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Equivalence of Characters in Deformation Quantization and Lie Theory

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Abstract. Let α_f be the Penney distribution associated to an element $f \in \mathfrak{g}^*$, where $\mathfrak{g}$ is a nilpotent Lie algebra. We prove that the analytical character of α_f coincides with the biquantization character of the zero degree cohomology of the Cattaneo-Felder A_∞ algebra in the linear case.
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1. Introduction.

1.1. Harmonic analysis on Lie groups is closely related to deformation quantization after the fundamental work of Kontsevich in [Kon03]. Among other things he proved an 1-1 correspondence between gauge equivalence classes of $*-$ products and gauge equivalence classes of Poisson structures $\{\cdot,\cdot\}$ on $\mathbb{R}^k$. Fixing such a structure, he also provided an explicit formula for the corresponding product $*_K$. In particular, considering the linear Poisson manifold $\mathfrak{g}^*$, where $\mathfrak{g}$ is a Lie algebra, he showed that there is an algebra isomorphism $(S(\mathfrak{g}), *_K) \simeq (U(\mathfrak{g}), \cdot)$, where $S(\mathfrak{g}), U(\mathfrak{g})$ are the symmetric and universal enveloping algebra respectively, of $\mathfrak{g}$. Furthermore, he reinterpreted Duflo's isomorphism, a major result in representation theory of Lie algebras, as a consequence of his Formality Theorem writing $(S(\mathfrak{g})^G, *_K) \simeq \mathcal{Z}(\mathfrak{g})$, where $\mathcal{Z}(\mathfrak{g})$ is the center of $U(\mathfrak{g})$ and G is the Lie group of $\mathfrak{g}$. The connection with harmonic analysis on Lie groups, comes from [KV78] where $U(\mathfrak{g})$ is considered as the distributions supported at the identity $e \in G$ and $S(\mathfrak{g})$ as the distributions supported at $0 \in \mathfrak{g}^*$. It is thus natural to address analytical problems on the group G using Kontsevich's deformation quantization techniques on the linear Poisson manifold $\mathfrak{g}^*$.

1.2. The goal of this paper is to compare a character constructed in the setting of harmonic analysis on Lie groups to a character from deformation quantization theory of linear Poisson manifolds. We prove that these characters, the character of the Penney vectors and the biquantization character, coincide on a certain algebra. By this we mean that we also prove isomorphisms of algebras appearing on

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the two sides. We briefly describe these characters below, the details are given in sections 2 and 3.

To define the analytic character, let G be a real nilpotent, connected and simply connected Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra with associated subgroup H, and $\lambda \in \mathfrak{h}^*$ a character of $\mathfrak{h}$. Set $\mathfrak{h}^\perp := \{l \in \mathfrak{g}^* \mid l(\mathfrak{h}) = 0\}$ and let $\mathfrak{b}$ be a polarization of $f \in \mathfrak{h}^\perp_\lambda := -\lambda + \mathfrak{h}^\perp$, that is a subalgebra of $\mathfrak{g}$ which is an isotropic subspace of maximal dimension for the Kostant form $f([\cdot,\cdot])$. Let B be the associated group of $\mathfrak{b}$ and $U_{\mathbb{C}}(\mathfrak{g})$ be the complexification of $U(\mathfrak{g})$. Throughout the text, $U(\mathfrak{L})\mathfrak{m}_\rho$ will denote the ideal of $U(\mathfrak{L})$ generated by elements of the form $\{X + \rho(X)\}$ for $X \in \mathfrak{m}$ a subalgebra of a Lie algebra $\mathfrak{L}$ and ρ a character of $\mathfrak{m}$. We denote by $(U(\mathfrak{L})/U(\mathfrak{L})\mathfrak{m}_\rho)^{\mathfrak{m}}$ the ad$\mathfrak{m}-$ invariants of the quotient space. Let now χ_f be the unitary character of B associated to f and denote as $C^\infty_c(G, B, \chi_f)$ the compactly supported, χ_f- invariant functions on G/B i.e functions $\phi : G \longrightarrow \mathbb{C}$ satisfying $\phi(gb) = (\chi_f(b))^{-1}\phi(g)$, $\forall b \in B, \forall g \in G$. Set then $\mathcal{H}^\infty_f$ to be the space of $C^\infty-$ functions of $L^2(G, B, \chi_f)$, the separable completion of $C^\infty_c(G, B, \chi_f)$ with respect to the appropriate L^2 norm. There is a natural action of $\mathfrak{g}$ on $\mathcal{H}^\infty_f$, which can be extended to an action of $U(\mathfrak{g})$, denoted by $\mathrm{d}\tau^\infty_f$. With these data, one defines an antilinear form $\alpha_f \in \mathcal{H}^{-\infty}_f$ on H^∞_f, setting $\langle \alpha_f, \phi \rangle = \int_{H/H\cap B} \overline{\phi(h)\chi_\lambda(h)} \mathrm{d}_{H/H\cap B}(h)$. This vector is called the *Penney vector* and it is H -semi-invariant. Because of this, the algebra $(U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{-if})^{\mathfrak{h}}$ acts on α_f and under suitable conditions, there is a character $\lambda_f : (U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{if})^{\mathfrak{h}} \longrightarrow \mathbb{C}$ such that $\mathrm{d}\tau^{-\infty}_f(\overline{A})(\alpha_f) = \overline{\lambda_f(A)}\alpha_f$. To define this character, as well as to state most of the results in this paper one requires the following *Lagrangian condition*:

$$\exists \mathcal{O} \subset \mathfrak{h}^\perp_\lambda, \text{ a non-empty Zariski-open set, such that} \tag{1}$$

$$\forall l \in \mathcal{O},\ \dim(\mathfrak{h} \cdot l) = \frac{1}{2} \dim(\mathfrak{g} \cdot l)$$

where the dot stands for the ad$\mathfrak{g}-$ action on $\mathfrak{g}^*$. Pointwisely, we will say that an $l \in \mathfrak{g}^*$ satisfies the Lagrangian condition (with respect to $\mathfrak{h}$) if $\dim(\mathfrak{h} \cdot l) = \frac{1}{2}\dim(\mathfrak{g} \cdot l)$. Alternatively and depending on the context, we will call $\mathfrak{h}$ *Lagrangian* with respect to l if it is simultaneously isotropic and coisotropic with respect to the Kostant form $l([\cdot,\cdot])$, that is, if it satisfies $l([\mathfrak{h},\mathfrak{h}]) = 0$ and $l([\mathfrak{h}^l, \mathfrak{h}^l]) = 0$, for $\mathfrak{h}^l = \{X \in \mathfrak{g} \mid\ l([X, Y]) = 0,\ \forall Y \in \mathfrak{h}\}$.

1.3. For the biquantization character one considers the deformation quantization theory of the linear Poisson manifold $X = \mathfrak{g}^*$, taking into account the existence of the coisotropic manifold $C = \mathfrak{h}^\perp_\lambda$. This case was studied in [CT08] following [Kon03],[CF04][CF07]. The main object of study is the degree 0 cohomology of a flat $A_\infty-$ algebra, equipped with a $*-$ product constructed following Kontsevich's techniques, the *Cattaneo-Felder product* $*_{CF}$. Fixing $\mathfrak{q}$ a complementary space of $\mathfrak{h}$ in $\mathfrak{g}$, the elements of this quantized algebra, called the *reduction algebra* and denoted by $H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}})$, are the solutions $F \in S(\mathfrak{q})[\epsilon]$ of the equation $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}}(F) = 0$, where $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}}$ is a differential described entirely with Kontsevich graphs in section 2.3. The biquantization character is a real character $\gamma_{CT} : H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}}) \longrightarrow \mathbb{R}[\epsilon]$ constructed in [CT08] (see also [Tor11]).

1.4. Set $T_{(\epsilon)}(\mathfrak{g}) = \mathbb{R}[\epsilon] \otimes T(\mathfrak{g})$ to be the deformed tensor algebra of $\mathfrak{g}$, $\mathcal{I}_\epsilon$ be its ideal generated by elements of the form $\{X\otimes Y - Y\otimes X - \epsilon[X,Y], X, Y \in \mathfrak{g}\}$, and set $U_{(\epsilon)}(\mathfrak{g}) = T_{(\epsilon)}(\mathfrak{g})/\mathcal{I}_\epsilon$. If $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$, define similarly, $S_{(\epsilon)}(\mathfrak{g}) = T_{(\epsilon)}(\mathfrak{g})/\langle X\otimes Y - Y\otimes X, X, Y \in \mathfrak{g}\rangle$. One of the main results of [Bat09], [Bat13] is that for every Lie algebra $\mathfrak{g}$, the reduction algebra is isomorphic to $\left(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_{\lambda+\rho}\right)^{\mathfrak{h}}$, where $\rho \in \mathfrak{h}^*$ is defined by $\rho(H) = -\omega_{\Gamma'}\mathrm{Tr}(\mathrm{ad}H)$, $H \in \mathfrak{h}$, and $\omega_{\Gamma'} \in \mathbb{R}$ is a certain Kontsevich coefficient. In the original version of this theorem, the character ρ is missing due to the fact that short loops were missing at the time from the original construction in [CF07] and [CT08], while now this is fixed by [CRT11]. In this paper however, we compare the two characters in the case where $\mathfrak{g}$ is nilpotent, so this isomorphism will be used here in the form

$$H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}}) \simeq \left(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda\right)^{\mathfrak{h}}. \tag{2}$$

1.5. To state and prove the results in this paper, we review Kontsevich's techniques in deformation quantization of a Poisson manifold X in Section 2.1. Section 2.2 contains the extension of these techniques in the presence of a coisotropic submanifold, and considers simultaneously the case $X = \mathfrak{g}^*$. The main objects of study, the reduction algebras and the biquantization diagrams, are defined in Section 2.3. The calculation of the biquantization character in our framework is done in Theorem 2.7. It constructs a character for the associative algebra $(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_f)^{\mathfrak{h}}$ and explains how the Lagrangian condition (1) permits us to construct many characters for this algebra. Section 3.1 is devoted in fixing the notation and framework for our analytic arguments. It defines the Penney vector α_f in the complex case, and recalls the analytic character $\lambda_l : (U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{il})^{\mathfrak{h}} \longrightarrow \mathbb{C}$ of this algebra. A problem appearing already is that the analytic character is complex, while the biquantization theory works over $\mathbb{R}$. We thus need to construct a real analytic character corresponding to λ_l. This is done in Section 3.2 and Theorem 3.3. This new character is using a real distribution $\alpha(f)$ and is defined only on a subalgebra $\mathcal{A}$ of $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$. This subalgebra is defined using a suitable deformation of $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$ and it is isomorphic to the specialization $H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}})$. In Section 4.1 we prove three Lemmata, 4.1, 4.2 and 4.3, that significally reduce the proof of the main result of this paper, Theorem 4.4 in Section 4.2. It states that the real analytic character of Theorem 3.3 coincides on $H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}})$ with the biquantization character $\gamma^{(\epsilon=1)}_{CT} : H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}}) \longrightarrow \mathbb{R}$ of Theorem 2.7.

2. Deformation Quantization.

2.1. General case. In [Kon03] Kontsevich solved the deformation quantization problem of Poisson manifolds, proving his Formality Theorem for the $L_\infty-$ algebras $\mathcal{T}_{poly}(\mathbb{R}^k)$ of polyvector fields and $\mathcal{D}_{poly}(\mathbb{R}^k)$ of polydifferential operators of bounded order on $\mathbb{R}^k$. The result states that choosing a Poisson structure $\{\cdot,\cdot\}$ on $\mathbb{R}^k$, the map $\mathcal{U} : \mathcal{T}_{poly}(\mathbb{R}^k) \longrightarrow \mathcal{D}_{poly}(\mathbb{R}^k)$ defined by its Taylor coefficients

$$\mathcal{U}_n := \sum_{\overline{m}\geq 0}\left(\sum_{\Gamma\in\mathbf{Q}_{n,\overline{m}}} \omega_\Gamma B_\Gamma\right) \tag{3}$$

is an $L_\infty-$ morphism and a quasi-isomorphism. Properties of this map prove that there is a bijection between the gauge equivalence classes of $*-$ products on $C^\infty(\mathbb{R}^k)$ and the gauge equivalence classes of Poisson structures on $\mathbb{R}^k$. Kontsevich also provided an explicit formula of the $*-$ product, denoted by $*_K$, associated to a Poisson structure. Letting ϵ be a deformation parameter, the operator $*_K : C^\infty(\mathbb{R}^k)[[\epsilon]] \times C^\infty(\mathbb{R}^k)[[\epsilon]] \longrightarrow C^\infty(\mathbb{R}^k)[[\epsilon]]$ defined for $f, g \in C^\infty(\mathbb{R}^k)$ by the formula

$$f *_K g := fg + \sum_{n=1}^{\infty} \epsilon^n \left(\frac{1}{n!} \sum_{\Gamma \in \mathbf{Q_{n,2}}} \omega_\Gamma B_\Gamma(f,g) \right) , \tag{4}$$

is an associative product. This formula was later globalised for a Poisson manifold X in [CFT02]. We explain its ingredients directly to our case, that is $X = \mathfrak{g}^*$, and $\{\cdot,\cdot\} = [\cdot,\cdot]$, the Lie bracket. The set $\mathbf{Q}_{n,2}$ denotes the *admissible graphs* Γ defined as follows: Let $V(\Gamma)$ be the set of vertices of Γ. It is the disjoint union of two ordered sets $V_1(\Gamma)$ and $V_2(\Gamma)$, isomorphic to $\{1,\dots,n\}$ and $\{\overline{1},\overline{2}\}$ respectively. The elements of $V_1(\Gamma)$ are called *type I* vertices, and the elements of $V_2(\Gamma)$ are called *type II* vertices. The set $E(\Gamma)$ of edges of Γ is finite and its elements are oriented. The set $S(r)$ of edges leaving from $r \in V_1(\Gamma)$ is ordered and has two elements, $S(r) = \{e_r^1, e_r^2\}$, while no loops or double edges are allowed and no edge leaves from a type II vertex. The set $E(\Gamma)$ is ordered in a compatible way with the orders in $V_1(\Gamma)$ and $S(r)$.
If $\{x_1,\dots,x_k\}$ is a basis of $\mathfrak{g}$, we associate a differential operator B_Γ to a graph $\Gamma \in \mathbf{Q}_{n,2}$ in the following way: Let $L: E(\Gamma) \longrightarrow [1,k] := \{1,\dots,k\}$ be a label function of the edges of the graph. To a type I vertex $r \in [1,n]$ associate the bracket $[x_{L(e_r^1)}, x_{L(e_r^2)}]$ and to each type II vertex, associate respectively a function $F, G \in C^\infty(\mathfrak{g}^*) \simeq S(\mathfrak{g})$. To the $p^{th}-$ edge of $S(r)$, associate the partial derivative $\partial_{L(e_r^p)}$ with respect to the basis variable $x_{L(e_r^p)}$. This derivative acts on the function associated to $w \in V_1(\Gamma) \cup V_2(\Gamma)$ where the edge e_r^p arrives. Let $(p,m) \in E(\Gamma)$ represent an oriented edge from p to m. Then the bidifferential operator associated to Γ is

$$B_\Gamma(F,G) = \sum_{L:E(\Gamma)\to[1,k]} \left[\prod_{r\in V_1(\Gamma)} \left(\prod_{\delta\in E(\Gamma),\, \delta=(\cdot,r)} \partial_{L(\delta)} \right) [x_{L(e_r^1)}, x_{L(e_r^2)}] \right] \times \tag{5}$$

$$\times \left(\prod_{\delta\in E(\Gamma),\, \delta=(\cdot,\overline{1})} \partial_{L(\delta)} \right)(F) \times \left(\prod_{\delta\in E(\Gamma),\, \delta=(\cdot,\overline{2})} \partial_{L(\delta)} \right)(G).$$

The last ingredient for (4) is the coefficient ω_Γ. Let $\mathcal{H} = \{z \in \mathbb{C} \mid \mathfrak{Im}(z) \geq 0\}$ be the upper-half plane and let $\mathcal{H}^+ = \{z \in \mathbb{C} \mid \mathfrak{Im}(z) > 0\}$. Embed an admissible graph Γ in $\mathcal{H}$ by putting the type II vertices on the real axis and letting the type I vertices move in $\mathcal{H}^+$. Let $\widehat{C}_{n,\overline{2}}$ be the configuration manifold of n type I and two type II vertices, invariant under horizontal translations and dilations. Set then $\widehat{C}^+_{n,\overline{2}} := \{(z_1,\dots,z_n,z_{\overline{1}},z_{\overline{2}}) \in \widehat{C}_{n,\overline{2}} \mid z_{\overline{1}} < z_{\overline{2}}\}$. Consider now the manifold $\widehat{C}_{2,0}$ and the so called *angle map*,

$$\phi : \widehat{C}_{2,0} \longrightarrow \mathbb{R}/2\pi\mathbb{Z},\ (z_1,z_2) \mapsto \mathrm{Arg}(\langle z_1, +i\infty\rangle, \langle z_1, z_2\rangle),$$

where $\langle z_1, +i\infty\rangle$ stands for the geodesic passing by z_1 and $+i\infty$, and $\langle z_1, z_2\rangle$ stands for the geodesic passing by z_1 and z_2. For $e = (z_i, z_j) \in E(\Gamma)$, let $p_e : \widehat{C}_{n,\overline{2}} \longrightarrow \widehat{C}_{2,0}$, $(z_1, \ldots, z_n, z_{\overline{1}}, z_{\overline{2}}) \mapsto (z_i, z_j)$ be the natural projection. One then defines the form $\mathrm{d}\phi_e := p_e^*(\mathrm{d}\phi) \in \Omega^1(\widehat{C}_{n,\overline{2}})$ and Ω_Γ to be the form $\Omega_\Gamma := \bigwedge_{e\in E(\Gamma)} \mathrm{d}\phi_e$. Then the Kontsevich coefficient is $\omega_\Gamma := \frac{1}{(2\pi)^{2n}} \int_{\widehat{C}^+_{n,\overline{2}}} \Omega_\Gamma$.

2.2. Biquantization. In [CF04] and [CF07], the authors expanded Kontsevich's Formality Theorem for the case of one coisotropic submanifold C of a Poisson manifold X. We adjust their results in our setting in the sense of [CT08]. Let $(X, \{\cdot,\cdot\})$ be a Poisson manifold and $C \subset X$ a submanifold. Then C is called *coisotropic* if the ideal $I(C) \subset C^\infty(X)$ of functions vanishing on C is a Poisson subalgebra of $C^\infty(X)$. It is clear that the annihilator $\mathfrak{h}^\perp$ of a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and, trivially, $\mathfrak{g}^*$, are coisotropic submanifolds of $X = \mathfrak{g}^*$. Let $\mathfrak{q}$ be a complementary space of $\mathfrak{h}$, that is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Let also $\{H_1, H_2, \ldots, H_t\}$ be a basis for $\mathfrak{h}$ and $\{Q_1, \ldots, Q_r\}$ a basis for $\mathfrak{q}$. We identify spaces $\mathfrak{q}^* \simeq \mathfrak{g}^*/\mathfrak{h}^* \simeq \mathfrak{h}^\perp$. By §2.1, to each edge of a graph Γ is associated a partial derivative with respect to a basis variable. We need now to distinguish from which part of the basis, the one of $\mathfrak{h}$ or the one of $\mathfrak{q}$, this variable comes. We thus consider an appropriate partition in two sets of basis vectors. One then says that there are two *colors* in the basis, and graphs labeled by such a basis are called *2-colored*. The corresponding set of graphs with n type I and two type II vertices satisfying the conditions of Section 2.1, will be denoted by $\mathbf{Q}^{(2)}_{n,2}$. For simplicity one associates a sign to each color, say $(-)$ for edges carrying derivatives ∂_{H_s}, and $(+)$ for derivatives ∂_{Q_l}. Graphically, the color $(-)$ will be represented with a dotted edge and the color $(+)$ will be represented with a straight edge (see Figure 1). To a 2-colored Γ is associated a 2-colored 1-form Ω_Γ and a 2-colored coefficient ω_Γ setting $\phi_+ : \widehat{C}_{2,0} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$ to be the function $\phi_+(z_1, z_2) := \mathrm{Arg}(z_1 - z_2) + \mathrm{Arg}(z_1 - \overline{z_2})$ and $\phi_- : \widehat{C}_{2,0} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$ be the function $\phi_-(z_1, z_2) := \mathrm{Arg}(z_1 - z_2) - \mathrm{Arg}(z_1 - \overline{z_2})$ (the bar here stands for the complex conjugate). The form Ω_Γ of a 2-colored graph Γ is similarly defined as $\Omega_\Gamma := \wedge_{e\in E(\Gamma)} \mathrm{d}\phi_{\cdot,e}$ where $\mathrm{d}\phi_{+,e} = p_e^*(\mathrm{d}\phi_+)$, $\mathrm{d}\phi_{-,e} = p_e^*(\mathrm{d}\phi_-)$, when $e \in E(\Gamma)$ has color $(+)/(-)$ respectively, and the 2-colored coefficient is $\omega_\Gamma := \frac{1}{(2\pi)^{2n}} \int_{\widehat{C}^+_{n,2}} \Omega_\Gamma$. Then the formula (5) of B_Γ in this 2-colored case has to be modified: For $e \in E(\Gamma)$, let $c_e \in \{+, -\}$ be its color. Let $L : E(\Gamma) \longrightarrow \{1, \ldots, t, t+1, \ldots t+r\}$, where $t = \dim(\mathfrak{h}), r = \dim(\mathfrak{q})$, satisfying $L(e) \in \{1, \ldots, t\}$ if $c_e = -$ and $L(e) \in \{t+1, \ldots, t+r\}$ if $c_e = +$, be a 2-colored label function. For $F, G \in S(\mathfrak{q}) \simeq C^\infty(\mathfrak{g}^*/\mathfrak{h}^\perp)$, the formula of *2-colored* bidifferential operators B_Γ is the same as in (5) but using the 2-colored label function L and the family $\mathbf{Q}^{(2)}_{n,2}$. The corresponding associative $*-$ product is $*_{CF,\epsilon} : S(\mathfrak{q})[\epsilon] \times S(\mathfrak{q})[\epsilon] \longrightarrow S(\mathfrak{q})[\epsilon]$ given by the formula $F *_{CF,\epsilon} G := F \cdot G + \sum_{n=1}^\infty \epsilon^n \left(\frac{1}{n!} \sum_{\Gamma\in\mathbf{Q^{(2)}_{n,2}}} \omega_\Gamma B_\Gamma(F, G)\right)$ and will be called the *Cattaneo-Felder* product.

2.3. Reduction algebras. If the basis of $\mathfrak{g}$ is separated in more than two parts, say k, we denote the set of $k-$ colored graphs in the product as $\mathbf{Q}^{(k)}_{n,2}$. If the number of colors is irrelevant, we refer to a graph simply as *colored*. We now specify some special colored graphs that we will use (see [CT08] § 1.3, 1.6 and [Bat09] § 2.3). They are 2-colored graphs, not in $\mathbf{Q}^{(2)}_{n,2}$ because they have an edge with no end, colored as $(-)$ and only one type II vertex. We'll say that edge with no end "points to ∞", and denote it by e_∞. To the type II vertex one again associates a function $F \in S(\mathfrak{q})[\epsilon]$. These are the graphs of the first and the third part of the next definition. Denote this family of graphs as $\mathbf{Q}^\infty_{n,1}$.

Definition 2.1. 1. **Bernoulli.** The Bernoulli type graphs with i type I vertices, $i \in \mathbb{N}, i \geq 2$, will be denoted by $\mathcal{B}_i$. They have $2i$ edges, i of them pointing to the type II vertex, and have an edge e_∞. These conditions imply the existence of a vertex $s \in V_1(\Gamma)$ that receives no edge, called the *root*.

2. **Wheels.** The wheel type graphs with i type I vertices, $i \in \mathbb{N}, i \geq 2$, will be denoted by $\mathcal{W}_i$. They have $2i$ edges i of them pointing to the type II vertex, and do not have an e_∞.

3. **Bernoulli attached to a wheel.** Graphs of this type with i type I vertices, $i \in \mathbb{N}, i \geq 4$, will be denoted by $\mathcal{BW}_i$. They have $i-1$ edges towards the type II vertex and an e_∞. For a $\mathcal{W}_m-$ type graph W_m attached to a $\mathcal{B}_l-$ type graph B_l, we will write $B_l W_m \in \mathcal{B}_l\mathcal{W}_m$. Obviously $\mathcal{B}_l\mathcal{W}_m \subset \mathcal{BW}_{l+m}$.

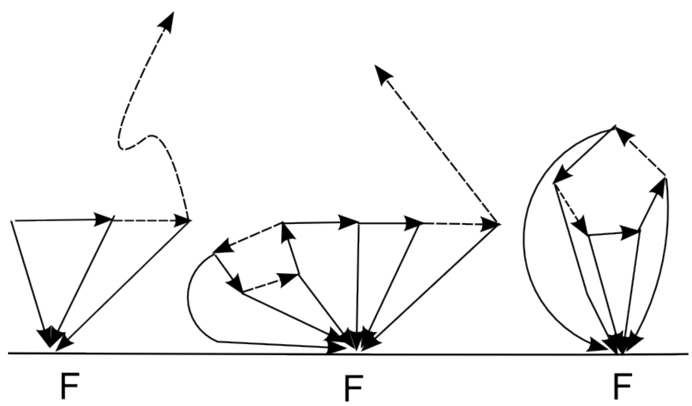

Figure 1:
From left to right: A $\mathcal{B}_3$-type graph, a $\mathcal{B}_3\mathcal{W}_4$-type graph, and a $\mathcal{W}_5$-type graph.

Let us now give the definition of the reduction algebra without character introduced in [CT08]. For a graph $\Gamma \in \mathbf{Q}^\infty_{n,1}$, and using the notation $H_i^* := \partial_i$, let $B_\Gamma\colon S(\mathfrak{q}) \to S(\mathfrak{q}) \otimes \mathfrak{h}^*$, $F \mapsto B_\Gamma(F)$ be the operator defined by the formula $B_\Gamma(F) =$

$$\sum_{\substack{L:E(\Gamma)\to[1,t+r] \\ \text{L is 2-colored}}} \left[\prod_{r=1}^{n} \left(\prod_{\substack{e\in E(\Gamma),\, e=(\cdot,r) \\ r\in V_1(\Gamma)}} \partial_{L(e)} \right) [x_{L(e_r^1)}, x_{L(e_r^2)}] \right] \times \left(\prod_{\substack{e\in E(\Gamma) \\ e=(\cdot,1)}} \partial_{L(e)} F \right) \otimes H^*_{L(e_\infty)} \tag{6}$$

Denote as $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp,\mathfrak{q}} : S(\mathfrak{q})[\epsilon] \longrightarrow S(\mathfrak{q})[\epsilon] \otimes \mathfrak{h}^*$ the operator $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp,\mathfrak{q}} = \sum_{i=1}^\infty \epsilon^i \mathrm{d}^{(i)}_{\mathfrak{h}^\perp,\mathfrak{q}}$ where $\mathrm{d}^{(i)}_{\mathfrak{h}^\perp,\mathfrak{q}} = \sum_{\Gamma\in\mathcal{B}_i\cup\mathcal{BW}_i} \omega_\Gamma B_\Gamma$.

Definition 2.2. The reduction algebra $H^0_{(\epsilon)}(\mathfrak{h}^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp,\mathfrak{q}})$ of polynomials in ϵ is the vector space of solutions $F_{(\epsilon)} \in S(\mathfrak{q})[\epsilon]$ of the equation

$$\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp,\mathfrak{q}}(F_{(\epsilon)}) = 0 \tag{7}$$

equipped with the $*_{CF,\epsilon}-$ product.

The system (7) can be turned into a homogeneous one, regrouping the left hand side with respect to the coefficients of degrees $\deg_\epsilon$ of ϵ. In this vector space case, each operator B_Γ has also a polynomial degree $\deg_\mathfrak{q}$ defined for a homogeneous $F \in S(\mathfrak{q})$ as $\deg_\mathfrak{q}(B_\Gamma) = \deg_\mathfrak{q}(F) - \deg_\mathfrak{q}(B_\Gamma(F))$. The system (7) can also be written in homogeneous equations without using the degree on ϵ, but using $\deg_\mathfrak{q}$ instead. This results in the same system of homogeneous equations, the solutions being functions $F = \sum_{i=0}^n F^{(n-i)} \in S(\mathfrak{q})$ with each $F^{(k)}$ being a homogeneous polynomial with $\deg_\mathfrak{q}(F^{(k)}) = k$. The most important reduction algebras for this paper are the following. Recall from (3.7) in [Bat09] that $S(\mathfrak{q})[\epsilon] \simeq S_{(\epsilon)}(\mathfrak{g})/S_{(\epsilon)}(\mathfrak{g})*_{CF,\epsilon}\mathfrak{h}_\lambda$ where $S_{(\epsilon)}(\mathfrak{g})/S_{(\epsilon)}(\mathfrak{g})*_{CF,\epsilon}\mathfrak{h}_\lambda$ is the quotient of the quantized algebra $(S_{(\epsilon)}(\mathfrak{g}), *_{CF,\epsilon})$ by the $*_{CF,\epsilon}-$ ideal generated by the elements $\{H + \lambda(H),\ H \in \mathfrak{h}\}$. In the text we denote as $A[[\epsilon]]/\langle \epsilon - 1\rangle$ the quotient of the deformed algebra $A[[\epsilon]]$ by the ideal $(\epsilon - 1)A[[\epsilon]]$.

Definition 2.3. **A.** Let λ be a character of $\mathfrak{h}$. Let $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}} : S(\mathfrak{q})[\epsilon] \longrightarrow S(\mathfrak{q})[\epsilon] \otimes \mathfrak{h}^*$ be the differential operator $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}} = \sum_{i=1}^\infty \epsilon^i \mathrm{d}^{(i)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}}$ where $\mathrm{d}^{(i)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}} = \sum_{\Gamma\in\mathcal{B}_i\cup\mathcal{BW}_i}\omega_\Gamma B_\Gamma$. Set $H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}})$ the algebra of $P_{(\epsilon)} \in S(\mathfrak{q})[\epsilon]$, solutions of the equation $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}}(P_{(\epsilon)}) = 0$, equipped with the $*_{CF,\epsilon}$ product.
B. Denote by $H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}}) := \left(H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}})/\langle\epsilon - 1\rangle\right)$ the corresponding specialized algebra. The $*-$ product on $H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}})$ is denoted as $*_{CF,(\epsilon=1)}$.

The difference between $H^0_{(\epsilon)}(\mathfrak{h}^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp,\mathfrak{q}})$ and $H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}})$ is that the second contains polynomials over the slice $\mathfrak{h}^\perp_\lambda$. In practice, one evaluates the solutions of $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}}(P_{(\epsilon)}) = 0$ at the quotient $S_{(\epsilon)}(\mathfrak{g})/S_{(\epsilon)}(\mathfrak{g}) *_{CF,\epsilon} \mathfrak{h}_\lambda$. The operators in $\mathrm{d}^{(i)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}}$ are no longer $\deg_\mathfrak{q}-$ homogeneous, so the system $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}}(P_{(\epsilon)}) = 0$ cannot be homogeneized with $\deg_\mathfrak{q}$ but only with $\deg_\epsilon$.

Example 2.4. For the total coisotropic submanifold $\mathfrak{g}^*$, one trivially has that its reduction algebra is $\left(H^0_{(\epsilon)}(\mathfrak{g}^*, \mathrm{d}^{(\epsilon)}_{\mathfrak{g}^*}), *_{CF,\epsilon}\right) \simeq \left(S_{(\epsilon)}(\mathfrak{g}), *_K\right) \simeq U_{(\epsilon)}(\mathfrak{g})$ from [Kon03] and [CT08]. This is because by (6), the outgoing edge e_∞ of a graph in the differential $\mathrm{d}^{(\epsilon)}_{\mathfrak{g}^*}$, has to belong to $(\mathfrak{g}^*)^\perp$. Thus $\mathrm{d}^{(\epsilon)}_{\mathfrak{g}^*} = 0$.

Example 2.5. Suppose there is an invariant complementary space, that is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$. Then $H^0_{(\epsilon)}(\mathfrak{h}^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp,\mathfrak{p}}) \simeq S(\mathfrak{p})^\mathfrak{h}[\epsilon]$. This is because the relation $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$ allows for no graphs in $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp,\mathfrak{p}}$ with type I vertices more than one. Thus the reduction equations are $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp,\mathfrak{p}}(F_{(\epsilon)}) = 0 \Leftrightarrow \mathrm{d}^{(1)}_{\mathfrak{h}^\perp,\mathfrak{p}}(F_{(\epsilon)}) = 0 \Leftrightarrow F_{(\epsilon)} \in S(\mathfrak{p})^\mathfrak{h}[\epsilon]$.

Biquantization diagrams. An important feature introduced in [CF04], [CF07] and [CT08] is the biquantization diagram. We present it again adjusted for our use. The biquantization diagram is modeled over the $(+,+)$ quadrant of the hyperbolic plane (see Figure 2 for example). One associates a coisotropic submanifold $C_1 \subset X$ on the vertical semi-axis, another coisotropic submanifold $C_2 \subset X$ on the horizontal one and then solves the corresponding reduction equations, using Definitions 2.2-2.3 and Figure 1, to compute the reduction algebras $H^0_{(\epsilon)}(C_1, \mathrm{d}^{(\epsilon)}_{C_1})$ and $H^0_{(\epsilon)}(C_2, \mathrm{d}^{(\epsilon)}_{C_2})$ at each axis respectively. There is also a third reduction space in this diagram, denoted as $H^0_{(\epsilon)}(C_1 \cap C_2, \mathrm{d}^{(\epsilon)}_{C_1\cap C_2})$. It is the one corresponding to the interSection $C_1 \cap C_2$. This vector space admits no $*-$ product and, graphically, we associate it to the corner of the biquantization diagram. It further has a $H^0_{(\epsilon)}(C_1, \mathrm{d}^{(\epsilon)}_{C_1}) - H^0_{(\epsilon)}(C_2, \mathrm{d}^{(\epsilon)}_{C_2})-$ bimodule structure. For $K \in H^0_{(\epsilon)}(C_1, \mathrm{d}^{(\epsilon)}_{C_1})$, $G \in H^0_{(\epsilon)}(C_1 \cap C_2, \mathrm{d}^{(\epsilon)}_{C_1\cap C_2})$, the left module structure $*_L$ is defined by $K *_L G = G \cdot K + \sum_{k=1}^{\infty} \frac{\epsilon^k}{k!} \sum_{\Gamma \in \mathbf{Q}^{(4)}_{k,2}} \omega_\Gamma B_\Gamma(K, G)$ while for $F \in H^0_{(\epsilon)}(C_2, \mathrm{d}^{(\epsilon)}_{C_2})$, $G \in H^0_{(\epsilon)}(C_1 \cap C_2, \mathrm{d}^{(\epsilon)}_{C_1\cap C_2})$, the right module structure $*_R$ is defined by $G *_R F = G \cdot F + \sum_{k=1}^{\infty} \frac{\epsilon^k}{k!} \sum_{\Gamma \in \mathbf{Q}^{(4)}_{k,2}} \omega_\Gamma B_\Gamma(G, F)$. Note that $\mathbf{Q}^{(4)}_{k,2}$ denotes is the family of 4-colored graphs, where the four colors correspond to a basis partition w.r.t whether the basis variable comes from $C_1 \cap C_2, C_1 \cap C_2^c, C_2 \cap C_1^c$, or $C_1^c \cap C_2^c$. This partition also adapts the calculation of the coefficient ω_Γ of each 4-colored graph Γ. In fact there is a 1-form of four colors at the corner of the diagram constructed as the 1-forms of two colors Ω_Γ at each of the axes of the diagram, generalizing the 2-color construction of § 2.2.

Let T_L, T_R be the operators $T_L : H^0_{(\epsilon)}(C_1, \mathrm{d}^{(\epsilon)}_{C_1}) \longrightarrow H^0_{(\epsilon)}(C_1 \cap C_2, \mathrm{d}^{(\epsilon)}_{C_1\cap C_2})$ such that $F \mapsto F *_L 1$, and $T_R : H^0_{(\epsilon)}(C_2, \mathrm{d}^{(\epsilon)}_{C_2}) \longrightarrow H^0_{(\epsilon)}(C_1 \cap C_2, \mathrm{d}^{(\epsilon)}_{C_1\cap C_2})$ such that $G \mapsto 1 *_R G$. Consider now the biquantization diagram with $C_1 = \mathfrak{g}^*$, $C_2 = \mathfrak{h}^\perp_\lambda$ and let $H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon)}_{\mathfrak{g}^*, \mathfrak{h}^\perp_\lambda, \mathfrak{q}})$ denote the reduction space at the origin of this diagram, it is isomorphic to $S(\mathfrak{q})[\epsilon]$. For $Y \in \mathfrak{g}$, set $q(Y) = \det_{\mathfrak{g}} \left(\frac{\sinh \frac{\mathrm{ad} Y}{2}}{\frac{adY}{2}} \right)$ and $q_{(\epsilon)}(Y) := q(\epsilon Y)$. Let $\beta : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ denote the PBW symmetrization, $\beta_{(\epsilon)} : S_{(\epsilon)}(\mathfrak{g}) \longrightarrow U_{(\epsilon)}(\mathfrak{g})$ the deformed symmetrization and $\overline{\beta}_{\mathfrak{q},(\epsilon)} : S(\mathfrak{q})[\epsilon] \longrightarrow U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda$ the quotient deformed symmetrization w.r.t a fixed complementary $\mathfrak{q}$ of $\mathfrak{h}$. Then by [Bat09], §3.4.2, (see also [Bat13] Theorem 5.1), if $\overline{T}_L := T_L|_{S(\mathfrak{q})[\epsilon]}$, the isomorphism (2) is given by the map

$$\overline{\beta}_{\mathfrak{q},(\epsilon)} \circ \partial_{q^{\frac{1}{2}}_{(\epsilon)}} \circ \overline{T}_L^{-1} T_R : H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda, \mathfrak{q}}) \xrightarrow{\sim} \left(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda\right)^{\mathfrak{h}} \tag{8}$$

2.4 Construction of the biquantization character. This Section constructs the biquantization character that we study. The main result is Theorem 2.7.
Fix $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$, a Lie subalgebra. Let $f \in \mathfrak{g}^*$ s.t $f([\mathfrak{h}, \mathfrak{h}]) = 0$. Suppose that there is a polarization $\mathfrak{b}$ with respect to f. We will call a vector space $\mathfrak{q} \subset \mathfrak{g}$, a transversal complementary of $\mathfrak{h}$ with respect to $\mathfrak{b}$, if it satisfies the relations $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and $\mathfrak{b} = \mathfrak{b} \cap \mathfrak{h} \oplus \mathfrak{b} \cap \mathfrak{q}$. It will be denoted as $\mathfrak{q}_{\mathfrak{b}}$. Consider the biquantization diagram of $C_1 = \mathfrak{b}^\perp_f, C_2 = \mathfrak{h}^\perp_f$ and denote as $H^0_{(\epsilon)}(-f + (\mathfrak{h} + \mathfrak{b})^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp, \mathfrak{b}^\perp})$ the reduction space at the corner of this diagram.

Proposition 2.6 ([CT08]). *Suppose that $\mathfrak{h}$ is Lagrangian w.r.t $f \in \mathfrak{g}^*$, there is a polarization $\mathfrak{b}$ of f and $\mathfrak{q}_\mathfrak{b}$ is a transversal complementary of $\mathfrak{h}$. Then T_R is a character of $H^0_{(\epsilon)}(\mathfrak{h}_f^\perp, d^{(\epsilon)}_{\mathfrak{h}_f^\perp, \mathfrak{q}_\mathfrak{b}})$.*

Proof. The idea of [CT08] is to show that
$$H^0_{(\epsilon)}(\mathfrak{b}_f^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{b}_f^\perp}) = H^0_{(\epsilon)}(-f + (\mathfrak{h}+\mathfrak{b})^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp, \mathfrak{b}^\perp}) = \mathbb{R}[\epsilon]. \quad \blacksquare$$

Extending the biquantization diagram, one can consider a triquantization diagram. It is modeled over the hyperbolic semi-band, which one constructs adding a vertical semi-axis to the right side of a biquantization diagram (see Figure 3 for example). Here we associate coisotropic submanifolds C_1, C_2, C_3 to the left, middle and right axis respectively, calculating the corresponding reduction algebras $H^0_{(\epsilon)}(C_i, \mathrm{d}^{(\epsilon)}_{C_i})$, $i = 1, 2, 3$. There are also the corresponding bimodule structures at each corner; these are module structures $*^L_L, *^L_R$ of $H^0_{(\epsilon)}(C_1 \cap C_2, \mathrm{d}^{(\epsilon)}_{C_1 \cap C_2})$ (the superscript L stands for the left corner) and $*^R_L, *^R_R$ of $H^0_{(\epsilon)}(C_2 \cap C_3, \mathrm{d}^{(\epsilon)}_{C_2 \cap C_3})$ at the right corner respectively. These bimodule structures are computed similarly to § 2.3, but using instead the family $\mathbf{Q}^{(8)}_{k,2}$ of 8− colored graphs. One may similarly to § 2.3, define maps T^L_L, T^L_R at the left corner, and T^R_L, T^R_R at the right corner.
In a triquantization diagram there is a 1-form of four colors at each corner of the diagram. In [CT08] § 6.2.1-2 the authors gave a precise definition of an 8-colored 1-form Θ, which in a triquantization diagram interpolates the two 4-colored 1-forms at the corners of the diagram. They proved that when Θ is used in a triquantization diagram of $C_1 = -f + \mathfrak{b}_1^\perp, C_2 = \mathfrak{h}^\perp, C_3 = -f + \mathfrak{b}_2^\perp$, where $\mathfrak{b}_1, \mathfrak{b}_2$ are polarizations of f in normal interSection, the character of Proposition 2.6 is independent of $\mathfrak{b}_1, \mathfrak{b}_2$. Let now

$$\begin{aligned}
c_e = (+,+,+) &\longrightarrow X_e^* \in (\mathfrak{g}\cap\mathfrak{h}\cap\mathfrak{b})^* \quad , \quad & c_e = (-,+,+) &\longrightarrow X_e^* \in (\mathfrak{h}\cap\mathfrak{b})^*/\mathfrak{g}^*, \\
c_e = (+,-,+) &\longrightarrow X_e^* \in (\mathfrak{g}\cap\mathfrak{b})^*/(\mathfrak{h}\cap\mathfrak{b})^*, & c_e = (-,-,+) &\longrightarrow X_e^* \in \mathfrak{b}^*/(\mathfrak{g}+\mathfrak{h})^*, \\
c_e = (+,+,-) &\longrightarrow X_e^* \in (\mathfrak{g}\cap\mathfrak{h})^*/(\mathfrak{h}\cap\mathfrak{b})^*, & c_e = (-,+,-) &\longrightarrow X_e^* \in \mathfrak{h}^*/(\mathfrak{g}+\mathfrak{b})^*, \\
c_e = (+,-,-) &\longrightarrow X_e^* \in \mathfrak{g}^*/(\mathfrak{h}+\mathfrak{b})^* \,, & c_e = (-,-,-) &\longrightarrow X_e^* \in \mathfrak{g}^*/(\mathfrak{g}+\mathfrak{h}+\mathfrak{b})^*.
\end{aligned} \tag{9}$$

be a coloring of graphs in the triquantization diagram of $C_1 = \mathfrak{g}^*$, $C_2 = \mathfrak{h}_f^\perp$ and $C_3 = \mathfrak{b}_f^\perp$. Obviously the colors in the right column of (9) do not appear, but initially we need to consider ourselves in a triquantization setting.

Theorem 2.7. *Fix $\mathfrak{g}, \mathfrak{h}$. Let $f \in \mathfrak{g}^*$ s.t $\mathfrak{h}$ is Lagrangian with respect to f and suppose there is a polarization $\mathfrak{b}$ of f and a transversal complementary $\mathfrak{q}_\mathfrak{b}$ of $\mathfrak{h}$. The linear map γ_{CT} : $\left(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_f\right)^\mathfrak{h} \longrightarrow \mathbb{R}[\epsilon]$ given by*

$$u \mapsto \overline{T}^L_L \circ \bar{\beta}^{-1}_{\mathfrak{q}_\mathfrak{b},(\epsilon)}(u)(f) \tag{10}$$

is a character of $\left(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_f\right)^\mathfrak{h}$.

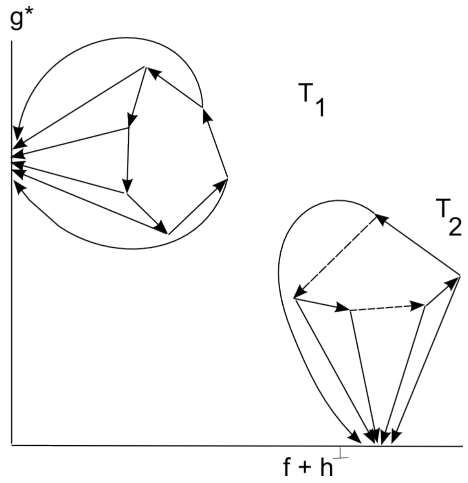

Figure 2:
Operators $\overline{T}_L^L$ and T_R^L at the left corner of the triquantization diagram of
$C_1 = \mathfrak{g}^*$, $C_2 = \mathfrak{h}_f^\perp$, $C_3 = \mathfrak{b}_f^\perp$

Proof. Consider the triquantization diagram of (9). By [CT08], and Proposition 2.6 we calculate at the right corner, the character $P \mapsto T_L^R(P) = P *_L^R 1$ of $H^0_{(\epsilon)}(\mathfrak{h}_f^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}_f^\perp,\mathfrak{q}_\mathfrak{b}})$. Move now P at the left corner of the diagram. Following our notation, $*_R^L$ is the right $H^0_{(\epsilon)}(\mathfrak{h}_f^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}_f^\perp,\mathfrak{q}_\mathfrak{b}})-$ module structure of $H^0_{(\epsilon)}(\mathfrak{h}_f^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{g}^*,\mathfrak{h}_f^\perp,\mathfrak{q}_\mathfrak{b}})$ and $T_R^L(P) = 1 *_R^L P$ is composed of $\mathcal{W}$-type graphs. Exterior graphs acting nontrivially on $T_R^L(P)$, necessarily have edges colored by $(+,+,-)$ and $(+,-,-)$ according to the coloring (9). The 1-form Θ of 8 colors that is used to calculate the weight of a graph in this triquantization diagram, is zero unless the graph has edges only of color $(+,-,-)$ deriving (the bracket at) the corner of the diagram. This color corresponds to variables in $(\mathfrak{h}+\mathfrak{b})^\perp$. Thus we have to restrict $T_R^L(P)$ in this direction and consider the restriction $T_R^L(P)|_{-f+(\mathfrak{h}+\mathfrak{b})^\perp}$. Let Γ_{int} be the family of possible graphs in T_R^L, and Γ_{ext} be the family of possible graphs deriving T_R^L. Let also $A = \exp(\sum_{\Delta\in\Gamma_{ext}} \Delta)$ and $B = \exp(\sum_{\Delta\in\Gamma_{int}} \Delta)$. The operator A consists of $\mathcal{W}$-type graphs because all the edges arriving at the corner have the same color. That is, the graphs in A correspond to differential operators with constant coefficients. In this special case, we can write $A(B(P))|_{-f+(\mathfrak{h}+\mathfrak{b})^\perp} = A(B(P)|_{-f+(\mathfrak{h}+\mathfrak{b})^\perp})$ because A derives in the same direction as the one where we have to restrict $A(B)$, that is $(\mathfrak{h}+\mathfrak{b})^\perp$. Let $A(B(P)|_{-f+(\mathfrak{h}+\mathfrak{b})^\perp}) = c_f$ be the result of the evaluation of A on $B(P)|_{-f+(\mathfrak{h}+\mathfrak{b})^\perp}$. By [CT08], c_f is a constant function on $-f+(\mathfrak{h}+\mathfrak{b})^\perp$ as it coincides with the evaluation $T_L^R(P)(f)$ of the character $P \mapsto T_L^R(P)$ of $H^0_{(\epsilon)}(\mathfrak{h}_f^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}_f^\perp,\mathfrak{q}_\mathfrak{b}})$ at the right corner of the diagram. Since A is an invertible differential operator of constant coefficients, the fact that c_f is a constant function means that A acts trivially on $B(P)|_{-f+(\mathfrak{h}+\mathfrak{b})^\perp}$. So at the left corner, the map

$$P \mapsto A(B(P)|_{-f+(\mathfrak{h}+\mathfrak{b})^\perp}) = B(P)|_{-f+(\mathfrak{h}+\mathfrak{b})^\perp} = T_R^L(P)|_{-f+(\mathfrak{h}+\mathfrak{b})^\perp} = (1 *_L^R P)|_{-f+(\mathfrak{h}+\mathfrak{b})^\perp}$$

is a character of $H^0_{(\epsilon)}(\mathfrak{h}_f^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}_f^\perp,\mathfrak{q}_\mathfrak{b}})$. By (8), we write

$$P = (T_R^L)^{-1}\overline{T}_L^L \circ \bar{\beta}^{-1}_{\mathfrak{q}_\mathfrak{b},(\epsilon)}(u) \in H^0_{(\epsilon)}(\mathfrak{h}_f^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}_f^\perp,\mathfrak{q}_\mathfrak{b}}) \text{ for some } u \in \left(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_f\right)^{\mathfrak{h}}.$$

Since the map $P \mapsto T_R^L(P)|_{-f+(\mathfrak{h}+\mathfrak{b})^\perp}$ is a character for $H^0_{(\epsilon)}(\mathfrak{h}_f^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}_f^\perp,\mathfrak{q}_\mathfrak{b}})$, the map $\gamma_{CT} : \left(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_f\right)^{\mathfrak{h}} \longrightarrow \mathbb{R}[\epsilon]$ with $\gamma_{CT}(u) = \overline{T}_L^L \circ \bar{\beta}^{-1}_{\mathfrak{q}_\mathfrak{b},(\epsilon)}(u)(f)$ is a character of $\left(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_f\right)^{\mathfrak{h}}$. ■

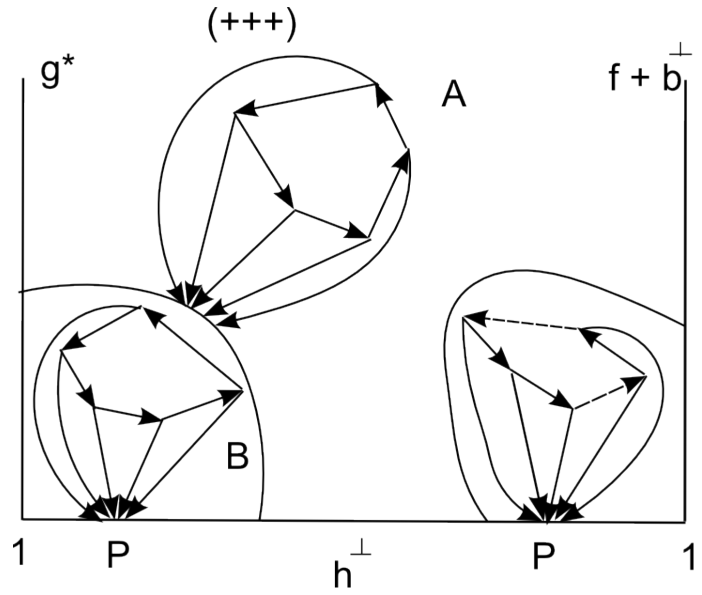

Figure 3:
The diagram constructing γ_{CT}.

Note that the function $(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_f)^{\mathfrak{h}} \ni u \mapsto \overline{T}_L^L \circ \overline{\beta}^{-1}_{\mathfrak{q}_\mathfrak{b},(\epsilon)}(u)$ is constant on $-f + (\mathfrak{h}+\mathfrak{b})^\perp$. In view of (1) we can construct a family of characters with the following:

Proposition 2.8. *Let $\mathfrak{g}$ be a real nilpotent Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra, $\lambda \in \mathfrak{h}^*$ a character of $\mathfrak{h}$, $f \in \mathfrak{g}^*$ s.t $f|_{\mathfrak{h}} = \lambda$, $\mathfrak{b}$ a polarization of f and $\mathfrak{q}_\mathfrak{b}$ a transversal complementary space. Suppose that (1) holds. Then for every such $l \in \mathcal{O}$, there is a character $\gamma^l_{CT} : \left(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda\right)^{\mathfrak{h}} \longrightarrow \mathbb{R}[\epsilon]$, $u \mapsto \overline{T}_L^L \circ \bar{\beta}^{-1}_{\mathfrak{q}_\mathfrak{b},(\epsilon)}(u)(l)$.*

Proof. Since we are in the nilpotent case, there is always a polarization for the element f. Due to (1), Theorem 2.7 gives us a character for each $l \in \mathcal{O}$. ■

3. Lie Theory.

3.1. Our framework. In this Section we recall the necessary facts and definitions from Lie theory so as to define the analytic character in Theorem 3.3. One may consult [Tor93a] and [Tor93b] for a more complete treatment of the analytical background. Let G be a real nilpotent, connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra, and $\lambda \in \mathfrak{h}^*$ a character of $\mathfrak{h}$. From λ, one defines a unitary character $\chi_\lambda : H \longrightarrow \mathbb{C}$ of the Lie subgroup H associated to

$\mathfrak{h}$, setting $\chi_\lambda(\exp Y) = e^{i\lambda(Y)}$, for $Y \in \mathfrak{h}$. Let $C^\infty(G, H, \chi_\lambda)$ be the vector space of complex smooth functions θ on G that satisfy $\theta(gh) = (\chi_\lambda(h))^{-1}\theta(g)$, $\forall h \in H, \forall g \in G$. Let also $\mathbb{D}(\mathfrak{g}, \mathfrak{h}, \lambda)$ be the algebra of linear differential operators, that leave the space $C^\infty(G, H, \chi_\lambda)$ invariant and commute with the left translation on G. Koornwinder in [Koorn81] proved that there is an algebra isomorphism

$$(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}} \tilde{\to} \mathbb{D}(\mathfrak{g}, \mathfrak{h}, \lambda). \tag{11}$$

The Corwin-Greenleaf Conjecture in [CG92a] relates the commutativity of this algebra with finiteness of multiplicities in the spectral decomposition of $\tau_\lambda := \mathrm{Ind}(G \uparrow H, \chi_\lambda)$, the representation induced from H, χ_λ. The Hilbert space of this representation is $\mathcal{H}_\lambda := L^2(G, H, \lambda)$, the separable completion of $C_c^\infty(G, H, \chi_\lambda)$, the compactly supported functions of $C^\infty(G, H, \chi_\lambda)$, with respect to the norm $||\phi||_2 = \int_{G/H} |\phi(g)|^2 \mathrm{d}_{G/H}(g)$. The action of G on $\phi \in L^2(G, H, \lambda)$ is through left translations: $\tau_\lambda(g)(\phi)(g') = \phi(g^{-1}g')$. Set also $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ and $U_{\mathbb{C}}(\mathfrak{g}) := U(\mathfrak{g}) \otimes \mathbb{C}$. Recall that a function $\phi \in \mathcal{H}_\lambda$ is said to be C^∞ if the map $G \ni g \mapsto \tau_\lambda(g)\phi$ is smooth with respect to $|| \cdot ||_2$. Denote as $\mathcal{H}_\lambda^\infty$ the dense subspace of $C^\infty-$ vectors of $\mathcal{H}_\lambda$ and as $\mathcal{H}_\lambda^{-\infty}$ the space of antilinear continuous forms on $\mathcal{H}_\lambda^\infty$. The action of $\mathfrak{g}$ on $\mathcal{H}_\lambda^\infty$ is denoted by $\mathrm{d}\tau_\lambda^\infty$ and defined for $X \in \mathfrak{g}$, $g \in G$, by $\mathrm{d}\tau_\lambda^\infty(X)(\phi)(g) = \frac{d}{dt}\phi(\exp -tX \cdot g)|_{t=0}$. It induces an action of $U_{\mathbb{C}}(\mathfrak{g})$ on $\mathcal{H}_\lambda^\infty$ denoted by the same symbol, $\mathrm{d}\tau_\lambda^\infty$, while the action of $U_{\mathbb{C}}(\mathfrak{g})$ on $\mathcal{H}_\lambda^{-\infty}$ is denoted by $\mathrm{d}\tau_\lambda^{-\infty}$. Let $\mathfrak{b}$ be a polarization of $f \in \mathfrak{h}_\lambda^\perp$ with Lie subgroup B and set $\tau_f = \mathrm{Ind}(G \uparrow B, \chi_f)$ with $\mathcal{H}_f = L^2(G, B, f)$. The norm used comes from the product $\langle \phi_1(x), \phi_2(x) \rangle := \int_{G/B} \phi_1(g)\overline{\phi_2(g)}\mathrm{d}_{G/B}(g)$ for $\phi_1, \phi_2 \in L^2(G, B, f)$ and $\mathrm{d}_{G/B}$ an invariant measure on G/B. Finally let $\mathcal{H}_f^{-\infty, H}$ be the $H-$ semi-invariant antilinear forms and $\mathrm{d}_{H/H\cap B}$ be a left-invariant measure on $H/H\cap B$. The *Penney vector* $\alpha_f \in \mathcal{H}_f^{-\infty}$ corresponding to f is defined as

$$\langle \alpha_f, \phi \rangle = \int_{H/H\cap B} \overline{\phi(h)\chi_\lambda(h)}\mathrm{d}_{H/H\cap B}(h) \quad \text{for} \quad \phi \in \mathcal{H}_f^\infty. \tag{12}$$

By [Ben84], $\alpha_f \in \mathcal{H}_f^{-\infty, H}$. Because of this, the algebra $(U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{-if})^{\mathfrak{h}}$ acts on α_f. Let now $l \in \mathfrak{h}_\lambda^\perp$ and $\mathfrak{g}(l)$ be the stabilizer of l. Such an l is called *regular* if $\dim(\mathfrak{g}(l))$ is minimal among $\dim(\mathfrak{g}(\xi))$, $\xi \in \mathfrak{h}_\lambda^\perp$. The regular elements form a non-empty Zariski-open set. An $l \in \mathfrak{h}_\lambda^\perp$ will be furthermore called *generic* if it is regular and satisfies $\dim(\mathfrak{h} \cdot l) = \frac{1}{2}\dim(\mathfrak{g} \cdot l)$. The set of generic elements is still a Zariski-open set. The next result constructs a complex character of the Penney vector. Its proof is in [F98], Theorem 1.

Theorem 3.1 ([F00]). *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra and λ a character of $\mathfrak{h}$. Suppose that (1) holds. Then for generic $l \in \mathfrak{h}_\lambda^\perp$ and for $A \in (U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{il})^{\mathfrak{h}}$, $\mathrm{d}\tau_l^{-\infty}(\overline{A})(\alpha_l)$ is a multiple of α_l, and there is defined a character $\lambda_l : (U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{il})^{\mathfrak{h}} \longrightarrow \mathbb{C}$ such that*

$$\mathrm{d}\tau_l^{-\infty}(\overline{A})(\alpha_l) = \overline{\lambda_l(A)}\alpha_l. \tag{13}$$

3.2. Construction of a real character. The main result of this Section is Theorem 3.3 constructing a real character similar to (13). We first define an appropriate real distribution on G/B. Let $\mathfrak{g}$ be a nilpotent Lie algebra, $\mathfrak{h}$ a subalgebra and λ a real character of $\mathfrak{h}$. Let $f \in \mathfrak{g}^*$ s.t $f|\mathfrak{h} = \lambda$ and $\mathfrak{b}$ be a polarization of f with corresponding group B. In this Section all characters are real, and if there's no confusion we will use the previous notations for them. Denote as $C_c^\infty(G, B, \chi_f)$ the $C^\infty-$ functions ψ with compact support satisfying the equivariance condition $\psi(gb) = e^{-f(X)}\psi(g)$ for $g \in G, X \in \mathfrak{b}$, $\exp(X) = b$. Define $\alpha(f)$ as

$$\langle \alpha(f), \psi \rangle := \int_{H/H\cap B} \psi(h) e^{f(Y)} \mathrm{d}_{H/H\cap B}(h), \quad \psi \in C_c^\infty(G, B, \chi_f), \tag{14}$$

for $\exp(Y) = h \in H$. Since the space $\langle X + f(X),\ X \in \mathfrak{h} \rangle$ acts by zero on $\alpha(f)$, the algebra $(U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$ acts on $\alpha(f)$. We need now to recall some facts from [Bat09]. Introduce a new variable T such that $[\mathfrak{g}, T] = 0$ and define $\mathfrak{g}_T := \mathfrak{g} \oplus \langle T \rangle$, that is $\dim(\mathfrak{g}_T) = \dim(\mathfrak{g}) + 1$. Let also $\mathfrak{h}_T := \mathfrak{h} \oplus \langle T \rangle$, $U_T(\mathfrak{g}) := U(\mathfrak{g}_T)$ and $U(\mathfrak{g}_T)\mathfrak{h}_\lambda^T$ be the ideal of $U(\mathfrak{g}_T)$ generated by $\mathfrak{h}_\lambda^T = \langle H + T\lambda(H),\ H \in \mathfrak{h} \rangle$. Setting $\mathbb{D}_T(\mathfrak{g}, \mathfrak{h}, \lambda) := \mathbb{D}(\mathfrak{g}_T, \mathfrak{h}_T, \lambda)$ we get following (11), that $(U(\mathfrak{g}_T)/U(\mathfrak{g}_T)\mathfrak{h}_\lambda^T)^{\mathfrak{h}_T} \simeq \mathbb{D}_T(\mathfrak{g}, \mathfrak{h}, \lambda)$. We denote by $\mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) := \mathbb{D}_T(\mathfrak{g}, \mathfrak{h}, \lambda)/\langle T - 1 \rangle$ the corresponding specialization algebra and by $\mathcal{P}_{(t=1)}\left((U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}\right)$ the values at $t = 1$ of polynomial in t families $t \longrightarrow u_t \in \left((U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_{t\lambda})^{\mathfrak{h}}\right)$. By Corollaries 4.4 and 3.3 of [Bat09], condition (1) implies that $H^0_{(\epsilon=1)}(\mathfrak{h}_\lambda^\perp, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}_\lambda^\perp, \mathfrak{q}})$ is commutative and that in general there is an injection $\mathfrak{i}_{(\epsilon=1)} : H^0_{(\epsilon=1)}(\mathfrak{h}_\lambda^\perp, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}_\lambda^\perp, \mathfrak{q}}) \hookrightarrow (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$. From [Bat09], Theorem 3.5 and Corollary 3.2, one furthermore has that the image of $\mathfrak{i}_{(\epsilon=1)}$ is $\mathcal{P}_{(t=1)}\left((U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}\right)$ and

$$H^0_{(\epsilon=1)}(\mathfrak{h}_\lambda^\perp, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}_\lambda^\perp, \mathfrak{q}}) \simeq \mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) \simeq \mathcal{P}_{(t=1)}\left((U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}\right) \tag{15}$$

For the proof of Theorem 3.3 we need the following intermediate Lemma. Fix a complementary space $\mathfrak{q}$ of $\mathfrak{h}$. Let $\pi_{it\lambda}$ denote the canonical projection $\pi_{it\lambda} : U_{\mathbb{C}}(\mathfrak{g}) \longrightarrow U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{it\lambda}$. For $H \in \mathfrak{h}$, let $\mathrm{d}_H : \ S_{\mathbb{C}}(\mathfrak{q}) \longrightarrow S_{\mathbb{C}}(\mathfrak{q})$ denote the linear map defined by $Q \mapsto \beta^{-1}\left(\pi_{it\lambda}([H, \beta(Q)])\right)$. Finally let $U(\mathfrak{g}) = \bigcup_n U_n(\mathfrak{g})$ be the PBW filtration and $\overline{\beta}_t : \ S(\mathfrak{q}) \longrightarrow U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_{t\lambda}$ denote the symmetrization isomorphism of these vector spaces.

Lemma 3.2. *The algebra $(U_n(\mathfrak{g})/U_n(\mathfrak{g})\mathfrak{h}_{it\lambda})^{\mathfrak{h}}$ depends rationally on t via the map $\overline{\beta}_t$, $\forall n \in \mathbb{N}$.*

Proof. Let $\{H_1, \ldots, H_t\}$ be a basis of $\mathfrak{h}$ such that $\lambda(H_t) = 1$ and $\{H_1, \ldots, H_{t-1}\}$ is a basis of $\mathfrak{h} \cap \ker(\lambda)$. Let $\{Q_1, \ldots, Q_r\}$ be a basis of $\mathfrak{q}$. The kernels of the maps d_{H_i}, are formed by the elements $Q^\alpha = Q_1^{\alpha_1} \cdots Q_r^{\alpha_r} \in S_{\mathbb{C}}(\mathfrak{q})$ such that $\pi_{it\lambda}([H_i, \beta(Q^\alpha)]) = 0$. We have that $\forall i$, $[H_i, Q^\alpha] = \sum_{\gamma,\delta} c^i_{\gamma\delta} Q^\delta H^\gamma$ and furthermore, for $z \in \mathbb{C}$, one has mod$U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{z\lambda}$ that

$$[H_i, Q^\alpha] \equiv \sum_\delta c^i_\delta Q^\delta + \sum_{\gamma=(0,\ldots,0,\gamma_t),\delta} c^i_{\gamma\delta} Q^\delta (-z)^{\gamma_t}.$$

Applying β^{-1} on $\pi_{z\lambda}([H_i, \beta(Q^\alpha)])$ one sees that the linear maps d_{H_i} correspond to matrices with polynomial coefficients, so we compute in the field of rational fractions $\mathbb{C}(z)$. Our algebra of invariants corresponds to the common kernel of d_{H_i} which depends in a rational way on z, with z generic. Indeed, let's first write $\mathrm{d}_{H_i}(z)$ meaning the dependance on z. Checking the dimension of $\ker(\mathrm{d}_{H_i}(z))$ for generic z, (actually proving that $\forall i, j$, $\dim_{\mathbb{C}}(\ker(\mathrm{d}_{H_i}) \cap \ker(\mathrm{d}_{H_j})) = \dim_{\mathbb{C}}(\ker(\mathrm{d}_{H_i}(z)) \cap \ker(\mathrm{d}_{H_j}(z)))$ and $[\ker(\mathrm{d}_{H_i}) \cap \ker(\mathrm{d}_{H_j})](z) = \ker(\mathrm{d}_{H_i}(z)) \cap \ker(\mathrm{d}_{H_j}(z))$), we conclude that the d_{H_i} depend rationally on generic values of z. ■

The proof of Theorem 3.1 was done by reducing the claim to the case $\mathfrak{z} \subset \mathfrak{h}$, $\ker(f) \cap \mathfrak{z} = \{0\}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$. In that case, $\dim(\mathfrak{z}) = 1$ and if $\mathfrak{z} = \langle Z \rangle$, then by [Dix96], § 4.7.7, there are $X, Y \in \ker(f) \cap \mathfrak{g}$ such that $[X, Y] = Z$, and $\mathfrak{g} = \langle X \rangle \oplus \mathfrak{g}_0$ where $\mathfrak{g}_0 := \{W \in \mathfrak{g}/[W, Y] = 0\}$. Also, Theorem 3.1 was proved for a unitary character f. The deformation quantization setting that we use, works over $\mathbb{R}$. It is thus necessary to establish a result similar to Theorem 3.1 but for a real character. Recall also that for $\mathfrak{g}_0 \subset \mathfrak{g}$ a codimension 1 ideal, the orbit $H \cdot f$ is said to be saturated with respect to $\mathfrak{g}_0^\perp$ iff $H \cdot f + \mathfrak{g}_0^\perp = H \cdot f$.

Theorem 3.3. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra and λ a character of $\mathfrak{h}$. Suppose that generically the $H-$ orbits are Lagrangian in $\mathfrak{h}_\lambda^\perp$. Then for generic $f \in \mathfrak{h}_\lambda^\perp$ and for $A \in D_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda)$, $\mathrm{d}\tau_f^{-\infty}(A)(\alpha(f))$ is a multiple of $\alpha(f)$, and there is defined a real character $\lambda^f_{(T=1)} : D_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) \longrightarrow \mathbb{R}$ such that*

$$\mathrm{d}\tau_f^{-\infty}(A)(\alpha(f)) = \lambda^f_{(T=1)}(A)(\alpha(f)). \tag{16}$$

Proof. The proof is based on the proof of [F98], Theorem 1. However, we will need some modifications in the arguments, very important for the rest of the paper. First, one can suppose that $\mathfrak{z} \subset \mathfrak{h}$. Indeed, let K be the corresponding group of the Lie subalgebra $\mathfrak{k} := \mathfrak{h} + \mathfrak{z}$. It is $\mathfrak{z} \subset \mathfrak{b}$, a polarization of f with corresponding group B. For $\psi \in C_c^\infty(G, B, \chi_f)$ we have

$$\langle \alpha(f), \psi \rangle = \int_{H/H\cap B} \psi(h) \chi_\lambda \mathrm{d}_{H/H\cap B}$$

$$= \int_{HZ/(HZ)\cap B} \psi(h) \chi_\mu \mathrm{d}_{H/H\cap B} = \int_{K/K\cap B} \psi(k) \chi_\mu \mathrm{d}_{K/K\cap B}$$

where $\mu := f|_{(\mathfrak{h}+\mathfrak{z})}$ and Z is the corresponding group of $\mathfrak{z}$. Then for $X \in \mathfrak{k}$, $\mathrm{d}\tau_f^{-\infty}(X)\alpha(f) = -\mu(X)\alpha(f)$ and thus $(U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{k}_\mu)^{\mathfrak{k}}$ acts on $\alpha(f)$. Also, there is a natural projection $\mathrm{J}_{\mathfrak{k}} : (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}} \longrightarrow (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k}_\mu)^{\mathfrak{k}}$ inducing a projection $\mathrm{j}_{\mathfrak{k}} : \mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) \longrightarrow \mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{k}, \mu)$. Thus for $A \in \mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda)$ and $\tilde{A} = \mathrm{j}_{\mathfrak{k}}(A)$, one has $\mathrm{d}\tau_f^{-\infty}(A)(\alpha(f)) = \mathrm{d}\tau_f^{-\infty}(\tilde{A})(\alpha(f))$. Since f is generic and $f \in \mathfrak{k}_\mu^\perp$, the induction hypothesis applies to define a real character $\tilde{\lambda}^f_{(T=1)} : \mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{k}, \mu) \longrightarrow \mathbb{R}$ which coincides with $\lambda^f_{(T=1)}$. So the proof of the Theorem is reduced to the case $\mathfrak{z} \subset \mathfrak{h}$.

One may further suppose that $\mathfrak{z} \cap \ker(f) = \{0\}$. Indeed, let $\mathfrak{z}' := \mathfrak{z} \cap \ker(f)$. Setting $\mathfrak{g}' := \mathfrak{g}/\mathfrak{z}'$, $\mathfrak{h}' := \mathfrak{h}/\mathfrak{z}'$, $f' := f|_{\mathfrak{g}'}$, the Theorem's conditions are satisfied by

$\mathfrak{g}', \mathfrak{h}', f'$. Also, $\mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) = \mathbb{D}_{(T=1)}(\mathfrak{g}', \mathfrak{h}', \lambda)$ and $\alpha(f)$ is $H'-$ semi-invariant. The character $\lambda^{f'}_{(T=1)} : \mathbb{D}_{(T=1)}(\mathfrak{g}', \mathfrak{h}', \lambda) \longrightarrow \mathbb{R}$ coincides with $\lambda^{f}_{(T=1)}$ so one further reduces the proof to the case $\mathfrak{z} \cap \ker(f) = \{0\}$. Then we distinguish two cases:

First case: $\mathfrak{h} \not\subset \mathfrak{g}_0$. Let $X \in \mathfrak{h} \cap \ker(f)$ such that $\mathfrak{g} = \langle X \rangle \oplus \mathfrak{g}_0$. Set also $\mathfrak{h}_0 := \mathfrak{h} \cap \mathfrak{g}_0$, $\lambda_0 = \lambda|_{\mathfrak{g}_0}$ and $\mathfrak{g}_{0T} := \mathfrak{g}_0 \oplus \mathbb{R}\langle T \rangle$. Let $u_T \in U(\mathfrak{g}_T)$. Then u_T can be written as $u_T = \sum_k v_T^{(k)} X^k$ where $v_T^{(k)} \in U(\mathfrak{g}_{0T})$. Since $X \in \mathfrak{h} \cap \ker(f)$ we have

$$\left(U(\mathfrak{g}_T)/U(\mathfrak{g}_T)\mathfrak{h}^T_\lambda\right)^{\mathfrak{h}_T} \hookrightarrow \left(U(\mathfrak{g}_{0T})/U(\mathfrak{g}_{0T})\mathfrak{h}^T_{0\lambda}\right)^{\mathfrak{h}_T} \subset \left(U(\mathfrak{g}_{0T})/U(\mathfrak{g}_{0T})\mathfrak{h}^T_{0\lambda}\right)^{\mathfrak{h}_{0T}} \quad (17)$$

Specializing (17) at $T = 1$ we get by (15) that for the corresponding algebras of polynomial families, it is

$$\mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) \hookrightarrow \mathbb{D}_{(T=1)}(\mathfrak{g}_0, \mathfrak{h}_0, \lambda_0) \quad (18)$$

Thus for $u_T \in \mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda)$, and for its image through (18) $u^0_T \in \mathbb{D}_{(T=1)}(\mathfrak{g}_0, \mathfrak{h}_0, \lambda_0)$ with $u_T \equiv u^0_T \mathrm{mod}[U(\mathfrak{g}_T)\mathfrak{h}^T_\lambda]$, we have $\lambda^f_{(T=1)}(u_T) = \lambda^{f_0}_{(T=1)}(u^0_T)$, following the corresponding computation at Theorem 3.1.

Second case: $\mathfrak{h} \subset \mathfrak{g}_0$. In this case, the condition of Corwin-Greenleaf (see (1) of equations (2.7) in [FLMM03]) holds for the character $it\lambda$ and we thus have

$$(U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{it\lambda})^{\mathfrak{h}} = (U_{\mathbb{C}}(\mathfrak{g}_0)/U_{\mathbb{C}}(\mathfrak{g}_0)\mathfrak{h}_{it\lambda})^{\mathfrak{h}}. \quad (19)$$

By Lemma 3.2, this equation depends rationally on it, for $t \in \mathbb{R}^*$. So if (19) holds for it, $t \in \mathbb{R}^*$, it holds also for t in a Zariski-open subset of $\mathbb{R}$ and we write

$$(U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_{t\lambda})^{\mathfrak{h}} = (U_{\mathbb{C}}(\mathfrak{g}_0)/U_{\mathbb{C}}(\mathfrak{g}_0)\mathfrak{h}_{t\lambda})^{\mathfrak{h}}. \quad (20)$$

However, we cannot conclude that a similar equation holds for the algebra $(U_{\mathbb{C}}(\mathfrak{g})/U_{\mathbb{C}}(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$. For this, we use polynomial families

$$t \mapsto u_t \in (U_{\mathbb{C}}(\mathfrak{g}_0)/U_{\mathbb{C}}(\mathfrak{g}_0)\mathfrak{h}_{t\lambda})^{\mathfrak{h}}$$

and argue as follows: Since $\mathfrak{h} \subset \mathfrak{g}_0$, it is also $\mathfrak{h}_T \subset \mathfrak{g}_{0T}$. Recall by [FLMM03], that for the generic element $f \in \mathfrak{h}^\perp_\lambda$ the orbits $H \cdot f$ are saturated with respect to $\mathfrak{g}^\perp_0$. We will prove that $(U(\mathfrak{g}_{0T})/U(\mathfrak{g}_{0T})\mathfrak{h}_T)^{\mathfrak{h}_T} = (U(\mathfrak{g}_T)/U(\mathfrak{g}_T)\mathfrak{h}_T)^{\mathfrak{h}_T}$: For $H \in \mathfrak{h}$ one has $l \in \mathfrak{h}^\perp_T \Leftrightarrow l(H + T\lambda(H)) = 0 \Leftrightarrow l(H) + l(T)\lambda(H) = 0 \Leftrightarrow l = f + \mu T^*,\ f \in \mathfrak{h}^\perp_{\mu\lambda}$. Thus for $l \in \mathfrak{h}^\perp_T$, $X \in \mathfrak{g}_T$, it is $X \in \mathfrak{h}_T(l) \Leftrightarrow \forall H' \in \mathfrak{h}_T,\ l([X, H']) = 0 \Leftrightarrow \forall H \in \mathfrak{h},\ f([X, H]) = 0 \Leftrightarrow X \in \mathfrak{h}(f) \oplus \langle T \rangle$. By (2.7)-(2.8) of [FLMM03] and since for the given $\mathfrak{g}, \mathfrak{h}, \lambda$ and $f \in \mathfrak{h}^\perp_\lambda$, the $H \cdot f$ orbits are saturated with respect to $\mathfrak{g}^\perp_0$, one gets $\dim(\mathfrak{h}(f)) = \dim(\mathfrak{h}(f_0)) - 1$. Let now $\mathcal{O}_T := \cup_\mu \mu\mathcal{O}$. The set $\mathcal{O}_T$ is Zariski - open since $\mathcal{O}$ is itself Zariski - open. Thus for $l \in \mathcal{O}_T$, it is $\dim(\mathfrak{h}_T(l)) = \dim(\mathfrak{h}_T(l|_{\mathfrak{g}_{0T}})) - 1$ and so the H_T- orbits are saturated with respect to $\mathfrak{g}^\perp_{0T} \subset \mathfrak{g}^*_T$. We conclude the proof of the Theorem using Theorem 5.2 and equivalences (2.7), (2.8) of [FLMM03], which prove that for the generic $f \in \mathfrak{h}^\perp_\lambda$,

$$\mathbb{D}(\mathfrak{g}_0, \mathfrak{h}, \lambda_0) = \mathbb{D}(\mathfrak{g}, \mathfrak{h}, \lambda) \Leftrightarrow H \cdot f \quad \text{are saturated with respect to } \mathfrak{g}^\perp_0. \quad (21)$$

Applying Theorem (21), for $\mathfrak{g}_T, \mathfrak{h}_T$ one gets

$$(U_{\mathbb{C}}(\mathfrak{g}_{0T})/U_{\mathbb{C}}(\mathfrak{g}_{0T})\mathfrak{h}_T)^{\mathfrak{h}_T} = (U_{\mathbb{C}}(\mathfrak{g}_T)/U_{\mathbb{C}}(\mathfrak{g}_T)\mathfrak{h}_T)^{\mathfrak{h}_T}. \quad (22)$$

Specializing (22) at $T=1$, we get an equality of the corresponding specialization algebras, namely

$$\mathbb{D}_{(T=1)}(\mathfrak{g}_0,\mathfrak{h},\lambda_0)=\mathbb{D}_{(T=1)}(\mathfrak{g},\mathfrak{h},\lambda). \tag{23}$$

The character is calculated according to Theorem 3.1. ■

4. Main result.

4.1. Three necessary Lemmata. For the proof of the main result of the paper, Theorem 4.4, we proceed in three steps: First we prove that one can suppose $\mathfrak{z}\subset\mathfrak{h}$ by showing that if instead of $\mathfrak{h}$ we consider $\mathfrak{k}:=\mathfrak{h}+\mathfrak{z}$ as the subalgebra, one computes the same biquantization character. For this step we will need Lemmata 4.1 an 4.2. We then prove that one can also suppose $\mathfrak{z}\cap\ker(f)=\{0\}$. For this step we will need Lemma 4.3. The third step is induction on $\dim(\mathfrak{g})$ where we distinguish the cases $\mathfrak{h}\subset\mathfrak{g}_0$ and $\mathfrak{h}\not\subset\mathfrak{g}_0$.

Lemma 4.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, $\mathfrak{h}\subset\mathfrak{g}$ a subalgebra, λ a character of $\mathfrak{h}$, and $f\in\mathfrak{h}^\perp_\lambda$. Let $\mathfrak{b}$ be a polarization with respect to f and $\mathfrak{q}_\mathfrak{b}$ a transversal complementary of $\mathfrak{h}$. Let $\mathfrak{z}_{\mathfrak{q}_\mathfrak{b}}:=\mathfrak{z}\cap\mathfrak{q}_\mathfrak{b}$, $\mathfrak{k}:=\mathfrak{h}\oplus\mathfrak{z}_{\mathfrak{q}_\mathfrak{b}}$ and V be a complementary of $\mathfrak{z}_{\mathfrak{q}_\mathfrak{b}}$ in $\mathfrak{q}_\mathfrak{b}$, that is $\mathfrak{q}_\mathfrak{b}=\mathfrak{z}_{\mathfrak{q}_\mathfrak{b}}\oplus V$. Then*

$$P\in H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda,\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}_\mathfrak{b}})\Rightarrow P|_{-f+\mathfrak{k}^\perp}\in H^0_{(\epsilon)}(\mathfrak{k}^\perp_f,\mathrm{d}^{(\epsilon)}_{\mathfrak{k}^\perp_f,V}).$$

Proof. Let $P=\sum z^\alpha P_\alpha$ where $z^\alpha\in S(\mathfrak{z}_{\mathfrak{q}_\mathfrak{b}})[\epsilon]$, $P_\alpha\in S(V)[\epsilon]$. Set $\overline{P}:=P|_{-f+\mathfrak{k}^\perp}$. By definition 2.3, only $\mathcal{B}-$ type and $\mathcal{BW}-$ type graphs appear in the differential defining a reduction algebra. To write the reduction equations for $H^0_{(\epsilon)}(\mathfrak{t}^\perp_f,\mathrm{d}^{(\epsilon)}_{\mathfrak{t}^\perp_f,V})$, let D_n be the differential operator corresponding to a possible $\mathcal{B}_n-$ type graph. Then $D_n(P)=\sum z^\alpha D_n(P_\alpha)$. Examining the possible coloring in such a $\mathcal{B}-$ type graph, the edges deriving $\overline{P}$ carry variables from V. Furthermore, no vertex except the root of the $\mathcal{B}-$ type graph can belong to $\mathfrak{z}_{\mathfrak{q}_\mathfrak{b}}$. It is also $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}_\mathfrak{b}=\mathfrak{k}\oplus V$. Let $\overline{D_n},\overline{P}$ be the respective objects defined on the new complementary V of $\mathfrak{k}$, by this decomposition. Checking all the possible colors for the decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}_\mathfrak{b}$ in a graph $\Gamma\in\mathcal{B}_n$ we see that

$$\overline{D_n(P)}=\overline{D_n}(\overline{P}). \tag{24}$$

With the same reasoning the Lemma holds for D_n being an operator coming from a $\mathcal{BW}_n-$ type graph. Finally, since $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}_\mathfrak{b}}=\sum_n D_n$, we have by (24)

$$P\in H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda,\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}_\mathfrak{b}})\Rightarrow\sum_i\epsilon^i\mathrm{d}^{(i)}_{\mathfrak{h}^\perp_\lambda,\mathfrak{q}_\mathfrak{b}}(P)=0\Rightarrow$$

$$\sum_i\epsilon^i\mathrm{d}^{(i)}_{\mathfrak{k}^\perp_l,V}(\overline{P})=0\Rightarrow\overline{P}\in H^0_{(\epsilon)}(\mathfrak{k}^\perp_f,\mathrm{d}^{(\epsilon)}_{\mathfrak{k}^\perp_f,V})$$

Thus in terms of reduction algebras, one may suppose that $\mathfrak{z}\subset\mathfrak{h}$. ■

Lemma 4.2. *Let $\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{z}$ be as before, V be a vector subspace such that $\mathfrak{g} = (\mathfrak{h}+\mathfrak{z}) \oplus V$ and $\mathfrak{q}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Let $f \in \mathfrak{h}_\lambda^\perp$ and suppose the Lagrangian condition holds for f. Let finally $\gamma_{CT}^{\mathfrak{z}} : (U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})(\mathfrak{h}+\mathfrak{z})_f)^{\mathfrak{h}+\mathfrak{z}} \longrightarrow \mathbb{R}[\epsilon]$, $u \mapsto \overline{T}_L^L \circ \overline{\beta}_{V,(\epsilon)}^{-1}(u)$ and $\gamma_{CT} : (U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_f)^{\mathfrak{h}} \longrightarrow \mathbb{R}[\epsilon]$, $u \mapsto \overline{T}_L^L \circ \overline{\beta}_{\mathfrak{q},(\epsilon)}^{-1}(u)$ be the characters following Theorem 2.7. Then*

$$\gamma_{CT}^{\mathfrak{z}}(\overline{u}) = \gamma_{CT}(u)|_{-f+(\mathfrak{h}+\mathfrak{z})^\perp} \tag{25}$$

Proof. Let $P = \sum_\alpha z_\alpha Q_\alpha \in S(\mathfrak{q})$ with $z_\alpha \in S(\mathfrak{z}_\mathfrak{q})[\epsilon], Q_\alpha \in S(V)[\epsilon]$. Let $u = \overline{\beta}_{\mathfrak{q},(\epsilon)}(P) + U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_f$. We have $\overline{\beta}_{\mathfrak{q},(\epsilon)}(P) = \sum_\alpha z_\alpha \beta_{(\epsilon)}(Q_\alpha)$. Set $\overline{u} = \sum_\alpha z_\alpha \beta(Q_\alpha)$. Then from Lemma 4.1, the two characters in (25) are well defined and equal. ■

Lemma 4.3. *Let $\mathfrak{g}, \mathfrak{h}, \lambda, \mathfrak{z}$ be as before and suppose $\mathfrak{z} \subset \mathfrak{h}$. Let $\mathfrak{q}$ be such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, and set $\mathfrak{z}' := \mathfrak{z} \cap \ker(\lambda)$. Let $\mathfrak{g}' := \mathfrak{g}/\mathfrak{z}'$, $\mathfrak{h}' := \mathfrak{h}/\mathfrak{z}'$ and let $\mathfrak{q}'$ be such that $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{q}'$. Then*

$$H^0_{(\epsilon)}(\mathfrak{h}_\lambda^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}}) \simeq H^0_{(\epsilon)}((\mathfrak{h}')_\lambda^\perp, \mathrm{d}^{(\epsilon)}_{(\mathfrak{h}')_\lambda^\perp,\mathfrak{q}'}).$$

Proof. Since $\mathfrak{z}' \subset \mathfrak{h}$ and $\mathfrak{q}'$ is such that $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{q}'$, we have $(\mathfrak{q}')^* \simeq \mathfrak{q}^*$. Recall that $\mathfrak{q}^* \simeq \mathfrak{h}^\perp$. Furthermore $(\mathfrak{h}')^\perp := \{X \in \mathfrak{g}/\forall Y \in \mathfrak{h}',\ \lambda([X,Y]) = 0\}$ and since again $\mathfrak{z}' \subset \mathfrak{z}$ and $\mathfrak{h} = \mathfrak{h}' + \mathfrak{z}'$, it is $(\mathfrak{h}')^\perp = \mathfrak{h}^\perp$. Thus there are vector space isomorphisms $\mathfrak{q}^* \overset{\sim}{\leftarrow} (\mathfrak{q}')^* \overset{\sim}{\leftarrow} (\mathfrak{h}')^\perp \overset{\sim}{\rightarrow} \mathfrak{h}^\perp \overset{\sim}{\rightarrow} \mathfrak{q}^*$. Check the differentials $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}}$ and $\mathrm{d}^{(\epsilon)}_{(\mathfrak{h}')_\lambda^\perp,\mathfrak{q}'}$. Since $\mathfrak{h}' = \mathfrak{h}/\mathfrak{z}'$, the graphs in $\mathrm{d}^{(\epsilon)}_{(\mathfrak{h}')_\lambda^\perp,\mathfrak{q}'}$ are the same with the graphs in the differential $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}'}$. By [CT08], a change of complementary space for $\mathfrak{h}_\lambda^\perp$ gives isomorphic reduction algebras so $H^0_{(\epsilon)}(\mathfrak{h}_\lambda^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}'}) \simeq H^0_{(\epsilon)}(\mathfrak{h}_\lambda^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}})$ and $H^0_{(\epsilon)}(\mathfrak{h}_\lambda^\perp, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}}) \simeq H^0_{(\epsilon)}((\mathfrak{h}')_\lambda^\perp, \mathrm{d}^{(\epsilon)}_{(\mathfrak{h}')_\lambda^\perp,\mathfrak{q}'})$. ■

4.2. The main result. We will use the previous Lemmata to prove the following.

Theorem 4.4. *Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra, λ a character of $\mathfrak{h}$. Let $P \in H^0_{(\epsilon=1)}(\mathfrak{h}_\lambda^\perp, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}})$ and $u \in \mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda)$ such that $u = \mathfrak{i}_{(\epsilon=1)}(P)$. Then for generic $f \in \mathfrak{h}_\lambda^\perp$, there is a pair $(\mathfrak{b}_f, \mathfrak{q}_f)$ of a polarization $\mathfrak{b}_f$ of f and a transversal to $\mathfrak{b}_f$ complementary space $\mathfrak{q}_f$ such that*

$$\overline{T}_L^L \circ \overline{\beta}_{\mathfrak{q}_f,(\epsilon)}^{-1}(P)|_{\epsilon=1}(-f) = \lambda^f_{(T=1)}(u). \tag{26}$$

Proof. **First step: The condition $\mathfrak{z} \subset \mathfrak{h}$.** Suppose that the center $\mathfrak{z} \not\subset \mathfrak{h}$ and let again $\mathfrak{k} = \mathfrak{h} + \mathfrak{z}$. Consider the characters of the specialized algebras $\gamma_{CT}^{(\epsilon=1)} : H^0_{(\epsilon=1)}(\mathfrak{h}_\lambda^\perp, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}_\lambda^\perp,\mathfrak{q}_\mathfrak{b}}) \longrightarrow \mathbb{R}$ and $\gamma_{CT}^{\mathfrak{z},(\epsilon=1)} : H^0_{(\epsilon=1)}(\mathfrak{k}_f^\perp, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{k}_f^\perp,V}) \longrightarrow \mathbb{R}$ for which it is $\gamma_{CT}^{(\epsilon=1)}(u) = \gamma_{CT}^{\mathfrak{z},(\epsilon=1)}(\overline{u})$ by Lemmata 4.1 and 4.2. Let $\mu := f|_\mathfrak{k}$ be character of $\mathfrak{k}$. There is an algebra morphism $\delta : \left(U(\mathfrak{g}_T)/U(\mathfrak{g}_T)\mathfrak{h}_\lambda^T\right)^{\mathfrak{h}_T} \longrightarrow \left(U(\mathfrak{g}_T)/U(\mathfrak{g}_T)\mathfrak{k}_\mu^T\right)^{\mathfrak{k}_T}$ and by specialization at $T=1$ we get an algebra morphism

$$\delta_{(T=1)} : \mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) \longrightarrow \mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{k}, \mu) \subset \left(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{k}_\mu\right)^{\mathfrak{k}}.$$

Let now $u \in \mathbb{D}_{(T=1)}(\mathfrak{g},\mathfrak{h},\lambda)$, $\tilde{u}$ be its image in $\mathbb{D}_{(T=1)}(\mathfrak{g},\mathfrak{k},\mu)$ and $\tilde{\lambda}^f_{(T=1)}$: $\mathbb{D}_{(T=1)}(\mathfrak{g},\mathfrak{k},\mu) \longrightarrow \mathbb{R}$ be the character of Theorem 3.3. By Theorems 3.1 and 3.3, it is $\tilde{\lambda}^f_{(T=1)}(\tilde{u}) = \lambda^f_{(T=1)}(u)$. So the constructions and computations from the analytic and biquantization side match. Thus if Theorem 4.4 is true for $\mathfrak{g}$, the subalgebra $\mathfrak{k} = \mathfrak{h} + \mathfrak{z}$, and a polarization $\mathfrak{b}$, then one can take

$$\mathfrak{b}' := \mathfrak{b} \cap (\mathfrak{h} + \mathfrak{z}_{\mathfrak{q}_\mathfrak{b}}) \oplus \mathfrak{b} \cap V = (\mathfrak{b} \cap \mathfrak{h} + \mathfrak{z}_{\mathfrak{q}_\mathfrak{b}}) \oplus \mathfrak{b} \cap V = \mathfrak{b} \cap \mathfrak{h} \oplus \mathfrak{b} \cap (\mathfrak{z}_{\mathfrak{q}_\mathfrak{b}} + V),$$

and prove the same fact for the subalgebra $\mathfrak{h}$ and the polarization $\mathfrak{b}'$ simply by taking for complementary the space $\mathfrak{q} = \mathfrak{z}_{\mathfrak{q}_\mathfrak{b}} + V$ of $\mathfrak{h}$. We can thus suppose $\mathfrak{z} \subset \mathfrak{h}$ without loss of generality.

Second step: The condition $\mathfrak{z} \cap \ker(\lambda) = \{0\}$. By Lemma 4.3, we can suppose that $\mathfrak{z}' = \mathfrak{z} \cap \ker(f) = \{0\}$ with no loss of generality for the computation of the biquantization character. For the analytic character this is also true from Theorems 3.1 and 3.3. Thus we can now proceed to the last step in the proof of Theorem 4.4 supposing that $\mathfrak{z} \subset \mathfrak{h}$ and $\mathfrak{z} \cap \ker(f) = \{0\}$.

The conditions $\mathfrak{z} \subset \mathfrak{h}$ and $\mathfrak{z} \cap \ker(\lambda) = \{0\}$ imply that $\dim(\mathfrak{z}) = 1$. Let $\mathfrak{z} = \mathbb{R}Z$. By [Dix96] § 4.7.7, and in non-trivial cases like $\dim(\mathfrak{g}) = \dim(\mathfrak{z}) = 1$, there are $X, Y \in \mathfrak{g} \cap \ker(f)$ such that $[X,Y] = Z$, $f(Z) = 1$. Set $\mathfrak{g}_0 := \{W \in \mathfrak{g}/[W,Y] = 0\}$. Again from [Dix96] § 4.7.7, $\mathfrak{g}_0$ is a codimension 1 ideal of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}_0 + \langle X \rangle$ and we proceed by induction. For $\dim(\mathfrak{g}) = 1$ the Theorem is true since $\mathfrak{h} = \mathfrak{g} = \mathbb{R}Z$ and so $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h} = \mathbb{R}$. Furthermore, the operators in Theorem (2.7) constructing the character from the biquantization side reduce to the identity, and the character is defined through the evaluation $Z \mapsto f(Z)$, which can be seen in the proof of (2.7). This is also trivially true from the analytic side, through the defined action of $U(\mathfrak{g})$ on α_f in Section 3.1. Suppose then that the Theorem is true for all nilpotent Lie algebras $\mathfrak{g}$ with $\dim(\mathfrak{g}) = n-1$. In particular, for $\mathfrak{g}_0$ we suppose that the induction hypothesis holds for $\mathfrak{g}_0, \mathfrak{h}, \lambda_0 = \lambda|_{\mathfrak{g}_0}$. For the final step of the induction, we distinguish two cases:

Case $\mathfrak{h} \subset \mathfrak{g}_0$. Suppose that $\mathfrak{h} \subset \mathfrak{g}_0$ and let V be such that $\mathfrak{g}_0 = \mathfrak{h} \oplus V$. Take $\mathfrak{q} := V \oplus \langle X \rangle$ as a complementary of $\mathfrak{h}$. By (15) and (23), it is

$$H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_\lambda, V}) = \mathbb{D}_{(T=1)}(\mathfrak{g}_0,\mathfrak{h},\lambda_0) = \mathbb{D}_{(T=1)}(\mathfrak{g},\mathfrak{h},\lambda) = H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_\lambda, \mathfrak{q}}) \quad (27)$$

where the far left algebra is considered as a subalgebra of $S(\mathfrak{g}_0)$ and the far right algebra as a subalgebra of $S(\mathfrak{g})$. Applying the induction hypothesis to $\mathfrak{g}_0, \mathfrak{h}, V$, we thus proved that all algebras are the same in this case of our induction. We prove now that the analytic and biquantization characters also coincide: Since $\mathfrak{b} \subset \mathfrak{g}_0$, $\alpha(f)$ can be considered as a distribution on G_0/B. Relation (27) implies that $H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_\lambda, \mathfrak{q}}) \subset S(V)$ as sets. Since $\mathfrak{h} \subset \mathfrak{g}_0$, $\mathfrak{b} \subset \mathfrak{g}_0$, all the operators and polynomials involved in the computation of γ_{CT} : $H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda, \mathfrak{q}}) \longrightarrow \mathbb{R}[\epsilon]$ are over $\mathfrak{g}_0$. So if $\gamma^{(\epsilon=1)'}_{CT}$: $H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_{\lambda_0}, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_{\lambda_0}, V}) \longrightarrow \mathbb{R}$ is the character for data $\mathfrak{g}_0, \mathfrak{h}, \lambda_0$, we have $\gamma^{(\epsilon=1)'}_{CT}(P) = \gamma^{(\epsilon=1)}_{CT}(P)$ for $P \in H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_\lambda, \mathfrak{q}})$. To conclude, apply the induction hypothesis for $\mathfrak{g}_0, \mathfrak{h}, \lambda_0$.

Case $\mathfrak{h} \not\subset \mathfrak{g}_0$. Set again $\mathfrak{h}_0 := \mathfrak{g}_0 \cap \mathfrak{h}$. Suppose that $\mathfrak{h}_0 \not\subset \mathfrak{h}$ and that the claim holds for $\mathfrak{g}_0, \mathfrak{h}_0, \lambda_0$. Since $\mathfrak{h} \not\subset \mathfrak{g}_0$, it is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{q}$ and there is a $Y \in \mathfrak{h}$ such that $Y \notin \mathfrak{g}_0$ and $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}_0 \oplus \langle Y \rangle$. Since $[\mathfrak{g}_0, \mathfrak{g}] \subset \mathfrak{g}_0$, it is $[Y, \mathfrak{q}] \subset \mathfrak{g}_0$, and $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda, \mathfrak{q}}$ contains all the possible graphs in $\mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_{0\lambda_0}, \mathfrak{q}}$ and additionally those that the variable Y is associated to the edge e_∞. By definition 2.3, one then has $H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda, \mathfrak{q}}) \subset H^0_{(\epsilon)}(\mathfrak{h}^\perp_{0\lambda_0}, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_{0\lambda_0}, \mathfrak{q}})$, and specializing at $\epsilon = 1$ we have $H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_\lambda, \mathfrak{q}}) \subset H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_{0\lambda_0}, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_{0\lambda_0}, \mathfrak{q}})$. We prove now the equality of characters for this induction step. Let $u_T \in \mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda)$, and $u^0_T \in \mathbb{D}_{(T=1)}(\mathfrak{g}_0, \mathfrak{h}_0, \lambda_0)$ be its image, i.e $u_T \equiv u^0_T \mathrm{mod}[U(\mathfrak{g}_T)\mathfrak{h}^T_\lambda]$ as in the first case of the proof of 3.3. Then $\lambda^f_{(T=1)}(u_T) = \lambda^{f_0}_{(T=1)}(u^0_T)$. For the biquantization character, since $\mathfrak{q} \subset \mathfrak{g}_0$ and $\mathfrak{b} \subset \mathfrak{g}_0$, the calculations for $\gamma_{CT} : H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_\lambda, \mathfrak{q}}) \longrightarrow \mathbb{R}[\epsilon]$ take place in $\mathfrak{g}_0$. So if $\gamma''_{CT} : H^0_{(\epsilon)}(\mathfrak{h}^\perp_{0\lambda_0}, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_{0\lambda_0}, \mathfrak{q}_0}) \longrightarrow \mathbb{R}[\epsilon]$, is the character computed for $\mathfrak{g}_0, \mathfrak{h}_0, \lambda_0$, and $\mathfrak{q}_0 \simeq \mathfrak{g}_0/\mathfrak{h}_0$, let $\gamma^{(\epsilon=1)''}_{CT} : H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_{0\lambda_0}, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_{0\lambda_0}, \mathfrak{q}_0}) \longrightarrow \mathbb{R}$ be the character defined on the specialization. Then for $P \in H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, \mathrm{d}^{(\epsilon=1)}_{\mathfrak{h}^\perp_\lambda, \mathfrak{q}}) \subset H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_{0\lambda_0}, \mathrm{d}^{(\epsilon)}_{\mathfrak{h}^\perp_{0\lambda_0}, \mathfrak{q}_0})$, it is $\gamma^{(\epsilon=1)''}_{CT}(P) = \gamma^{(\epsilon=1)}_{CT}(P)$. ■

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