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Applications of Index Sets and Nikolayevsky Derivations to Positive Rank Nilpotent Lie Algebras

Tracy L. Payne

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Abstract. We consider real nilpotent Lie algebras of positive rank. We fix a set Λ indexing the nonzero structure constants for a Lie algebra $\mathfrak{g}$ with respect to a basis of eigenvectors for an $\mathbb{R}$-split torus in the derivation algebra of $\mathfrak{g}$. We give criteria for when two Lie algebras with the same index set are isomorphic. We present a criterion for when there is a nilsoliton metric Lie algebra having a given index set, and we determine which nilsoliton metric Lie algebras have a given index set, up to isometric isomorphism and rescaling, in some common situations. We study the Nikolayevsky derivation, showing that it commutes with automorphisms that preserve certain inner products, and we find conditions on the Nikolayevsky derivation that insure that the isometry group of a metric Lie algebra is finite. We give examples showing that index sets and the Nikolayevsky derivation are useful invariants for nilpotent Lie algebras.
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1. Introduction

Let $\mathfrak{g}$ be a real Lie algebra. Let $\mathrm{Aut}(\mathfrak{g})$ be the automorphism group of $\mathfrak{g}$ and let $\mathrm{Der}(\mathfrak{g})$ be the derivation algebra of $\mathfrak{g}$. Let

$$\mathrm{Der}(\mathfrak{g}) = \mathfrak{s} \oplus (\mathfrak{t}^{\mathbb{R}} \oplus \mathfrak{t}^{i\mathbb{R}}) \oplus \mathfrak{n} \tag{1}$$

be the Levi-Mal'cev decomposition of the derivation algebra, where $\mathfrak{s}$ is semisimple and the solvable radical $\mathfrak{t} \oplus \mathfrak{n}$ is the direct sum of its nilradical $\mathfrak{n}$ and a torus $\mathfrak{t}$ that is maximal in the solvable radical (all such tori are conjugate by elements of $\mathrm{Aut}(\mathfrak{g})$). The torus further decomposes as the sum $\mathfrak{t} = \mathfrak{t}^{\mathbb{R}} \oplus \mathfrak{t}^{i\mathbb{R}}$ of an $\mathbb{R}$-split torus $\mathfrak{t}^{\mathbb{R}}$ and a compact torus $\mathfrak{t}^{i\mathbb{R}}$ (See [25], proof of Theorem 1.1). The dimension of $\mathfrak{t}^{\mathbb{R}}$ is called the *real rank* or *rank* of $\mathfrak{n}$. We say that $\mathfrak{g}$ has *positive rank* if its real rank is greater than zero.

Here, we are interested in positive rank nilpotent Lie algebras. In particular, we are interested in nilpotent Lie algebras that admit a nonsingular semisimple

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derivation with real eigenvalues called the *Nikolayevsky derivation.* Any nilpotent Lie algebra $\mathfrak{n}$ admits a basis $\mathcal{B}$ that simultaneously diagonalizes the elements of an $\mathbb{R}$-split torus in the derivation algebra $\mathrm{Der}(\mathfrak{n})$. For such a basis $\mathcal{B} = \{X_i\}_{i=1}^n$ the *index set* Λ is the set consisting of all triples (i, j, k) such that $i < j$ and so that the structure constant α_{ij}^k for $\mathfrak{n}$ with respect to $\mathcal{B}$ is nonzero. We also consider distinguished inner products for Lie algebras, especially *nilsoliton inner products.* Our primary goal is to provide simple, usable tools that can be used to analyze and relate the Nikolayevsky derivation, index sets, and distinguished inner products for specific classes of nilpotent Lie algebras.

A *metric Lie algebra* $(\mathfrak{g}, Q)$ is a Lie algebra $\mathfrak{g}$ endowed with an inner product Q. If $\{X_i\}_{i=1}^n$ is an orthonormal basis of a nilpotent metric Lie algebra $(\mathfrak{n}, Q)$, then the *Ricci form* for $(\mathfrak{n}, Q)$ is the symmetric form defined by

$$\mathsf{ric}(X, Y) = \frac{1}{4}\sum_{i,j=1}^{n} \langle [X_i, X_j], X\rangle\langle [X_i, X_j], Y\rangle - \frac{1}{2}\sum_{i=1}^{n}\langle [X, X_i], [Y, X_i]\rangle, \qquad (2)$$

where $X, Y \in \mathfrak{n}$ (this follows from [2], Corollary 7.38). The Ricci endomorphism Ric is the unique endomorphism of $\mathfrak{n}$ such that

$$\langle \mathrm{Ric}(X), Y\rangle = \mathsf{ric}(X, Y) \text{ for all } X, Y \in \mathfrak{n}.$$

A nilpotent metric Lie algebra $(\mathfrak{n}, Q)$ is called *nilsoliton* if the Ricci endomorphism Ric of $\mathfrak{n}$ defined by Q differs from a derivation $\hat{D}$ of $\mathfrak{n}$, called the *nilsoliton derivation*, by a scalar multiple of the identity:

$$\hat{D} = \mathrm{Ric} - \beta\,\mathrm{Id} \in \mathrm{Der}(\mathfrak{n}). \qquad (3)$$

The constant β is called the *nilsoliton constant.* The nilsoliton derivation may be considered a distinguished derivation for $\mathfrak{n}$, and the nilsoliton inner product may be viewed as preferred inner product on $\mathfrak{n}$.

Nilsoliton inner products on nilpotent Lie algebras have been intensively studied by geometers, as they are intimately related to Einstein inner products on solvable Lie algebras and soliton metrics for the Ricci flow ([20]). In addition, distinguished inner products such as nilsoliton inner products or inner products with respect to which elements of an $\mathbb{R}$-split torus $\mathfrak{t}^{\mathbb{R}}$ are diagonal may be useful in the algebraic setting, e.g. for providing bases that give efficient or canonical presentations of a Lie algebra.

Nikolayevsky defined a derivation associated to each real Lie algebra ([25]). He was motivated by an interest in noncompact simply connected Einstein homogeneous spaces, which, if Alekseevsky's conjecture is true, are in bijective correspondence with Einstein solvable metric Lie algebras. For this reason Nikolayevsky called the derivation a *pre-Einstein derivation.* However, the derivation he defined is a purely algebraic object and has applications beyond the study of nilsoliton and Einstein inner products. For this reason we prefer to call his derivation the *Nikolayevsky derivation.*

If a nilpotent Lie algebra $\mathfrak{n}$ admits a nilsoliton inner product Q with associated nilsoliton derivation $\hat{D}$, then the nilsoliton derivation is symmetric with respect to Q and has positive eigenvalues, and the nilsoliton constant β is negative.

If Ric is the Ricci endomorphism for Q, then $\operatorname{trace}(\operatorname{Ric} \circ F) = 0$ for any derivation F of $\mathfrak{n}$. It follows that $\operatorname{trace}(\hat{D} \circ F) = -\beta \operatorname{trace}(F)$ for all derivations F of $\mathfrak{n}$, where β is the nilsoliton constant. See Section 2 of [25] for justifications of these properties.

In view of these properties of the nilsoliton derivation, it is natural to define a generalization as follows. A *Nikolayevsky derivation* is a derivation D^N of $\mathfrak{g}$ that is semisimple with real eigenvalues such that

$$\operatorname{trace}(D^N \circ F) = \operatorname{trace}(F) \tag{4}$$

for all $F \in \operatorname{Der}(\mathfrak{g})$. Note that it is possible for a Nikolayevsky derivation to be the zero map; this occurs if $\mathfrak{n}$ is characteristically nilpotent.

We summarize some properties of the Nikolayevsky derivation D^N.

(D1) Every Lie algebra admits a Nikolayevsky derivation that is unique up to automorphism (Theorem 1, [25]).

(D2) If $\phi : \mathfrak{n}_1 \to \mathfrak{n}_2$ is an isomorphism of Lie algebras and D^N is a Nikolayevsky derivation of $\mathfrak{n}_1$, then it is an immediate consequence of the definition that $\phi \circ D^N \circ \phi^{-1}$ is a Nikolayevsky derivation of $\mathfrak{n}_2$. Hence, the eigenvalues and multiplicities of a Nikolayevsky derivation are isomorphism invariants of a Lie algebra.

(D3) Any Nikolayevsky derivation is contained in an $\mathbb{R}$-split torus, and any $\mathbb{R}$-split torus contains a Nikolayevsky derivation. Moreover, D^N is a Nikolayevsky derivation if and only if the relation in Equation (4) holds for all F in an $\mathbb{R}$-split torus $\mathfrak{t}^{\mathbb{R}}$. (These properties follow from the proof of Theorem 1.1(a) of [25].)

(D4) If a nilpotent Lie algebra $\mathfrak{n}$ admits a nilsoliton inner product, then

- (a) the nilsoliton derivation $\hat{D}$ is a positive multiple of the Nikolayevsky derivation D^N (Theorem 1.2, [25]),
- (b) $D^N, \hat{D}$ and $\operatorname{Ric} = \hat{D} + \beta \operatorname{Id}$ have the same eigenvectors; and
- (c) D^N has positive eigenvalues and is symmetric relative to the soliton inner product on $\mathfrak{n}$.

(D5) The eigenvalues of the Nikolayevsky derivation are rational (Theorem 1, [25]).

(D6) Given an inner product Q on a Lie algebra $\mathfrak{g}$, there is at most one Nikolayevsky derivation which is symmetric with respect to Q (Theorem 3.3).

The Nikolayevsky derivation can be used to nicely characterize the nilpotent Lie algebras which admit nilsoliton inner products in terms of critical points of natural functionals on associated Lie groups (Theorem 2, [25]). Nikolayevsky also used the Nikolayevsky derivation to study nilsoliton inner products for filiform nilpotent Lie algebras and nilpotent Lie algebras whose Nikolayevsky derivation is simple ([24]).

The research culminating in the definition of the Nikolayevsky derivation evolved from Heber's analysis of derivations of nilpotent Lie algebras associated to Einstein solvable metric Lie algebras in the seminal paper [13] and Lauret's analysis of nilsoliton inner products, starting with [20]. Lauret showed that a nilsoliton derivation of a nilsoliton nilpotent metric Lie algebra $\mathfrak{n}$ can be used to define an Einstein inner product on a solvable extension of $\mathfrak{n}$, and conversely all rank one Einstein solvable metric Lie algebras arise this way. By Property (D4) above, a Nikolayevsky derivation is a generalization of a nilsoliton derivation, and many of the results from [13] and [20] generalize to Nikolayevsky derivations. For example, property (D5) generalizes a result of Heber's on derivations associated to Einstein solvable Lie algebras (Theorem C, [13]).

Several of our results assume in the hypotheses that the Nikolayevsky derivation of a Lie algebra has all eigenvalues positive. Jacobson showed that if a Lie algebra admits a nonsingular derivation, then it is nilpotent ([16]). Thus many of our results implicitly assume that the underlying Lie algebra is nilpotent.

We are interested in index sets Λ associated to a basis of a Lie algebra $\mathfrak{g}$. If $\mathcal{B} = \{x_i\}_{i=1}^n$ is a basis of $\mathfrak{g}$, and α_{ij}^k denotes the structure constants for $\mathfrak{g}$ relative to $\mathcal{B}$, then the *index set* Λ *for* $\mathfrak{g}$ *relative to* $\mathcal{B}$ is defined to be

$$\Lambda = \{(i,j,k) \,:\, i < j \text{ and } \alpha_{ij}^k \neq 0\}.$$

The set Λ indexes the nonzero structure constants modulo skew-symmetry. Let $[n]$ denote the set $\{1,2,\ldots,n\}$. If $\mathfrak{g}$ is n-dimensional, then $\Lambda \subseteq [n]^3$. We will consider arbitrary subsets of $[n]^3$ with the property that if $(i,j,k) \in \Lambda$, then $i < j$, and we will determine whether the subsets can be index sets for Lie algebras, in particular, nilpotent Lie algebras whose Nikolayevsky derivation is simple with positive eigenvalues. We will see in Theorem 3.1 that under certain reasonable restrictions on Λ, the index set Λ for a Lie algebra determines the Nikolayevsky derivation for the Lie algebra.

The index set determines matrices U, Y and $\hat{Y}$ as follows. For $i,j,k \in [n]$, define the $1 \times n$ *root vector* $\boldsymbol{y}_{ij}^k$ by $\boldsymbol{y}_{ij}^k = E_i + E_j - E_k$, where $\{E_i\}_{i=1}^n$ is the standard basis for $\mathbb{R}^n$. To an index set $\Lambda \subseteq [n]^3$ of cardinality m we associate an $m \times n$ *root matrix* Y as follows. We enumerate the root vectors $\boldsymbol{y}_{ij}^k$ for $(i,j,k) \in \Lambda$ in dictionary order, and we define Y so that the lth row of Y is the lth root vector in the list. We define the $\mathbb{Z}_2$*-root matrix* $\hat{Y}$ to be the same as the root matrix Y, except now we view its entries of $-1, 0, 1$ as elements of the finite field $\mathbb{Z}_2$. The *Gram matrix* U associated to Λ is the real $m \times m$ integral matrix $U = YY^T$.

We will frequently assume that Λ contains no elements of the form (i,j,j) or (i,j,i), and that $i < j < k$ for elements (i,j,k) of Λ. If these conditions hold, then the root vectors $\boldsymbol{y}_{ij}^k = E_i + E_j - E_k$ for $(i,j,k) \in \Lambda$ have entries 1, 1, and -1, in that order, with all other entries zero (when written with respect to the standard basis).

Important properties of the matrices Y and U associated to $(\mathfrak{n}, Q)$ are:

(U1) If Λ contains no elements of the form (i,j,j) or (i,j,i), then U is symmetric, positive semidefinite, with entries of 3 on the diagonal and off-diagonal entries from the set $\{-2,-1,0,1,2\}$.

(U2) The equation $U\boldsymbol{v} = [1]_{m\times 1}$ is consistent; i.e., there is at least one solution (Theorem 2, [26]).

(U3) If Λ contains no elements of the form (i, j, j) or (i, j, i), the Nikolayesky derivation is determined by Y (Theorem 3.1).

We use $[1]_{r\times s}$ to denote an $r \times s$ matrix or vector all of whose entries are 1.

The properties of U listed in (U1) follow from the fact that U is a Gram matrix for the set of root vectors for Λ, and the fact that when these root vectors are represented with respect to the standard basis, each vector has two 1's, one -1, and the remaining entries being zero. Regarding item (U2) above, the hypotheses of Theorem 2 of [26]) require that $U = cA + d[1]_{m\times m}$ where A is a "generalized Cartan matrix," and c and d are positive. The definition of generalized Cartan matrix is not important here (see [19] or Section 2.1 of [26] for the definition), but it is indeed true that the Gram matrix for U is of the required form (by letting $A = 2U - 4[1]_{m\times m}$ and using the properties of U listed in (U1)). The conclusion of Theorem 2 of [26]) lists five mutually exclusive cases. The equation $U\boldsymbol{v} = [1]_{m\times 1}$ has at least one solution in the first four cases. In the fifth case, the equation is not necessarily consistent, but the theorem asserts that that case may not occur if U satisfies a certain condition. However, the proof that the fifth case may not occur under that certain condition carries over verbatim to our situation, relying only on the facts that $U = YY^T$ and that $Y[1]_{n\times 1} = [1]_{m\times 1}$.

Obstructions to whether or not a nilpotent Lie algebra $\mathfrak{n}$ admits a nilsoliton inner product can be phrased in terms of the matrices U and Y :

(O1) If the equation $U\boldsymbol{v} = [1]_{m\times 1}$ has no solutions with all entries positive, and the basis $\mathcal{B}$ simultaneously diagonalizes a torus $\mathfrak{t}^{\mathbb{R}}$ in $\mathrm{Der}(\mathfrak{n})$, then $\mathfrak{n}$ does not admit a nilsoliton inner product (Theorem 1, [26]).

(O2) If the Nikolayesky derivation does not have positive eigenvalues, then $\mathfrak{n}$ does not admit a nilsoliton inner product.

For any index set $\Lambda \subseteq [n]^3$ we define $\mathcal{F}_\Lambda$ to be the family of n-dimensional Lie algebras $\mathfrak{g}$ that admit an ordered basis that diagonalizes the action of a torus in $\mathrm{Der}(\mathfrak{g})$ and such that the index set for that basis is Λ. In Theorem 3.8 we provide a necessary and sufficient criterion for two elements of $\mathcal{F}_\Lambda$ to be isomorphic via an isomorphism that maps basis vectors to basis vectors. In Theorem 4.2 we provide criteria for which, if any, nilpotent Lie algebras in certain subfamilies $\mathcal{F}_\Lambda$ admit a nilsoliton inner product. We derive a simple formula for computing the Nikolayevsky derivation for elements of $\mathcal{F}_\Lambda$; this expression is presented in Theorem 3.1.

For an inner product Q on a Lie algebra $\mathfrak{g}$ with respect to which the Nikolayevsky derivation is symmetric, let $\mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$ denote the group of inner-product preserving automorphisms of a metric nilpotent Lie algebra $(\mathfrak{g}, Q)$. Wilson showed that when $\mathfrak{g}$ is nilpotent, $\mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$ is the stabilizer of the identity in the isometry group of the simply connected nilmanifold corresponding to $(\mathfrak{g}, Q)$ ([36]). In Corollary 3.5 we show that if the Nikolayevsky derivation D^N is symmetric with respect to Q, then maps in $\mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$ commute with D^N. Corollary 4.5 gives a criterion, in terms of the Nikolayevsky derivation, for the group $\mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$ to be finite.

Finally, we demonstrate that the Nikolayevsky derivation is an effective, easily computable invariant for nilpotent Lie algebras of positive real rank. In contrast to the semisimple case, a powerful general structure theory for the nilpotent Lie algebras does not exist. There are few algebraic invariants so it may be difficult to tell when two nilpotent Lie algebras are nonisomorphic, or to match up a given nilpotent Lie algebra with a nilpotent Lie algebra isomorphic to it in a list of nilpotent Lie algebras. We describe invariants associated to the index set of a Nikolayevsky derivation in Proposition 3.6. In Examples 3.11 and 3.12, we present simplified proofs of some of the results from the classification of 7-dimensional nilpotent Lie algebras in [33] and [9] using these invariants.

Theorem 4.2 is a cornerstone for the work in [17] and [27], in which algorithms are implemented to find all nilpotent Lie algebras of dimension 7 and 8 such that the Nikolayevsky derivation has distinct positive eigenvalues. We will now give a rough description of that procedure. (The algorithm is actually more complicated than the one described here because recursive constructions, pruning procedures and other efficiencies are required in order to allow the program to run in a realistic amount of time.) The output of the procedure, for any fixed n, is a list of all n-dimensional nilpotent Lie algebras whose Nikolayevsky derivation D^N has distinct positive eigenvalues $0 < \lambda_1 < \cdots < \lambda_n$, up to isomorphism, and a determination for each Lie algebra in the list, whether or not it admits a nilsoliton inner product. A Lie algebra in this class will be represented with respect to a basis $\mathcal{B} = \{X_i\}_{i=1}^n$ of eigenvectors for D^N such that $D^N(X_i) = \lambda_i X_i$ for all i. It is an essential point that the basis is ordered so that the eigenvalues for X_i ascend with the index i.

1. Enumerate all possible index sets $\Lambda \subseteq [n]^3$ such that elements (i, j, k) of Λ satisfy $i < j < k$. (This condition is necessary because the eigenvalues of D^N are assumed to be positive.)

2. For each such Λ,

   (a) Compute Y, and use Y to find the Nikolayevsky derivation D^N associated with Λ according to the formula in Theorem 3.1.

   (b) If the eigenvalues of D^N are not distinct, positive and in ascending order, stop and go to the next Λ in the list.

   (c) Find the solution space to $Uv = [1]_{m\times 1}$. If none of the solutions have all entries positive, save Λ in the list of nonsoliton Λs. Otherwise, save Λs in the list of soliton Λs.

3. For each Λ in the list of soliton Λs and each Λ in the list of nonsoliton Λs, find associated structure constants. For each Λ,

   (a) The values of $(\alpha_{ij}^k)^2$ are parametrized by the set

   $$\{\boldsymbol{v} = (v_1, \ldots, v_n) \,:\, U\boldsymbol{v} = [1]_{m\times 1} \,\text{and}\, v_1, \ldots, v_n > 0\}.$$

   Find possible sign choices for α_{ij}^k, for $(i, j, k) \in \Lambda$, by finding elements of $\mathbb{Z}_2^{|\Lambda|}$ in a transversal for the action defined in Equation (5). This determines a set S_Λ of structure constants.

(b) Check the Jacobi identity. Remove any choices of structure constants in S_Λ that do not satisfy the Jacobi identity.

At the end of the procedure, we are left with a list of Λs, and a set S_Λ of structure constants for each Λ. Each Λ and one choice of structure constants determines a nilpotent Lie algebra. These will parametrize up to isomorphism all nilpotent Lie algebras of the type we sought.

Not all nilpotent Lie algebras admit nonsingular derivations; in fact there are huge families of characteristically nilpotent Lie algebras in dimensions 8 and more ([11],[18]). Even so, nilpotent Lie algebras that do admit nonsingular derivations are often of algebraic and geometric interest. As already mentioned in Property (D4) of Nikolayevsky derivations listed in Section 1, if a nilpotent Lie algebra $\mathfrak{n}$ admits a nilsoliton inner product, then the Nikolayevsky derivation for $\mathfrak{n}$ has positive eigenvalues. There also exist many nilpotent Lie algebras that do not admit nilsoliton inner products but whose Nikolayevsky derivations still are nonsingular ([15], [35], [28]). In addition, in the study of graded nilpotent Lie algebras and nonunimodular solvable Lie algebras one often encounters nonsingular derivations of nilpotent Lie algebras.

Many of our results assume that in addition to being nonsingular, the Nikolayevsky derivation has eigenvalues all of multiplicity one. This condition may not be generic in certain classes of nilpotent Lie algebras: for example, among two-step nilpotent Lie algebras, the generic Lie algebra has Nikolayevsky derivation with only two eigenvalues (This is shown for all but two cases in [6], and later [25]; it is shown for the remaining cases in [14]). On the other hand, among higher step nilpotent Lie algebras, such as $\mathbb{N}$-graded filiform nilpotent Lie algebras, Nikolayevsky derivations with simple eigenvalues are seen more commonly (See [24]).

The paper is organized as follows. In Section 2, we review necessary background material. In Section 3 we focus on index sets and the Nikolayevsky derivation in the purely algebraic domain. Section 4 includes applications of the Nikolayevsky derivation to the setting of nilpotent Lie algebras endowed with inner products.

2. Preliminaries

In this section we present some necessary prerequisite material. Much of this material was first described in [26] and [29]; the reader may refer to those references for a more detailed exposition on these topics.

2.1. Definitions of root vectors, root matrices and Gram matrices. Let $\mathfrak{g}$ be a nonabelian Lie algebra, and let $\mathrm{Der}(\mathfrak{g}) = \mathfrak{s} \oplus (\mathfrak{t}^{\mathbb{R}} \oplus \mathfrak{t}^{i\mathbb{R}}) \oplus \mathfrak{n}$ be a decomposition of its derivation algebra as in Equation (1). Let $\mathcal{B} = \{X_i\}_{i=1}^n$ be an ordered basis. Let

$$\Lambda = \{(i,j,k) \,:\, i < j, \alpha_{ij}^k \neq 0\}$$

index the nonzero structure constants α_{ij}^k for $\mathfrak{g}$ relative to $\mathcal{B}$, ignoring repetitions due to skew-symmetry, as described in the introduction.

For a fixed $\Lambda \subseteq [n]^3$ we define a family $\mathcal{F}_\Lambda$ of Lie algebras as follows.

Definition 2.1. Suppose a skew-symmetric bilinear map $\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defines a Lie bracket on $\mathbb{R}^n$, yielding a n-dimensional Lie algebra $\mathfrak{g}_\alpha$. Fix the standard basis $\mathcal{B} = \{E_i\}_{i=1}^n$ of $\mathbb{R}^n$. Then $\mathfrak{g}_\alpha \in \mathcal{F}_\Lambda$ if and only if

- the basis $\mathcal{B}$ simultaneously diagonalizes the elements of an $\mathbb{R}$-split torus $\mathfrak{t}^{\mathbb{R}}$ of $\mathrm{Der}(\mathfrak{g})$ and
- the index set for the basis $\mathcal{B}$ is Λ.

A natural topology on $\mathcal{F}_\Lambda$ comes from viewing the defining maps α for elements $\mathfrak{g}_\alpha$ of $\mathcal{F}_\Lambda$ as points in the vector space $\wedge^2\mathbb{R}^* \otimes \mathbb{R}$.

By property (D3) of Nikolayevsky derivations in the introduction, every torus $\mathfrak{t}^{\mathbb{R}}$ contains a Nikolayevsky derivation, so for every $\mathfrak{g}_\alpha$ of $\mathcal{F}_\Lambda$ there is a Nikolayevsky derivation diagonalized by the basis $\mathcal{B}$. If all of the eigenvalues of D^N have multiplicity one, then any basis consisting of D^N-eigenvectors diagonalizes any $\mathbb{R}$-split torus $\mathfrak{t}^{\mathbb{R}}$ that contains D^N. We will see later in Theorem 3.1 that if the index set Λ contains no elements of the form (i, j, j) or (i, j, i), then all of the Lie algebras in $\mathcal{F}_\Lambda$ have the same Nikolayevsky derivation.

Now let $\mathfrak{g}$ be an arbitrary Lie algebra with basis $\mathcal{B}$ and index set Λ of cardinality m, and let α_{ij}^k denote the structure constants for $\mathfrak{g}$ with respect to $\mathcal{B}$. Let $\boldsymbol{v} = [(\alpha_{ij}^k)^2]_{(i,j,k)\in\Lambda}$ be the $m \times 1$ vector listing the squares of the structure constants according to the dictionary ordering of Λ : the lth entry of v_l is $(\alpha_{(i,j,k)})^2$, where (i, j, k) is the lth element in Λ. This vector is called the *structure vector* for $\mathfrak{g}$ with respect to $\mathcal{B}$. Let $\boldsymbol{s} = [s_{(i,j,k)}]_{(i,j,k)\in\Lambda}$ be the $m \times 1$ vector with entries in $\mathbb{Z}_2$ so that the lth entry s_l of $\boldsymbol{s}$ is

$$s_l = \begin{cases} 0 & \text{if } \alpha_{ij}^k > 0, \text{ where } (i,j,k) \text{ is the } l\text{th element of } \Lambda \\ 1 & \text{if } \alpha_{ij}^k < 0, \text{ where } (i,j,k) \text{ is the } l\text{th element of } \Lambda \end{cases}.$$

This vector is called the *sign vector* for $\mathfrak{g}$ with respect to $\mathcal{B}$. Observe that the nonzero structure constants for $\mathfrak{g}$ and their indices, hence the definition of the Lie bracket on $\mathbb{R}^n$, may be recovered from the basis $\mathcal{B}$, the index set Λ, the structure vector $\boldsymbol{v}$ and the sign vector $\boldsymbol{s}$.

Fix a $m \times n$ root matrix Y and the corresponding $\mathbb{Z}_2$ root matrix $\hat{Y}$. The matrix $\hat{Y}$ defines an action of $\mathbb{Z}_2^n$ on $\mathbb{Z}_2^m$ by

$$\boldsymbol{c} \cdot \boldsymbol{z} = \boldsymbol{z} + \hat{Y}\boldsymbol{c}, \tag{5}$$

where $\boldsymbol{c} \in \mathbb{Z}_2^n$ and $\boldsymbol{z} \in \mathbb{Z}_2^m$. A subset S of $\mathbb{Z}_2^m$ is said to be a *global transversal* for the action $\mathbb{Z}_2^n \cdot \mathbb{Z}_2^m \to \mathbb{Z}_2^m$ defined by the $\mathbb{Z}_2$ root matrix $\hat{Y}$ if for all vectors $\boldsymbol{z} \in S$, the orbit

$$\mathcal{O}_{\boldsymbol{z}} = \{\boldsymbol{z} + \hat{Y}\boldsymbol{c} \,:\, \boldsymbol{c} \in \mathbb{Z}_2^n\},$$

meets the set S exactly once.

The matrix Y defines an action of $\mathbb{R}^n$ on $\mathbb{R}^m$ by

$$\boldsymbol{d} \cdot \boldsymbol{z} = \boldsymbol{z} + Y\boldsymbol{d},$$

for $\boldsymbol{d} \in \mathbb{R}^n$ and $\boldsymbol{z} \in \mathbb{R}^m$. We let $\mathcal{O}_{\boldsymbol{v}}$ denote the orbit of a vector $\boldsymbol{v} \in \mathbb{R}^m$ under the $\mathbb{R}^n$ action:

$$\mathcal{O}_{\boldsymbol{v}} = \{\boldsymbol{v} + Y\boldsymbol{c} \,:\, \boldsymbol{c} \in \mathbb{R}^n\} = \boldsymbol{v} + \mathrm{col}(Y), \tag{6}$$

where $\mathrm{col}(Y)$ denotes the column space of the matrix Y.

An m-tuple of nonzero structure constants $[\alpha_{ij}^k]_{(i.j,k)\in\Lambda} \in (\mathbb{R}_{\neq 0})^m$ may be encoded in the vector $[\ln(\alpha_{ij}^k)^2]_{(i.j,k)\in\Lambda} \in \mathbb{R}^m$ and the sign vector $[\mathrm{sgn}(\alpha_{ij}^k)]_{(i.j,k)\in\Lambda} \in \mathbb{Z}_2^m$. We will see in Theorem 3.8 that the product

$$(\mathbb{Z}_2^n \times \mathbb{R}^n) \cdot (\mathbb{Z}_2^m \times \mathbb{R}^m) \to (\mathbb{Z}_2^m \times \mathbb{R}^m) \tag{7}$$

of the two actions defined above provides a convenient way of representing the action of $(\mathbb{R}_{\neq 0})^n \cong \mathbb{Z}_2^n \times \mathbb{R}^n$ on nonzero structure constants in $(\mathbb{R}_{\neq 0})^m$ by rescaling.

2.2. Results involving index sets, root matrices, Gram matrices and the two actions. The following lemma reformulates an elementary fact from [26].

Lemma 2.2. *(Lemma 2, [26]) Let $\mathfrak{g}$ be a nonabelian metric Lie algebra and let $\mathcal{B} = \{X_i\}_{i=1}^n$ be a basis for $\mathfrak{n}$. Let Λ denote the index set for $\mathcal{B}$ and let Y be the associated root matrix. Let D be an endomorphism of $\mathfrak{g}$ such that $D(X_i) = \lambda_i X_i$ with $\lambda_i \in \mathbb{R}$, for $i \in [n]$. Then D is a derivation if and only if the vector $(\lambda_1, \ldots, \lambda_n)^T$ is in the null space of Y.*

The following theorem gives a useful criterion for the existence of a nilsoliton inner product on a nilpotent Lie algebra.

Theorem 2.3. *(Theorem 1, [26]) Let $(\mathfrak{n}, Q)$ be a nonabelian metric nilpotent Lie algebra with orthonormal Ricci eigenvector basis $\mathcal{B}$. Let U and $\boldsymbol{v}$ be the Gram matrix and the structure vector for $(\mathfrak{n}, Q)$ with respect to $\mathcal{B}$. Then $(\mathfrak{n}, Q)$ satisfies the nilsoliton condition (3) with nilsoliton constant β if and only if $U\boldsymbol{v} = -2\beta[1]_{m\times 1}$.*

A stronger version of this theorem was later proved in [25] (Theorem 3).

The next proposition may be used to confirm that a given basis for a metric nilpotent Lie algebra is a Ricci eigenvector basis so that the hypotheses in the previous theorem are met.

Proposition 2.4. *Let $(\mathfrak{n}, Q)$ be a nonabelian metric nilpotent Lie algebra. Let $\mathcal{B} = \{X_i\}_{i=1}^n$ be an orthogonal basis, let Λ index the nonzero structure constants for $\mathfrak{n}$ with respect to $\mathcal{B}$, and let U be the Gram matrix for Λ. If the Gram matrix for Λ has no entries of 2, then the basis $\mathcal{B}$ is a Ricci eigenvector basis.*

Proof. If we rescale the basis so it is orthonormal, the index set remains the same, so we may assume without loss of generality that $\mathcal{B}$ is orthonormal. The hypothesis that the Gram matrix has no entries of 2 is equivalent to the condition that Λ contains no two pairs (i_1, j_1, k_1) and (i_2, j_2, k_2) such that the corresponding root vectors $\boldsymbol{y}_{i_1 j_1}^{k_1}$ and $\boldsymbol{y}_{i_2 j_2}^{k_2}$ have inner product 2. This in turn is equivalent to the condition that there are no two pairs (i_1, j_1, k_1) and (i_2, j_2, k_2) in Λ such that $i_1 \in \{i_2, j_2\}$ and $k_1 = k_2$; or so that $\{i_1, j_1\} = \{i_2, j_2\}$.

It follows from Equation (2) that for $i, j = 1, \dots, n$,

$$\mathsf{ric}(X_i, X_j) = \frac{1}{4} \sum_{k,l=1}^{n} \alpha_{kl}^i \alpha_{kl}^j - \frac{1}{2} \sum_{k,l=1}^{n} \alpha_{ik}^l \alpha_{jk}^l.$$

The conditions on Λ insure that it is never possible for pairs of structure constants of the form α_{kl}^i and α_{kl}^j or of the form α_{ik}^l and α_{jk}^l to simultaneously be nonzero when $i \neq j$. Hence the Ricci endomorphism is diagonal when represented with respect to $\mathcal{B}$, and $\mathcal{B}$ is a Ricci eigenvector basis. ■

One case in which the Gram matrix has no entries of 2 is when the Nikolayevsky derivation has positive eigenvalues, all of multiplicity one:

Lemma 2.5. *Let $\mathfrak{n}$ be a nonabelian Lie algebra such that the Nikolayevsky derivation D^N for $\mathfrak{n}$ has positive eigenvalues, all of multiplicity one. Let $\mathcal{B}$ be a basis of $\mathfrak{n}$ consisting of eigenvectors for D^N. Then the Gram matrix U associated to $\mathcal{B}$ has no entries of* 2.

Proof. Suppose that $\mathcal{B} = \{X_i\}_{i=1}^n$ is a basis for $\mathfrak{n}$ and $D^N(X_i) = \lambda_i X_i$, for $i = 1, \dots, n$, where $0 < \lambda_1 < \cdots < \lambda_n$.

Because D^N is a derivation, the λ_i-eigenspace and the λ_j-eigenspace bracket into the $\lambda_i + \lambda_j$ eigenspace, for all $i, j = 1, \dots, n$. It follows that if $(i, j, k) \in \Lambda$, then $\lambda_i + \lambda_j = \lambda_k$. Because the eigenspaces are all one-dimensional and $\mathcal{B}$ is an eigenvector basis, the Lie bracket of any two basis vectors is a scalar multiple of another basis vector. Hence Λ contains no pairs of elements of the form (i, j, k_1) and (i, j, k_2) with $k_1 \neq k_2$. On the other hand the condition that if $(i, j, k) \in \Lambda$, then $\lambda_i + \lambda_j = \lambda_k$ precludes the possibility that Λ contains distinct elements (i_1, j_1, k_1) and (i_2, j_2, k_2) such that $k_1 = k_2$ and i_1 or j_1 is in $\{i_2, j_2\}$. Therefore, if (i_1, j_1, k_1) and (i_2, j_2, k_2) are distinct elements of Λ, the inner product of the corresponding root vectors is not 2. Hence none of the entries of the Gram matrix U are 2. ■

The following proposition allows one to determine when two Lie algebras with nearly identical presentations (in which only the signs of structure constants differ) are isomorphic. We will use it in the proof of Theorem 3.8. The proof of the proposition is the same as the proof of Theorem B of [29].

Proposition 2.6. *(Theorem B, [29]) Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras. For $j = 1, 2$, let $\mathcal{B}_j = \{X_i^j\}_{i=1}^n$ be a basis for $\mathfrak{g}_j$, let Λ_j be the index set for $\mathcal{B}_j$, let $\boldsymbol{v}_j$ be the structure vector for $\mathcal{B}_j$, and let $\boldsymbol{s}_j$ be the sign vector for $\mathcal{B}_j$. Suppose that $\Lambda_1 = \Lambda_2$. There exists an isomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\phi(X_i^1) = \pm X_i^2$ for all $i = 1, \dots, n$ if and only if $\boldsymbol{v}_1 = \boldsymbol{v}_2$ and the sign vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are in the same orbit of the $\mathbb{Z}_2^n \cdot \mathbb{Z}_2^m \to \mathbb{Z}_2^m$ action defined in Section 2.*

3. Algebra

In this section we address algebraic properties of the Nikolayevsky derivation and we apply the Nikolayevsky derivation to algebraic problems.

3.1. The Nikolayevsky derivation.

The next theorem gives an explicit formula for the Nikolayevsky derivation. The formula depends on the index set Λ, but not the underlying nilpotent Lie algebra. The virtue of this perspective is that index sets in any dimension may be enumerated in a finite list, while enumerations of nilpotent Lie algebras do not exist in general for two reasons. First, there do not exist classifications of nilpotent Lie algebras in dimension 8 and higher, and second, there are (uncountably) infinitely many nilpotent Lie algebras in dimension of 7 and higher.

Theorem 3.1. *Fix a nonempty index set $\Lambda \subseteq [n]^3$ containing no elements of the form (i,j,j) or (i,j,i). Let $m = |\Lambda|$. Let Y be the root matrix associated to Λ, and let U be the Gram matrix associated to Λ. Let $\mathfrak{g}_\alpha$ be in the family $\mathcal{F}_\Lambda$ as defined in Section 2, such that $\mathcal{B}$ diagonalizes an $\mathbb{R}$-split torus $\mathfrak{t}^{\mathbb{R}}$. Then:*

1. *The equation $U\boldsymbol{v} = [1]_{m\times 1}$ is consistent; i.e, there exists at least one solution to the equation.*

2. *For all choices of $\boldsymbol{b}$ in the solution space to $U\boldsymbol{v} = [1]_{m\times 1}$, the vector $\boldsymbol{v}_{D^N}$ defined by $\boldsymbol{v}_{D^N} = -\boldsymbol{b}^T Y + [1]_{1\times n}$ is independent of $\boldsymbol{b}$.*

3. *Let D^N be the endomorphism of $\mathbb{R}^n$ given with respect to the basis $\mathcal{B}$ by the diagonal matrix $[D^N]_{\mathcal{B}} = \operatorname{diag}(\boldsymbol{v}_{D^N})$. Then D^N is the Nikolayevsky derivation for all Lie algebras in the family $\mathcal{F}_\Lambda$.*

Proof. Consider the equation $U\boldsymbol{v} = [1]_{m\times 1}$. By property (U2) of Gram matrices from the introduction, the system $U\boldsymbol{v} = [1]_{m\times 1}$ has at least one solution.

Let $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ be solutions to $U\boldsymbol{v} = [1]_{m\times 1}$. Then $\boldsymbol{b}_1 - \boldsymbol{b}_2$ is in the null space of U. Since $U = YY^T$, the null space of U coincides with the null space of Y^T. We have $-\boldsymbol{b}_1^T Y = -\boldsymbol{b}_2^T Y$, and consequently the vector $\boldsymbol{v}_{D^N}$ is independent of the choice of $\boldsymbol{b}$ as claimed.

Now let D^N be as in the statement of the theorem. Since it is represented by a real diagonal matrix relative to $\mathcal{B}$, the derivation D^N is semisimple with real eigenvalues. It is a derivation by Lemma 2.2:

$$Y(-\boldsymbol{b}^T Y + [1]_{1\times n})^T = -[1]_{m\times 1} + [1]_{m\times 1} = [0]_{m\times 1},$$

where we have used the fact that the inner product of the vector $[1]_{1\times n}$ and any row of a root matrix Y is 1, and $[0]_{m\times 1}$ denotes the $m \times 1$ vector with every entry a zero.

Now we need to show that $\operatorname{trace}(D^N \circ F) = \operatorname{trace}(F)$ for all F in $\mathfrak{t}^{\mathbb{R}} \subseteq \operatorname{Der}(\mathfrak{g}_\alpha)$. Fix $F \in \mathfrak{t}^{\mathbb{R}}$ and let $\boldsymbol{v}_F$ be the $n \times 1$ vector such that $[F]_{\mathcal{B}} = \operatorname{diag}(\boldsymbol{v}_F)$. For any such derivation F, the vector $\boldsymbol{v}_F$ is in the null space of Y, by Lemma 2.2.

Now we compute $\operatorname{trace}(D^N \circ F)$ using the basis $\mathcal{B}$:

$$\begin{aligned}\operatorname{trace}(D^N \circ F) &= \operatorname{trace}([D^N \circ F]_{\mathcal{B}}) \\ &= \operatorname{diag}(\boldsymbol{v}_{D^N}) \cdot \operatorname{diag}(\boldsymbol{v}_F) \\ &= (-\boldsymbol{b}^T Y + [1]_{1\times n}) \cdot \boldsymbol{v}_F \\ &= [1]_{1\times n} \cdot \boldsymbol{v}_F \\ &= \operatorname{trace}([F]_{\mathcal{B}}) \\ &= \operatorname{trace}(F).\end{aligned}$$

Thus D^N is the Nikolayevsky derivation as desired. ■

The following example demonstrates an application of the previous theorem.

Example 3.2. Let $n = 7$ and let

$$\Lambda = \{(1,2,4),(1,3,5),(1,4,6),(1,6,7),(2,3,6),(3,4,7)\}.$$

Let $\mathcal{F}_\Lambda$ denote the family of nilpotent metric Lie algebras whose index set with respect to the standard basis $\mathcal{B} = \{E_i\}_{i=1}^7$ is Λ. We will see in Example 3.10 that the family $\mathcal{F}_\Lambda$ is nonempty. The structure constants α_{ij}^k may vary freely so long as the Jacobi Identity is satisfied, or equivalently, $\alpha_{23}^6\alpha_{16}^7 + \alpha_{12}^4\alpha_{34}^7 = 0$ (See Theorem 7 of [26]). The root matrix Y is

$$Y = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{bmatrix}$$

and $U = YY^T$ is given by

$$U = \begin{bmatrix} 3 & 1 & 0 & 1 & 1 & -1 \\ 1 & 3 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 & -1 & 1 \\ 1 & 1 & 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 1 & 1 & 3 \end{bmatrix}$$

Solving the equation $U\boldsymbol{v} = [1]_{6\times 1}$ we get

$$\boldsymbol{v} = \frac{1}{19}(5,-1,4,5,4,4)^T + t(1,0,0,-1,-1,1),$$

for any $t \in \mathbb{R}$. We let $t = 0$ to get one solution $\boldsymbol{b} = \frac{1}{19}(5,-1,4,5,4,4)^T$. Now we let

$$\boldsymbol{v}_{D^N} = -\boldsymbol{b}^T Y + [1]_{1\times 7} = \tfrac{2}{19}(3,5,6,8,9,11,14).$$

By Theorem 3.1, the derivation D^N defined relative to $\mathcal{B}$ by the matrix $[D^N]_{\mathcal{B}} = \operatorname{diag}(\boldsymbol{v}_{D^N})$ is the Nikolayevsky derivation for all elements of the family $\mathcal{F}_\Lambda$.

Next we see that there is at most one Nikolayevsky derivation for a Lie algebra which is symmetric relative to a given inner product.

Theorem 3.3. *Let Q be an inner product on a Lie algebra $\mathfrak{g}$. There exists at most one Nikolayevsky derivation of $\mathfrak{g}$ which is symmetric with respect to Q.*

The proof of Theorem 3.3 requires the following lemma.

Lemma 3.4. *Let $\mathfrak{g}$ be a (real) solvable subalgebra of* $\mathrm{End}(\mathbb{R}^n)$, *and let* $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$. *Then all of the elements of $\mathfrak{n}$ are nilpotent linear transformations.*

Proof. Let $\mathfrak{g}'$ denote the complexification of $\mathfrak{g}$ in $\mathrm{End}(\mathbb{C}^n)$, and let $\mathfrak{n}' = [\mathfrak{g}', \mathfrak{g}']$. By Lie's Theorem, there is a basis $\mathcal{B}$ of $\mathbb{C}^n$ such that the elements of $\mathfrak{n}'$ are all upper triangular with zeros on the diagonal, which shows that the elements of $\mathfrak{n}'$ are nilpotent linear transformations on $\mathbb{C}^n$. In particular, the elements of $\mathfrak{n} \subseteq \mathfrak{n}'$ are nilpotent linear transformations on $\mathbb{C}^n$ and $\mathbb{R}^n$. ■

Now we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. Let $\mathfrak{g}$ be a Lie algebra endowed with an inner product Q. Suppose that D_1 and D_2 are Nikolayevsky derivations of $\mathfrak{g}$, both of which are symmetric relative to Q.

It follows that the linear transformation $N = [D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is a skew-symmetric derivation of $\mathfrak{g}$ relative to Q. By definition of the Nikolayevsky derivation, D_1 and D_2 are elements of the solvable radical $\mathfrak{s}$ of $\mathrm{Der}(\mathfrak{g})$. Hence, N is an element of $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and must be a nilpotent linear transformation by Lemma 3.4. Since N is both skew symmetric and nilpotent, it must be zero. Thus we have shown that D_1 and D_2 commute.

Now let $\mathfrak{g} = W_1 \oplus \cdots \oplus W_k$ be an orthogonal direct sum decomposition of $\mathfrak{g}$ into common eigenspaces of the commuting maps D_1 and D_2. For $0 \leq t \leq 1$, let $D_t = (1-t)D_1 + tD_2$. Fix $i \in \{1, \ldots, k\}$. If $D_1 = \lambda_i \,\mathrm{Id}$ on W_i and $D_2 = \mu_i \,\mathrm{Id}$ on W_i, then the restriction of D_t to W_i may be expressed as

$$D_t|_{W_i} = ((1-t)\lambda_i + t\mu_i)\,\mathrm{Id}\,.$$

The transformations D_t are all semisimple with real eigenvalues. In addition, they satisfy the condition $\mathrm{trace}(D_t \circ F) = \mathrm{trace}(F)$ for all derivations F of $\mathfrak{g}$, since D_1 and D_2 satisfy this trace condition. Hence, D_t is a Nikolayevsky derivation for $0 \leq t \leq 1$. It follows that the eigenvalues of D_t are the same for all t, but this is only possible if $\lambda_i = \mu_i$ for all $i = 1, \ldots, k$. Therefore, $D_1 = D_2$. ■

As a corollary to Theorem 3.3, we obtain some more properties of the Nikolayevsky derivation.

Corollary 3.5. *Let D^N be a Nikolayevsky derivation of a Lie algebra that is symmetric with respect to an inner product Q on $\mathfrak{g}$. Let* $\mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$ *denote the group of automorphisms of $\mathfrak{g}$ that preserve the inner product Q.*

1. *D^N commutes with elements of* $\mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$.

2. *Suppose that* $\phi \in \mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$. *Then* ϕ *fixes every eigenspace of* D^N.

3. D^N *commutes with all derivations of* $\mathfrak{g}$ *that are skew-symmetric with respect to* Q.

Proof. If $\phi \in \mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$, then $\phi \circ D^N \circ \phi^{-1}$ is also a Nikolayevsky derivation that is symmetric relative to Q. By Theorem 3.3, $\phi \circ D^N \circ \phi^{-1} = D^N$. This proves Part (1).

The second part follows immediately from the fact that if two linear transformations commute, then each one preserves the eigenspaces of the other.

If D is a skew-symmetric derivation of $\mathfrak{g}$, then $e^{tD} \in \mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$ for all $t \in \mathbb{R}$. Differentiating the equation $e^{tD} \circ D^N \circ e^{-tD} = D^N$ at $t = 0$ shows that $D \circ D^N - D^N \circ D = 0$. ■

3.2. Isomorphism invariants.

Recall that the spectrum with multiplicities of the Nikolayevsky derivation D^N for a Lie algebra is an isomorphism invariant of the Lie algebra. Example 3.12 will show that the spectrum with multiplicities of D^N is not sufficient to determine the isomorphism class of the Lie algebra.

The following proposition asserts that when the eigenvalues of the Nikolayevsky derivation have multiplicity one, the index set and the root matrix for an eigenvector basis serve as algebraic invariants (property (D2) from the introduction). This result may be used to show that two nilpotent Lie algebras with the same Nikolayevsky derivation are not isomorphic, as we will see in Example 3.12.

Proposition 3.6. *Let* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *be Lie algebras, and let* $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ *be an isomorphism. For* $s = 1$ *and* 2, *let* $\mathcal{B}_s = \{X_i^s\}_{i=1}^n$ *be a basis consisting of eigenvectors of the Nikolayevsky derivation* D_s^N *for* $\mathfrak{g}_s$, *where* $D_s^N(X_i^s) = \lambda_i X_i^s$ *for all* $i = 1, \ldots, n$. *For* $s = 1, 2$, *let the index set* Λ_s *index the set of nonzero structure constants for* $\mathfrak{g}_s$ *with respect to the basis* $\mathcal{B}_s$, *and let* Y_s *and* U_s *denote the root matrix and Gram matrix associated to* Λ_s. *If the Nikolayevsky derivations for* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *both have eigenvalues all of multiplicity one, then* $\Lambda_1 = \Lambda_2, Y_1 = Y_2$ *and* $U_1 = U_2$.

Proof. Suppose that $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is an isomorphism. Then $\phi \circ D_1^N \circ \phi^{-1}$ is a Nikolayevsky derivation of $\mathfrak{g}_2$. A Nikolayevsky derivation is unique up to automorphism, so there exists an automorphism ψ of $\mathfrak{g}_2$ so that $\phi \circ D_1^N \circ \phi^{-1} = \psi \circ D_2^N \circ \psi^{-1}$. The set $\mathcal{B}^\phi = \{X_i' = \psi^{-1}(\phi(X_i^1))\}$ is a basis of $\mathfrak{g}_2$ such that X_i' is an eigenvector of D_2^N with eigenvalue λ_i for all $i \in [n]$. Because the eigenspaces are one-dimensional, X_i' is a nonzero scalar multiple of X_i^2 for all i. Therefore $\Lambda_1 = \Lambda_2$. The root matrices depend only on the index set so $Y_1 = Y_2$. It follows that $U_1 = Y_1 Y_1^T = Y_2 Y_2^T = U_2$. ■

As a corollary, we obtain a description of isomorphisms between elements of the family $\mathcal{F}_\Lambda$ defined in Section 2.

Corollary 3.7. *Let $\Lambda \subseteq [n]^3$ be an index set and let $\mathcal{F}_\Lambda$ be the associated family of Lie algebras as defined in Section 2. Recall that we may assume that the standard basis $\{E_i\}_{i=1}^n$ consists of eigenvectors for the Nikolayevsky derivation D^N. Suppose that all of the eigenvalues of D^N have multiplicity one. If a map $\phi : \mathfrak{g}_\alpha \to \mathfrak{g}_\beta$ between two elements of $\mathcal{F}_\Lambda$ is an isomorphism, then $\phi(E_i)$ is a nonzero scalar multiple of E_i for all $i = 1, \ldots, n$.*

Proof. By Proposition 3.6, eigenspaces of D^N are preserved. As these eigenspaces are one-dimensional, eigenvectors are simply rescaled by ϕ. By definition of $\mathcal{F}_\Lambda$, the standard basis $\mathcal{B}$ is an ordered eigenvector basis, so elements of $\mathcal{B}$ rescaled by ϕ. ■

One might ask about the converse to Proposition 3.6. That is, how can we determine whether two elements of $\mathcal{F}_\Lambda$ are isomorphic? The next theorem gives a partial answer. In some cases, isomorphism classes are orbits of the product action defined in Equation (7). This theorem tells us not just whether two elements of $\mathcal{F}_\Lambda$ are isomorphic, but it allows us to find the full set of elements of $\mathcal{F}_\Lambda$ which are isomorphic to a given Lie algebra in $\mathcal{F}_\Lambda$.

For a vector $\boldsymbol{v} = (v_i)$ with positive entries, we define a vector $\ln(\boldsymbol{v})$ of the same size by letting $(\ln(\boldsymbol{v}))_i = \ln v_i$ for all i.

Theorem 3.8. *Let $\Lambda \subseteq [n]^3$ be an index set with associated family of Lie algebras $\mathcal{F}_\Lambda$ as in Definition 2.1. Recall that the standard basis $\{E_i\}_{i=1}^n$ consists of eigenvectors for the Nikolayevsky derivation D^N. Let $m = |\Lambda|$. Let Y be the root matrix associated to Λ, and let $\hat{Y}$ be the $\mathbb{Z}_2$ root matrix associated to Λ. Let $\mathbb{Z}_2^n \cdot \mathbb{Z}_2^m \to \mathbb{Z}_2^m$ and $\mathbb{R}^n \cdot \mathbb{R}^m \to \mathbb{R}^m$ be the actions determined by $\hat{Y}$ and Y respectively, as defined in Section 2.*

Let $\mathfrak{g}_\alpha$ be a Lie algebra in $\mathcal{F}_\Lambda$ with α_{ij}^k denoting the structure constants for $\mathfrak{g}_\alpha$ relative to the standard basis $\mathcal{B}$, and let $\mathfrak{g}_\beta$ be in $\mathcal{F}_\Lambda$ with β_{ij}^k denoting the structure constants for $\mathfrak{g}_\beta$ relative to $\mathcal{B}$. Let $\boldsymbol{a} = [(\alpha_{ij}^k)^2]$ and $\boldsymbol{b} = [(\beta_{ij}^k)^2]$ denote the structure vectors for $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ respectively, and let $\boldsymbol{s}_\alpha$ and $\boldsymbol{s}_\beta$ denote the sign vectors for $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ respectively.

Then $\mathfrak{g}_\beta$ is isomorphic to $\mathfrak{g}_\alpha$ via an isomorphism that maps each vector E_i in the basis to a scalar multiple of itself if and only if the two vectors $\ln(\boldsymbol{a})$ and $\ln(\boldsymbol{b})$ are in the same orbit for the first action $\mathbb{R}^n \cdot \mathbb{R}^m \to \mathbb{R}^m$, and the two sign vectors $\boldsymbol{s}_\alpha$ and $\boldsymbol{s}_\beta$ are in the same orbit for the second action $\mathbb{Z}_2^n \cdot \mathbb{Z}_2^m \to \mathbb{Z}_2^m$.

As a consequence, if the rank of $\hat{Y}$ is maximal over $\mathbb{Z}_2$ and $\boldsymbol{a} = \boldsymbol{b}$, then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are isomorphic.

Proof. Suppose that an isomorphism $\rho : \mathfrak{g}_\alpha \to \mathfrak{g}_\beta$ is given by

$$\rho(E_i) = (-1)^{d_i} \exp(c_i) E_i, \tag{8}$$

where $i = 1, \ldots, n$; $c_1, \ldots, c_n \in \mathbb{R}$; and $d_i \in \{0, 1\}$. For $i = 1, \ldots, n$, let $E_i' = \rho(E_i) = \exp(c_i)E_i$, and let $\mathcal{B}'$ denote the basis $\{E_i'\}_{i=1}^n$. Suppose that instead of using the basis $\mathcal{B}$, we use $\mathcal{B}'$ to present $\mathfrak{g}$. Let α_{ij}^k and β_{ij}^k denote the structure constants for $\mathfrak{g}$ with respect to $\mathcal{B}$ and $\mathcal{B}'$, respectively. We find β_{ij}^k in terms of α_{ij}^k from the equality

$$[(-1)^{d_i}e^{c_i}E_i, (-1)^{d_j}e^{d_j}E_j] = \sum_{l=1}^{k}(-1)^{d_i+d_j-d_k}e^{c_i+c_j-c_k}\alpha_{ij}^k\left((-1)^{d_k}e^{c_k}E_k\right), \quad (9)$$

obtaining

$$\beta_{ij}^k = (-1)^{d_i+d_j-d_k}\exp\left((c_1,\ldots,c_n)\cdot(E_i+E_j-E_k)\right)\alpha_{ij}^k, \quad (10)$$

for all i, j, k.

From Equation (10) we obtain

$$\ln|\beta_{ij}^k| = (E_i+E_j-E_k)\cdot(c_1E_1+\ldots+c_nE_n)+\ln|\alpha_{ij}^k|. \quad (11)$$

It follows that α_{ij}^k is nonzero if and only if β_{ij}^k is nonzero, and in the case that both are nonzero,

$$\ln(\beta_{ij}^k)^2 = (E_i+E_j-E_k)\cdot(2c_1E_1+\ldots+2c_nE_n)+\ln(\alpha_{ij}^k)^2. \quad (12)$$

Therefore, the vector $\ln(\boldsymbol{b}) = [\ln(\beta_{ij}^k)^2]_{(i,j,k)\in\Lambda}$ is related to the vector $\ln(\boldsymbol{a}) = [\ln(\alpha_{ij}^k)^2]_{(i,j,k)\in\Lambda}$ by the linear equation

$$\ln(\boldsymbol{b}) = \ln(\boldsymbol{a}) + Y\boldsymbol{c},$$

where Y is the root matrix for $(\mathfrak{g}, Q)$ with respect to $\mathcal{B}$, and $\boldsymbol{c} = (2c_1,\ldots,2c_n)^T$. Hence the vectors $\ln(\boldsymbol{a})$ and $\ln(\boldsymbol{b})$ are in the same orbit of the $\mathbb{R}^n$ action.

The sign of nonzero β_{ij}^k is given by

$$\operatorname{sgn}(\beta_{i,j}^k) = d_i + d_k - d_k + \operatorname{sgn}(\alpha_{ij}^k),$$

where now we view d_i, d_j and d_k as elements of $\mathbb{Z}_2$, and the sign vectors $\boldsymbol{s}_\alpha$ and $\boldsymbol{s}_\beta$ are related by

$$\boldsymbol{s}_\beta = \boldsymbol{s}_\alpha + \hat{Y}\boldsymbol{d}, \quad (13)$$

where $\boldsymbol{d} = (d_1,\ldots,d_n)^T \in \mathbb{Z}_2^n$. Hence the sign vectors $\boldsymbol{s}_\alpha$ and $\boldsymbol{s}_\beta$ are in the same orbit of the $\mathbb{Z}_2^n$ action.

Now, to prove the converse, suppose that that $\ln(\boldsymbol{a}) = \ln(\boldsymbol{b})$ are in the same orbit for the action defined by Y and that the sign vectors $\boldsymbol{s}_\alpha$ and $\boldsymbol{s}_\beta$ are in the same orbit for the action defined by $\hat{Y}$. Then there exists a vector $\boldsymbol{c} = (2c_i) \in \mathbb{R}^n$ so that Equation (12) holds, and there exists $\boldsymbol{d} = (d_i) \in \mathbb{Z}_2^n$ so that Equation (13) holds. Let Equation (8) define $\rho : \mathfrak{g}_\alpha \to \mathfrak{g}_\beta$. It follows from Equation (9) that ρ is a homomorphism.

Now suppose that $\hat{Y}$ has maximal rank. Then the orbit of the $\mathbb{Z}_2^n$ action for any sign vector is all of $\mathbb{Z}_2^m$. It is elementary to show by reduction mod 2 that since $\hat{Y}$ has independent rows over $\mathbb{Z}_2$, Y has independent rows over $\mathbb{Z}$. Because Y has independent rows, the column space of Y is $\mathbb{R}^m$, and the orbit of the $\mathbb{R}^n$ action for any structure vector is all of $\mathbb{R}^m$. Therefore, so long as at least one element of $\mathcal{F}_\Lambda$ satisfies the Jacobi identity and defines a Lie algebra, any pair of elements of $\mathcal{F}_\Lambda$ with the same structure vector are isomorphic Lie algebras. ■

Remark 3.9. Orbits of the $\mathbb{R}^n$ actions are parametrized by elements of the left null space of the matrix Y. This fact does not carry over the the $\mathbb{Z}_2^n$ action defined by $\hat{Y}$, as next example will show.

We illustrate the previous theorem with a continuation of Example 3.2.

Example 3.10. Let $\mathfrak{n}$ and $\mathcal{B}$ be as in Example 3.2. The root matrix Y and the $\mathbb{Z}_2$ root matrix are given by 6×7 matrices over $\mathbb{R}$ and $\mathbb{Z}_2$ respectively. The ranks of these matrices are $\operatorname{rank}_{\mathbb{R}} Y = 5$ and $\operatorname{rank}_{\mathbb{Z}_2} \hat{Y} = 5$. We have nonzero structure constants

$$\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5, x_6) = (\alpha_{12}^4, \alpha_{13}^5, \alpha_{14}^6, \alpha_{16}^7, \alpha_{23}^6, \alpha_{34}^7).$$

The Jacobi identity translates to the constraint $x_4 x_5 + x_1 x_6 = 0$. One admissible choice of structure constants is

$$\boldsymbol{x}_0 = (1, 1, 1, 1, 1, -1).$$

We'd like to know which other elements in the family $\mathcal{F}_\Lambda$ are isomorphic to the nilpotent Lie algebra defined by this choice of structure constants. We will show that all nilpotent Lie algebras in $\mathcal{F}_\Lambda$ are isomorphic to the Lie algebra defined by the structure constants listed in $\boldsymbol{x}_0$.

As the eigenvalues of the common Nikolayevsky derivation are simple, by Corollary 3.7, any isomorphism $\phi : \mathfrak{n}_\alpha \to \mathfrak{n}_\beta$ between two elements in $\mathcal{F}_\Lambda$ must be of the form $\phi(E_i) = c_i E_i$ for some nonzero c_i, for all $i = 1, \ldots, 7$. Therefore, Theorem 3.8 applies.

Orbits for the action $\mathbb{R}^7 \cdot \mathbb{R}^6 \to \mathbb{R}^6$ defined by Y on vectors of form

$$(\ln(x_1^2), \ln(x_2^2), \ln(x_3^2), \ln(x_4^2), \ln(x_5^2), \ln(x_6^2))$$

are parametrized by the left null space of Y. The null space is spanned by $(1, 0, 0, -1, -1, 1)$, and the vector $t(1, 0, 0, -1, -1, 1)$ corresponds to the structure constants $(x_1, x_2, x_3, x_4, x_5, x_6)$ such that

$$(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2) = (e^t, 1, 1, e^{-t}, e^{-t}, e^t).$$

For a nontrivial vector in the left null space of Y, the value of t is nonzero and $x_4^2 x_5^2 = e^{-2t} \neq e^{2t} = x_1^2 x_6^2$. Hence a vector in the $\mathbb{R}^n$ orbit of $t(1, 0, 0, -1, -1, 1)$ will not satisfy the Jacobi identity unless $t = 0$, and only the $t = 0$ orbit is contained in $\mathcal{F}_\Lambda$.

Now we consider signs for vectors

$$(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2) = (1, 1, 1, 1, 1, 1)$$

in that $t = 0$ orbit. We know from the Jacobi Identity that $|x_4 x_5| = |x_1 x_6|$, whence it follows that the sign of $x_4 x_5$ is opposite that of $x_1 x_6$. The column space of $\hat{Y}$ is a 5-dimensional subspace of $\mathbb{Z}_2^6$ and an easy computation shows transversal for the $\mathbb{Z}_2^n$ action defined by $\hat{Y}$ is the set

$$\{(0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 1)\} \subseteq \mathbb{Z}_2^6.$$

(Here, the left null space for $\hat{Y}$ is a subspace of the column space, so it can not be a transversal for the $\mathbb{Z}_2^n$ action.) The Jacobi identity fails for all elements of the orbit of $(0,0,0,0,0,0)$, so only the orbit of $(0,0,0,0,0,1)$ is in $\mathcal{F}_\Lambda$.

Thus we have demonstrated our claim that there is exactly one isomorphism class of nilpotent Lie algebras in $\mathcal{F}_\Lambda$ represented by the elements $(0,0,0,0,0,0) \in \mathbb{R}^6$ and $(0,0,0,0,0,1) \in \mathbb{Z}_2^m$, corresponding to the structure constants in the vector $\boldsymbol{x}_0$ chosen at the outset.

3.3. Applications to classification of Lie algebras.

The Nikolayevsky derivation can be very useful in classification problems. An eigenvector basis gives an efficient presentation and the associated eigenvalues and index sets provide effective invariants. In this section, we justify these assertions by giving examples of the application of the Nikolayevsky derivation to problems arising in the classification of nilpotent Lie algebras of dimension 7.

First we review the state of the classification of nilpotent Lie algebras. Nilpotent Lie algebras of dimension 6 and less are classified ([4], [23], [34], [3]). Several classifications exist in dimension 7, based on different approaches ([30],[1],[9], [33], [12]). See also [21], [22]. Seeley presented a classification of nilpotent Lie algebras of dimension 7 in [33]; Gong found a few minor errors in the work of Seeley ([9]). To our knowledge, there are no errors in Seeley's classification after Gong's corrections are made.

Now we survey some of the existing methods for distinguishing nonisomorphic nilpotent Lie algebras. Dimensions and lengths from the upper and lower central series are often used as rough invariants. In other applications, more precise invariants are quite particular, such as the existence of a special basis with certain algebraic properties (rank of adjoint map, commutation with central subgroups, etc.). Orbits of group actions are often used to separate nonisomorphic nilpotent Lie algebras (e.g. [5], [8],[9]). Adjoint cohomology sequences may serve as effective invariants (See [21]). Scheuneman proved that a function related to the universal enveloping algebra is an isomorphism invariant for two-step nilpotent Lie algebras ([32].) Favre, Santharoubane, Goze, and Hakimjanov and others have used the structure of the derivation algebra of a complex nilpotent Lie algebra for classification problems ([7], [31], [11]).

The algebras in the classifications by Seeley and Gong are collected into subclasses based on the sequence $(d_1, \ldots, d_k)$ of dimensions of the central subgroups.

Example 3.11. Seeley and Gong find 18 nonisomorphic indecomposable nilpotent Lie algebras of type $(2,4,7)$. Seeley enumerates these as $(2,4,7)_A, (2,4,7)_B$, etc. He presents these with using bases with few nonzero structure constants. Most of these are bases for $\mathbb{R}$-split tori. In some cases, the eigenvectors for the Nikolayevsky derivation give a more efficient presentation than that used by Seeley ($(2,4,7)_E$ and $(2,4,7)_R$, for example). We computed the Nikolayevsky derivation in each case using Theorem 3.1, following the basic outline in Example 3.2. For easy reference, we rescaled the eigenvalues to be integral with no common factors, and we ordered them smallest to largest. The nilpotent Lie algebras along with

Label in [33]	Eigenvalues of D^N (rescaled and ordered)
$(2,4,7)_A$	$(1,4,4,5,5,6,6)$
$(2,4,7)_B$	$(6,11,15,17,21,27,28)$
$(2,4,7)_C$	$(11,20,29,31,40,42,51)$
$(2,4,7)_D$	$(7,10,12,17,19,24,29)$
$(2,4,7)_E$	$(3,5,5,8,8,11,13)$
$(2,4,7)_F$	$(5,5,6,11,11,16,16)$
$(2,4,7)_G$	$(20,21,22,42,43,62,64)$
$(2,4,7)_H$	$(1,1,1,2,2,3,3)$
$(2,4,7)_I$	$(7,10,11,17,21,24,28)$
$(2,4,7)_J$	$(7,7,10,14,17,21,24)$
$(2,4,7)_K$	$(5,6,7,11,12,17,18)$
$(2,4,7)_L$	$(5,10,11,15,16,20,21)$
$(2,4,7)_M$	$(11,22,30,33,41,52,55)$
$(2,4,7)_N$	$(15,19,23,34,38,42,53)$
$(2,4,7)_O$	$(2,3,4,5,6,7,8)$
$(2,4,7)_P$	$(1,1,1,2,2,2,3)$
$(2,4,7)_Q$	$(3,5,6,8,9,11,14)$
$(2,4,7)_R$	$(1,2,2,3,3,4,5)$

Table 1: Indecomposable nilpotent Lie algebras of type $(2,4,7)$ and the eigenvalue types of their Nikolayevsky derivations (from pages 485-6 of [33]).

their eigenvalue types are listed in Table 1.

Some of these derivations were first computed in [17]. No two Lie algebras in the table have the same eigenvalue lists, and it is immediately clear from this fact and Proposition 3.6 that none of the underlying Lie algebras can be isomorphic. In contrast, to separate the Lie algebras in the list, Seeley relies on finding unique commutation properties of special basis vectors for each Lie algebra, while Gong computes orbits for group actions with the assistance of MAPLE (page 114, [9]).

Even when two nilpotent Lie algebras have the same dimensions of central subgroups and the same eigenvalue type for the Nikolayevsky derivation, we still get an invariant through the index set, as we see in the next example.

Example 3.12. We consider the Lie algebras $(1,3,4,5,7)_C$ and $(1,3,4,5,7)_E$ from [33]. Relative to the standard basis $\mathcal{B} = \{E_i\}_{i=1}^7$ of $\mathbb{R}^7$, the Lie bracket for $(1,3,4,5,7)_C$ is described by

$$[E_1, E_2] = E_3 \qquad [E_1, E_3] = E_4 \qquad [E_1, E_4] = E_5$$

$$[E_1, E_6] = E_7 \qquad [E_2, E_5] = E_7 \qquad [E_3, E_4] = -E_7.$$

Relative to the basis $\mathcal{B} = \{E_i\}_{i=1}^7$, the Lie bracket for $(1,3,4,5,7)_E$ is given by

$$[E_1, E_2] = E_3 \qquad [E_1, E_3] = E_4 \qquad [E_1, E_4] = E_5 \qquad [E_2, E_3] = E_5$$

$$[E_1, E_6] = E_7 \qquad [E_2, E_5] = E_7 \qquad [E_3, E_4] = -E_7.$$

We claim the Nikolayevsky derivations for the two Lie algebras both have the same D^N-eigenvalue list $1, 2, 3, 4, 5, 6, 7$. For the first Lie algebra, the index set with respect to $\mathcal{B}$ is

$$\Lambda = \{(1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 6, 7), (2, 5, 7), (3, 4, 7)\}.$$

The root matrix Y is given by

$$Y = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{bmatrix}$$

and the Gram matrix U is given by

$$U = \begin{bmatrix} 3 & 0 & 1 & 1 & 1 & -1 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 1 & -1 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 0 & -1 & 1 & 3 & 1 \\ -1 & 0 & 1 & 1 & 1 & 3 \end{bmatrix}.$$

The solution space for $U\boldsymbol{v} = [1]_{6\times 1}$ is the set of all

$$\boldsymbol{v}_t = \tfrac{1}{10}(3, 4, 3, -2, 3, 3)^T + t(1, 0, -1, 0, -1, 1),$$

for $t \in \mathbb{R}$. There is no value of t for which the entries of $\boldsymbol{v}_t$ are all positive. One solution is $\boldsymbol{b} = \boldsymbol{v}_0$, and

$$\boldsymbol{v}_{D^N} = -\boldsymbol{b}^T Y + [1]_{1\times 7} = \tfrac{1}{5}(1, 2, 3, 4, 5, 6, 7).$$

By Theorem 3.1, the derivation D^N defined relative to $\mathcal{B}$ by the matrix $[D^N]_{\mathcal{B}} = \operatorname{diag}(\boldsymbol{v}_{D^N})$ is the Nikolayevsky derivation for the Lie algebra $(1, 3, 4, 5, 7)_C$.

For the second Lie algebra, the index set Λ with respect to $\mathcal{B}$ is

$$\Lambda = \{(1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 6, 7), (2, 3, 5), (2, 5, 7), (3, 4, 7)\}.$$

The root matrix Y is given by

$$Y = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{bmatrix}$$

and the Gram matrix U is given by

$$U = \begin{bmatrix} 3 & 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 3 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 3 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 & 3 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 & 3 \end{bmatrix}.$$

The solution set for the matrix equation $U\boldsymbol{v} = [1]_{7\times 1}$ is the set of all

$$\boldsymbol{v}_t = \tfrac{1}{10}(3, 4, 3, 0, -1, 3, 3)^T + t(1, 0, -1, 0, 0, -1, 1),$$

for $t \in \mathbb{R}$. Note that none of the vectors $\boldsymbol{v}_t$ have all entries positive. One solution is $\boldsymbol{b} = \boldsymbol{v}_0$, and

$$\boldsymbol{v}_{D^N} = -\boldsymbol{b}^T Y + [1]_{1\times 7} = \tfrac{1}{5}(1, 2, 3, 4, 5, 6, 7).$$

By Theorem 3.1, the derivation D^N defined relative to $\mathcal{B}$ by the matrix $[D^N]_{\mathcal{B}} = \mathrm{diag}(\boldsymbol{v}_{D^N})$ is the Nikolayevsky derivation for the Lie algebra.

We will see in Example 4.4 that neither Lie algebra admits a nilsoliton inner product since the vectors $\boldsymbol{v}_t$ for both Lie algebras never have positive entries for any value of t.

For both Lie algebras, $\mathcal{B}$ is an ordered eigenvector basis for the Nikolayevsky derivation, where the vector E_i has eigenvalue $\frac{i}{5}$, for $i = 1, \ldots, n$. Both Lie algebras have the same dimensions and lengths for both the lower- and upper-central series. However, the index sets for $(1, 3, 4, 5, 7)_C$ and $(1, 3, 4, 5, 7)_E$ relative to $\mathcal{B}$ clearly differ– one has cardinality 6 whereas the other has cardinality 7. Proposition 3.6 implies that the two Lie algebras are not isomorphic.

4. Geometry

4.1. Classification problems.

In Definition 2.1 we defined families of Lie algebras $\mathcal{F}_\Lambda$ relative to index sets Λ. In this section we present criteria for determining which elements of such a family admit nilsoliton inner products, up to equivalence. In the category of nilpotent Lie algebras, the best notion of equivalence is isometric isomorphism. We say that two nilpotent metric Lie algebras $(\mathfrak{n}_1, Q_1)$ and $(\mathfrak{n}_2, Q_2)$ are *isometrically isomorphic* if there is an isomorphism $\phi : \mathfrak{n}_1 \to \mathfrak{n}_2$ so $\phi^* Q_2 = Q_1$.

First we prove a lemma that follows in a straightforward way from Theorem 2.3.

Lemma 4.1. *Let $\Lambda \subseteq [n]^3$ be a nonempty index set of cardinality m, relative to the standard basis $\mathcal{B} = \{E_i\}_{i=1}^n$ of $\mathbb{R}^n$. Let $\mathcal{F}_\Lambda$ be as defined in Definition 2.1. Let U be the Gram matrix associated to Λ.*

1. *If some nilpotent Lie algebra in $\mathcal{F}_\Lambda$ admits a nilsoliton inner product with respect to which $\mathcal{B}$ is orthonormal, then there is a solution to $U\boldsymbol{v} = [1]_{m\times 1}$ which has all entries positive.*

2. *If the Gram matrix U has no entries of 2, then for all solutions $\boldsymbol{a} = [a_{ij}^k]_{(i,j,k)\in\Lambda}$ to $U\boldsymbol{v} = [1]_{m\times 1}$ with all positive entries, the element $(\mathfrak{n}_\alpha, Q)$ of $\mathcal{F}_\Lambda$ defined by taking $\alpha_{ij}^k = \pm\sqrt{a_{ij}^k}$ for all $(i,j,k) \in \Lambda$, and defining Q so that $\mathcal{B}$ is orthonormal, is a nilsoliton metric nilpotent Lie algebra in $\mathcal{F}_\alpha$, so long as the Jacobi identity and the nilpotency condition are satisfied.*

Proof. First we prove Part (1). Suppose that $\mathfrak{n}_\alpha$ in $\mathcal{F}_\Lambda$ admits a nilsoliton inner product with respect to which $\mathcal{B}$ is orthonormal. By definition of $\mathcal{F}_\Lambda$, the basis $\mathcal{B}$ consists of D^N-eigenvectors for some Nikolayevsky derivation D^N, and since the Nikolayevsky derivation D^N is a positive scalar multiple of the nilsoliton derivation $\hat{D}$, $\mathcal{B}$ consists of $\hat{D}$-eigenvectors. Therefore, since it diagonalizes D^N, the basis $\mathcal{B}$ also diagonalizes $\mathrm{Ric} = \hat{D} + \beta\,\mathrm{Id}$, hence is an orthonormal Ricci eigenvector basis for $(\mathfrak{n}_\alpha, Q)$. Let $\boldsymbol{a}$ be the structure vector for $\mathfrak{n}_\alpha$. By definition of $\mathcal{F}_\Lambda$, all entries of $\boldsymbol{a}$ are nonzero. By Theorem 2.3, $U\boldsymbol{a}$ is a scalar multiple of $[1]_{m\times 1}$. Thus $U\boldsymbol{v} = [1]_{m\times 1}$ has a solution with all entries positive.

Now we show that Part (2) holds. Suppose that U has no entries of 2, and that $\boldsymbol{a}$ is a solution to $U\boldsymbol{v} = [1]_{m\times 1}$ with all the entries of $\boldsymbol{a}$ are all positive. Define a Lie algebra $\mathfrak{n}_\alpha$ using the structure constants $\alpha_{ij}^k = \pm\sqrt{a_{ij}^k}$ for all $(i,j,k) \in \Lambda$, and define the inner product Q so that $\mathcal{B}$ is orthonormal. Assume that $\mathfrak{n}_\alpha$ is nilpotent. By Proposition 2.4, $\mathcal{B}$ is a Ricci eigenvector basis. If $\mathfrak{n}_\alpha$ satisfies the Jacobi identity, then it is in $\mathcal{F}_\Lambda$, and by Theorem 2.3, $(\mathfrak{n}_\alpha, Q)$ is a nilsoliton metric Lie algebra. ■

Now we are ready to state the main theorem of this section, a practical result that allows one to classify up to isomorphism, all those nilpotent Lie algebras in certain families $\mathcal{F}_\Lambda$ admitting nilsoliton inner products, and those in the family not admitting nilsoliton inner products. We use the notation $\mathcal{O}_{\boldsymbol{v}}$ as in Equation (6) to denote the orbit of a vector $\boldsymbol{v}$ under the $\mathbb{R}^n$ action defined in Section 2. We say that a subset $\mathcal{T}$ of $\mathcal{F}_\Lambda$ is a *global transversal* for the $\mathbb{R}^n \cdot \mathbb{R}^m \to \mathbb{R}^m$ action defined by the root matrix Y for $\mathcal{F}_\Lambda$ if for all structure vectors $\boldsymbol{v}$ for elements of $\mathcal{F}_\Lambda$, the orbit $\mathcal{O}_{\ln(\boldsymbol{v})}$ meets the set $\mathcal{T}$ exactly once.

Theorem 4.2. *Let $\Lambda \subseteq [n]^3$ be an index set of cardinality $m \geq 1$, and let $\mathcal{F}_\Lambda$ be as defined in Section 2, relative to the standard basis $\mathcal{B} = \{E_i\}_{i=1}^n$. Let Y be the root matrix associated to Λ, let U be the Gram matrix associated to Λ, and let $\hat{Y}$ be the $\mathbb{Z}_2$ root matrix associated to Λ. Suppose further that all of the eigenvalues of the Nikolayevsky derivation D^N are positive with multiplicity one.*

Let X_Λ equal the set of all $\boldsymbol{v} = (v_i) \in \mathbb{R}^m$ such that $U\boldsymbol{v} = [1]_{m\times 1}, v_i > 0$ for all i, and $\boldsymbol{v}$ is the structure vector for an element $\mathfrak{n}_\alpha$ of $\mathcal{F}_\Lambda$.

1. *Suppose $X_\Lambda \neq \emptyset$. Define the subset $\mathcal{S}_\Lambda$ of $\mathcal{F}_\Lambda$ to be all nilpotent Lie algebras $\mathfrak{n}_\alpha$ in $\mathcal{F}_\Lambda$ such that*

   (a) *The structure vector $\boldsymbol{v}$ for $\mathfrak{n}_\alpha$ is in X_Λ.*

   (b) *The sign vector $\boldsymbol{s}$ for $\mathfrak{n}_\alpha$ is in a transversal for the action $\mathbb{Z}_2^n \cdot \mathbb{Z}_2^m \to \mathbb{Z}_2^m$ defined by $\hat{Y}$.*

Then for every element of $\mathcal{S}_\Lambda$, the inner product $\hat{Q}$ for which $\mathcal{B}$ is orthonormal is nilsoliton. Furthermore, no two distinct elements of $\mathcal{S}_\Lambda$ are isomorphic. If in addition, the closed submanifold $\mathcal{T} = \{\ln(\boldsymbol{x}) : \boldsymbol{x} = (x_i) \in X_\Lambda\}$ is a global transversal for the $\mathbb{R}^n \cdot \mathbb{R}^m \to \mathbb{R}^m$ action defined by Y, then every element of $\mathcal{F}_\Lambda$ is isomorphic to a unique element of $\mathcal{S}_\Lambda$, hence every element of $\mathcal{F}_\Lambda$ admits a nilsoliton inner product.

2. *If $X_\Lambda = \emptyset$, then no element of $\mathcal{F}_\Lambda$ admits a nilsoliton inner product.*

   Let T be a subset of the set of structure vectors for elements of $\mathcal{F}_\Lambda$ relative to the basis $\mathcal{B}$, such that $\mathcal{T} = \{\ln(\boldsymbol{x}) : \boldsymbol{x} \in T\}$ is a global transversal for the $\mathbb{R}^n \cdot \mathbb{R}^m \to \mathbb{R}^m$ action defined by Y.

   Then the Lie algebras in $\mathcal{F}_\Lambda$ are parametrized up to isomorphism by the subset $\mathcal{A}_\Lambda$ of $\mathcal{F}_\Lambda$ consisting of all nilpotent Lie algebras $\mathfrak{n}_\alpha$ in $\mathcal{F}_\Lambda$ such that the structure vector $\boldsymbol{a}$ for $\mathfrak{n}_\alpha$ is in T, and the sign vector $\boldsymbol{s}$ is in a transversal for the action $\mathbb{Z}_2^n \cdot \mathbb{Z}_2^m \to \mathbb{Z}_2^m$ defined by $\hat{Y}$.

Remark 4.3. In the statement of Part (1) of the theorem we require that the closed submanifold $\mathcal{T}$ of $\mathbb{R}^m$ meet every leaf of the foliation of $\mathbb{R}^m$ by planes parallel to the column space $\operatorname{col} U$ exactly once; i.e. $\mathcal{T}$ is a global section for the foliation. This condition holds trivially when U is nonsingular and $\operatorname{col} U = \mathbb{R}^m$.

Proof. Suppose that $D^N(E_i) = \lambda_i E_i$, for $i = 1, \ldots, n$, where $0 < \lambda_1 < \cdots < \lambda_n$. All Lie algebras in $\mathcal{F}_\Lambda$ admit a nonsingular derivation, so by Jacobson's theorem, all are nilpotent. By Lemma 2.5, the Gram matrix for Λ has no entries of 2.

Let us first consider Part (1) of the theorem. Suppose that there is a vector $\boldsymbol{v} = (v_i)$ in X_Λ with $U\boldsymbol{v} = [1]_{m\times 1}$ with $v_i^0 > 0$ for all i, and that $\boldsymbol{v}$ is the structure vector for $\mathfrak{n}_\alpha \in \mathcal{F}_\Lambda$. By Part (2) of Lemma 4.1, $(\mathfrak{n}_\alpha, Q)$ is a nilsoliton metric Lie algebra.

Now we show that if two elements $\mathfrak{n}_\alpha$ and $\mathfrak{n}_\beta$ of $\mathcal{S}_\Lambda$ are isomorphic, then their structure vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are the same and their sign vectors $\boldsymbol{s_a}$ and $\boldsymbol{s_b}$ are in the same orbit for the $\mathbb{Z}_2^n$ action on sign vectors. Suppose that $\phi : \mathfrak{n}_\alpha \to \mathfrak{n}_\beta$ is an isomorphism. By Corollary 3.7, there exist nonzero constants $c_1, \ldots, c_n$ so that $\phi(E_i) = c_i E_i$ for all $i = 1, \ldots, n$. The pushforward $\phi_* Q$ of Q is a nilsoliton inner product on $\mathfrak{n}_\beta$. Lauret showed that nilsoliton inner products are unique up to scaling, so $\phi_* Q$ is a scalar multiple of Q. Say $\phi_* Q = cQ$ for $c > 0$. Then for all vectors $E_i, i = 1, \ldots, n$, in the orthonormal basis, we have

$$\begin{aligned} c = cQ(E_i, E_i) &= (\phi_* Q)(E_i, E_i) \\ &= Q(\phi(E_i), \phi(E_i)) \\ &= Q(c_i E_i, c_i E_i) \\ &= c_i^2 Q(E_i, E_i i) \\ &= c_i^2. \end{aligned}$$

Hence $\phi(X) = cX$ for all X.

Then the structure vector $\boldsymbol{a}$ is a nonzero multiple of the structure vector $\boldsymbol{b}$. But since $U\boldsymbol{a} = [1]_{m\times 1}$ and $U\boldsymbol{b} = [1]_{m\times 1}$, that constant must be 1 and $\boldsymbol{a} = \boldsymbol{b}$. By Proposition 2.6, since $\mathfrak{n}_\alpha$ and $\mathfrak{n}_\beta$ are isomorphic, their sign vectors are in the same orbit for the $\mathbb{Z}_2^n$ action. Therefore, no two distinct nilpotent Lie algebras in the set $\mathcal{S}_\Lambda$ are isomorphic, as claimed.

By Theorem 3.8, the orbit $\mathcal{O}_{\ln(\boldsymbol{v})}$ meets $\mathcal{T}$ exactly once for all $\boldsymbol{v}$, if and only if each element $\mathfrak{n}_\alpha$ of $\mathcal{F}_\Lambda$ is isomorphic to precisely one element of $\mathcal{S}_\Lambda$.

Now consider Case (2). By Lemma 4.1, no Lie algebras in $\mathcal{F}_\Lambda$ admit a nilsoliton inner product. Again by Theorem 3.8, the orbit $\mathcal{O}_{\ln(\boldsymbol{v})}$ meets $\mathcal{T}$ exactly once for all $\boldsymbol{v}$, if and only if each element $\mathfrak{n}_\alpha$ of $\mathcal{F}_\Lambda$ is isomorphic to precisely one element of $\mathcal{A}_\Lambda$. ■

Example 4.4. Let the Lie algebra $(1, 3, 4, 5, 7)_C$ be as defined relative to the basis $\mathcal{B}$ as in Example 3.12. In Example 3.12 we found the index set Λ, the root matrix Y, and the Gram matrix U with respect to $\mathcal{B}$, and we saw that all of the eigenvalues of the Nikolayevsky derivation D^N have multiplicity one. Let $\mathfrak{t}^{\mathbb{R}}$ be an $\mathbb{R}$-split torus containing D^N. As D^N is simple, the only bases that diagonalize it are $\mathcal{B}$ and bases obtained from $\mathcal{B}$ be rescaling elements. Hence $\mathcal{B}$ diagonalizes $\mathfrak{t}^{\mathbb{R}}$ and the Lie algebra $(1, 3, 4, 5, 7)_C$ is in $\mathcal{F}_\Lambda$. It was shown that the equation $U\boldsymbol{v} = [1]_{6\times 1}$ has no solutions with all positive entries. Hence, by Theorem 4.2, the Lie algebra $(1, 3, 4, 5, 7)_C$ does not admit a nilsoliton inner product.

An analogous argument shows that the Lie algebra $(1, 3, 4, 5, 7)_E$ does not admit a nilsoliton inner product.

4.2. Isometry groups.

Let $(\mathfrak{g}, Q)$ be a metric Lie algebra. Gorbatsevich considered the questions of when $\mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$ is finite, and which possible finite groups $\mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$ might be for given $\mathfrak{g}$. He showed that when $(\mathfrak{g}, Q)$ is a filiform metric nilpotent Lie algebra, the group $\mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$ is isomorphic to a subgroup of $\mathbb{Z}_2^2$ (Theorem 3, [10]). The following corollary yields a similar type of result for Lie algebras whose Nikolayevsky derivation has positive eigenvalues, all of multiplicity one.

Corollary 4.5. *Let $(\mathfrak{g}, Q)$ be a metric Lie algebra. Suppose that the Nikolayevsky derivation D^N is symmetric with respect to Q. If $\lambda_1, \ldots, \lambda_k$ denote the distinct eigenvalues of the restriction of D^N to $[\mathfrak{g}, \mathfrak{g}]^\perp$, and these eigenvalues have corresponding multiplicities $m_1, \ldots, m_k$, then the isometry group of $(\mathfrak{g}, Q)$ imbeds isomorphically in the product $O(m_1) \times \cdots \times O(m_k)$ of orthogonal groups.*

In particular, if the eigenvalues for the restriction of the Nikolayevsky derivation to $[\mathfrak{g}, \mathfrak{g}]^\perp$ all have multiplicity one, then $\mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$ is finite and is isomorphic to a subgroup of $\mathbb{Z}_2^k$, where k is the codimension of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$.

Proof. An isometric isomorphism $\phi \in \mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q)$ preserves $[\mathfrak{g}, \mathfrak{g}]$ and $[\mathfrak{g}, \mathfrak{g}]^\perp$. Hence, the decomposition of $\mathfrak{g}$ into eigenspaces for D^N is compatible with the decomposition $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]^\perp \oplus [\mathfrak{g}, \mathfrak{g}]$ in the sense that $[\mathfrak{g}, \mathfrak{g}]^\perp$ may be written as the direct sum

$$[\mathfrak{g},\mathfrak{g}]^\perp = \bigoplus_{i=1}^{k} \left([\mathfrak{g},\mathfrak{g}]^\perp \cap E(\lambda_i)\right),$$

where $E(\lambda_i)$ is the λ_i-eigenspace for D^N for $i = 1,\ldots,k$. As ϕ is an automorphism, the restriction of ϕ to $[\mathfrak{g},\mathfrak{g}]^\perp$ completely determines ϕ on all of $\mathfrak{g}$. Corollary 3.5, Part (2) implies that ϕ fixes $E(\lambda_i)$, and hence $[\mathfrak{g},\mathfrak{g}]^\perp \cap E(\lambda_i)$, for all i. Thus, for $i = 1,\ldots,k$, the restriction of the isometry ϕ to the m_i-dimensional subspace $[\mathfrak{g},\mathfrak{g}]^\perp \cap E(\lambda_i)$ may be represented as an element of the orthogonal group $O(m_i)$ relative to an orthonormal eigenvector basis. In the case that all the eigenspaces are one-dimensional, $O(m_i) = O(1) \cong \mathbb{Z}_2$ for all i, and

$$\mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Isom}(Q) \subseteq O(m_1) \times \cdots \times O(m_k) \cong \mathbb{Z}_2^k$$

as claimed. ■

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Tracy L. Payne
Department of Mathematics
Idaho State University
921 S. 8th Ave.
Pocatello, ID 83209-8085, USA
payntrac@isu.edu

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