite-Dimensional Simple Lie Superalgebras*

Bintao Cao and Li Luo

Communicated by J. Hilgert

Abstract. Hom-Lie superalgebras, which can be considered as deformations of Lie superalgebras, are $\mathbb{Z}_2$-graded generalization of Hom-Lie algebras, In this paper, we prove that there only exists the trivial Hom-Lie superalgebra structure on a finite-dimensional simple Lie superalgebra.
Mathematics Subject Classification 2010: 17B05, 17B40, 17B60.
Key Words and Phrases: Simple Lie superalgebra, Hom-Lie superalgebra.

1. Introduction

Hom-Lie algebras have been studied by many mathematicians before they were introduced formally by Hartwig, Larsson and Silverstrov [10] . We refer to [2, 9, 13, 4, 5, 6] for those pioneering works. Part of the reason why they study Hom-Lie algebras is to deal with the q-deformations of the Witt and the Virasoro algebras.

Closely related to discrete and deformed vector fields and differential calculus, Hom-Lie algebras have recently been paid much attention in the works [3, 11, 14, 16, 17, 18, 19, 21]. In particular, Jin and Li [11] proved that such an algebra structure on a finite-dimensional simple Lie algebra is equivalent to the trivial one. They also classified all of the nontrivial Hom-Lie algebra structures on finite-dimensional semi-simple Lie algebras.

Recently, Hom-Lie algebras were generalized to Hom-Lie superalgebras by Ammar and Makhlouf [1] and to Hom-Lie color algebras by Yuan [20].

Definition 1.1. A *Hom-Lie superalgebra* $(\mathfrak{g}, [-,-], \sigma)$ consists of a $\mathbb{Z}_2$-graded vector space $\mathfrak{g}$, a bilinear map $[-,-] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ and an even linear map $\sigma : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$\sigma[x, y] = [\sigma(x), \sigma(y)] \quad \text{(multiplicativity)}, \tag{1.1}$$

$$[x, y] = -(-1)^{|x||y|}[y, x] \quad \text{(graded skew-symmetry)} \tag{1.2}$$

*Work supported by NSFC Grants (11101436, 11101151), and the Fundamental Research Funds for the Central Universities.

ISSN 0949–5932 / \$2.50

and

$$(-1)^{|x||z|}[\sigma(x),[y,z]]+(-1)^{|y||x|}[\sigma(y),[z,x]]+(-1)^{|z||y|}[\sigma(z),[x,y]]=0 \quad (1.3)$$

(graded Hom-Jacobi identity, or graded σ-twisted Jacobi identity),

where x, y and z are homogeneous elements in $\mathfrak{g}$.

Definition 1.2. Let $\mathfrak{g}$ be an Lie superalgebra, $[-,-]$ be its Lie bracket, and σ be a linear transformation on $\mathfrak{g}$. Then $(\mathfrak{g},\sigma)$ is called a *Hom-Lie superalgebra structure on the Lie superalgebra* $\mathfrak{g}$ if $(\mathfrak{g},[-,-],\sigma)$ is a Hom-Lie superalgebra.

It is obvious that the Hom-Lie superalgebra $(\mathfrak{g},[-,-],\mathrm{id})$ is just the Lie superalgebra $\mathfrak{g}$. We call $(\mathfrak{g},\mathrm{id})$ *the trivial Hom-Lie superalgebra structure on* $\mathfrak{g}$. Thanks to the multiplicativity (1.1), one sees that σ has to be a homomorphism of $\mathfrak{g}$ if $(\mathfrak{g},\sigma)$ is a Hom-Lie superalgebra structure on the Lie superalgebra $\mathfrak{g}$. For any simple Lie superalgebra $\mathfrak{g}$, a homomorphism $\sigma:\mathfrak{g}\to\mathfrak{g}$ is either an automorphism or the zero map. The graded Hom-Jacobi identity (1.3) will become "0 = 0" if $\sigma=0$. In this case, the Hom-Lie superalgebra $(\mathfrak{g},\sigma)$ is too simple to say anything. Therefore, we always assume that σ is an automorphism of $\mathfrak{g}$ in the sequel.

See [11] for the definition of Hom-Lie algebra structures on a Lie algebra. We point out that they made a slight change for the definition of Hom-Lie algebras so that both $(\mathfrak{g},\mathrm{id})$ and $(\mathfrak{g},0)$ are the Lie algebra $\mathfrak{g}$ itself in their case.

Inspired by the work [11], it is natural to ask whether there exist nontrivial Hom-Lie superalgebra structures on finite-dimensional simple Lie superalgebras. The answer is no. The following theorem is the main result of this paper.

Main Theorem: The Hom-Lie superalgebra structures on a finite-dimensional simple Lie superalgebra $\mathfrak{g}$ are trivial. That is, if $(\mathfrak{g},\sigma)$ is a Hom-Lie superalgebra then $\sigma=\mathrm{id}$.

It is well known that there are two families of complex finite-dimensional simple Lie superalgebras. One consists of classical Lie superalgebras and the other one consists of Cartan type Lie superalgebras. For a classical Lie superalgebra of the form $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$, we first restrict the Hom-Lie superalgebra structure $(\mathfrak{g},\sigma)$ to the even part $\mathfrak{g}_{\bar{0}}$ so that we can use the result in [11] to obtain that $\sigma|_{\mathfrak{g}_{\bar{0}}}=\mathrm{id}$. Then we combine the description of the automorphisms of classical Lie superalgebras (cf. [15, 8]) to check that $\sigma=\mathrm{id}$ case by case. For a Cartan type Lie superalgebra of the form $\mathfrak{g}=\bigoplus_{j=-1}^{n-1}\mathfrak{g}_i$, we first show that the isomorphism must be identity on the subspace $\mathfrak{g}_{-1}\bigoplus\mathfrak{g}_0$ via the transitivity of $W(n)$, $S(n)$ or $H(n)$. Then the conclusion is obtained by the nontriviality and the irreducibility of $\mathfrak{g}_{-1}$ as a $\mathfrak{g}_0$-module.

The paper is organized as follows. In Section 2, we recall the definition of Hom-Lie superalgebras and list some endomorphisms of $\mathfrak{gl}(m|n)$, which are used to describe the automorphism groups of classical simple Lie superalgebras. We also list some properties of Cartan type Lie superalgebras, which will be used in Section 4. Section 3 is devoted to the proof of the main theorem in the classical Lie superalgebras cases, while Section 4 is devoted to the proof in the Cartan type Lie superalgebras cases.

Throughout this paper, the base field is assumed to be the complex number field $\mathbb{C}$. The symbol $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ stands for the group of two elements. When $|x|$ appears in an expression, we always implicitly assume that x is a $\mathbb{Z}_2$-homogeneous element and automatically extend the relevant formulae by linearity (whenever applicable). For any superalgebra in this paper, the homomorphisms always mean even homomorphisms.

2. Preliminaries

In this section, we first list some endomorphisms of $\mathfrak{gl}(m|n)$, which are used to describe the automorphism groups of classical simple Lie superalgebras in Section 3. We also list some properties of Cartan type Lie superalgebras, which will be used to prove the Cartan type cases of the main theorem in Section 4.

2.1. Some endomorphisms of $\mathfrak{gl}(m|n)$. There are some endomorphisms of the general linear Lie superalgebra

$$\mathfrak{gl}(m|n) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} | A \in M_m, B \in M_{m,n}, C \in M_{n,m}, D \in M_n \right\}, \tag{2.1}$$

where $M_{p,q}$ is the set of all $p \times q$ matrices and $M_p := M_{p,p}$.

For any $(X, Y) \in SL_m \times SL_n$, define $\mathrm{Ad}(X, Y) \in \mathrm{End}\, \mathfrak{gl}(m|n)$ by

$$\mathrm{Ad}(X,Y) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} XAX^{-1} & XBY^{-1} \\ YCX^{-1} & YDY^{-1} \end{pmatrix}. \tag{2.2}$$

It implies a group homomorphism $\mathrm{Ad} : SL_m \times SL_n \to \mathbf{Aut}\mathfrak{gl}(m|n)$. That is,

$$\mathrm{Ad}(X_1, Y_1)\mathrm{Ad}(X_2, Y_2) = \mathrm{Ad}(X_1X_2, Y_1Y_2) \tag{2.3}$$

for any $(X_1, Y_1), (X_2, Y_2) \in SL_m \times SL_n$.

For any $\lambda \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$, there is an endomorphism defined by

$$\jmath(\lambda) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & \lambda B \\ \lambda^{-1}C & D \end{pmatrix}. \tag{2.4}$$

It is clear that

$$\jmath(\lambda_1)\jmath(\lambda_2) = \jmath(\lambda_1\lambda_2) \quad \text{for any } \lambda_1, \lambda_2 \in \mathbb{C}^\times. \tag{2.5}$$

The *supertransposition* τ is the endomorphism of $\mathfrak{gl}(m|n)$ given by

$$\tau : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} -A^t & C^t \\ -B^t & -D^t \end{pmatrix}. \tag{2.6}$$

It satisfies that

$$\tau^2 = \jmath(-1) \quad \text{and} \quad \tau^4 = 1. \tag{2.7}$$

Furthermore, when $m = n$, there is another endomorphism π of $\mathfrak{gl}(n|n)$ given by

$$\pi : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} D & C \\ B & A \end{pmatrix}, \tag{2.8}$$

which satisfies that

$$\pi^2 = 1. \tag{2.9}$$

At last, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2$, we define the endomorphism of $\mathfrak{gl}(2|2)$ as follows

$$\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & aB + b\Psi(C) \\ c\Psi(B) + dC & D \end{pmatrix}, \tag{2.10}$$

where $\Psi(F) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} F^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

One can check that $\rho : SL_2 \to \mathbf{Aut}\mathfrak{gl}(2|2)$ is a group homomorphism. That is,

$$\rho(AB) = \rho(A)\rho(B) \quad \text{for any } A, B \in SL_2. \tag{2.11}$$

Moreover, one can see easily that for any $(X, Y) \in SL_m \times SL_n$ and $\lambda \in \mathbb{C}^\times$

$$\mathrm{Ad}(X, Y)\jmath(\lambda) = \jmath(\lambda)\mathrm{Ad}(X, Y), \tag{2.12}$$

$$\mathrm{Ad}(X, Y)\tau = \tau\mathrm{Ad}((X^t)^{-1}, (Y^t)^{-1}) \tag{2.13}$$

and

$$\jmath(\lambda)\tau = \tau\jmath(\lambda^{-1}). \tag{2.14}$$

In the case of $m = n$, we have

$$\mathrm{Ad}(X, Y)\pi = \pi\mathrm{Ad}(Y, X), \tag{2.15}$$

$$\jmath(\lambda)\pi = \pi\jmath(\lambda^{-1}) \tag{2.16}$$

and

$$\tau\pi = \pi\tau\jmath(-1) = \pi\tau^3. \tag{2.17}$$

Finally, if $m = n = 2$, then for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2$, we have

$$\mathrm{Ad}(X, Y)\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathrm{Ad}(X, Y) \tag{2.18}$$

and

$$\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix}\pi = \pi\rho\begin{pmatrix} d & c \\ b & a \end{pmatrix}. \tag{2.19}$$

2.2. Some properties for Cartan type Lie superalgebras.

Let $\Lambda(n)$ be the exterior algebra in n indeterminates $\xi_1, \xi_2, \ldots, \xi_n$. There are four simple Lie superalgebras consisting of its derivations:

$$W(n) := \{\sum_{j=1}^{n} f_j \frac{\partial}{\partial \xi_j} | f_j \in \Lambda(n)\} \quad (n \geq 3), \tag{2.20}$$

$$S(n) := \{\sum_{j=1}^{n} f_j \frac{\partial}{\partial \xi_j} | f_j \in \Lambda(n),\ \sum_{j=1}^{n} \frac{\partial f_j}{\partial \xi_j} = 0\} \quad (n \geq 3), \tag{2.21}$$

$$\widetilde{S}(n) := \{(1-\xi_1\xi_2\cdots\xi_n)\sum_{j=1}^{n} f_j\frac{\partial}{\partial\xi_j}|f_j \in \Lambda(n),\ \sum_{j=1}^{n}\frac{\partial f_j}{\partial \xi_j} = 0\} \tag{2.22}$$
$$(n \geq 4 \text{ is an even number}),$$

and

$$H(n) := \{\sum_{j=1}^{n}\frac{\partial f_j}{\partial\xi_j}\frac{\partial}{\partial\xi_j}|f_j \in \Lambda(n)\} \quad (n\geq 4). \tag{2.23}$$

There is a natural $\mathbb{Z}$-gradation $W(n) = \bigoplus_{j=-1}^{n-1} W(n)_j$ by setting $\deg\xi_i = 1$ and $\deg\frac{\partial}{\partial\xi_i} = -1$ for any $i = 1, 2, \ldots, n$, which is called the *principal gradation*. It induces the $\mathbb{Z}_2$-gradation on $W(n)$:

$$W(n)_{\bar{s}} := \bigoplus_{j\equiv s (\text{mod } 2)} W(n)_j \quad \text{for} \quad s \in \{0,1\}. \tag{2.24}$$

The $\mathbb{Z}_2$-gradation of $S(n)$, $\widetilde{S}(n)$ and $H(n)$ is inherited from $W(n)$. Furthermore, the algebras $S(n)$ and $H(n)$ are $\mathbb{Z}$-graded subalgebras of $W(n)$. The subalgebra $\widetilde{S}(n)$ is no longer $\mathbb{Z}$-graded, but it has the following vector spaces direct sum decomposition in $W(n)$:

$$\widetilde{S}(n) = \bigoplus_{j=-1}^{n-2} \widetilde{S}(n)_j, \tag{2.25}$$

where $\widetilde{S}(n)_{-1} = \text{span}\{(1-\xi_1\xi_2\cdots\xi_n)\frac{\partial}{\partial\xi_i}|\ i = 1,\ldots,n\}$ and $\widetilde{S}(n)_j = S(n)_j$ for $j > -1$.

The 0-degree components of these superalgebras are Lie algebras. Precisely, one has that $W(n)_0 \cong \mathfrak{gl}(n)$, $S(n)_0 = \widetilde{S}(n)_0 \cong \mathfrak{sl}(n)$ and $H(n)_0 \cong \mathfrak{so}(n)$. The subspaces with other degrees are all irreducible modules of the 0-degree component. In particular, the (-1)-degree subspace is a nontrivial module.

The following proposition, which is called the *transitivity* of $W(n)$, $S(n)$ and $H(n)$, will be used in Section 4.

Proposition 2.1. (cf.[7]) *Let $\mathfrak{g} = W(n)$, $S(n)$ or $H(n)$. For any $g \in \mathfrak{g}$, if $[g, \mathfrak{g}_{-1}] = 0$ then $g \in \mathfrak{g}_{-1}$.*

3. Proof of the Main Theorem for Classical Lie Superalgebras

In this section, we first prove that $\sigma|_{\mathfrak{g}_{\bar{0}}} = \text{id}$ for any Hom-Lie superalgebra structure $(\mathfrak{g}, \sigma)$ on a classical Lie superalgebra $\mathfrak{g}$. Then thanks to the classification for classical simple Lie superalgebras due to Kac [12], we are able to prove the main theorem case by case.

3.1. Hom-Lie algebra structure on $\mathfrak{g}_{\bar{0}}$. The following lemma is clear by the definition 1.1.

Lemma 3.1. *If $(\mathfrak{g}, \sigma)$ is a Hom-Lie superalgebra structure on the Lie superalgebra $\mathfrak{g}$, then $(\mathfrak{g}_{\bar{0}}, \sigma|_{\mathfrak{g}_{\bar{0}}})$ is a Hom-Lie algebra structure on the Lie algebra $\mathfrak{g}_{\bar{0}}$.*

The following theorem is obtained directly by the main results in [11]. We will use it to study the Hom-Lie superalgebra structures on a classical Lie superalgebra.

Theorem 3.2. (Jin-Li[11]) *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and σ an automorphism of $\mathfrak{g}$, then $(\mathfrak{g}, \sigma)$ is a Hom-Lie algebra if and only if $\sigma = \mathrm{id}$.*

Proof. If $\mathfrak{g}$ is a finite-dimensional Lie algebra with non-isomorphic simple summands, then $\sigma = \mathrm{id}$ by Corollary 3.1 in [11].

If $\mathfrak{g}$ has the isomorphic simple summands, then Theorem 3.1 in [11] implies that for any non-trivial Hom-Lie algebra $(\mathfrak{g}, \sigma)$ on $\mathfrak{g}$, the homomorphism σ cannot be an automorphism because a projection homomorphism appears as its factor. ■

Corollary 3.3. *For any classical simple Lie superalgebra $\mathfrak{g}$, if $(\mathfrak{g}, \sigma)$ is a Hom-Lie superalgebra, then $\sigma|_{\mathfrak{g}_{\bar{0}}} = \mathrm{id}$.*

Proof. For any classical simple Lie superalgebra $\mathfrak{g}$, its even part $\mathfrak{g}_{\bar{0}}$ is a semisimple Lie algebra. Thus the statement follows from Lemma 3.1 and Theorem 3.2. ■

3.2. Hom-Lie superalgebra structures on $\mathfrak{sl}(m|n), (m, n \geq 1, m \neq n)$.

The special linear Lie superalgebra $\mathfrak{sl}(m|n)$ consists of those matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(m|n)$ such that $\mathrm{tr}A - \mathrm{tr}D = 0$.

It was proved in [15, 8] that $\mathbf{Aut}\mathfrak{sl}(m|n)$ is generated by $\mathrm{Ad}(SL_m \times SL_n)$, $\jmath(\mathbb{C}^\times)$ and τ.

If $(\mathfrak{sl}(m|n), \sigma)$ is a Hom-Lie superalgebra, then $\sigma|_{\mathfrak{sl}(m|n)_{\bar{0}}} = \mathrm{id}$ by Corollary 3.3, where $\mathfrak{sl}(m|n)_{\bar{0}}$ consists of those matrices $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{sl}(m|n)$. So it should be that $\sigma = \jmath(\lambda)$ for some $\lambda \in \mathbb{C}^\times$ via (2.2)-(2.7) and (2.12)-(2.14).

We always use $e_{i,j} \in \mathfrak{gl}(m|n)$ to denote the matrix whose (i,j)-th entry is 1 and other entries are 0 in the following text. Setting $x = e_{1,m+1}$, $y = e_{1,2}$ and $z = e_{2,1}$ in (1.3), we have that $\lambda[x, [y, z]] = [x, [y, z]]$. Thus it must be that $\lambda = 1$ and hence $\sigma = \jmath(1) = \mathrm{id}$.

3.3. Hom-Lie superalgebra structure on $\mathfrak{psl}(n|n), (n > 2)$.

Recall that

$$\mathfrak{psl}(n|n) := \mathfrak{sl}(n|n)/\{\lambda I_{2n} | \lambda \in \mathbb{C}\}. \tag{3.1}$$

The automorphism group $\mathbf{Aut}\mathfrak{psl}(n|n)$ is generated by $\mathrm{Ad}(SL_n \times SL_n)$, $\jmath(\mathbb{C}^\times)$, τ and π (cf. [15, 8]).

If $(\mathfrak{psl}(n|n), \sigma)$ is a Hom-Lie superalgebra, then $\sigma = \jmath(\lambda)$ for some $\lambda \in \mathbb{C}^\times$ via Corollary 3.3, and, (2.2)-(2.9) and (2.12)-(2.17). Thus by the same arguments as in the case of $\mathfrak{sl}(m|n)$, we have $\sigma = \jmath(1) = \mathrm{id}$.

3.4. Hom-Lie superalgebra structure on $\mathfrak{psl}(2|2)$.

It was proved in [15, 8] that $\mathbf{Aut}\mathfrak{psl}(2|2)$ is generated by $\mathrm{Ad}(SL_2 \times SL_2)$, $\rho(SL_2)$ and π.

If $(\mathfrak{psl}(2|2), \sigma)$ is a Hom-Lie superalgebra, then $\sigma = \rho \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2$ by Corollary 3.3, and, (2.2),(2.3),(2.8)-(2.11),(2.15),(2.18) and (2.19).

Now let $x = e_{2,3}$, $y = e_{1,2}$ and $z = e_{2,1}$, then we have $\sigma(x) = ae_{2,3} - ce_{4,1}$, $\sigma(y) = y$ and $\sigma(z) = z$, which leads to $[\sigma(x), [y, z]] = [x, [y, z]]$. This implies that $a = 1$ and $c = 0$. Similarly, we can set $x = e_{3,2}$, $y = e_{3,4}$ and $z = e_{4,3}$ to show that $d = 1$ and $b = 0$ by $[\sigma(x), [y, z]] = [x, [y, z]]$ again.

3.5. Hom-Lie superalgebra structure on $P(n-1)$.

The simple Lie superalgebra $P(n-1)$ is a subsuperalgebra of $\mathfrak{sl}(n|n)$, consisting of the matrices of the following form.

$$P(n-1) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} | A \in \mathfrak{sl}_n, B = B^t, C = -C^t \right\} \tag{3.2}$$

There is a group homomorphism

$$\mathrm{Ad} : SL_n \to \mathbf{Aut}P(n-1), \qquad X \mapsto \mathrm{Ad}(X, (X^t)^{-1}). \tag{3.3}$$

Automorphism group $\mathbf{Aut}P(n-1)$ is generated by $\mathrm{Ad}(SL_n)$ and $\jmath(\mathbb{C}^\times)$.

A plausible automorphism σ with $(P(n-1), \sigma)$ being a Hom-Lie superalgebra, should be that $\sigma = \jmath(\lambda)$ for some $\lambda \in \mathbb{C}^\times$ by Corollary 3.3, and, (2.2)-(2.5) and (2.12).

Let $x = e_{1,1} - e_{2,2} - e_{n+1,n+1} + e_{n+2,n+2}$, $y = e_{1,2}$ and $z = e_{1,n+2} + e_{2,n+1}$. Then $\sigma(x) = x$, $\sigma(y) = y$ and $\sigma(z) = \lambda z$. One has $\lambda[[x, y], z] = [x, y], z]$ by σ-twisted Jacobi identity. Hence $\lambda[e_{1,2}, e_{1,n+2} + e_{2,n+1}] = [e_{1,2}, e_{1,n+2} + e_{2,n+1}]$. It follows that $\lambda = 1$, i.e. $\sigma = \mathrm{id}$.

3.6. Hom-Lie superalgebra structure on $Q(n-1)$.

First the subsuperalgebra of $\mathfrak{sl}(n|n)$ called $\tilde{Q}(n-1)$ is given by

$$\tilde{Q}(n-1) := \{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} | \mathrm{tr} B = 0 \}. \tag{3.4}$$

The simple Lie superalgebra $Q(n-1)$ is the quotient

$$Q(n-1) := \tilde{Q}(n-1)/\{\lambda I_{2n} | \lambda \in \mathbb{C}\}. \tag{3.5}$$

This simple Lie superalgebra is not invariant under the supertransposition τ, but it is invariant under the q-supertransposition

$$\sigma_q : \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto \begin{pmatrix} A^t & \zeta B^t \\ \zeta B^t & A^t \end{pmatrix}, \tag{3.6}$$

where ζ is a fixed primitive 4-th root of unity.

There is a group homomorphism

$$\mathrm{Ad}: SL_n \to \mathbf{Aut}Q(n-1), \qquad X \mapsto \mathrm{Ad}(X,X). \tag{3.7}$$

Automorphism group $\mathbf{Aut}Q(n-1)$ is generated by $\mathrm{Ad}(SL_n)$ and σ_q. One can check easily that

$$\sigma_q^2 = \jmath(-1) \quad \text{and} \quad \sigma_q^4 = 1, \tag{3.8}$$

and

$$\sigma_q \mathrm{Ad}(X) = \mathrm{Ad}((X^t)^{-1})\sigma_q. \tag{3.9}$$

Thus if $(Q(n-1),\sigma)$ is a Hom-Lie superalgebra, then the plausible automorphism σ must be the identity or σ_q^2 by Corollary 3.3, (2.2),(2.3),(3.8) and (3.9). One can show that $\sigma = \mathrm{id}$ by setting $x = e_{1,2} + e_{n+1,n+2}$, $y = e_{1,1} - e_{2,2} + e_{n+1,n+1} - e_{n+2,n+2}$ and $z = e_{2,n+1} + e_{n+2,1}$ in the σ-twisted Jacobi identity (1.3).

3.7. Hom-Lie superalgebra structure on $\mathfrak{osp}(m|2n)$.

Recall that the orthosymplectic Lie superalgebra $\mathfrak{osp}(m|2n)$ is the subsuperalgebra of $\mathfrak{sl}(m|2n)$ defined by

$$\mathfrak{osp}(m|2n) := \left\{ \begin{pmatrix} A & B \\ J_n B^t & D \end{pmatrix} | A \in \mathfrak{so}_m, B \in M_{m,2n}, D \in \mathfrak{sp}_{2n} \right\} \tag{3.10}$$

where $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

When m is even, we take $\gamma_m \in O_m$ such that $\det \gamma_m = -1$ and $\gamma_m^2 = I_m$.

The automorphism group $\mathbf{Aut}\mathfrak{osp}(m|2n)$ is generated by $\mathrm{Ad}(SO_m \times Sp_{2n})$ if m is odd, or by $\mathrm{Ad}(SO_m \times Sp_{2n})$ and $\mathrm{Ad}(\gamma_m, I_{2n})$ if m is even.

Obviously, $\sigma = \mathrm{id}$ if $(\mathfrak{osp}(m|2n),\sigma)$ is a Hom-Lie superalgebra by Corollary 3.3.

3.8. Hom-Lie superalgebra structure on $G(3)$.

For Lie superalgebra $G(3)$, its even part $G(3)_{\bar 0} \simeq G_2 \oplus \mathfrak{sl}_2$ and its automorphism group $\mathbf{Aut}G(3)$ is generated by $\mathrm{Ad}(G_2 \times SL_2)$. Hence $\sigma = \mathrm{id}$ if $(G(3),\sigma)$ is a Hom-Lie superalgebra by Corollary 3.3.

3.9. Hom-Lie superalgebra structure on $F(4)$.

For Lie superalgebra $F(4)$, its even part $F(4)_{\bar 0} \simeq \mathfrak{so}_7 \oplus \mathfrak{sl}_2$ and its automorphism group $\mathbf{Aut}F(4)$ is generated by $\mathrm{Ad}(\mathrm{Spin}_7 \times SL_2)$. Hence $\sigma = \mathrm{id}$ if $(F(4),\sigma)$ is a Hom-Lie superalgebra by Corollary 3.3.

3.10. Hom-Lie superalgebra structure on $D(2,1,\alpha)$.

For Lie superalgebra $\mathfrak{g} = D(2,1,\alpha)$, its even part is $\mathfrak{g}_{\bar 0} \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and its odd part is $\mathfrak{g}_{\bar 1} \simeq V_2 \otimes V_2 \otimes V_2$, where V_2 is the natural module of $\mathfrak{sl}_2$.

Given $\sigma \in \mathfrak{S}_3$ and $\lambda \in \mathbb{C}^\times$, we define $\theta(\sigma,\lambda) \in GL(D(2,1,\alpha))$ by

$$\theta(\sigma,\lambda)((x_1,x_2,x_3),(u_1 \otimes u_2 \otimes u_3)) = ((x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}),\lambda(u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes u_{\sigma(3)})). \tag{3.11}$$

Thus, in the case $\alpha \notin \{1,-\frac{1}{2},-2\}$ and $\alpha^3 \neq 1$, $\mathbf{Aut}D(2,1,\alpha)$ is generated by $\mathrm{Ad}(SL_2 \times SL_2 \times SL_2)$. In the case $\alpha \in \{1,-\frac{1}{2},-2\}$, $\mathbf{Aut}D(2,1,\alpha)$ is generated by $\mathrm{Ad}(SL_2 \times SL_2 \times SL_2)$ and $\theta((1,2),1)$. And in the case $\alpha^3 = 1$ and $\alpha \neq 1$, $\mathbf{Aut}D(2,1,\alpha)$ is generated by $\mathrm{Ad}(SL_2 \times SL_2 \times SL_2)$ and $\theta((1,2,3),\lambda)$, where $\lambda^2 = \frac{1}{\alpha}$. These together show that the Hom-Lie superalgebra struct on $D(2,1,\alpha)$ is trivial.

4. Proof of the Main Theorem for Cartan Type Lie Superalgebras

In this section, we will show the main theorem in the case of Cartan type Lie superalgebras, i.e., $\sigma = \mathrm{id}$ for any Hom-Lie super algebra structure $(\mathfrak{g}, \sigma)$ on a Cartan type Lie superalgebra. There are two key points in the proof. One is the transitity of $W(n)$, $S(n)$ and $H(n)$. The other one is that the (-1)-degree subspaces are nontrivial irreducible modules of the 0-degree components for these superalgebras.

4.1. Hom-Lie superalgebra structure on $W(n)$.

Let $x = \frac{\partial}{\partial\xi_i}$, $y = \frac{\partial}{\partial\xi_j}$ and $z = \xi_j\frac{\partial}{\partial\xi_l}$. Then one has $[x,y] = [x,z] = 0$ if $i \neq j$, and $[y,z] = \frac{\partial}{\partial\xi_l}$. The σ-twisted Jacobi identity implies that

$$[\sigma(\frac{\partial}{\partial\xi_i}), \frac{\partial}{\partial\xi_l}] = 0 \quad \text{for any } l. \tag{4.1}$$

Since $W(n)$ is transitive by Proposition 2.1, we have $\sigma(W(n)_{-1}) = W(n)_{-1}$. Set $x = \frac{\partial}{\partial\xi_i}$, $y = \xi_s\frac{\partial}{\partial\xi_t}$ and $z = \xi_p\frac{\partial}{\partial\xi_q}$, where $s \neq i$ and $p \neq i$. Then $[x,y] = [x,z] = 0$ and $[y,z] = \delta_{t,p}\xi_s\frac{\partial}{\partial\xi_q} - \delta_{s,q}\xi_p\frac{\partial}{\partial\xi_t}$, which leads to $[\sigma(x),[y,z]] = 0$. We write $\sigma(\frac{\partial}{\partial\xi_i}) = \sum_{k=1}^n a_{ki}\frac{\partial}{\partial\xi_k}$. It follows that $a_{ki} = 0$ for $k \neq i$, i.e. $\sigma(\frac{\partial}{\partial\xi_i}) = a_i\frac{\partial}{\partial\xi_i}$ for $i = 1,\dots,n$.

Now let $x = \xi_i\frac{\partial}{\partial\xi_j}$, $y = \frac{\partial}{\partial\xi_k}$ and $z = \xi_k\frac{\partial}{\partial\xi_l}$, where $k \neq j$, $l \neq i$ and $k \neq i$. Then one gets $[x,y] = [x,z] = 0$. It implies that

$$[\sigma(\xi_i\frac{\partial}{\partial\xi_j}), \frac{\partial}{\partial\xi_l}] = 0 \quad \text{for any } l \neq i. \tag{4.2}$$

For $l = i$, one has

$$[\sigma(\xi_i\frac{\partial}{\partial\xi_j}), \frac{\partial}{\partial\xi_i}] = [\sigma(\frac{\partial}{\partial\xi_k}), -\xi_k\frac{\partial}{\partial\xi_j}] = -a_k\frac{\partial}{\partial\xi_j} = a_k[\xi_i\frac{\partial}{\partial\xi_j}, \frac{\partial}{\partial\xi_i}]. \tag{4.3}$$

Thus

$$\sigma(\xi_i\frac{\partial}{\partial\xi_j}) = a_k\xi_i\frac{\partial}{\partial\xi_j}, \tag{4.4}$$

which also implies that $a_1 = a_2 = \cdots = a_n$. It follows that $\sigma|_{W(n)_0} = \mathrm{id}$. Then one has $a_k = 1$ for $k = 1,2,\dots,n$ in (4.4). Hence

$$\sigma|_{W(n)_{-1}\bigoplus W(n)_0} = \mathrm{id}. \tag{4.5}$$

Set $y \in W(n)_{-1}$ and $z \in W(n)_0$. By the σ-twisted Jacobi identity we have that

$$[\sigma(x) - x, [y,z]] = 0 \quad \text{for any } x \in W(n)_l, \tag{4.6}$$

where $l=1,2,\ldots,n-1$. Thus $\sigma(x)-x\in W(n)_{-1}$ by the transitivity of $W(n)$. Finally, setting $y\in W(n)_0$ and $z\in W(n)_0$, we get that $\sigma(x)=x$, i.e. $\sigma=\mathrm{id}$.

4.2. Hom-Lie superalgebra structure on $S(n)$.

Since $S(n)_{-1}=W(n)_{-1}$, $S(n)_0\subset W(n)_0$ and $S(n)_0\cong\mathfrak{sl}(n)$, we obtain that

$$\sigma|_{S(n)_{-1}\bigoplus S(n)_0}=\mathrm{id}. \tag{4.7}$$

Note that $S(n)$ is also transitive. Now the same argumentation as in the last paragraph of above subsection implies that $\sigma=\mathrm{id}$.

4.3. Hom-Lie superalgebra structure on $\widetilde{S}(n)$.

In this subsection, we always assume that $n(\geq 4)$ is an even integer. Denote $A=1-\xi_1\cdots\xi_n$, and $A_i=\xi_1\cdots\xi_{i-1}\xi_{i+1}\cdots\xi_n$. Note that $\widetilde{S}(n)=\bigoplus_{l=-1}^{n-2}\widetilde{S}(n)_l$ where $\widetilde{S}(n)_{-1}=\mathrm{span}\{A\frac{\partial}{\partial\xi_i}|i=1,\ldots,n\}$ and $\widetilde{S}(n)_l=S(n)_l$ for $l\geq 0$.

Let $x=A\frac{\partial}{\partial\xi_i}$, $y=A\frac{\partial}{\partial\xi_j}$ and $z=\xi_p\frac{\partial}{\partial\xi_q}$, where $i\neq j$ and $p\neq q$. Then $[x,y]=(-1)^iA_i\frac{\partial}{\partial\xi_j}+(-1)^jA_j\frac{\partial}{\partial\xi_i}$, $[x,z]=\delta_{p,i}A\frac{\partial}{\partial\xi_q}$ and $[y,z]=\delta_{p,j}A\frac{\partial}{\partial\xi_q}$. Thus one has

$$[\sigma(A\frac{\partial}{\partial\xi_i}),\delta_{p,j}A\frac{\partial}{\partial\xi_q}]=[(-1)^iA_i\frac{\partial}{\partial\xi_j}+(-1)^jA_j\frac{\partial}{\partial\xi_i},\sigma(\xi_p\frac{\partial}{\partial\xi_q})]-[\sigma(A\frac{\partial}{\partial\xi_j}),\delta_{p,i}A\frac{\partial}{\partial\xi_q}], \tag{4.8}$$

by the σ-twisted Jacobi identity. Now let $p=j$. We write $\sigma(A\frac{\partial}{\partial\xi_i})=\alpha_i+\beta_i$ where $\alpha_i\in\widetilde{S}(n)_{-1}$ and $\beta_i\in\bigoplus_{l\geq 1}\widetilde{S}(n)_l$. Then the equation (4.8) becomes

$$[\alpha_i+\beta_i,A\frac{\partial}{\partial\xi_q}]=[(-1)^iA_i\frac{\partial}{\partial\xi_j}+(-1)^jA_j\frac{\partial}{\partial\xi_i},\sigma(\xi_j\frac{\partial}{\partial\xi_q})]. \tag{4.9}$$

One observes that $[\alpha_i,A\frac{\partial}{\partial\xi_q}]\in W(n)_{n-2}$ and $[(-1)^iA_i\frac{\partial}{\partial\xi_j}+(-1)^jA_j\frac{\partial}{\partial\xi_i},\sigma(\xi_j\frac{\partial}{\partial\xi_q})]\in W(n)_{n-2}$. Thus $[\beta_i,A\frac{\partial}{\partial\xi_q}]$ lies also in $W(n)_{n-2}$. But $\beta_i\in\bigoplus_{l=1}^{n-2}\widetilde{S}(n)_l$. In these subspaces, the action of operator $A\frac{\partial}{\partial\xi_q}$ is just as the same as $\frac{\partial}{\partial\xi_q}$, which implies that $[\beta_i,\frac{\partial}{\partial\xi_q}]=0$ for $q\neq j$. These show that $\beta_i=0$, and hence

$$\sigma(\widetilde{S}(n)_{-1})=\widetilde{S}(n)_{-1}. \tag{4.10}$$

Now let $x=A\frac{\partial}{\partial\xi_i}$, $y=\xi_s\frac{\partial}{\partial\xi_t}$ and $z=\xi_p\frac{\partial}{\partial\xi_q}$, where $s\neq i$ and $p\neq i$. Then $[x,y]=[x,z]=0$ and $[y,z]=\delta_{p,t}\xi_s\frac{\partial}{\partial\xi_q}-\delta_{q,s}\xi_p\frac{\partial}{\partial\xi_t}$. It follows that

$$[\sigma(A\frac{\partial}{\partial\xi_i})-A\frac{\partial}{\partial\xi_i},\delta_{p,t}\xi_s\frac{\partial}{\partial\xi_q}-\delta_{q,s}\xi_p\frac{\partial}{\partial\xi_t}]=0. \tag{4.11}$$

It shows that

$$\sigma(A\frac{\partial}{\partial\xi_i})=a_iA\frac{\partial}{\partial\xi_i}\quad\text{for}\quad i=1,\ldots,n. \tag{4.12}$$

Set $x=\xi_i\frac{\partial}{\partial\xi_j}$, $y=A\frac{\partial}{\partial\xi_k}$ and $z=\xi_k\frac{\partial}{\partial\xi_l}$, where $k\neq j$ and $k\neq i$. If $l\neq i$, we have $[x,y]=[x,z]=0$ and $[y,z]=A\frac{\partial}{\partial\xi_l}$. Hence $[\sigma(\xi_i\frac{\partial}{\partial\xi_j}),A\frac{\partial}{\partial\xi_l}]=0$ via the σ-twisted Jacobi identity. For $l = i$, we have $[x, y] = 0$, $[y, z] = A\frac{\partial}{\partial \xi_i}$ and $[x, z] = -\xi_k \frac{\partial}{\partial \xi_j}$. This implies that

$$[\sigma(\xi_i \frac{\partial}{\partial \xi_j}), A\frac{\partial}{\partial \xi_i}] = [\sigma(A\frac{\partial}{\partial \xi_k}), -\xi_k \frac{\partial}{\partial \xi_j}] = -a_k A\frac{\partial}{\partial \xi_j} = a_k[\xi_i \frac{\partial}{\partial \xi_j}, A\frac{\partial}{\partial \xi_i}]. \tag{4.13}$$

Thus

$$[\sigma(\xi_i \frac{\partial}{\partial \xi_j}) - a_k \xi_i \frac{\partial}{\partial \xi_j}, A\frac{\partial}{\partial \xi_l}] = 0 \quad \text{for any } l = 1, \ldots, n. \tag{4.14}$$

Hence $\sigma(\xi_i \frac{\partial}{\partial \xi_j}) = a_k \xi_i \frac{\partial}{\partial \xi_j}$. Since $\widetilde{S}(n)_0 \cong \mathfrak{sl}(n)$, we have that

$$\sigma|_{\widetilde{S}(n)_{-1} \bigoplus \widetilde{S}(n)_0} = \text{id}. \tag{4.15}$$

Now let $x \in \widetilde{S}(n)_l$ for $l > 0$. Then

$$[\sigma(x), [y, z]] = [x, [y, z]] \quad \text{for } y \in \widetilde{S}(n)_{-1} \text{ and } z \in \widetilde{S}(n)_0. \tag{4.16}$$

Thus $\sigma(x) - x \in \widetilde{S}(n)_{-1}$. Since $\widetilde{S}(n)_{-1}$ is a nontrivial irreducible module of $\widetilde{S}(n)_0 \cong \mathfrak{sl}(n)$, one shows that $\sigma(x) = x$ via setting $y \in \widetilde{S}(n)_0$ and $z \in \widetilde{S}(n)_0$. In a word, $\sigma = \text{id}$ on the superalgebras $\widetilde{S}(n)$.

4.4. Hom-Lie superalgebra structure on $H(n)$.

First, we note that $H(n)_{-1} = W(n)_{-1}$ and $H(n)_0 \cong \mathfrak{so}(n)$.

Let $x = \frac{\partial}{\partial \xi_i}$, $y = \frac{\partial}{\partial \xi_j}$ and $z = \xi_j \frac{\partial}{\partial \xi_l} - \xi_l \frac{\partial}{\partial \xi_j}$, where $i \neq j$, $i \neq l$ and $j \neq l$. Then $[x, y] = [x, z] = 0$ and $[y, z] = \frac{\partial}{\partial \xi_l}$. We obtain that

$$[\sigma(\frac{\partial}{\partial \xi_i}), \frac{\partial}{\partial \xi_l}] = [\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_l}] = 0 \tag{4.17}$$

for $l \neq i$.

Now set $l = i$, i.e. $x = \frac{\partial}{\partial \xi_i}$, $y = \frac{\partial}{\partial \xi_j}$ and $z = \xi_j \frac{\partial}{\partial \xi_i} - \xi_i \frac{\partial}{\partial \xi_j}$. We have that

$$[\sigma(\frac{\partial}{\partial \xi_i}), [\frac{\partial}{\partial \xi_j}, \xi_j \frac{\partial}{\partial \xi_i} - \xi_i \frac{\partial}{\partial \xi_j}]] = -[\sigma(\frac{\partial}{\partial \xi_j}), [\frac{\partial}{\partial \xi_i}, \xi_j \frac{\partial}{\partial \xi_i} - \xi_i \frac{\partial}{\partial \xi_j}]], \tag{4.18}$$

i.e.

$$[\sigma(\frac{\partial}{\partial \xi_i}), \frac{\partial}{\partial \xi_i}] = [\sigma(\frac{\partial}{\partial \xi_j}), \frac{\partial}{\partial \xi_j}]. \tag{4.19}$$

We write $\sigma(\frac{\partial}{\partial \xi_i}) = \alpha_i + \beta_i$ where $\alpha_i \in H(n)_{-1}$ and $\beta_i \in \bigoplus_{l \geq 1} H(n)_l$. Moreover, β_i can be written as $\beta_i = \sum_{k=1}^{n} \frac{\partial g_i}{\partial \xi_k} \frac{\partial}{\partial \xi_k}$ for $i = 1, \ldots, n$. Thus

$$[\sum_{k=1}^{n} \frac{\partial g_i}{\partial \xi_k} \frac{\partial}{\partial \xi_k}, \frac{\partial}{\partial \xi_i}] = [\sum_{k=1}^{n} \frac{\partial g_j}{\partial \xi_k} \frac{\partial}{\partial \xi_k}, \frac{\partial}{\partial \xi_j}]. \tag{4.20}$$

This gives that

$$[\frac{\partial g_i}{\partial \xi_j} \frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_i}] = [\frac{\partial g_j}{\partial \xi_j} \frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_j}] = 0. \tag{4.21}$$

Thus

$$[\beta_i, \frac{\partial}{\partial \xi_i}] = 0. \tag{4.22}$$

It implies that $\sigma(H(n)_{-1}) = H(n)_{-1}$. Moreover, let $x = \frac{\partial}{\partial \xi_i}$, $y = \xi_s \frac{\partial}{\partial \xi_t} - \xi_t \frac{\partial}{\partial \xi_s}$ and $z = \xi_p \frac{\partial}{\partial \xi_q} - \xi_q \frac{\partial}{\partial \xi_p}$ with $s, t, p, q \neq i$. Then $[\sigma(x), [y, z]] = 0$. Writing $\sigma(\frac{\partial}{\partial \xi_i}) = \sum_{k=1}^{n} a_{ik} \frac{\partial}{\partial \xi_k}$, we have

$$[\sum_{k=1}^{n} a_{ik} \frac{\partial}{\partial \xi_k}, \delta_{t,p}(\xi_s \frac{\partial}{\partial \xi_q} - \xi_q \frac{\partial}{\partial \xi_s}) - \delta_{s,q}(\xi_p \frac{\partial}{\partial \xi_t} - \xi_t \frac{\partial}{\partial \xi_p}) - \delta_{t,q}(\xi_s \frac{\partial}{\partial \xi_p} - \xi_p \frac{\partial}{\partial \xi_s}) + \delta_{s,p}(\xi_q \frac{\partial}{\partial \xi_t} - \xi_t \frac{\partial}{\partial \xi_q})] = 0 \tag{4.23}$$

for $s, t, p, q \neq i$. It shows that

$$\sigma(\frac{\partial}{\partial \xi_i}) = a_i \frac{\partial}{\partial \xi_i} \quad \text{for } i = 1, \ldots, n. \tag{4.24}$$

Let $x = \xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i}$, $y = \frac{\partial}{\partial \xi_s}$ and $z = \xi_s \frac{\partial}{\partial \xi_t} - \xi_t \frac{\partial}{\partial \xi_s}$. Then

$$[\sigma(\xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i}), \frac{\partial}{\partial \xi_t}] = 0 \quad \text{for } t \neq i \text{ and } t \neq j. \tag{4.25}$$

Let $x = \xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i}$, $y = \frac{\partial}{\partial \xi_s}$ and $z = \xi_s \frac{\partial}{\partial \xi_i} - \xi_i \frac{\partial}{\partial \xi_s}$ with $s \neq i$. Then

$$\begin{aligned} [\sigma(\xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i}), \frac{\partial}{\partial \xi_i}] &= [\sigma(\frac{\partial}{\partial \xi_s}), -(\xi_s \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_s})] \\ &= a_s[\xi_s \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_s}, \frac{\partial}{\partial \xi_s}] \\ &= -a_s \frac{\partial}{\partial \xi_j} \\ &= [a_s(\xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i}), \frac{\partial}{\partial \xi_i}] \end{aligned} \tag{4.26}$$

for $s \neq i$. We also show that

$$[\sigma(\xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i}), \frac{\partial}{\partial \xi_i}] = [a_s(\xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i}), \frac{\partial}{\partial \xi_j}] \quad \text{for } s \neq j. \tag{4.27}$$

Hence

$$[\sigma(\xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i}) - a_s(\xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i}), \frac{\partial}{\partial \xi_k}] = 0 \quad \text{for any } k = 1, \ldots, n. \tag{4.28}$$

These tell us that $\sigma(H(n)_0) = H(n)_0$. Thus $\sigma|_{H(n)_0} = \text{id}$. Since $\sigma(\xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i}) = a_s(\xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i})$, we have $a_s = 1$. Hence

$$\sigma|_{H(n)_{-1} \bigoplus H(n)_0} = \text{id}. \tag{4.29}$$

For $x \in H(n)_l$, where $l > 0$, we have

$$[\sigma(x) - x, [y, z]] = 0 \tag{4.30}$$

whenever $y \in H(n)_{-1}$ and $z \in H(n)_0$. This shows that $\sigma(x) - x \in H(n)_{-1}$, and hence $\sigma(x) - x = 0$ by setting $y, z \in H(n)_0$. That is, $\sigma = \text{id}$.

References

[1] Ammar, F., and A. Makhlouf, *Hom-Lie superalgebras and Hom-Lie admissible superalgebras*, J. Alg. **324** (2010), 1513–1528.

[2] Aizawa, N., and H. Sato, *q-Deformation of the Virasoro algebra with central extension*, Phys. Lett. B **256** (1991), 185–190.

[3] Benayadi, S., and A. Makhlouf, *Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms*, arXiv:1009.4226.

[4] Chaichian, M., P. Kulish, and J. Lukierski, *q-Deformed Jacobi identity, q-oscillators and q-deformed infinite-dimensional algebras*, Phys. Lett. B **237** (1990), 401–406.

[5] Chaichian, M., A. P. Isaev, J. Lukierski, Z. Popowicz, and P. Prešnajder, *q-Deformations of Virasoro algebra and conformal dimensions*, Phys. Lett. B **262** (1991), 32–38.

[6] Chaichian, M., Z. Popowicz, and P. Prešnajder, *q-Virasoro algebra and its relation to the q-deformed KdV system*, Phys. Lett. B **249** (1990), 63–65.

[7] Frappat, L., A. Sciarrino, and P. Sorba, "Dictionary on Lie Algebras and Superalgebras," Academic Press, 2000.

[8] Grantcharov, D., and A. Pianzola, *Automorphisms and twisted loop algebras of finite-dimensional simple Lie superalgebras*, Int. Math. Res. Not. **73** (2004), 3937–3962.

[9] Hu, N., *q-Witt algebras, q-Lie algebras, q-holomorph structure and representations*, Alg. Colloq. **6** (1999), 51–70.

[10] Hartwig, J. T., D. Larsson and S. D. Silvesrov, *Derformations of the Lie algebras using σ-derivations*, J. Alg. **295** (2006), 314–361.

[11] Jin, Q., and X. Li, *Hom-Lie algebra structures on semi-simple Lie algebras*, J. Alg. **319** (2008), 1398–1408.

[12] Kac, V. G., *Lie superalgebras*, Adv. Math. **26** (1977), 8–96.

[13] Liu, K., *Characterizations of quantum Witt algebra*, Lett. Math. Phys. **24** (1992), 257–265.

[14] Makhlouf, A., and S. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. **2** (2008), 51–64.

[15] Serganova, V., *Automorphisms of simple Lie superalgebras*, Math. USSR-Izv **24** (1985), 539–551.

[16] Sheng, Y., *Representations of hom-Lie algebras*, Alg. and Rep. Theory **15** (2012), 1081–1098.

[17] Yau, D., *Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras*, J. Phys. A **42** (2009), 165202.

[18] —, *Hom-algebras and homology*, J. Lie Theory **19** (2009), 409–421.

[19] —, *Hom-Novikov algebras*, J. Phys. A **44** (2011), 085202.

[20] Yuan, L., *Hom-Lie color algebra structures*, Comm. Alg., **40** (2012), 575–592.

[21] Zhang, R., D. Hou, and C. Bai, *A Hom-version of the affinizations of Balinskii-Novikov and Novikov superalgebras*, J. Math. Phys. **52** (2011), 023505.

Bintao Cao
School of Mathematics
and Computational Science
Sun Yat-sen University
Guangzhou, 510275, China
caobt@mail.sysu.edu.cn

Li Luo
Department of Mathematics
East China Normal University
Shanghai, 200241, China
lluo@math.ecnu.edu.cn

Received September 28, 2012
and in final form March 26, 2013